Pricing Parisian-Type Options

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By Tung, Ya-Ching
Graduate Institute of Finance
National Taiwan University
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Abstract

A derivative is a financial instrument whose payoff depends on the underlying assets, such as stocks or futures contracts. As the financial market becomes more prosperous, various new derivatives are designed to fulfill the needs of investors. Some derivatives are complicated in their terms, and give rise to problems in valuation.

A path-dependent option is among all the most complicated derivative in its valuation. The terminal payoff for an option of such type depends critically on the price path of its underlying financial instrument. In this thesis we develop pricing techniques for Parisian type options. These options have path-dependent feature and their terms vary. We focus on the pricing of cumulative Parisian options and consecutive Parisian options.

Tree models are standard approaches to pricing path-dependent derivatives. Few researches focus on the pricing of Parisian type options, and the discussions on the applications of tree models for options of such type is rare. We develop a modified trinomial tree model and adjust the tree lattice to fit the special excursion figure of such options. The numerical convergence behavior is much smoother, and the price obtained from the approach is more accurate. Monte Carlo simulation is used to obtain price approximations for the Parisian type options. It is less efficient than the tree methods in computation time.

Some numerical results are also tested to demonstrate the characteristics of cumulative Parisian options and consecutive Parisian options.

Chapter 1

Introduction

1.1 Setting the Ground

Options are financial contracts that give buyer the right to buy or sell the underlying asset at a certain price on a certain date. Investors pay "premium" for options. There are two basic types of options. A call option gives its buyer the right to buy the underlying asset, and a put option offers the right to sell the underlying asset. The price at which the underlying asset is bought or sold is called **exercise price** or **strike price**, and the date specified on the that the option expires is called **expiration date**. Their payoff is defined as:

$$\max(S_T - X, 0)$$

for a call option, and

$$\max(X - S_T, 0)$$

for a put option. S_T is the underlying asset's price at the expiration date T, and X is the stride price.

These options described above are sometimes called **plain vanilla options**. They are the simplest type of option contracts. They are widely traded financial securities, and depending on the purpose of the contract, their underlying asset could be stock, stock index, futures contract, interest rate, etc. They are **derivative securities** or **derivatives** because their values depend on the underlying asset.

Options that can be exercised only at the expiration date are called **"European,"** whereas those that can be exercised at or before the expiration date (the duration period of an option) are called **"American."**

Other than plain vanilla options, there are many kinds of non-standard options designed to meet a variety of needs in the markets. They are called **exotic options**. **The barrier option** is a very popular type of exotic option. Its payoff depends on whether the price of the underlying asset reaches a certain level at a certain period of time during the duration period of the option. Barrier options can have two forms: knock-out and knock-in. Holder of a knock-out option would see his right to exercise the option terminated if the underlying asset price reaches a certain barrier; holder of a knock-in option receives the right to exercise when underlying asset price reaches a barrier. Figure 1.1 demonstrates two possible stock price process with the same initial stock price $S_0 = 90$. In stock process 2, the stock price goes below 88 dollars during the process, whereas stock process 1 stays above 88 dollars at the time. A knock-out barrier option with a barrier level $H = 88$ would stay alive if its underlying stock price follows process 1 and would be knocked out if the stock price follows process 2.

Barrier option is less expensive than a plain vanilla option with the same parameters, and is therefore attractive to buyers. Take a down-and-out call option as an example. It is one type of barrier option, and would cease to exist if the underlying asset price reaches a certain barrier level, H. H is below the initial stock price. Since the option might cease to exist, the buyer

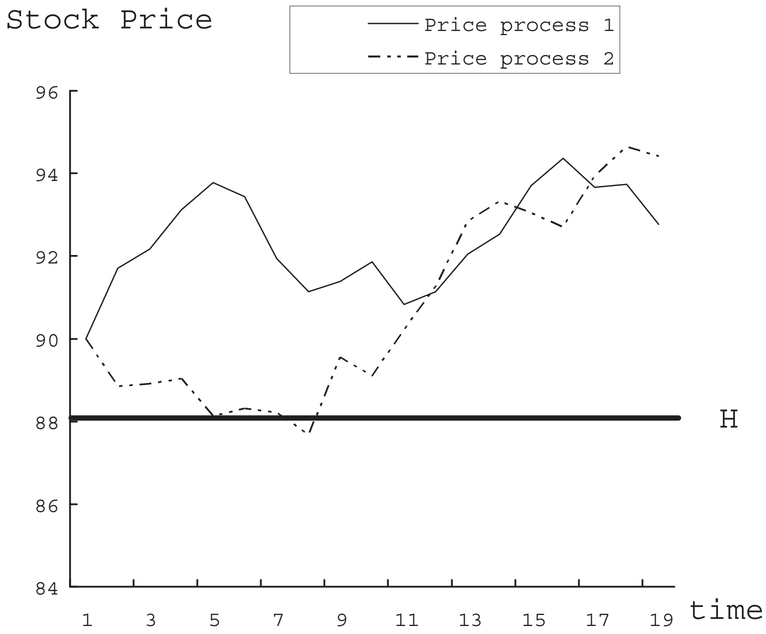

Figure 1.1: Feature of a Barrier Option. The two stock price processes have the same initial stock price $S = 90$. Using the stock price as the underlying asset, a knock-out barrier option with a barrier level $H = 88$ would stay alive under stock process 1. The option would be terminated under stock process 2.

loses his call privilege under this situation, his right is "restricted"; thus the premium must be less than its plain vanilla counterpart. By choosing an option with a proper barrier, buyers do not need to pay for the situation that they believe the stock price would not reach and are still able to meet their hedging needs.

The barrier option has a major drawback: price manipulation is possible around the barrier level. Consider the down-and-out feature again. Since the option is knocked out once the stock price touches the barrier, when the stock price is very close to the barrier, it may be pulled downward deliberately to activate the knock-out feature. Define the **delta** of an option, Δ, be the rate of change of the option price with respect to the price of the underlying asset, and the **gamma** of an option, Γ be the rate of change of the option's delta with respect to the price of the underlying asset. We have:

$$\Delta = \frac{\partial P}{\partial S},$$

and

$$\Gamma = \frac{\partial^2 P}{\partial S^2}.$$

P and S stand for the option price and underlying stock price. Notice that the delta of the option would be discontinuous around the barrier, and the gamma would approach infinity. Hedging therefore would be difficult.

Derivatives with various modifications have been introduced to address the flaw of barrier options. **Parisian type options** are one of such kinds that are widely practiced in the OTC market. This is an option whose value depends on whether the underlying asset reaches or goes beyond a certain price level (the excursion level) within a certain time interval for a pre-determined period of time. Like the barrier option, it can have the form of a down-and-out, up-and-out, down-and-in, up-and-in call or put. It contains a great variety of features, and can be adapted to different needs. For example, a down-and-out Parisian option will have a premium lower than a plain vanilla option but higher than a barrier option with the same parameters. Because the stock price would have to stay on or below the excursion level for a period of time rather than just touch it, the chance of being knocked out is lower than the barrier one; therefore, a higher premium is charged. Also, since more than mere touching is required to activate the knock-out feature, price manipulation is harder.

This thesis will focus on the pricing of two Parisian type options: cumulative Parisian option and consecutive Parisian option. The former's value depends on the cumulative time the underlying asset's price spends beyond the excursion level, and the latter's depends on the consecutive time its underlying asset's price spends. These type of options are **"path-dependent"** since their values rely critically on the price path of the underlying asset. The pricing for options with path-dependent feature could be difficult since their value depend not only on the terminal value of the underlying assets, but also on the assets' prices path during the duration period of the options. We will use the tree model to price the options. This method can capture the path-dependent feature of such options, and when analytical solutions are difficult to obtain, the tree model is still able to find approximation for the prices. To handle the Parisian feature, we use trinomial trees instead of the commonly practiced binomial trees and introduce a stretch parameter discussed by Ritchken (1995) in the process. As to the period that is required to stimulate knock-out or knock-in, or the **excursion period,** since the tree model is a discrete approximation of the asset price process, when the time to expiration is divided into some small interval Δt, the excursion period may not contain an integer number of interval Δt, and will result in unsmooth convergence of the option value. In this thesis, we introduce a new technique to adjust the lattice of the tree to solve the problem. Our model is a generalization of Kwok and Lau (2001) and can be used to price Parisian type contracts with different excursion periods.

1.2 Survey of Literature

Several researches address the pricing of Parisian options. Analytical, quasi-analytical, or numerical approaches have been used. Chesney, Jeanblanc-Picque, and Yor (1997) use the theory of Brownian excursions and define the

Parisian option value in terms of an integral expressed as an inverse Laplace transform. Fusai, and Tagliani (2001) use different numerical methods, including the multidimensional inverse Laplace transform, finite difference partial differential equation (PDE) solution, and Monte Carlo simulation, to price occupation time derivatives. They examine the effect of continuous and discrete monitoring of the underlying asset. Hugonnier (1999) obtains closed-form formulae for the cumulative Parisian option that is monitored continuously. Haber, Schonbucher, and Wilmott (1999) develop PDE for continuously monitored consecutive and cumulative options and provide finite difference approach to evaluate them.

Tree-based algorithms are popular in pricing options with path-dependence characteristics. Babbs (2000) presents a straightforward binomial approximation scheme to evaluate lookback options. It also allows for the pricing of American versions. Hsueh and Liu (2002) propose the step-reset option and offer a trinomial tree model to account for the discrete nature of reset monitoring. Tian (1999) develops a flexible binomial model with a tilt parameter. The model calibrates nodes on a binomial tree to fit the features of different contracts and improve the rate of convergence.

1.3 Thesis Structure

The remainder of this thesis will be organized as follows. Chapter 2 reviews the pricing technologies and option features. Chapter 3 covers the pricing of cumulative Parisian options and consecutive Parisian options. The numerical results are presented in chapter 4. Chapter 5 concludes.

Chapter 2

Fundamental Concepts

In this chapter, we review the background concepts and numerical pricing techniques.

2.1 Option Pricing Preliminaries

Options or other derivatives are financial products whose value depend on the price of the underlying asset. Therefore, a good description of the underlying asset's price process during the life of a option is critical to its valuation. We describe a widely used model for stock price process. It can be applied to the process of other assets with similar behavior. Also, when evaluating the present value of an option that expires at a future date, one needs to find the appropriate discount factor. We introduce the concept of risk-neutral valuation. Under a risk-neutral world, one can find the present value of a financial product by simply using the risk-free interest rate to discount it.

2.1.1 The Process for Stock Prices

We first state the Markov property and Itô's Lemma below without proof. A definition of standard Brownian motion is also given here. They are very useful in the modelling of stock price process.

Lemma 1 (The Markov Property) *For any time t, let $f : Z^{T-t+1} \to \mathcal{R}$ be arbitrary. Then there exists a fixed function $g{:}Z \to \mathcal{R}$ such that for any i in Z,*

$$E^i[f(X_t, ..., X_T)|\mathcal{F}_t] = E^i[f(X_t, ..., X_T)|X_t] = g(X_t),$$

where E^i denotes expectation under P_i.

A stochastic process that has the Markov property is called a Markov process.

Definition 1 (Standard Brownian Motion) *In a probability space $(\Omega, \mathcal{F}, P)$, a process is a measurable function on $\Omega \times [0, \infty)$ into $\mathcal{R}$. A standard Brownian motion B is a process defined by the following properties:*

(a) $B(0) = 0$ *almost surely;*

(b) *for any times t and $s > t$, $B(s) - B(t)$ is normally distributed with mean zero and variance $s - t$;*

(c) *for any times $t_0, \ldots, t_n$ such that $0 \leq t_0 < t_1 < \cdots < t_n < \infty$, the random variables $B(t_0), B(t_1) - B(t_0), \ldots, B(t_n) - B(t_{n-1})$ are independently distributed; and*

(d) *for each w in Ω, the sample path $t \to B(w, t)$ is continuous.*

Lemma 2 (Itô's Lemma) *Suppose x is an Itô process with $dx(t) = \mu(x(t), t)\,dt + \sigma(x(t), t)\,dB(t)$, and let $f : \mathcal{R}^2 \to \mathcal{R}$ be twice continuously differentiable.*

Then the process Y defined by $Y(t) = f(x(t), t)$ is an Itô process with

$$dY(t) = \left(\frac{\partial Y(t)}{\partial x(t)}\mu(x(t), t) + \frac{\partial Y(t)}{\partial t} + \frac{1}{2}\frac{\partial^2 Y(t)}{\partial x^2(t)}\sigma^2(x(t), t)\right)dt + \frac{\partial Y(t)}{\partial x}\sigma(x(t), t)\,dB(t).$$

Stock price is usually assumed to follow a Markov process. It implies that the market is efficient in the weak form. That is to say, all past information is reflected in the current stock price, and no arbitrage opportunities would exist.

An appropriate model for the stock price process must capture the following properties. First, the expected rate of return required by an investor must be independent of the current stock price. Therefore for some constant parameter μ, a stock that worth S at time t must have an increment of value by $\mu S\Delta t$ after a small interval of time Δt, where μ is the expected rate of return to the stock. The model can be illustrated as follows:

$$\Delta S = \mu S \Delta t.$$

As Δt approaches zero, the model can be written as

$$\mathrm{d}S = \mu S \mathrm{dt}.$$

Therefore

$$S_t = S_0 \exp(\mu t),$$

where S_t and S_0 are the stock price at time t and time zero.

Second, the variability of the expected rate of return over a small time interval Δt must be the same regardless of the current stock price. Thus the volatility (or the standard deviation) of the stock price change over a short period of time Δt must be proportional to the price itself. Define σ as the stock price volatility. We have the model

$$\mathrm{d}S = \mu S \mathrm{dt} + \sigma S \mathrm{d}B. \tag{2.1}$$

This model captures the two characteristics of the stock price return rate, and is most widely used for the modelling of price process. The process is called a geometric Brownian motion.

In stead of a normal distribution, stock price is usually modelled to follow a log-normal distribution. It is because a variable that has a log-normal distribution takes value between zero and infinity. A variable that follows normal distribution can have value from negative infinity to positive infinity, which is not consistent with the real world stock price behavior. The log-normal distribution is more appropriate.

Assume the stock price process follows the geometric Brownian motion in equation (2.1). Let $Y = \ln S$. We can derive the process of $\ln S$ by Itô's lemma, thus

$$\ln S = \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma \mathrm{d}B.$$

Since the drift rate $\mu - \sigma^2/2$ and variance rate σ^2 are both constant, it also follows a geometric Brownian motion. The change in $\ln S$ between time zero and t is normally distributed, where

$$\ln S_t - \ln S_0 \sim \Phi\left[\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma\sqrt{t}\right].$$

Thus we know that S_t follows a log-normal distribution. The stock price process becomes

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B\right]. \tag{2.2}$$

2.1.2 Risk-Neutral Valuation

We state the definition of an *equivalent martingale measure* and a resulting theorem without proof.

Definition 2 (Equivalent Martingale Measure) *Define a new probability measure Q to be equivalent to P if Q and P assign zero probabilities to the same events. A probability measure Q is equivalent to P is an equivalent martingale measure for (δ, S) if and only if $S_T = 0$ and the discounted gain process G^r is a martingale with respect to Q.*

Here, δ is the dividend process $\delta = (\delta^1, \ldots, \delta^N)$ for N securities, and $S = (S^1, \ldots, S^N)$ is their adapted price process. G is the gain process for (δ, S), G^r is the deflated gain process under a deflator r.

Theorem 1 (The Equivalent Martingale Measure Result) *There is no arbitrage if and only if there exists an equivalent martingale measure. Moreover, π is a state-price deflator if and only if an equivalent martingale measure Q has the density process ξ defined by $\xi_t = E_t(\xi_T) = R_{0,t}\pi_t/\pi_0$, where π is a state-price deflator, $R_{0,t}$ the payback at time t of one unit of account borrowed risklessly at time 0 and rolled over in short-term borrowing repeatedly until date t.*

Define a dollar money market account D that worth \$1 at time zero and earns a instantaneous risk-free rate r at any given time in the future. Its process can be written as follows:

$$\mathrm{d}D = rD\mathrm{d}t$$

The equivalent martingale measure result shows that, when there is no arbitrage opportunity, for a given numeraire security D, $\theta = f/D$ is a martingale for all securities f if the market price of risk is set equal to the volatility of D. Since the market price of risk for a dollar money market account D is zero, the world that used D as numeraire has zero market price of risk, and is referred to as the traditional risk-neutral world.

Under the traditional risk-neutral world, f/D is a martingale and therefore has the following property:

$$\frac{f_0}{D_0} = \mathrm{E}^Q\left(\frac{f_T}{D_T}\right)$$

or

$$f_0 = D_0 \mathrm{E}^Q\left(\frac{f_T}{D_T}\right) \tag{2.3}$$

The short-term interest rate r can be stochastic, but here we will assume it to be a constant r. Thus we have

$$D_T = e^{rT}.$$

Use the above result, and the fact that $D_0 = 1$, equation (2.3) is reduced to

$$\begin{aligned} f_0 &= \mathrm{E}^Q(e^{-rT} f_T) \\ &= e^{-rT}\mathrm{E}^Q(f_T). \end{aligned} \tag{2.4}$$

From equation (2.4), we can conclude that under a traditional risk-neutral world, the discount factor used to price a security that mature at some future time t is the market risk-free rate r.

2.2 Tree Model

Tree models are widely practiced numerical approaches to price options. The main idea of a tree model is to simulate the stock price process in a discrete-time and discrete-state version, and calculate the value of options underlain by the stock. In this thesis the trinomial version of tree models will be used as the valuation tool.

We assume the underlying stock price follows a geometric Brownian motion $S = e^X$ so that $\mathrm{d}S = \mu S\mathrm{d}t + \sigma\mathrm{d}B$. From equation (2.2) we know $X = \ln S$ is normally distributed with parameters $(\mu - \sigma^2/2, \sigma)$; thus S follows a log-normal distribution. Under the traditional risk-neutral world, the expected drift rate μ is equal to the risk-free interest rate r. The trinomial approximation of the process can be described as follows. Let the current time be time zero, and τ be the time to maturity of an option. Divide τ into n period such that $\Delta t = \tau/n$. Let S_t be the stock price at time $t, 0 \leq t \leq \tau$, then for a small time interval Δt, we have

$$\ln \frac{S_{t+\Delta t}}{S_t} \sim \Phi\Big[\Big(\mu - \frac{\sigma^2}{2}\Big)\Delta t, \sigma\Delta t\Big].$$

At any time t, $0 \leq t \leq \tau$, the stock price is assumed to have three possible drift directions. It can either move up by u, stay unchanged, or move down by d after Δt time period. The point on a tree that demonstrates a possible stock price at some time t is called a **node** of the tree. Therefore we have node (i, j) represent the j^{th} possible stock price at time $i\Delta t$, where $j = 0, 1, \ldots, 2i$ for any time $i\Delta t$. See Figure 2.1 for a two-period trinomial tree.

Let $ud = 1$ and p_u, p_m, p_d be the probabilities of an up, middle, down movements of stock price from time t to time $t + \Delta t$. The three possible stock price after time Δt would be

$$S_{t+\Delta t} = \begin{cases} S_t u & \text{with probability } p_u \\ S_t & \text{with probability } p_m \\ S_t d & \text{with probability } p_d. \end{cases}$$

From equation (2.2) and properties of the log-normal distribution, we have

$$\mathrm{E}(S_{t+\Delta t}) = S_t e^{r\Delta t},$$

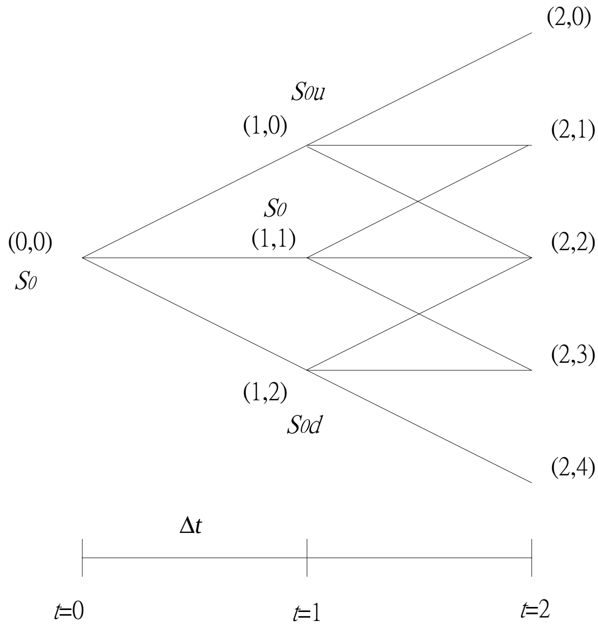

Figure 2.1: A Two-Period Trinomial Tree.

and

$$\mathrm{Var}(S_{t+\Delta t}) = S_t^2 e^{2r\Delta t}(e^{\sigma^2 \Delta t} - 1).$$

Now we let the trinomial model match the mean and variance of the stock price at time $t+\Delta t$. Also, sum of the probabilities must be equal to 1. Thus we have

$$\begin{aligned} 1 &= p_u + p_m + p_d, \\ \mathrm{E}(S) &= S\Big(p_u u + p_m + p_d d\Big), \\ \mathrm{Var}(S) &= p_u\Big(S_t u - \mathrm{E}(S)\Big)^2 + p_m\Big(S_t - \mathrm{E}(S)\Big)^2 + p_d\Big(S_t d - \mathrm{E}(S)\Big)^2. \end{aligned}$$

The probabilities can be derived as follows:

$$\begin{aligned} p_u &= \frac{u(V + M^2 - M) - (M-1)}{(u-1)(u^2-1)}, \\ p_d &= \frac{u^2(V + M^2 - M) - u^3(M-1)}{(u-1)(u^2-1)}, \end{aligned}$$

$$p_m \;=\; 1 - p_u - p_d,$$

where $M \equiv e^{r\Delta t}$ and $V \equiv M^2(e^{\sigma^2 \Delta t} - 1)$.

We need to make sure that the probabilities lie between zero and one. Use

$$u = e^{\lambda\sigma\sqrt{\Delta t}},$$

where $\lambda \geq 1$ is a parameter to adjust the tree lattice.

As the time interval Δt becomes smaller, that is, as n becomes larger, a trinomial tree will become a better approximation to the real stock price process. When a trinomial tree model is used to price options, satisfactory numerical results will be obtained as n becomes sufficiently large.

2.3 Tree Model for Option with Barrier Feature

A tree model can be used to illustrate stock price process and evaluate option price. When the model is used to price options with the barrier features, like the Parisian options, some problems will arise. Since we cannot assure the barrier to lie exactly on a layer of nodes on a tree, convergence will be very slow when a tree algorithm is used to price option of this type. The results will be significantly erractic even after a large number of time steps n.

Ritchken (1995) suggests a good way to solve the problem. By adjusting the value of λ, we allow the barrier to be set exactly on a layer of nodes in a trinomial tree.

Consider a downward barrier of a knock-out (knock-in) option for instance. The idea is to take

$$h = \frac{\ln(S/H)}{\lambda\sigma\sqrt{\Delta t}}$$

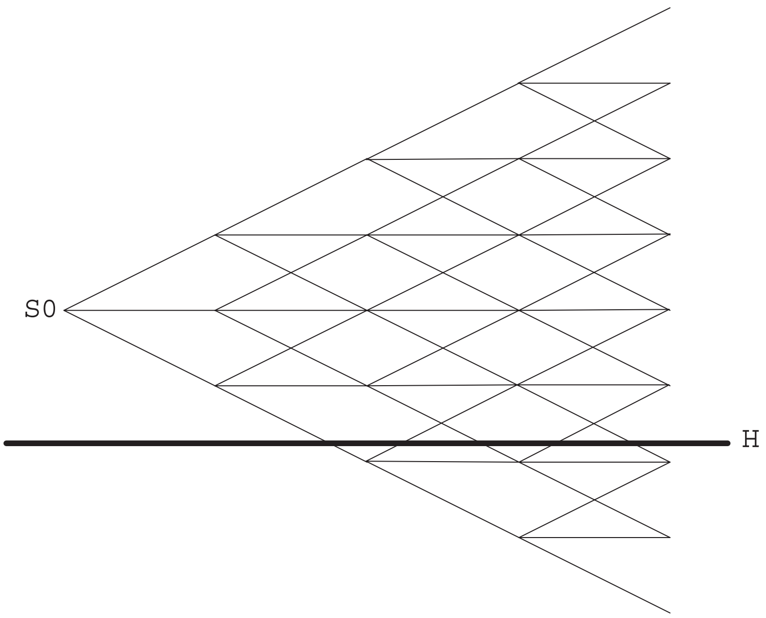

Figure 2.2: TRINOMIAL TREE NOT ADJUSTED FOR THE BARRIER FEATURE. H is the barrier of an option, S_0 is the initial stock price.

consecutive down moves to go from S to H if h is an integer. Here, H is the barrier of an option, and S is the current stock price and is above the barrier level. The idea is easy to achieve by adjusting λ, and find the smallest $\lambda \geq 1$ such that h is an integer, that is,

$$\lambda = \max_{j=1,2,3,\dots} \frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}}.$$

Note λ must be greater than one so that the probabilities would not turn to negative. Also when $\lambda = 1$, p_m becomes zero and the trinomial tree reduces to a binomial tree. A binomial tree is a tree model that allows the stock price to move upward or downward after a time step.

Match the mean and variance of the stock price process, together with the third condition that $p_u + p_m + p_d = 1$, we approximate the three probabilities:

$$\begin{aligned} p_u &= \frac{1}{2\lambda^2} + \frac{\mu\sqrt{\Delta t}}{2\lambda\sigma}, \\ p_m &= 1 - \frac{1}{\lambda^2}, \end{aligned}$$

$$p_d = \frac{1}{2\lambda^2} - \frac{\mu\sqrt{\Delta t}}{2\lambda\sigma},$$

where $\mu \equiv r - q - \sigma^2/2$, q is the continuous dividend rate of S.

The adjusted tree is demonstrated in figure 2.3.

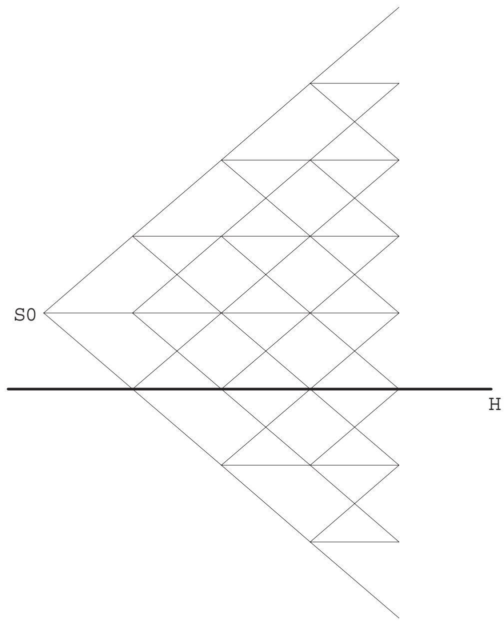

Figure 2.3: THE ADJUSTED TRINOMIAL TREE. H is the barrier of an option, S_0 is the initial stock price.

2.4 Auxiliary State Vector

In order to price an exotic path-dependent derivative, we need to capture the specific path-dependent feature of the option. An auxiliary state vector is used to record the characteristics of an option at each lattice node in a tree model. This method is especially appreciable when a derivative's partial differential equation is difficult to derive. Numerical methods such as the finite-difference algorithms, which require one to deal with the PDE function, are not applicable in such cases. In this section we draw on Kwok, and Lau (2001).

Consider a path-dependent option. Let F be the path-dependent variable associated with the contract, V denote the auxiliary state vector function between F and the underlying stock price S over a small time interval Δt. Therefore we have:

$$F_{t+\Delta t} = V(F_t, S_{t+\Delta t}), t = 0, 1, 2, \ldots, \tau.$$

Define (i, j) as the lattice node on a trinomial tree that corresponds to the i^{th} time step starting from time zero and the j^{th} stock price at time $i\Delta t$, $j = 0, 1, 2, \ldots, 2i$. The time step length is equal to Δt, and the stock price difference is ΔX, where $X = \ln S$. Let $P(i, j, k)$ denote option value at the $(i, j)^{th}$ node at which k is an integer that records the value of F at the node. The probabilities of an upward, middle or downward stock price movement are as described in the previous section.

The auxiliary state vector determines values of k in a forward way. At time t, the value of k at time $t-1$ together with the current stock price are used to decide the k value at time t. Option price $P(i, j, k)$ is then calculated using backward induction. The algorithm can be represented abstractly as follows:

$$\begin{aligned} P(i-1, j, k) = \quad & \Big\{ \quad p_u \times P\Big(i, j, V(k, j)\Big) \\ & + \quad p_m \times P\Big(i, j+1, V(k, j+1)\Big) \\ & + \quad p_d \times P\Big(i, j+2, V(k, j+2)\Big)\Big\} \times e^{-r\Delta t}, \end{aligned} \tag{2.5}$$

where $e^{-r\Delta t}$ is the discount factor under the risk-neutral world.

From algorithm (2.5) we note that once the auxiliary state vector function $V(k, j)$ is determined, the option value can be derived through the algorithm. We state some examples on how to determine the $V(k, j)$ function.

Cumulative Parisian Options

First we define the cumulative Parisian option. A cumulative Parisian option has the feature of a regular plain vanilla option but adds an extra clause to the determination of its price. It consider a predetermined period of time B— the excursion period. Price of the underlying stock has to stay beyond a certain excursion level H for B period of time cumulatively to initiate or terminate the option. Take a down-and-out cumulative Parisian call for instance. The auxiliary state vector $V(k, j)$ is the function to decide the cumulative time the stock price has spent below the excursion level H. At the expiration date, if the cumulated excursion time is less than the required period B, the option value is equal to $\max(S_\tau - X, 0)$. If the cumulated excursion period is greater or equal to B, the option ceases to exist.

In a trinomial tree model, the time to maturity of an option is divide into n discrete periods of time Δt; therefore, the excursion period B is divided into

$$N = \left[\frac{B}{\Delta t}\right]$$

time periods. $V(k, j)$ will now become a function to calculate the number of Δt intervals the stock price stays beyond the excursion level at time $i\Delta t$. Define $P_{cum}(i, j, k)$ as the value of a cumulative Parisian option at the $(i, j)^{th}$ node. Integer k is an integer that records the time period stock price has spent below H before or at time $i\Delta t$.

At each monitoring instant, the stock price is observed to see if it is below the excursion level. When S moves on or below H, the value of k will be increased by 1. Define $1_{\{S \le H\}}$ to be an indicator function:

$$1_{\{S \le H\}} = \begin{cases} 1 & \text{if } S \le H \\ 0 & \text{if } S > H. \end{cases}$$

The auxiliary state vector function $V_{cum}(k, j)$ for a cumulative Parisian option therefore will be

$$V_{cum}(k, j) = k + 1_{\{S \leq H\}}.$$

The trinomial tree algorithm for a cumulative Parisian option at the monitoring instance is as follows:

$$\begin{aligned} P_{cum}(i-1, j, k) = & \Big\{ \ p_u \times P_{cum}\Big(i, j, V_{cum}(k, j)\Big) \\ & + \ p_m \times P_{cum}\Big(i, j+1, V_{cum}(k, j+1)\Big) \\ & + \ p_d \times P_{cum}\Big(i, j+2, V_{cum}(k, j+2)\Big) \Big\} \times e^{-r\Delta t}. \end{aligned} \tag{2.6}$$

Consecutive Parisian Options

A consecutive Parisian option is defined slightly differently from its cumulative counterpart. It considers the consecutive time the price of the underlying stock spends beyond a certain excursion level H. The auxiliary state vector function for a consecutive Parisian option is used to determine the length of its consecutive excursion period. Again we consider a down-and-out consecutive Parisian call option. The required excursion period B is divided into N time intervals in a trinomial tree model, as described before. Function $V_{con}(k, j)$ is used to calculate the consecutive time periods the stock price spends below H at or before time $i\Delta t$. Integer k is an integer to record the time periods stock price has spent below H at the time.

At each monitoring instant, the stock price is observed. The value of k will be increased by 1 if stock price stays below H, and reset to 0 if the price moves above the excursion level. The auxiliary state vector for a consecutive Parisian option can be demonstrated to be

$$V_{con}(k, j) = (k + 1)1_{\{S \leq H\}},$$

where $1_{\{S \leq H\}}$ is an indicator function. The trinomial tree algorithm for such an option is similar to algorithm (2.6), but the state vector function is replaced by $V_{con}(k, j)$.

Chapter 3

Pricing Parisian Options

3.1 Pricing Cumulative Parisian Options

In this section we investigate the valuation of cumulative Parisian options. First we define the contract, then a trinomial tree as described in the previous chapter will be used to do the valuation.

A cumulative Parisian option is designed to have a knock-out or knock-in provision. For a knock-out feature, if the underlying stock price goes beyond a certain price level cumulatively for a pre-determined period of time B, the option ceases to exist and has a zero value. Otherwise, it stays alive, and the terminal value will be $\max(S_\tau - X, 0)$ for a call or $\max(X - S_\tau, 0)$ for a put. B is called the **excursion period.** When the length of B approaches zero, the contract reduce to an ordinary barrier option. When the length of B approaches the duration period τ of the derivative, the contract turns into a plain vanilla option. For a knock-in feature, the option comes into existence when the underlying stock price stays beyond a certain price level cumulatively for a pre-determined period of time.

The difficulties of Parisian option valuation lies in the path-dependent feature of the instrument, and the problem of its excursion period. The former can be solved by using an auxiliary state vector. The latter is more complicated. We propose a binomial branching technique to modify the original trinomial tree. The method can reduce the unsmooth convergence behavior of option prices and allows for the pricing of contracts with general excursion periods.

Consider a down-and-out cumulative Parisian call for example. A trinomial tree is used to price the option. On a trinomial tree, the calculation is operated in a backward fashion. First we set $k = N - 1$ for all the stock price nodes at time $\tau - \Delta t$. Then the option price $P_{cum}(\tau - \Delta t, j, N - 1)$ for all j can be derived from option prices at time τ. Consider nodes (τ, j), $(\tau, j + 1)$, and $(\tau, j + 2)$. We investigate the stock price at each of these nodes. If the stock price is below or at the excursion level H, integer k would be incremented by 1; otherwise, it stays unchanged. Since $P_{cum}(i, j, N) = 0$ for all node (i, j) and $P_{cum}(i, j, N - 1) = \max(S - X, 0)$ at time τ, we set $k = N - 1$ in equation (2.6) and obtain $P_{cum}(\tau - \Delta t)$ in a backward fashion. Then we move on to time $\tau - 2\Delta t$, and proceed with the same operations. The process is repeated until we find $P_{cum}(0, 0, N - 1)$. This way we generate the whole option price trinomial tree with $k = N - 1$. This option price tree is then used to derive the option price tree with $k = N - 2$. The process continues until the integer k hits 0, and the value of $P_{cum}(0, 0, 0)$ is obtained. Discounting the price with the market risk-free rate will yield the current value of the cumulative Parisian option.

Now we consider problems with the excursion period. A trinomial tree model approximates the option value in a discrete way. When the time to maturity τ of an option is divided into n sub-intervals, the excursion period

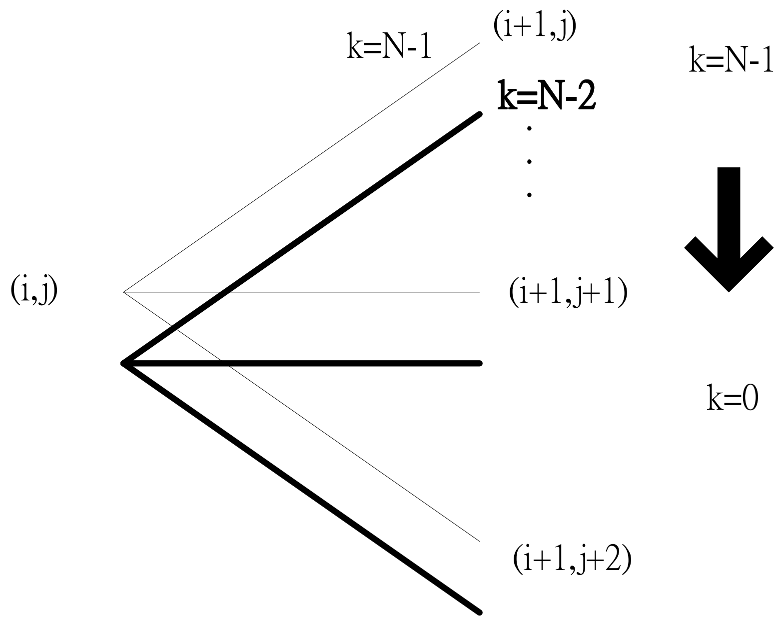

Figure 3.1: Cumulative Parisian Option Pricing Algorithm.

would be divided into

$$N = \lceil \frac{B}{\Delta t} \rceil$$

sub-intervals, where $\Delta t = \tau / n$. The the last sub-interval of the excursion period may not have a full interval length. When the original trinomial tree is used to calculate the option value, the problem of unsmooth convergence will occur. We impose binomial branches on the trinomial tree at the final fractional interval to adjust for the nodes.

Figure (3.2) demonstrates the tree lattice at time $t = i\Delta t$, when we have $N - 1$ cumulative excursion periods. Consider node O and assume it lies on the excursion level H. Let the length of the fractional interval be ε,

$$\epsilon = N - \left(\frac{B}{\Delta t} \right).$$

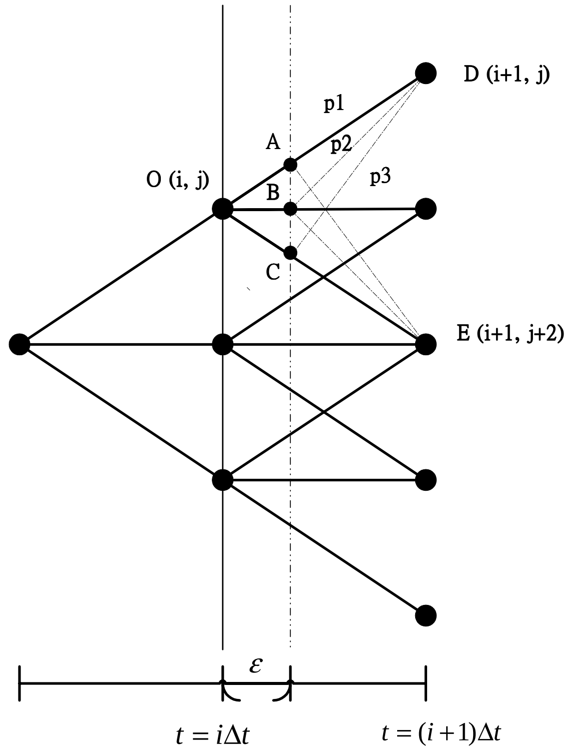

Figure 3.2: TRINOMIAL TREE WHEN $k = N - 1$.

Node A, node B, and node C represent three possible directions of the stock price movement after ε period of time. For a down-and-out feature, if the stock price moves up to node A, the option continues to exist, and the state integer k remains equal to $N - 1$. If the stock price moves to node B or C, $k = N$ and the contract would be terminated. The probability for the stock price to move from node O to node A, B, or C are

$$\begin{aligned} p_A &= \frac{1}{2\lambda^2} + \frac{\mu\sqrt{\varepsilon}}{2\lambda\sigma}, \\ p_B &= 1 - \frac{1}{\lambda^2}, \\ p_C &= \frac{1}{2\lambda^2} - \frac{\mu\sqrt{\varepsilon}}{2\lambda\sigma}, \end{aligned}$$

respectively.

Consider node A. Using the binomial branching technique, the stock price may move to node D or E at the next time period. After matching the mean and variance of stock price at time $(i+1)\Delta t$, the probability p_1 for stock price to move from node A to node D will be:

$$\begin{aligned} p_1 &= \frac{e^{r(\Delta t-\epsilon)} - d_1}{u_1 - d_1}, \\ u_1 &= e^{\lambda\sigma(\sqrt{\Delta t}-\sqrt{\epsilon})}, \\ d_1 &= e^{-\lambda\sigma(\sqrt{\Delta t}+\sqrt{\epsilon})}. \end{aligned}$$

The probability to move from node A to node E is $1-p_1$. With similar calculation we can find the probability for stock price to move from node B to node D be p_2, where

$$\begin{aligned} p_2 &= \frac{e^{r(\Delta t-\epsilon)} - d_2}{u_2 - d_2}, \\ u_2 &= e^{\lambda\sigma\sqrt{\Delta t-\epsilon}}, \\ d_2 &= e^{-\lambda\sigma\sqrt{\Delta t-\epsilon}}. \end{aligned}$$

The probability to move from node B to node E is $1-p_2$. The probability to move from node C to node D is p_3, where

$$\begin{aligned} p_3 &= \frac{e^{r(\Delta t-\epsilon)} - d_3}{u_3 - d_3}, \\ u_3 &= e^{\lambda\sigma(\sqrt{\Delta t}+\sqrt{\epsilon})}, \\ d_3 &= e^{-\lambda\sigma(\sqrt{\Delta t}-\sqrt{\epsilon})}. \end{aligned}$$

The probability to move from node C to node E is $1-p_3$.

Since the stock price has stayed below H for $N-1$ times cumulatively at or before node O, at time $i+\epsilon$, the option will stay alive at node A, but be terminated at node B or C. Backward induction is used to derive the option value at node A from node D and E. The algorithm is described below:

$$P_{cum}(i+\epsilon, j, N-1) = \left\{ \quad p_1 \times P_{cum}\Big(i+1, j, V_{cum}(N-1, j)\Big) \right.$$

$$
\begin{aligned}
&+ \quad (1-p_1) \times P_{cum}\Big(i+1, j+2, V_{cum}(N-1, j+2)\Big)\Big\} \\
&\times \quad e^{-r(\Delta t - \epsilon)}. \qquad (3.1)
\end{aligned}
$$

If the stock price moves from node O to node B or C, the option price would be zero. Finally, the option value at node O is derived from nodes A, B, and C, using equation (2.6). This algorithm reduces the sawtooth pattern incurred by the fractional interval, and option values will converge more smoothly with the technique.

The algorithm will use up to $O(n^2)$ memory space and $O(n^2)$ computational steps.

3.2 Pricing Consecutive Parisian Options

A consecutive Parisian option has a knock-out or knock-in feature slightly different from its cumulative counterpart. It calculates the time underlying stock price stays beyond the excursion level consecutively. For the otherwise identical parameters, a consecutive Parisian option requires a higher premium than a cumulative Parisian option because it is more difficult to activate the knock-out or knock-in feature of a consecutive Parisian option.

A consecutive Parisian option is path-dependent. We use the trinomial tree model with the auxiliary state vector to do the pricing.

To determine the option value on a trinomial tree, consider a down-and-out consecutive Parisian call. Let $P_{con}(i, j, k)$ be the option value at the $(i, j)^{th}$ node. Integer k stands for the consecutive number of periods the stock price has spent below the excursion level H. Since the integer variable k will be reset to zero whenever stock price moves above the excursion level, we need to know $P_{con}(i, j, 0)$ at all nodes (i, j) before backward induction is proceeded. Set $k = N - 1$ for all price nodes at time $\tau - \Delta t$. At time

τ, if the stock price is above the excursion level, k is reset to zero. If the stock price is at or below the excursion level, k is incremented by 1, and we have $k = N$. Option prices at nodes (τ, j), $(\tau, j + 1)$, and $(\tau, j + 2)$ are then used to calculate $P_{con}(i, j, N - 1)$. Since $P_{con}(i, j, N) = 0$ and $P_{con}(i, j, 0) = \max(S - X, 0)$ at time τ, the option values at node (τ, j), $(\tau, j+1)$, and $(\tau, j+2)$ can be determined. Backward induction is then used to determine $P_{con}(i, j, N-1)$. The same operations are repeated at time $\tau - \Delta t$, with $k = N - 2, k = N - 3, \ldots, k = 0$, until values of $P_{con}(\tau - \Delta t, j, 0)$ are found for all j. A cross sectional price lattice at time $\tau - \Delta t$ will all k values is generated. We then move one time period ahead, and repeat the same operations to find the price lattice at time $\tau - 2\Delta t$. The process continues until we move to time 0, and find $P_{con}(0, 0, 0)$. $P_{con}(0, 0, 0)$ is then discounted with the market risk-free rate to yield the current value of the consecutive Parisian option.

The excursion interval B may contain a fractional interval of length less than Δt, when the time to maturity of an option is divided into τ/n small intervals. The binomial branching technique described in the previous section is used to eliminate the problem.

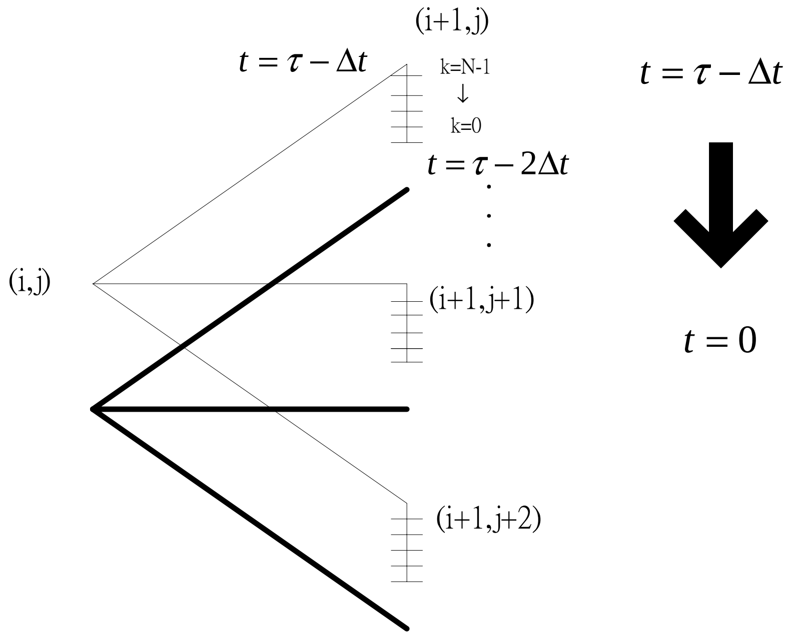

Figure 3.3: Consecutive Parisian Option Pricing Algorithm.

Chapter 4

Numerical Evaluation

A cumulative Parisian or a consecutive Parisian option whose excursion period contains a fractional interval will result in unsmooth convergence of the option value. This section investigates the numerical results of our model. First we compare the convergence behavior of option values obtained from the regular trinomial and those from the adjusted tree. Then we examine the characteristics of cumulative Parisian options and consecutive Parisian options.

In Figure 4.1 we demonstrate the convergence behavior of a down-and-out cumulative Parisian call before and after we adjust for the fractional interval of the excursion period. When the excursion period contains non-integer periods of time Δt, we round up to the greatest smaller integer periods of time Δt, and use it to derive the lower bounds for option values. The smallest greater integer periods of time Δt is used to derive the upper bounds for option values. The contract used here has one year maturity τ and one month excursion period B. The option parameters are as follows: $S = 95, X = 100, H = 90, r = 10\%, \sigma = 25\%, \tau = 1$ (year), and $B = 1$ (month). It is shown that the values obtained from the adjusted trinomial tree is bounded by the lower and upper bounds. The adjusted trinomial tree algorithm can be used to price the options more accurately, and can be applied more generally to price cumulation Parisian option with different excursion periods.

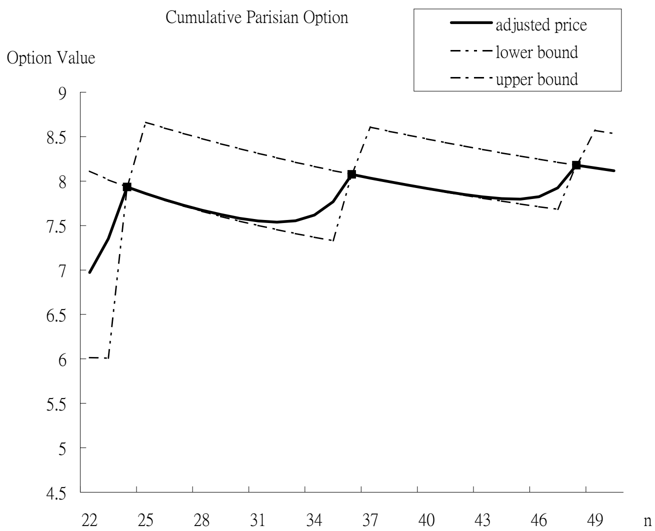

Figure 4.1: The Convergence Behavior of a Adjusted Trinomial Tree—A Cumulative Parisian Option. This figure demonstrates the convergence behavior of a cumulative Parisian option.

The application of our algorithm to price consecutive Parisian options can yield satisfactory results as well. Figure 4.2 is the convergence behavior for a consecutive Parisian option. The same option parameters are used here. It is demonstrated that after we adjust the trinomial tree for the fractional excursion period, more accurate option prices can be obtained. Therefore we can apply the algorithm to price consecutive Parisian option with different excursion periods.

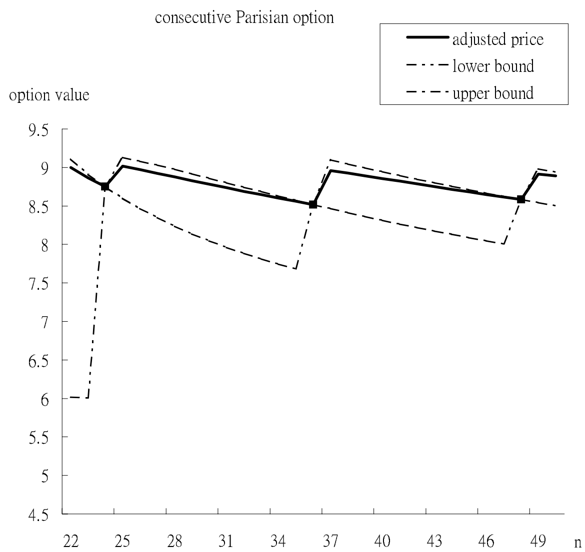

Figure 4.2: The Convergence Behavior of a Adjusted Trinomial Tree—A Consecutive Parisian Option. This figure demonstrates the convergence behavior of a consecutive Parisian Option.

Figures 4.3, 4.4, 4.5, and 4.6 depict the convergence behavior of the cumulative Parisian option and consecutive Parisian option. As n becomes sufficiently large, option prices approach the correct value in a smooth way.

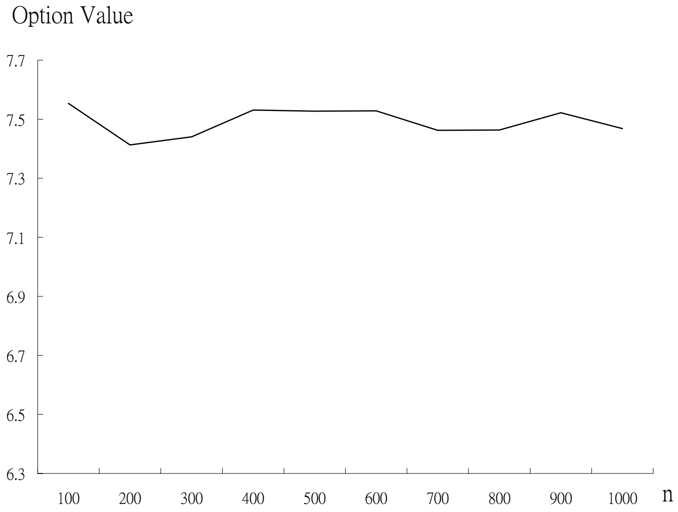

Figure 4.3: Numerical Results for a Cumulative Parisian Option. The option parameters are specified as follow: $S = 60, X = 60, H = 50, \tau =$ 1(year), $B =$ 1(month), $r = 5\%, q = 2\%, and \sigma = 30\%$. Time to maturity τ is divided into n sub-intervals, and n is from 100 to 1000. The price obtained from simulation is 7.5217.

Next we investigate the behaviors of cumulative Parisian calls and consecutive Parisian calls with various market risk. Table 4.1 lists option values with different market risk σ. Prices obtained from simulation are also displayed in the table. As the market risk becomes larger, investors are more uncertain about the future stock price, and option values are higher. Table 4.1 confirms the claim.

Since it is more difficult for a consecutive Parisian option to activate knock-out or knock-in than its cumulative counterpart, a consecutive Parisian

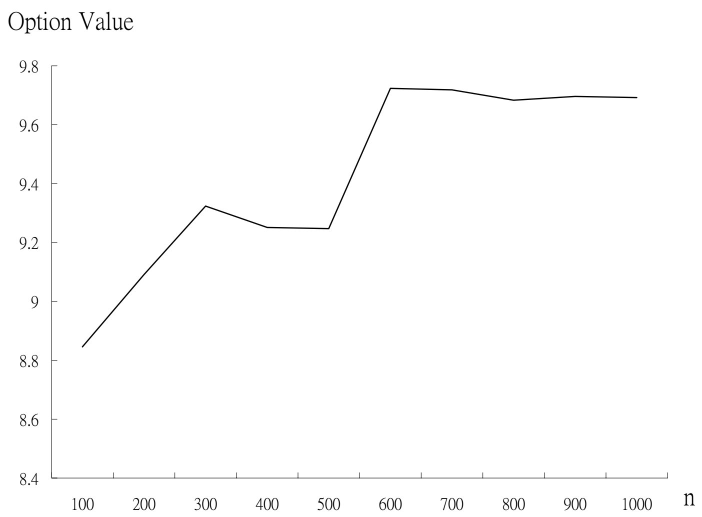

Figure 4.4: NUMERICAL RESULTS FOR A CUMULATIVE PARISIAN OPTION. The option parameters are specified as follow: $S = 95, X = 100, H = 90, \tau = 1$(year), $B = 1$(month), $r = 10\%, q = 0\%, and \sigma = 25\%$. Time to maturity τ is divided into n sub-intervals, and n is from 100 to 1000. The price obtained from simulation is 9.484.

Table 4.1: OPTION VALUES UNDER DIFFERENT MARKET RISK. Option parameters: $S = 80, X = 80, H = 70, \tau = 1$ (year), $B = 1$ (month), $r = 10\%$, and $q = 0$.

	$\sigma = 20\%$	$\sigma = 30\%$	$\sigma = 40\%$	$\sigma = 50\%$	$\sigma = 60\%$
cumulative Parisian	10.3320	12.4388	13.8137	15.2568	16.6355
simulated value	10.3598	12.4080	14.0989	15.3041	17.1665
consecutive Parisian	10.4394	12.7591	14.6218	16.4702	18.2623
simulated value	10.5773	13.3986	16.2172	19.0773	21.825

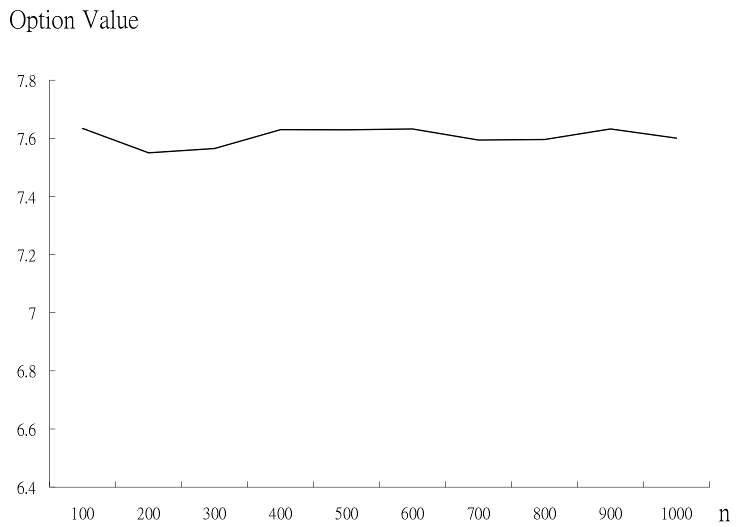

Figure 4.5: NUMERICAL RESULTS FOR A CONSECUTIVE PARISIAN OPTION. The option parameters are specified as follow: $S = 60, X = 60, H = 50, \tau =$ 1(year), $B =$ 1(month), $r = 5\%, q = 2\%, and \sigma = 30\%$. Time to maturity τ is divided into n sub-intervals, and n is from 100 to 1000. The price obtained from simulation is 7.7795.

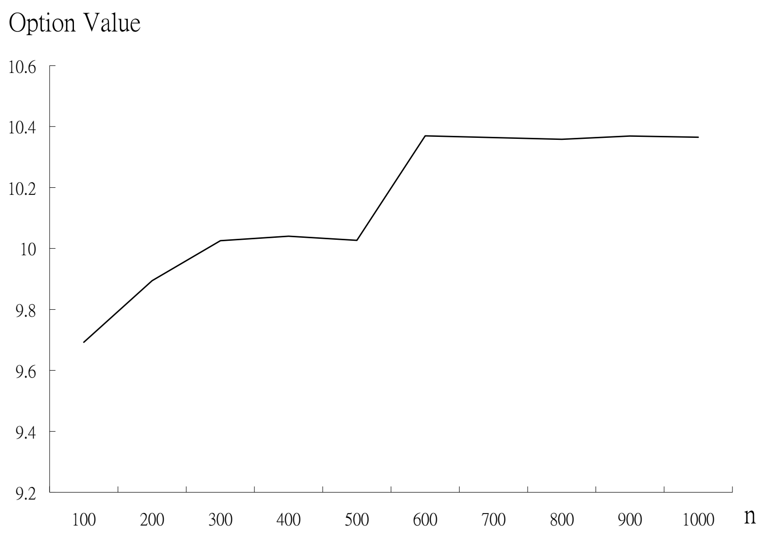

Figure 4.6: NUMERICAL RESULTS FOR A CONSECUTIVE PARISIAN OPTION. The option parameters are specified as follow: $S = 95, X = 100, H = 90, \tau = 1$(year), $B = 1$(month), $r = 10\%, q = 0\%, and \sigma = 25\%$. Time to maturity τ is divided into n sub-intervals, and n is from 100 to 1000. The price obtained from simulation is 11.5311.

always has a higher premium than a cumulative Parisian option. Consecutive and cumulative Parisian option values are increasing functions of the excursion period B. When the length of the option excursion period approaches its time to maturity τ, the value of a cumulative Parisian option and that of a consecutive Parisian option will both approximate the value of the plain vanilla option with identical parameters. Table 4.1 demonstrates that consecutive Parisian options always have a higher value than cumulative Parisian options, as we have expected. Figure 4.7 shows the numerical results for a consecutive Parisian call and a cumulative Parisian call with different excursion periods. The value of a plain vanilla option with the same parameters is obtained from the Black-Scholes formula.

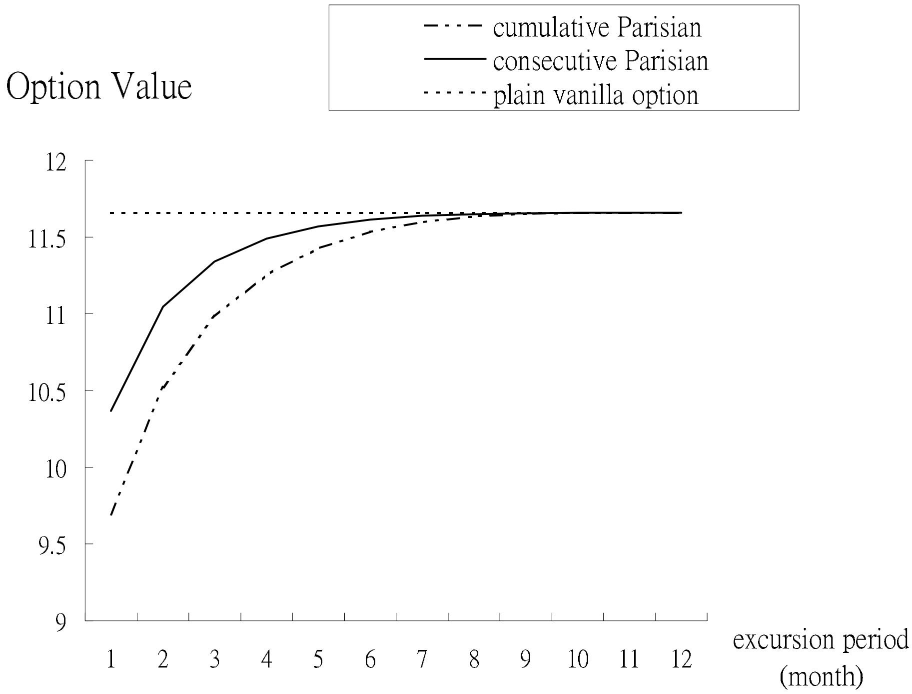

Figure 4.7: Option Values of Consecutive Parisian Call Option and Cumulative Parisian Call Option. Option parameters are specified as follow: $S = 95, X = 100, H = 90, \tau = 1$(year), $r = 10\%, q = 0\%, \sigma = 25\%$, $n = 500$, and the excursion period B is from 1 to 12 months. We also calculate the plain vanilla option price with identical parameters for comparison. The price is obtained from the Black-Scholes pricing formula.

Chapter 5

Conclusions

This thesis investigates the path-dependent Parisian type options. Efficient and accurate pricing techniques are introduced. A cumulative Parisian option would be knocked-out or knocked-in if the price of its underlying asset stays beyond the excursion level for a pre-determined cumulative period of time. The knock-out or knock-in of a consecutive Parisian option would be activated if price the underlying asset stays beyond the excursion level consecutively for a pre-determined period of time.

We show that trinomial trees with auxiliary state vectors can be used to price Parisian type options efficiently. A binomial branching technique can be used to eliminate the problem with the fractional excursion period. The pricing algorithm would be more accurate after we implement binomial branches to adjust the tree. Characteristics of the Parisian type options are also examined.

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