# A General Analytic Formula for Path-Dependent Options:

## Theory and Applications

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**Abstract**

As Taiwan became a member of the WTO, the positive of the regulatory agencies toward the financial markets is more proactive. Through efficient management, the government is open to creating new market. This sets in motion the acceleration of internationalization. More financial derivatives which provide the necessary risk-management tools are expected in the future. In the environment with diverse arrays of derivatives, the crucial issue is how to price them accurately and efficiently.

Under the assumption of perfect market, this thesis proposes a novel systematic approach to deal with the pricing problem of complex path-dependent derivatives. By this method, not only the pricing formula be derived for these derivatives, but pricing can also be programmed. Besides European-style vanilla options, this thesis investigates reset options, compounded options, rainbow options, etc. After successfully establishing pricing formulas for the above-mentioned options, we are convinced of the generality and the power of our approach.

Furthermore, we compare our formula for the European-style geometric average reset option and the one published in *Journal of Derivatives* by Cheng and Zhang with Monte Carlo simulation. We find the significant difference between the Monte Carlo result and the claim of Cheng and Zhang. Therefore their formula is incorrect. Finally, important theoretical properties of this option is proved by Brownian bridge in this thesis.

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# Chapter 1

# Introduction

## 1.1 Research Motivations

Recently, large volumes of trading on financial derivatives and contingent claims have been witnessed in financial markets all over the world. These derivatives are invented by the decomposition and synthesis of some basic financial tools, such as options and futures, combined with some property such as the reset property. Actually, these financial innovations result in new problems concerning pricing and hedging. These problems will become more and more complex in the future as the investors in the financial markets need a variety of derivatives to fit their cash flows and hedge their positions. Knowledge of computer and mathematics plays a more and more important role in financial engineering field.

## 1.2 Solutions to Varieties of Options

A call option holder has the right to buy a stock at a prespecified strike price at maturity. However, investors always want to buy the option at a fair price. Consequently, there are different kinds of path-dependent options that allow the exercise price to change. The reset option is typical one of the considerations. For the simplest reset option, on prespecified date, the original strike price will be reset to the prevailing stock price if the reset option is out of the money. There still exist a variety of reset properties: full or partial period reset, continuous or discrete monitoring, single or multiple reset barriers, single or average stock price reset, etc. Average reset can be classified as arithmetic average or geometric average and, on one hand, as moving-window average or cumulative average on each other. The reset feature has been applied to the financial derivatives for many years. In the end of 1996, the NYSE and CBOE started to trade the S&P 500 index bear-market warrants with the following reset specification: In three months, if the bear-market warrant is out of the money, the strike price will be reset upward once automatically. In 1997, Taiwan's

Stock Exchange began to trade the reset call warrant. For example, the Bloomberg 0522TT has the following covenants: the strike price (initial=$57.25) will be reset to $52.65 if the six-day average price of 2323TT falls below $52.65 any time during the first three months after the issuing of warrant. At the same time, Morgan Stanley issued a warrant on the Hong Kong Stock with the reset feature. In Taiwan and Japan, resettable convertible bonds are very popular among the convertible arbitrage traders as well.

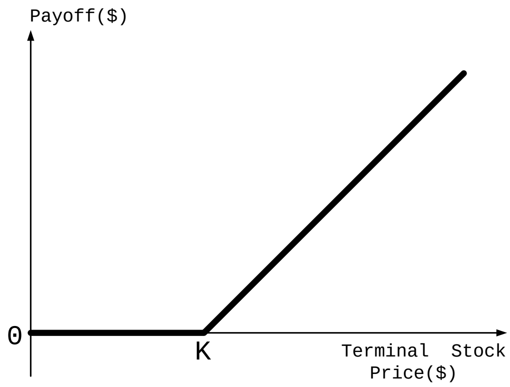


Figure 1.1: Call Option.

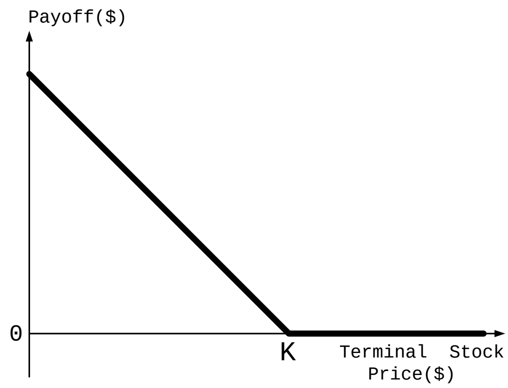


Figure 1.2: Put Option.

## 1.3 Research Structure

The main topic is a systematic approach in deriving the prices of derivatives. In Chapter 2, we review the assumptions in pricing and introduce our approach. Some examples are also presented in this chapter. Next, we focus on the geometric average reset options in Chapter 3. The formula of geometric average multi-reset option also can be derived by the approach and find some properties of this option. We price the quanto option, compound option and rainbow option using the same approach as interesting applications. Finally, we make some conclusions in Chapter 5.

# Chapter 2

# A General Analytic Pricing Formula

## 2.1 Brownian Motion

Brownian motion is a useful model in pricing derivatives. It has many properties. One of them is the Markov property. The Markov property implies that the probability distribution of the asset price at any future particular time does not depend on the history. Before Brownian motion is mentioned, we introduce the Winner process first.

**Definition 2.1.1**
*A Wiener process* $W(t) = (W_1(t), W_2(t), \ldots, W_n(t))$ *in* $n$ *dimensions is a stochastic process with following properties:*

1. *The path begins with* $0$ *at* $t = 0$*, and* $W(t)$ *is continuous with respect to* $t$*.*
2. *For* $0 \leq s < t$*,* $W(t) - W(s)$ *has a stationary independent increment.*
3. *For* $0 \leq s < t$*,* $W(t) - W(s)$ *follows distribution* $\mathrm{N}(0, (t-s)\mathrm{I}_n)$*, where* $\mathrm{I}_n$ *is the* $n \times n$ *identity matrix.*

The process in each dimension of the Wiener process in $n$ dimension is not only a Wiener process in 1 dimension but although independent of the processes of other dimensions. That means the distribution of each component conditional on history is a standard normal distribution.

**Definition 2.1.2**
*A* $(\mu, \sigma)$ *Brownian motion* $B(t) = (B_1(t), B_2(t), \ldots, B_n(t))$ *in* $n$ *dimensions is a stochastic process with following properties:*

1. *The path with* $0$ *at* $t = 0$*, and* $B(t)$ *is continuous with respect to* $t$*.*
2. *For* $0 \leq s < t$*,* $B(t) - B(s)$ *has a stationary independent increment.*

3. *For $0 \leq s < t$, $B(t) - B(s)$ follows distribution $\mathrm{N}((t-s)\mu, (t-s)\Sigma)$, where $\mu$ is the drift vector denoted as $(\mu_1, \mu_2, \ldots, \mu_n)$, and $\Sigma = (\Sigma_{i,j})$ is the volatility matrix denoted as $\Sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$, where $\sigma_k$ is the volatility of $B_k(t)$, $(k = 1, 2, \ldots, n)$, and $\rho_{i,j} = \rho_{j,i}$ is the correlation coefficient between $B_i(t)$ and $B_j(t)$.*

If the Brownian motion is not degenerate, then $\det\Sigma \neq 0$. Besides, $\Sigma$ is symmetric and positive definition so we can find a unique $n \times n$ matrix $\sigma$ such that $\Sigma = \sigma\sigma^*$ and $\det\,\sigma \neq 0$, where the superscript $^*$ denotes the transpose of a matrix. The matrix $\sigma = (\sigma_{i,j})$ is called the diffusion matrix, and $\sigma_{i,j}$ is the diffusion of $B_i(t)$ with respect to $W_j(t)$. The diffusion matrix $\sigma$ can be viewed as a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$ and the transformation is 1-1. The process can be written in matrix as follows:

$$\begin{bmatrix} B_1(t) \\ B_2(t) \\ \vdots \\ B_n(t) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} t + \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} & \cdots & \sigma_{1,n} \\ \sigma_{2,1} & \sigma_{2,2} & \cdots & \sigma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n,1} & \sigma_{n,2} & \cdots & \sigma_{n,n} \end{bmatrix} \begin{bmatrix} W_1(t) \\ W_2(t) \\ \vdots \\ W_n(t) \end{bmatrix}$$

From the above equation, the relationship between $\sigma_i$ and $\sigma_{i,j}$ is

$$\begin{aligned} \sigma_i &= \sqrt{\textstyle\sum_{j=1}^n \sigma_{i,j}^2} \\ \rho_{i,j} &= \sum_{k=1}^n \frac{\sigma_{i,k}\sigma_{j,k}}{\sigma_i\sigma_j} \end{aligned}$$

Another important topic is the joint p.d.f. of Brownian motion at times $t_1$,$t_2$,...,$t_n$. The theorem below can be derived easily since the p.d.f. of the $n$-dimension normal distribution is

$$f(x) = \frac{1}{(\sqrt{2\pi})^n(\det\,\Sigma)^{\frac{1}{2}}}\; e^{-\left[(x-\mu)\Sigma^{-1}(x-\mu)^*\right]/2}.$$

In the above equation, $x = (x_1, x_2, \ldots, x_n)$ is a vector; $\mu$ is the mean vector, and $\Sigma$ is the covariance matrix.

**Theorem 2.1.3** *The joint probability density function of the $(\mu, \sigma)$ Brownian motion $B(t)$, $f(B(t_1), B(t_2), \ldots,, B(t_n))$, in $n$ dimensions at time $t_1$,$t_2$,...,$t_m$, where $0 = t_0 < t_1 < t_2 < \ldots < t_m$, is*

$$\begin{aligned} & f(B(t_1), B(t_2), \ldots, B(t_n)) \\ = & \prod_{i=1}^m \left( \frac{e^{-\left\{[B(t_i)-B(t_{i-1})-\mu(t_i-t_{i-1})][(t_i-t_{i-1})\Sigma]^{-1}[B(t_i)-B(t_{i-1})-\mu(t_i-t_{i-1})]^*\right\}/2}}{(\sqrt{2\pi})^n\left\{\det\,[(t_i-t_{i-1})\Sigma]\right\}^{\frac{1}{2}}} \right)^i \end{aligned}$$

## 2.2 Behavior of Asset Prices

Suppose there are $n$ assets (stock price or foreign exchange ratio, for example) in the world and $A_i(t)$ is the $i$th asset price at time $t$. Define asset price vector $A(t)$, with assets' prices at time $t$ included, is a column vector. In matrix notation, $A(t)$ can be written as

$$A(t) = \begin{bmatrix} A_1(t) \\ A_2(t) \\ \vdots \\ A_n(t) \end{bmatrix}.$$

Assume the process of each asset price in the world follows geometric Brownian motion. It means that the asset price vector satisfies

$$A(t) = \begin{bmatrix} A_1(t) \\ A_2(t) \\ \vdots \\ A_n(t) \end{bmatrix} = \begin{bmatrix} A_1(0)\ e^{B_1(t)} \\ A_2(0)\ e^{B_2(t)} \\ \vdots \\ A_n(0)\ e^{B_n(t)} \end{bmatrix}$$

for some Brownian motion $B^*(t) = (B_1(t), B_2(t), \ldots, B_n(t))$. The stochastic process can be expressd in some Wiener process as

$$\begin{bmatrix} \ln(A_1(t)/A_1(0)) \\ \ln(A_2(t)/A_2(0)) \\ \vdots \\ \ln(A_n(t)/A_n(0)) \end{bmatrix} = \begin{bmatrix} B_1(t) \\ B_2(t) \\ \vdots \\ B_n(t) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} t + \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} & \ldots & \sigma_{1,n} \\ \sigma_{2,1} & \sigma_{2,2} & \ldots & \sigma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n,1} & \sigma_{n,2} & \ldots & \sigma_{n,n} \end{bmatrix} \begin{bmatrix} W_1(t) \\ W_2(t) \\ \vdots \\ W_n(t) \end{bmatrix}.$$

It also can be written in differential form by Itô's lemma:

$$dA(t) = \begin{bmatrix} A_1(t)(\mu_1 + \sigma_1^2/2) \\ A_2(t)(\mu_2 + \sigma_2^2/2) \\ \vdots \\ A_n(t)(\mu_n + \sigma_n^2/2) \end{bmatrix} dt + \begin{bmatrix} A_1(t)\sigma_{1,1} & A_1(t)\sigma_{1,2} & \ldots & A_1(t)\sigma_{1,n} \\ A_2(t)\sigma_{2,1} & A_2(t)\sigma_{2,2} & \ldots & A_2(t)\sigma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_n(t)\sigma_{n,1} & A_n(t)\sigma_{n,2} & \ldots & A_n(t)\sigma_{n,n} \end{bmatrix} dW(t).$$

In the risk-neutral world, people only care about the expected return rate rather than what the risk is and how much the risk is. Moreover, there is absence of arbitrage in efficient market. So the expected return rate must be equal to $r$ where $r$ is the risk-free interest rate. Thus,

$$(\mu_1, \mu_2, \ldots, \mu_n) = (r - \sigma_1^2/2, r - \sigma_2^2/2, \ldots, r - \sigma_n^2/2)$$

## 2.3 A Systematic Approach to Pricing

Now, we present a general analytic pricing approach for pricing a large class of options. The systematic approach applies when the payoff of a derivative is a linear combination of $e^{bX^*}1_{\{X\in A\}}$, where $b$ is a given $m$-dimensional row vector, $X$ is the $m$-dimensional normal distribution, and $A$ is a given subspace of $\mathbf{R}^m$. The payoff of many sophisticated derivatives can be expressed as a linear combination of the form $e^{bX^*}1_{\{X\in A\}}$. Therefore, they can be all priced by our approach. This approach takes two steps:

- completing the squares, and
- change of random variables.

The first step makes the expected value $\mathrm{E}(\mathrm{e}^{\mathrm{bX}^*}1_{\{\mathrm{X}\in\mathrm{A}\}})$ the probability of another random variable with the $m$-dimensional normal distribution in $A$ multiplied by an adjustment. The second step makes the probability of a random variable with $m$-dimensional normal distribution in $A$ be the CDF, $\mathrm{N}(\cdot,\Sigma)$, of $m$-dimensional normal distribution with mean 0 and covariance matrix $\Sigma$.

### Completing the Squares

As mentioned above, $X$ has $m$-dimensional normal distribution. Let $\mu$ and $\Sigma$ ($\det\Sigma \neq 0$) be the mean vector and the covariance matrix of $X$, respectively. By the definition, the expected value of $E(e^{bX^*}1_{\{X\in A\}})$ can be written as

$$\begin{aligned} & \int_A e^{bx^*}\frac{1}{(\sqrt{2\pi})^m(\det\Sigma)^{\frac{1}{2}}}e^{-[(x-\mu)\Sigma^{-1}(x-\mu)^*]/2}dx \\ = & \int_A \frac{1}{(\sqrt{2\pi})^m(\det\Sigma)^{\frac{1}{2}}}e^{bx^*-[(x-\mu)\Sigma^{-1}(x-\mu)^*]/2}dx. \end{aligned} \tag{2.1}$$

To achieve the goal, we complete the squares in the index of the exponent in (2.1). The trick is to define $a = b\Sigma$. Then $b = a\Sigma^{-1}$. Therefore, the index of the exponent equals

$$\begin{aligned} & bx^* - \left[(x-\mu)\Sigma^{-1}(x-\mu)^*\right]/2 \\ = & -[(x-\mu)\Sigma^{-1}(x-\mu)^* - 2bx^*]/2 \\ = & -[x\Sigma^{-1}x^* - x\Sigma^{-1}\mu^* - \mu\Sigma^{-1}x^* + \mu\Sigma^{-1}\mu^* - 2a\Sigma^{-1}x^*]/2 \\ = & -[x\Sigma^{-1}x^* - (\mu+a)\Sigma^{-1}x^* - x\Sigma^{-1}(\mu+a)^* + \mu\Sigma^{-1}\mu^*]/2 \\ = & -[(x-(\mu+a))\Sigma^{-1}(x-(\mu+a))^* + \mu\Sigma^{-1}\mu^* - (\mu+a)\Sigma^{-1}(\mu+a)^*]/2 \\ = & -[(x-(\mu+a))\Sigma^{-1}(x-(\mu+a))^* + \mu\Sigma^{-1}\mu^* - 2\mu\Sigma^{-1}a)^* - a\Sigma^{-1}a^*]/2 \\ = & \mu\Sigma^{-1}a^* + a\Sigma^{-1}a^*/2 - [(x-(\mu+a))\Sigma^{-1}(x-(\mu+a))^*]/2 \\ = & b\mu^* + b\Sigma b^*/2 - [(x-(\mu+a))\Sigma^{-1}(x-(\mu+a))^*]/2. \end{aligned}$$

Because $a\Sigma^{-1}x^*$ and $\mu\Sigma^{-1}a^*$ are numbers. We reduce that $a\Sigma^{-1}x^* = x\Sigma^{-1}a^*$ and $\mu\Sigma^{-1}a^* = a\Sigma^{-1}\mu^*$. As a result, (2.1) can be rewritten as

$$e^{b\mu^*+b\Sigma b^*/2}\int_A \frac{1}{(\sqrt{2\pi})^m(\det\Sigma)^{\frac{1}{2}}} e^{-[(x-(\mu+b\Sigma))\Sigma^{-1}(x-(\mu+b\Sigma))^*]/2}dx.$$

The adjustment factor is $e^{\mu b^*+b\Sigma b^*/2}$. The resulting is random variable $X'$ has an $m$-dimensional normal distribution $\mathrm{N}(\mu+b\Sigma,\Sigma)$ over $A$. Interestingly, $b\mu^*$ and $b\Sigma b^*$ are the mean and the variance of $bX^*$. Hence

$$\mathrm{E}(e^{bX^*}) = e^{b\mu^*+b\Sigma b^*/2} = e^{\mathrm{E}(bX^*)+\mathrm{Var}(bX^*)/2}.$$

If $X(t)$ is a $(\mu,\Sigma)$ Brownian motion, then the the adjustment is the expected value of geometric Brownian motion $e^{bX^*(t)}$. We summarize our result below.

**Theorem 2.3.1** *Let $X$ be an $m$-dimensional normal distribution random variable with mean vector $\mu$ and covariance matrix $\Sigma$. Then*

$$\mathrm{E}(e^{bX^*}1_{\{X\in A\}}) = \mathrm{E}(e^{bX^*})\int_A \frac{1}{(\sqrt{2\pi})^m(\det\Sigma)^{\frac{1}{2}}} e^{-[(x-(\mu+b\Sigma))\Sigma^{-1}(x-(\mu+b\Sigma))^*]/2}dx. \quad (2.2)$$

The right-hand side of (2.2) is called the completed-square form.

### Change of Random Variables

Theorem 2.3.1 reduces the problem of evaluating $\mathrm{E}(e^{bX^*}1_{\{X\in A\}})$ to a multiple integration of a multi-dimensional normal distribution on the same area $A$. However, the area of $A$ is usually a non-orthogonal polyhedron, making it to calculate the integration. By change of random variables, the area $A$ can be mapped into a rectangular area $A'$. Suppose $Y^* = CX^*$ and $\mathrm{C} \neq 0$. Then,

$$\begin{aligned}
& e^{b\mu^*+b\Sigma b^*/2}\int_A \frac{1}{(\sqrt{2\pi})^m(\det\Sigma)^{\frac{1}{2}}} e^{-[(x-(\mu+b\Sigma))\Sigma^{-1}(x-(\mu+b\Sigma))^*]/2}dx \\
= \; & e^{b\mu^*+b\Sigma b^*/2}\int_A \frac{1}{(\sqrt{2\pi})^m(\det\Sigma)^{\frac{1}{2}}} e^{-[(x-(\mu+b\Sigma))C^*(C^*)^{-1}\Sigma^{-1}C^{-1}C(x-(\mu+b\Sigma))^*]/2}dx \\
= \; & e^{b\mu^*+b\Sigma b^*/2}\int_A \frac{1}{(\sqrt{2\pi})^m(\det\Sigma)^{\frac{1}{2}}} e^{-[(xC^*-(\mu+b\Sigma)C^*)(C^*)^{-1}\Sigma^{-1}C^{-1}(xC^*-(\mu+b\Sigma)C^*)^*]/2}dx \\
= \; & e^{b\mu^*+b\Sigma b^*/2}\int_{A'} \frac{1}{(\sqrt{2\pi})^m(\det\Sigma)^{\frac{1}{2}}} e^{-[(y-(\mu+b\Sigma)C^*)(C\Sigma C^*)^{-1}(y-(\mu+b\Sigma)C^*)^*]/2}\left|\frac{dx}{dy}\right|dy \\
= \; & e^{b\mu^*+b\Sigma b^*/2}\int_{A'} \frac{1}{(\sqrt{2\pi})^m(\det\Sigma')^{\frac{1}{2}}} e^{-[(y-(\mu+b\Sigma)C^*)\Sigma'^{-1}(y-(\mu+b\Sigma)C^*)^*]/2}dy,
\end{aligned}$$

where $\Sigma' = C\Sigma C^*$. In fact, the above equation applies for any linear transformation $C$, which $\det\Sigma \neq 0$. We usually find a linear transformation $C$ which satisfies the two requirements:

1. $A'$ is a rectangle $[-\infty,d_1]\times[-\infty,d_2]\times\cdots\times[-\infty,d_n]$.

2. The variance of each components of $Y$ is 1.

Then, above equation becomes

$$\mathrm{N}(d-(\mu+b\Sigma)C^*,\Sigma'),$$

where $d=(d_1,d_2,\cdots,d_n)$.

## 2.4 A Simple Application

The pricing formula for some basic derivatives, like the Black-Scholes formula for vanilla call options, can be derived easily by the systematic approach. The current price of underlying asset is $S(0)$ and its volatility is $\sigma$. Consider a vanilla call option on the underlying asset with strike price $K$ and time to maturity $T$. Assume the risk-free interest rate is $r$. The payoff of the vanilla call option is $\max\{S(T)-K,0\}$ or written as $(S(T)-K)^+$. The call option value is the expected payoff multiplied by the discount factor, so the price of call option can be expressed by a linear combination of the form $\mathrm{E}(\mathrm{e}^{\mathrm{bX}*}1_{\{\mathrm{X}\in\mathrm{A}\}})$ as follows:

$$\begin{aligned}
& \mathrm{E}\left[(S(T)-K)^+\right] \\
= & \mathrm{E}\left[(S(T)-K)1_{\{S(T)>K\}}\right] \\
= & \mathrm{E}\left[S(T)1_{\{S(T)>K\}}\right]-E\left[K1_{\{S(T)>K\}}\right] \\
= & S(0)\mathrm{E}\left[(S(T)/S(0))1_{\{(S(T)/S(0))>K/S(0)\}}\right]-K\mathrm{E}\left[1_{\{(S(T)/S(0))>K/S(0)\}}\right]
\end{aligned}$$

Since $S(t)/S(0)$ is a geometric Brownian motion in a risk-neutral world,

$$S(t)/S(0)=e^{X(t)}=e^{(r-\sigma^2/2)t+\sigma W(t)}$$

Then $X(T)=\ln(S(T)/S(0))$ has normal distribution with mean $(r-\sigma^2/2)T$ and variance $\sigma^2T$. Therefore,

$$A=\{X(T)|e^{X(T)}\geq(K/S(0))\}=\{X(T)|X(T)\geq\ln(K/S(0))\}.$$

The price of the vanilla call equals

$$e^{-rT}\mathrm{E}(S(T)-K)^+=e^{-rT}S(0)\mathrm{E}(e^{X(T)}1_{\{X(T)\in A\}})-e^{-rT}K\mathrm{E}(1_{\{X(T)\in A\}}). \tag{2.3}$$

Since $X(T)$ is one-dimensional, $bX^*(T)=bX(T)$. The right-hand side of (2.3) can be evaluated by applying Theorem 2.3.1 as follows:

$$\begin{aligned}
& e^{-rT}S(0)\mathrm{E}(e^{X(T)}1_{\{X(T)\in A\}})-e^{-rT}K\mathrm{E}(1_{\{X(T)\in A\}}) \\
= & e^{-rT}S(0)e^{rT}\int_A\frac{1}{\sqrt{2\pi}\sigma\sqrt{T}}e^{-\frac{\left[x-(r+\sigma^2/2)T\right]^2}{2\sigma^2T}}dx-e^{-rT}K\int_A\frac{1}{\sqrt{2\pi}\sigma\sqrt{T}}e^{-\frac{\left[x-(r-\sigma^2/2)T\right]^2}{2\sigma^2T}}dx \\
= & S(0)\int_{\ln(K/S(0))}^{\infty}\frac{1}{\sqrt{2\pi}\sigma\sqrt{T}}e^{-\frac{\left[x-(r+\sigma^2/2)T\right]^2}{2\sigma^2T}}dx-e^{-rT}K\int_{\ln(K/S(0))}^{\infty}\frac{1}{\sqrt{2\pi}\sigma\sqrt{T}}e^{-\frac{\left[x-(r-\sigma^2/2)T\right]^2}{2\sigma^2T}}dx
\end{aligned}$$

Let $Y = -X/\sqrt{\text{Var}(X)} = -X/(\sigma\sqrt{T})$, which is a linear transformation. Apply the change of random variables in the first term. Then, the right hand side of (2.3) becomes

$$\begin{aligned} & S(0)\int_{-\infty}^{\frac{\ln(S(0)/K)}{\sigma\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-\left[y+\frac{(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right]^2/2} dy - e^{-rT}K \int_{-\infty}^{\frac{\ln(S(0)/K)}{\sigma\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-\left[y+\frac{(r-\sigma^2/2)T}{\sigma\sqrt{T}}\right]^2/2} dy \\ = \; & S(0)\mathrm{N}\left(\frac{\ln(S(0)/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right) - e^{-rT}K\mathrm{N}\left(\frac{\ln(S(0)/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}}\right), \end{aligned}$$

where $\mathrm{N}(\cdot)$ denotes the CDF of the standard normal distribution. This is the celebrated Black-Scholes formula, as desired.

# Chapter 3

# Pricing of Geometric Average Reset Options

## 3.1 The Geometric Average Reset Option

Geometric average reset options are reset options whose strike price can be reset to the geometric average price of underlying asset over a monitoring interval. Suppose $S(t)$ is the underlying asset price and the current price is $S(0)$. Consider a general reset option with $m$ reset dates: $0 \leq t_1 < t_2 < \ldots < t_{m-1} < t_m \leq T$. Assume the $m$ monitoring intervals are $[t_1 - \ell_1, t_1]$, $[t_2 - \ell_2, t_2]$, $\cdots$, $[t_m - \ell_m, t_m]$, where $\ell_i$ denotes the length of the $i$th monitoring interval. Define $\text{avg}^G(t_i)$ as the geometric average price of the underlying asset during the $i$th monitoring interval. Let $K(t)$ be the strike price prevailing at time $t$ and set $K(0)$ as $K$, the original strike price. The reset procedure at time $t_i$ is:

$$K(t_i) = \begin{cases} K(t_{i-1}), & \text{if } \text{avg}^G(t_i) \geq K(t_{i-1}) \\ \text{avg}^G(t_i), & \text{if } \text{avg}^G(t_i) < K(t_{i-1}) \end{cases}$$

The payoff of the call is $\max\{S(T) - K(T), 0\}$. Obviously,

$$K(T) = \min\{K, \text{avg}^G(t_1), \text{avg}^G(t_2), \cdots, \text{avg}^G(t_m)\}.$$

Similarly, the reset procedure for the put is

$$K(t_i) = \begin{cases} K(t_{i-1}), & \text{if } \text{avg}^G(t_i) \leq K(t_{i-1}) \\ \text{avg}^G(t_i), & \text{if } \text{avg}^G(t_i) > K(t_{i-1}) \end{cases}$$

and the payoff of the put is $\max\{K(T) - S(T), 0\}$. Obviously,

$$K(T) = \max\{K, \text{avg}^G(t_1), \text{avg}^G(t_2), \cdots, \text{avg}^G(t_m)\}.$$

## 3.2 Analytic Formula for Geometric Average Reset Options

Assume the volatility of the underlying asset is $\sigma$ and the risk-free interest rate is $r$. We also assume that the monitoring intervals are disjoint to simplify the presentation. To price the geometric average reset call option with $m$ monitoring intervals, define a row vector $X = (X_1, X_2, X_3 \ldots X_{m+1})$, where $X_i = \ln(\text{avg}^G(t_i)/S(0))$, for $i = 1, 2, \ldots, m$, and $X_{m+1} = \ln(S(T)/S(0))$. Obviously, $X$ has a $(m+1)$-dimensional normal distribution with mean $\mu$ and covariance matrix $\Sigma$. In the risk-neutral world, $\mu$ and each element $\Sigma(i,j)$ in $\Sigma$ can be expressed as follows[1]:

$$\mu = \left[(r-\sigma^2/2)(t_1-\ell_1/2), \ldots, (r-\sigma^2/2)(t_m-\ell_m/2), (r-\sigma^2/2)T\right]$$

and

$$\begin{aligned}
\Sigma(i,i) &= \sigma^2(t_i - 2\ell_i/3) \quad \text{if } 1 \le i \le m \\
\Sigma(i,j) = \Sigma(j,i) &= \sigma^2(t_i - \ell_i/2) \quad \text{if } 1 \le i < j \le m+1 \\
\Sigma(m+1,m+1) &= \sigma^2 T
\end{aligned}$$

Then the payoff of the reset option can now be expressed in terms of $e^{bX^*}1_{\{X\in A\}}$:

$$\begin{aligned}
& \sum_{i=1}^{m} [(S(T)-S(t_i))^+ \, 1_{\{K(t_m)=S(t_i)\}}] + (S(T)-K)^+ 1_{\{K(t_m)=K\}} \qquad (3.1)\\
= & \sum_{i=1}^{m} \left[S(0)e^{b_{m+1}X^*}1_{\{X\in A_i\}} - S(0)e^{b_iX^*}1_{\{X\in A_i\}}\right] \\
& + S(0)e^{b_{m+1}X^*}1_{\{X\in A_{m+1}\}} - K1_{\{X\in A_{m+1}\}},
\end{aligned}$$

where

$$\begin{aligned}
b_i &= [\overbrace{0,\ldots,0,}^{i-1}1,\overbrace{0,\ldots,0,}^{m+1-i}] \qquad \text{for} \quad 1 \le i \le m+1, \\
A_i &= \{X | K(T) = S(0)e^{X_i}, S(T) \ge S(0)e^{X_i}\} = \{X | K(T) = \text{avg}(t_i), S(T) \ge \text{avg}(t_i)\},
\end{aligned}$$

for $1 \le i \le m$ and $A_{m+1} = \{X | K(T) = K, S(T) \ge K\}$.

Consequently, each term in (3.1) can be reduced to a completed-square form. To obtain an analytic formula, a linear transformation matrix $C$ is needed to transform the polyhedron into an rectangle like $[-\infty, d_1] \times [-\infty, d_2] \times \cdots \times [-\infty, d_{m+1}]$ in $\mathbf{R}^{m+1}$. There are two cases.

**Case 1 : Area $A_k$ $(1 \le k \le m)$**

Each point $X = (X_1, \ldots, X_{m+1})$ in $A_k$ should satisfy the following $m+1$ inequalities:

$$\begin{aligned}
X_k &\le \ln(K/S(0)) \\
X_k - X_i &\le 0 \qquad \text{for } 1 \le i \le m+1 \text{ and } i \ne k.
\end{aligned}$$

[1]see Kemna and Vorst [1990].

Since $A_k$ is not a rectangle, a linear transformation matrix $C_k$ is needed to convert the non-orthogonal polyhedron into a rectangle. Let

$$Y_i = \begin{cases} X_k/\sqrt{\mathrm{Var}(X_k)} & \text{for } i = k \\ (X_k - X_i)/\sqrt{\mathrm{Var}(X_k - X_i)} & \text{for } 1 \le i \le m+1 \text{ and } i \ne k. \end{cases}$$

Equivalently,

$$Y_i = \begin{cases} X_k/\sqrt{\Sigma_{k,k}} & \text{for } i = k \\ (X_k - X_i)/\sqrt{\Sigma_{k,k} + \Sigma_{i,i} - 2\Sigma_{k,i}} & \text{for } 1 \le i \le m+1 \text{ and } i \ne k. \end{cases}$$

For convenience, assume $C_k(i,j)$ is the element allocated at the $i$th row and the $j$th column in $C_k$. Then

$$\begin{array}{ll} C_k(k,k) = 1/\sqrt{\Sigma_{k,k}}, & \text{when } 1 \le i \le m+1 \\ C_k(i,k) = 1/\sqrt{\Sigma_{k,k} + \Sigma_{i,i} - 2\Sigma_{k,i}}, & \text{when } 1 \le i \le m+1 \text{ and } \ i \ne k \\ C_k(i,i) = -1/\sqrt{\Sigma_{k,k} + \Sigma_{i,i} - 2\Sigma_{k,i}}, & \text{when } 1 \le i \le m+1 \text{ and } \ i \ne k \\ C_k(i,j) = 0, & \text{otherwise .} \end{array}$$

By change of random variables, the area $A_k$ is transformed into the area $A'_k$,

$$A'_k = \{Y \mid Y_k \le \ln(K/S(0))/\sqrt{\Sigma_{k,k}} \text{ and } Y_i \le 0, \text{ for } 1 \le i \le m+1 \text{ and } i \ne k\}.$$

**Case 2 : Area** $A_{m+1}$

Each point $X = (X_1, \ldots, X_{m+1})$ in $A_{m+1}$ should satisfy the following $m+1$ inequalities:

$$X_i \ge \ln(K/S(0)) \text{ for } \ 1 \le i \le m+1.$$

Although $A_{m+1}$ is a rectangle, but it is not required. We still need the linear transform matrix, $C_{m+1}$. Let

$$Y_i = -X_i/\sqrt{\mathrm{Var}(X_i)} = -X_i/\sqrt{\Sigma_{i,i}} \quad \text{for } \ 1 \le i \le m+1.$$

Then,

$$\begin{array}{ll} C_{m+1}(i,i) = -1/\sqrt{\Sigma_{i,i}}, & \text{when } \ 1 \le i \le m+1 \\ C_{m+1}(i,j) = 0, & \text{when } i \ne j. \end{array}$$

By change of random variables again, the area $A_{m+1}$ is transformed into the area $A'_{m+1}$,

$$A'_{m+1} = \{Y \mid Y_i \le -\ln(K/S(0))/\sqrt{\Sigma_{i,i}}, \text{ for } \ 1 \le i \le m+1\}.$$

To simplify the notation, let

$$\Sigma_i = C_i \Sigma C_i^*$$

The expected payoff can be finally expressed as follows:

$$\begin{aligned}
& S(0)\sum_{i=1}^{m}\left\{ e^{rT}\int_{A_i'} \frac{e^{-[(y-(\mu+b_{m+1}\Sigma)C_i^*)\Sigma_i^{-1}(y-(\mu+b_{m+1}\Sigma)C_i^*)^*]/2}}{(\sqrt{2\pi})^{m+1}(\det\Sigma_i)^{\frac{1}{2}}}dy \right.\\
- & \left. e^{r(t_i-\ell_i/2)}\int_{A_i'} \frac{e^{-[(y-(\mu+b_i\Sigma)C_i^*)\Sigma_i^{-1}(y-(\mu+b_i\Sigma)C_i^*)^*]/2}}{(\sqrt{2\pi})^{m+1}(\det\Sigma_i)^{\frac{1}{2}}}dy \right\}\\
+ & S(0)e^{rT}\int_{A_{m+1}'} \frac{e^{-[(y-(\mu+b_{m+1}\Sigma)C_{m+1}^*)\Sigma_{m+1}^{-1}(y-(\mu+b_{m+1}\Sigma)C_{m+1}^*)^*]/2}}{(\sqrt{2\pi})^{m+1}(\det\Sigma_{m+1})^{\frac{1}{2}}}dy\\
- & K\int_{A_{m+1}'} \frac{e^{-[(y-\mu C_{m+1}^*)\Sigma_{m+1}^{-1}(y-\mu C_{m+1}^*)^*]/2}}{(\sqrt{2\pi})^{m+1}(\det\Sigma_{m+1})^{\frac{1}{2}}}dy
\end{aligned}$$

So the pricing formula is

$$\begin{aligned}
& S(0)\sum_{i=1}^{m}\left\{ \int_{A_i'} \frac{e^{-[(y-(\mu+b_{m+1}\Sigma)C_i^*)\Sigma_i^{-1}(y-(\mu+b_{m+1}\Sigma)C_i^*)^*]/2}}{(\sqrt{2\pi})^{m+1}(\det\Sigma_i)^{\frac{1}{2}}}dy \right.\\
- & \left. e^{r(t_i-\ell_i/2-T)}\int_{A_i'} \frac{e^{-[(y-(\mu+b_i\Sigma)C_i^*)\Sigma_i^{-1}(y-(\mu+b_i\Sigma)C_i^*)^*]/2}}{(\sqrt{2\pi})^{m+1}(\det\Sigma_i)^{\frac{1}{2}}}dy \right\}\\
+ & S(0)\int_{A_{m+1}'} \frac{e^{-[(y-(\mu+b_{m+1}\Sigma)C_{m+1}^*)\Sigma_{m+1}^{-1}(y-(\mu+b_{m+1}\Sigma)C_{m+1}^*)^*]/2}}{(\sqrt{2\pi})^{m+1}(\det\Sigma_{m+1})^{\frac{1}{2}}}dy\\
- & Ke^{-rT}\int_{A_{m+1}'} \frac{e^{-[(y-\mu C_{m+1}^*)\Sigma_{m+1}^{-1}(y-\mu C_{m+1}^*)^*]/2}}{(\sqrt{2\pi})^{m+1}(\det\Sigma_{m+1})^{\frac{1}{2}}}dy
\end{aligned}$$

## 3.3 The Special Case of Single Reset Options

Obviously, the analytic formula for the geometric average reset option with arbitrary monitoring intervals can be derived by the above systematic approach. We now take the case of one monitoring interval as an example to demonstrate the power of our approach. This is a geometric average reset option with $m = 1$. Assume reset date $t_1 = t(t \leq T)$, monitoring interval $\ell$ and the other parameters are the same. Then the mean vector of $X$ is $\mu = ((r-\sigma^2/2)(t-\ell/2), (r-\sigma^2/2)T)$. The covariance matrix of $X$ is:

$$\Sigma = \begin{bmatrix} \sigma^2(t-2\ell/3) & \sigma^2(t-\ell/2) \\ \sigma^2(t-\ell/2) & \sigma^2 T \end{bmatrix}.$$

The transformation matrices for integration area $A_1$ and $A_2$, called $C_1$ and $C_2$, respectively, are

$$\begin{aligned} C_1 &= \begin{bmatrix} 1/\sqrt{\sigma^2 T} & 0 \\ 1/\sqrt{\sigma^2(T-t-5\ell/6)} & -1/\sqrt{\sigma^2(T-t-5\ell/6)} \end{bmatrix}, \\ C_2 &= \begin{bmatrix} -1/\sqrt{\sigma^2(t-2\ell/3)} & 0 \\ 0 & -1/\sqrt{\sigma^2 T} \end{bmatrix}. \end{aligned}$$

Therefore, $A_1$ and $A_2$ can be transformed into $A_1'$ and $A_2'$ by change of random variables, where

$$\begin{aligned} &A_1' = \{Y|Y_1 \leq 0, Y_2 \leq \ln(K/S(0))/\sqrt{\sigma^2(T-t-5\ell/6)}\}, \\ &A_2' = \{Y|Y_1 \leq -\ln(K/S(0))/\sqrt{\sigma^2(t-2\ell/3)}, Y_2 \leq -\ln(K/S(0))/\sqrt{\sigma^2 T}\}. \end{aligned}$$

Let $b_1 = (1, 0)$ and $b_2 = (0, 1)$. The price of the option can be expressed as follows:

$$\begin{aligned} & S(0)\int_{A_1'} \frac{e^{-[(y-(\mu+b_2\Sigma)C_1^*)\Sigma_1^{-1}(y-(\mu+b_2\Sigma)C_1^*)^*]/2}}{2\pi(\det\Sigma_1)^{\frac{1}{2}}}dy \\ & -S(0)e^{-r(T-t+\ell/2)}\int_{A_1'} \frac{e^{-[(y-(\mu+b_1\Sigma)C_1^*)\Sigma_1^{-1}(y-(\mu+b_1\Sigma)C_1^*)^*]/2}}{2\pi(\det\Sigma_1)^{\frac{1}{2}}}dy \\ & +S(0)\int_{A_2'} \frac{e^{-[(y-(\mu+b_2\Sigma)C_2^*)\Sigma_2^{-1}(y-(\mu+b_2\Sigma)C_2^*)^*]/2}}{2\pi(\det\Sigma_2)^{\frac{1}{2}}}dy \\ & -Ke^{-rT}\int_{A_2'} \frac{e^{-[(y-\mu C_2^*)\Sigma_2^{-1}(y-\mu C_2^*)^*]/2}}{2\pi(\det\Sigma_2)^{\frac{1}{2}}}dy \\ = \; & S(0)\mathrm{N}(d_1-(\mu+b_2\Sigma)C_1^*, \Sigma_1) \\ & -S(0)e^{-r(T-t+\ell/2)}\mathrm{N}(d_1-(\mu+b_1\Sigma)C_1^*, \Sigma_1) \\ & +S(0)\mathrm{N}(d_2-(\mu+b_2\Sigma)C_2^*, \Sigma_2) \\ & -Ke^{-rT}\mathrm{N}(d_2-\mu C_2^*, \Sigma_2), \end{aligned}$$

where

$$\begin{aligned} d_1 &= (0, \ln(K/S(0))/\sqrt{\sigma^2(T-t-5\ell/6)}), \\ d_2 &= (-\ln(K/S(0))/\sqrt{\sigma^2(t-2\ell/3)}, -\ln(K/S(0))/\sqrt{\sigma^2 T}), \end{aligned}$$

and $\mathrm{N}(\cdot, \Sigma)$ is the CDF of normal distribution with mean vector 0 and covariance matrix $\Sigma$.

## 3.4 Properties of Geometric Average Reset Options

Some special properties for reset options are discussed in this section. First, an American-style reset call will be proved not to be exercised early if the underlying asset does not pay dividends. Second, the relations between the time span of monitoring intervals and the option value are also discussed.

**Theorem 3.4.1** *An American-style reset call option will not be exercised early if the underlying asset does not pay dividends.*

***Proof.*** It is well-known that an American-style vanilla call option will not be exercised early if the underlying assets do not pay dividends. That is to say, at any arbitrary time $t$ before maturity date $T$, the following should always be satisfied:

$$\mathrm{E}_t[C_v] \geq S(t) - K(t)$$

where $\mathrm{E}_t[C_v]$ denotes the option price at time $t$ and $S(t)$ denotes the underlying asset's value at time $t$. Since the strike price of a reset call could be reset to a lower level, a reset call is not less valuable then the vanilla call. It is observed that

$$\mathrm{E}_t[C_r] \geq \mathrm{E}_t[C_v] \geq S(t) - K(t)$$

where $E_t[C_r]$ denotes the option value of a reset option at time $t$. As a result, a reset call will not be exercised early. $\square$

Most average reset options in real markets are triggered by the arithmetic average price instead of a geometric one. Pricing arithmetic average reset options is not easy, but it is surprisingly easy to derive the relationship between these two kinds of options.

**Theorem 3.4.2** *The price of the geometric average reset call option is higher than the price of the arithmetic average reset call option.*

***Proof.*** Let $\mathrm{avg}^G(t_i)$ and $\mathrm{avg}^A(t_i)$ denote the geometric and the arithmetic average price of the $i$th monitoring interval, respectively. Since $\mathrm{avg}^G(t_i) \leq \mathrm{avg}^A(t_i)$, we have

$$\min\{K, \mathrm{avg}^G(t_1), \mathrm{avg}^G(t_2), \ldots, \mathrm{avg}^G(t_m)\} \leq \min\{K, \mathrm{avg}^A(t_1), \mathrm{avg}^A(t_2), \ldots, \mathrm{avg}^A(t_m)\}.$$

Consequently,

$$\begin{aligned} & \mathrm{E}(S(T) - \min\{K, \mathrm{avg}^G(t_1), \mathrm{avg}^G(t_2), \ldots, \mathrm{avg}^G(t_m)\})^+ \\ \geq\ & \mathrm{E}(S(T) - \min\{K, \mathrm{avg}^A(t_1), \mathrm{avg}^A(t_2), \ldots, \mathrm{avg}^A(t_m)\})^+. \end{aligned}$$

$\square$

**Theorem 3.4.3** *The price of the geometric average reset put option is lower than the price of the arithmetic average reset put option.*

***Proof.*** Obviously,

$$\max\{K, \text{avg}^G(t_1), \text{avg}^G(t_2), \ldots, \text{avg}^G(t_m)\} \leq \max\{K, \text{avg}^A(t_1), \text{avg}^A(t_2), \ldots, \text{avg}^A(t_m)\}.$$

Consequently,

$$\begin{aligned} & \mathrm{E}(\max\{K, \text{avg}^G(t_1), \text{avg}^G(t_2), \ldots, \text{avg}^G(t_m)\} - S(T))^+ \\ \leq \ & \mathrm{E}(\max\{K, \text{avg}^A(t_1), \text{avg}^A(t_2), \ldots, \text{avg}^A(t_m)\} - S(T))^+. \end{aligned}$$

$\square$

In real markets, we compute the average price of some representative prices like closing prices instead of the continuous geometric average price during the monitoring intervals. The relationship between the sampling frequencies and the option value can be explored by taking advantages of Brownian bridge. We sample $n$ points, including the beginning and the end points, in each monitoring interval. These $n$ points divide the monitoring interval into $n-1$ equal sub-intervals. The average of the $n$ prices of sampling points is denoted as $\text{avg}_n^G(t_i)$. Some required properties of Brownian bridge are explored in the following lemma.

**Definition 3.4.4** *Assume that a Brownian motion $\{B(t)\}$ which begins at time 0 with $B(0) = 0$ has drift $\mu$, and volatility $\sigma$. The sample space of the Brownian motion satisfying the condition $B(T) = B_T$ is called a Brownian bridge process.*

**Lemma 3.4.5** *Define $A_n(\tau, \tau + \ell)$ as $\sum_{i=0}^{n-1} B(\tau + i\ell/(n-1))/n$, the discrete average on time $[\tau, \tau + \ell]$, where $\ell > 0$ and $\tau + \ell \leq T$. Then, the mean and the variance of $A_n$ conditioning on $B(T) = B_T$, are*

$$\mathrm{E}(A_n | B_T) = (\tau + \ell/2) B_T / T$$

$$Var(A_n | B_T) = \left[\tau + \frac{(2n-1)\ell}{6n} - \frac{(\tau + \ell/2)^2}{T}\right] \sigma^2.$$

***Proof.*** $\mathrm{E}(A_n | B_T)$, can be computed as follows:

$$\begin{aligned} & E\left(\frac{1}{n}\sum_{i=0}^{n-1} B(\tau + i\ell/(n-1)) | B_T\right) \\ = \ & \frac{1}{n}\sum_{i=0}^{n-1} (\tau + i\ell/(n-1))\, B_T/T \\ = \ & (\tau + \ell/2)\, B_T/T. \end{aligned}$$

$A_n$ can be expressed as

$$A_n = \frac{1}{n}\sum_{i=0}^{n-1} B(\tau + i\ell/(n-1)) = \frac{1}{n}\sum_{i=1}^{n-1}(n-i)Y_i + B(\tau),$$

where $Y_i = B(\tau + i\ell/n-1) - B(\tau + (i-1)\ell/(n-1))$ and $\mathrm{Var}(Y_i) = \sigma^2\ell/(n-1)$ for $1 \leq i \leq n-1$. Therefore, the variance of $A_n$, $\mathrm{Var}(A_n)$, can be computed as follows:

$$\begin{aligned}
\mathrm{Var}(A_n) &= \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n-1}(n-i)Y_i + B(\tau)\right) \\
&= \sum_{i=1}^{n-1}\mathrm{Var}\left(\frac{1}{n}(n-i)Y_i\right) + \mathrm{Var}(B(\tau)) \\
&= \frac{1}{n^2}\sum_{i=1}^{n-1}\big((n-i)^2\sigma^2\ell/(n-1)\big) + \tau\sigma^2 \\
&= \frac{2n-1}{6n}\ell\sigma^2 + \tau\sigma^2,
\end{aligned}$$

Since $\mathrm{Var}(A_n) = \mathrm{E}(\mathrm{Var}(A_n|B_T)) + \mathrm{Var}(\mathrm{E}(A_n|B_T))$ [2] and $\mathrm{E}(\mathrm{Var}(A_n|B_T)) = \mathrm{Var}(A_n|B_T)$,

$$\begin{aligned}
\mathrm{Var}(A_n|B_T) &= \mathrm{E}(\mathrm{Var}(A_n|B_T)) \\
&= \mathrm{Var}(A_n) - \mathrm{Var}(\mathrm{E}(A_n|B_T)) \\
&= (\tau + (2n-1)\ell/(6n))\,\sigma^2 - \mathrm{Var}((\tau + \ell/2)B_T/T) \\
&= \left[\tau + \frac{(2n-1)\ell}{6n} - \frac{(\tau+\ell/2)^2}{T}\right]\sigma^2.
\end{aligned}$$

□

By the assumption of stock behavior, it is obvious that $\mathrm{avg}_n^G(t_i) = S\mathrm{e}^{A_n(t_i-\ell,t_i)}$. We can analyze the relation between the sampling frequency and the average price by the above lemma.

**Lemma 3.4.6** *The conditional variance of $A_n$ and the conditional mean of $avg_n^G$ become larger as the number of samples, $n$, increases.*

***Proof.*** The conditional mean of $\mathrm{avg}_n^G$, which has a log-normal distribution, is $S\mathrm{e}^{\mathrm{E}(A_n|B_T)+\mathrm{Var}(A_n|B_T)/2}$. Since $\partial\mathrm{Var}(A_n|B_T)/\partial n = \sigma^2\ell/6n^2 > 0$, $\mathrm{Var}(A_n|B_T)$ tends larger as $n$ increases. Besides, $\mathrm{E}(A_n|B_T)$ is a constant for $n$. Therefore, the conditional mean of $\mathrm{avg}_n^G$ increases with $n$. □

[2] See Ross [1994]

**Theorem 3.4.7** (*Sampling frequency theorem: One monitoring interval version*). *Without dividends, a European-style geometric average reset put with one reset date tends to become more valuable as the monitoring frequency increases.*

***Proof.*** Let $t$ be the reset date. The payoff of the put can be rewritten as

$$(\max\{K, \mathrm{avg}_n^G(t)\} - S(T))^+ = \begin{cases} (K - S(T)) + (\mathrm{avg}_n^G(t) - K)^+ & \text{if } K \geq S(T), \\ (\mathrm{avg}_n^G(t) - S(T))^+ & \text{if } S(T) > K. \end{cases}$$

Therefore, the expected payoff of the options is

$$\begin{aligned} & \mathrm{E}\left(\max\{K, \mathrm{avg}_n^G(t)\} - S(T)\right)^+ \\ = \ & \mathrm{E}\left(\mathrm{E}\left(\left(\max\{K, \mathrm{avg}_n^G(t)\} - S(T)\right)^+ |S(T)\right)\right) \\ = \ & \mathrm{E}\left(\mathrm{E}\left(\left[(K - S(T)) + \left(\mathrm{avg}_n^G(t) - K\right)^+\right] 1_{\{K \geq S(T)\}}\right) |S(T)\right) \\ + \ & \mathrm{E}\left(\mathrm{E}\left(\left(\mathrm{avg}_n^G(t) - S(T)\right)^+ 1_{\{S(T) > K\}}\right) |S(T)\right). \end{aligned}$$

Assume $K'$ is a known constant. Some terms take the form $(\mathrm{avg}_n^G(t) - K')^+$, which is similar to the payoff of a call option that takes $\mathrm{avg}_n^G(t)$ as its underlying asset and $C$ as its strike price, in the above formula. Recall that the price process, $\{\ln(S(t)/S(0))\}$, is a Brownian motion; hence, $\{\ln(S(t)/S(0))\}$ conditional on $S(T)$ must be a Brownian bridge. Note that $\ln(\mathrm{avg}_n^G(t)/S(0))$ conditional on $S(T)$ is a normal random variable. Lemma 3.4.5 says that both the mean of $\mathrm{avg}_n^G(t)/S(0)$ and the variance of $\ln(\mathrm{avg}_n^G(t)/S(0))$ increase as the monitoring frequency increases. Since *rho* and *vega* (the Greek letters) of a call are positive, the value $\mathrm{E}(\mathrm{avg}_n^G(t) - K')^+$ increases as the sampling frequency increases. Consequently, a reset put become more valuable as the sampling frequency increases. □

On the other hand, a reset call might be less variable when the monitoring frequency becomes larger. This is because the expected payoff of a reset call can be rewritten as

$$\begin{aligned} & \mathrm{E}\left(S(T) - \min\{K, \mathrm{avg}_n^G(t)\}\right)^+ \\ = \ & \mathrm{E}\left(\mathrm{E}\left(\left[(S(T) - K) + \left(K - \mathrm{avg}_n^G(t)\right)^+\right] 1_{\{S(T) \geq K\}}\right) |S(T)\right) \\ + \ & \mathrm{E}\left(\mathrm{E}\left(\left(S(T) - \mathrm{avg}_n^G(t)\right)^+ 1_{\{S(T) < K\}}\right) |S(T)\right), \end{aligned}$$

which contains some terms like the payoff of a vanilla put. Since the $\rho$ of a vanilla put is negative, these terms might contribute less value as the monitoring frequency increases. Some extreme experimental results are given in the following section to verify this fact. The sampling theorem can be extended to puts with multiple monitoring intervals.

**Corollary 3.4.8** (*Sampling frequency theorem: Multiple monitoring intervals version*). *Without dividends, a European-style geometric average reset put with the same multiple reset dates tends to be more valuable as the monitoring frequency increases if the monitoring intervals are disjoint.*

***Proof.*** Assume there are $m$ geometric average reset put. The $j$th ($j = 0, 1, \cdots, m$) put has $L$ sampling points in the first $j$ monitoring intervals and $H$ sampling points in the other $m-j$ monitoring intervals, where the integer $H$ is a given positive larger than integer $L$. $P(\overbrace{L, \ldots, L}^{j}, \overbrace{H, \ldots, H}^{m-j})$ denotes the price of the $j$th put. We prove the goal by showing that

$$P(\overbrace{L, \ldots, L}^{m}) \leq P(\overbrace{L, \ldots, L}^{m-1}, H) \leq \cdots \leq P(\overbrace{H, \cdots, H}^{m}).$$

Suppose that

$$K_j = \max\left\{\text{avg}_L^G(t_1), \ldots, \text{avg}_L^G(t_{j-1}), \text{avg}_H^G(t_{j+1}), \ldots, \text{avg}_H^G(t_m)\right\}.$$

Comparing the expected payoff of the $m$th put with that of the $(m-1)$th put, we have

$$\begin{aligned}
& \text{E}\left(\max\{K_m, \text{avg}_L^G(t_m)\} - S(T)\right)^+ \\
= \; & \text{E}\left(\text{E}\left(\left(\max\{K_m, \text{avg}_L^G(t_m)\} - S(T)\right)^+ |S(T), S(t_{m-1}), K_m\right)\right) \\
\leq \; & \text{E}\left(\text{E}\left(\left(\max\{K_m, \text{avg}_H^G(t_m)\} - S(T)\right)^+ |S(T), S(t_{m-1}), K_m\right)\right) \\
= \; & \text{E}\left(\max\{K_m, \text{avg}_H^G(t_m)\} - S(T)\right)^+
\end{aligned}$$

According to Theorem 3.4.7, conditional on $S(T)$, $S(t_{m-1})$ and $K_m$, the value

$$\text{E}\left(\left(\max\{K_m, \text{avg}_n^G(t_m)\} - S(T)\right)^+ |S(T), S(t_{m-1}), K_m\right)$$

increases with $n$. So we conclude that $P(\overbrace{L, \cdots, L}^{m}) \leq P(\overbrace{L, \cdots, L}^{m-1}, H)$. In the same way, it is also true that

$$\begin{aligned}
& \text{E}\left(\max\{K_{m-1}, \text{avg}_L^G(t_{m-1})\} - S(T)\right)^+ \\
= \; & \text{E}\left(\text{E}\left(\left(\max\{K_{m-1}, \text{avg}_L^G(t_{m-1})\} - S(T)\right)^+ |S(T), S(t_{m-2}), S(t_{m-1}), K_{m-1}\right)\right) \\
\leq \; & \text{E}\left(\text{E}\left(\left(\max\{K_{m-1}, \text{avg}_H^G(t_{m-1})\} - S(T)\right)^+ |S(T), S(t_{m-2}), S(t_{m-1}), K_{m-1}\right)\right) \\
= \; & \text{E}\left(\max\{K_{m-1}, \text{avg}_H^G(t_{m-1})\} - S(T)\right)^+.
\end{aligned}$$

According to Theorem 3.4.7, conditional on $S(T)$, $S(t_{m-2})$, $S(t_{m-1})$ and $K_{m-1}$, the value

$$\mathrm{E}\left(\left(\max\{K_{m-1}, \mathrm{avg}_H^G(t_{m-1})\} - S(T)\right)^+ |S(T), S(t_{m-2}), S(t_{m-1}), K_{m-1}\right)$$

increases with $n$. We conclude that

$$P(\overbrace{L, \ldots, L}^{m-1}, \overbrace{H}^{1}) \leq P(\overbrace{L, \ldots, L}^{m-2}, \overbrace{H, H}^{2}).$$

Similarly, we can also show that

$$P(\overbrace{L, \ldots, L}^{m-2}, \overbrace{H, H}^{2}) \leq P(\overbrace{L, \ldots, L}^{m-3}, \overbrace{H, \ldots, H}^{3}) \leq \cdots \leq P(\overbrace{H, \ldots, H}^{m})$$

by induction, so the proof is complete. □

We turn to the influence of the length of monitoring interval on option values. The price of put or call is not have clear relationship with the length of monitoring interval. We will take examples in next section.

## 3.5 Experimental Results

| Reset Date | Exact | M.C. | CZ |
|---|---|---|---|
| 1.00 | 17.254 | 17.189 | 10.866 |
| 0.75 | 18.141 | 17.990 | 10.553 |
| 0.50 | 18.226 | 18.133 | 9.139 |
| 0.25 | 17.847 | 17.936 | 5.567 |

Table 3.1: **Comparison of the analytic formula in Cheng and Zhang [2000] and the ones in our paper.**

First, we compare our analytic formula with the one suggested by Cheng and Zhang [2000]. The result is illustrated in Fig. 3.5 where "CZ" denotes the formula in Cheng and Zhang [2000][3] and "Exact" denotes ours. We use Monte Carlo simulations, denoted as "M.C.", as a benchmark to verify the correctness of both analytic formula. Suppose that the initial stock price is 100, the initial strike price is 95, the interest rate 5%, the volatility is 30%, the time to maturity for the option is 1 year, and the length of monitoring interval is 0.06 year in Table 3.5. "Reset Date" denotes the the ending time of monitoring interval. Obviously, our results are consistent with

| Case 1: $S_0 = 100$, $K = 250$, $r = 8\%$, $\sigma = 8\%$, $T = 4$, $\ell = 1$, Reset Date=1. | | |
|---|---|---|
| number of samples | Call | Put |
| Cont. | 24.5946 | 81.5378 |
| 250 | 24.5947 | 81.5378 |
| 24 | 24.5954 | 81.5378 |
| 12 | 24.5961 | 81.5378 |
| 4 | 24.5985 | 81.5378 |
| 2 | 24.6010 | 81.5378 |
| Case 2: $S_0 = 100$, $K = 100$, $r = 10\%$, $\sigma = 60\%$, $T = 4$, $\ell = 1$, Reset Date=1. | | |
| number of samples | Call | Put |
| Cont. | 58.2813 | 27.4527 |
| 250 | 58.2776 | 27.4429 |
| 24 | 58.2443 | 27.3540 |
| 12 | 58.2098 | 27.2626 |
| 4 | 58.0917 | 26.9565 |
| 2 | 57.9580 | 26.6217 |

Table 3.2: **Verifications on Sampling Theorem.**

simulation results, whereas Cheng and Zhang produce incorrect ones, the correctness of the especially when the reset date is close to the beginning of the option.

The Table3.2 shows some evidence for sampling theorems. All the data in this table are computed by the analytic formula. “Cont.” denotes samples in the monitoring interval is continuous. The data show that the price of the reset put increases as the number of samples increases. But the reset call fails to satisfy this claims in case 1 where the exercise price is extremely high and the volatility is extremely small. This is consistent with our previous analysis.

[3]The Complete formulae are listed in Appendix. A.

| Case 1: $S_0 = 100$, $K = 100$, $r = 1\%$, $\sigma = 60\%$, $T = 4$, Reset Date=4. | | | | |
|---|---|---|---|---|
| | Call | | Put | |
| interval length | $m = 12$ | Cont. | $m = 12$ | Cont. |
| $\ell = 0.1$ | 47.2605 | 47.2831 | 45.1673 | 45.2411 |
| $\ell = 0.3$ | 47.8625 | 47.9024 | 46.8209 | 46.9624 |
| $\ell = 0.5$ | 48.2088 | 48.2613 | 47.7313 | 47.9254 |
| $\ell = 0.7$ | 48.4482 | 48.5117 | 48.3250 | 48.5652 |
| $\ell = 0.9$ | 48.6238 | 48.6973 | 48.7268 | 49.0090 |
| Case 2: $S_0 = 100$, $K = 100$, $r = 10\%$, $\sigma = 60\%$, $T = 4$, Reset Date=1. | | | | |
| | Call | | Put | |
| interval length | $m = 12$ | Cont. | $m = 12$ | Cont. |
| $\ell = 0.1$ | 59.5782 | 59.5823 | 32.0651 | 32.0823 |
| $\ell = 0.3$ | 59.3682 | 59.3816 | 31.1173 | 31.1690 |
| $\ell = 0.5$ | 59.1194 | 59.1440 | 30.1218 | 30.2086 |
| $\ell = 0.7$ | 58.8181 | 58.8569 | 29.0596 | 29.1834 |
| $\ell = 0.9$ | 58.4405 | 58.4987 | 27.8982 | 28.0639 |

Table 3.3: **Results of Different Monitoring Intervals.**

Table 3.3 shows that the relationship between the price and the length of monitoring interval. There, $m$ denotes the number of samples in the monitoring interval. In case 1, the data suggest that both the prices of call and put increase with monitoring interval longer; in case 2, the data suggest that both the prices of call and put decrease with monitoring interval longer.

# Chapter 4

# Pricing of Other Exotic Options

## 4.1 Analytic Formula for Foreign Domestic Options

A foreign domestic options are concerned with foreign exchange rates. Usually, the case is a call option on the foreign asset evaluated in foreign currency but its strike price is given in domestic currency. $S_f(t)$ denotes the foreign asset's price evaluated in foreign currency at time $t$ and the underlying asset has continuous dividend rate $q$ and volatility $\sigma_{S_f}$. The foreign exchange rate at time $t$ is denoted as $F(t)$ and its volatility is $\sigma_F$. Assume $r_f$ is the foreign risk-free interest rate and $r_d$ is the domestic risk-free interest rate. Consider a European-style call option on the foreign asset with maturity date at time $T$ and the strike price $K$ given in domestic currency. Then the payoff of the option is

$$\begin{cases} S_f(T)F(T) - K, & \text{if} \quad S_f(T)F(T) \geq K, \\ 0, & \text{if} \quad S_f(T)F(T) < K. \end{cases}$$

In other words, $S_f(t)F(t) = S_d(t)$ can be thought as the price of foreign asset evaluated in domestic currency at time $t$. By assumption, the price process of the two assets is

$$\begin{bmatrix} dF(t) \\ dS_f(t) \end{bmatrix} = \begin{bmatrix} r_d - r_f \\ \mu - q \end{bmatrix} dt + \begin{bmatrix} \sigma_F & 0 \\ 0 & \sigma_{S_f} \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix},$$

for some Brownian motion $B = (B_1, B_2)$. Or

$$\begin{bmatrix} dF(t) \\ dS_f(t) \end{bmatrix} = \begin{bmatrix} r_d - r_f \\ \mu - q \end{bmatrix} dt + \begin{bmatrix} \sigma_{F,1} & \sigma_{F,2} \\ \sigma_{S_f,1} & \sigma_{S_f,2} \end{bmatrix} \begin{bmatrix} dW_1 \\ dW_2 \end{bmatrix},$$

where $\rho$ is the correlative coefficient between $B_1$ and $B_2$ and $W = (W_1, W_2)$ is a two-dimensional Wiener process. Besides, $\sigma_F^2 = \sigma_{F,1}^2 + \sigma_{F,2}^2$ and $\sigma_{S_f}^2 = \sigma_{S_f,1}^2 + \sigma_{S_f,2}^2$. Define $X = (X_1, X_2)$ where $X_1 = \ln(F(T)/F(0))$ and $X_2 = \ln(S_f(T)/S_f(0))$. The joint distribution of $X_1$ and $X_2$ is two-dimensional normal distribution, since the process of $S_f(t)$ and $F(t)$ are geometric Brownian motion as follows:

$$\begin{aligned} dF &= F(r_d - r_f)dt + F\sigma_F dB_1, \\ dS_f &= S_f(r_f - q - \rho\sigma_{S_f}\sigma_F)dt + S_f\sigma_{S_f}dB_2. \end{aligned}$$

Then the payoff also can be expressed as

$$\begin{aligned} & (S_f(T)F(T) - K)1_{\{S_f(T)F(T)\geq K\}} \\ = \;& S_f(0)F(0)\left[(S_f(T)/S_f(0))(F(T)/F(0)) - K\right]1_{\{[S_f(T)/S_f(0)][F(T)/F(0)]\geq \ln(K/[S_f(0)F(0)])\}} \\ = \;& S_f(0)F(0)e^{X_1+X_2}1_{\{X_1+X_2\geq \ln(K/[S_f(0)F(0)])\}} - K1_{\{X_1+X_2\geq \ln(K/[S_f(0)F(0)])\}}. \end{aligned}$$

The mean and covariance matrix of $X$ are as follows:

$$\begin{aligned} \mu = \mathrm{E}(X) &= \left((r_d - r_f - \sigma_F^2/2)T, (r_f - q - \rho\sigma_F\sigma_{S_f} - \sigma_{S_f}^2/2)T\right), \\ \Sigma = \mathrm{Var}(X) &= \begin{bmatrix} \sigma_F^2 T & \rho\sigma_F\sigma_{S_f}T \\ \rho\sigma_F\sigma_{S_f}T & \sigma_{S_f}^2 T \end{bmatrix}. \end{aligned}$$

By Theorem 2.3.1, let $b = (1, 1)$, then the expected value of the payoff is

$$\begin{aligned} & S_f(0)F(0)\int_A e^{X_1+X_2}\frac{e^{-\left\{[x-(\mu+b\Sigma)]\Sigma^{-1}[x-(\mu+b\Sigma)]^*\right\}/2}}{2\pi(\det\Sigma)^{\frac{1}{2}}}dx - K\int_A \frac{e^{-[(x-\mu)\Sigma^{-1}(x-\mu)^*]/2}}{2\pi(\det\Sigma)^{\frac{1}{2}}}dx \\ = \;& S_d(0)e^{r_d-q}\int_A \frac{e^{-\left\{[x-(\mu+b\Sigma)]\Sigma^{-1}[x-(\mu+b\Sigma)]^*\right\}/2}}{2\pi(\det\Sigma)^{\frac{1}{2}}}dx - K\int_A \frac{e^{-[(x-\mu)\Sigma^{-1}(x-\mu)^*]/2}}{2\pi(\det\Sigma)^{\frac{1}{2}}}dx, \quad (4.1) \end{aligned}$$

where

$$A = \{X | X_1 + X_2 \geq \ln(K/[S_f(0)F(0)])\}.$$

Suppose that

$$\begin{aligned} Y_1 = (-X_1 - X_2)/\sqrt{\mathrm{Var}(X_1 + X_2)} = (-X_1 - X_2)/\sqrt{(\sigma_F^2 + \sigma_{S_f}^2 + 2\rho\sigma_F\sigma_{S_f})T}, \\ Y_2 = (X_1 - X_2)/\sqrt{\mathrm{Var}(X_1 + X_2)} = (X_1 - X_2)/\sqrt{(\sigma_F^2 + \sigma_{S_f}^2 - 2\rho\sigma_F\sigma_{S_f})T}. \end{aligned}$$

Next, change of random variables is needed to make $A$ become a rectangle. Suppose that the transformation required is

$$C = \begin{bmatrix} -1/\sqrt{(\sigma_F^2 + \sigma_{S_f}^2 + 2\rho\sigma_F\sigma_{S_f})T} & -1/\sqrt{(\sigma_F^2 + \sigma_{S_f}^2 + 2\rho\sigma_F\sigma_{S_f})T} \\ 1/\sqrt{(\sigma_F^2 + \sigma_{S_f}^2 - 2\rho\sigma_F\sigma_{S_f})T} & -1/\sqrt{(\sigma_F^2 + \sigma_{S_f}^2 - 2\rho\sigma_F\sigma_{S_f})T} \end{bmatrix}.$$

$A'$, the image of $A$ under the linear transformation $C$, is a half-plane as

$$A' = \left\{Y | Y_1 \leq \ln(S_f(0)F(0)/K)/\sqrt{(\sigma_F^2 + \sigma_{S_f}^2 + 2\rho\sigma_F\sigma_{S_f})T}\right\}.$$

Then equation (4.1) becomes

$$\begin{aligned} & S_d(0)e^{r_d-q}\int_{A'}\frac{e^{-\left\{[y-(\mu+b\Sigma)C^*]\Sigma^{-1}[y-(\mu+b\Sigma)C^*]^*\right\}/2}}{2\pi[\det(C\Sigma C^*)]^{\frac{1}{2}}}dy \\ - & K\int_{A'}\frac{e^{-[(y-\mu C^*)\Sigma^{-1}(y-\mu C^*)^*]/2}}{2\pi[\det(C\Sigma C^*)]^{\frac{1}{2}}}dy. \end{aligned}$$

Besides,

$$\begin{aligned} \mu C^* &= \left(\frac{(-r_d+q+\sigma_{S_d}^2/2)T}{\sigma_{S_d}\sqrt{T}}, \frac{(r_d-2r_f+q-\sigma_F^2/2+\rho\sigma_F\sigma_{S_f}+\sigma_{S_f}^2/2)T}{\sqrt{(\sigma_F^2+\sigma_{S_f}^2-2\rho\sigma_F\sigma_{S_f})T}}\right), \\ (\mu+b\Sigma)C^* &= \left(\frac{(-r_d+q-\sigma_{S_d}^2/2)T}{\sigma_{S_d}\sqrt{T}}, \frac{(r_d-2r_f+q+\sigma_F^2/2+\rho\sigma_F\sigma_{S_f}-\sigma_{S_f}^2/2)T}{\sqrt{(\sigma_F^2+\sigma_{S_f}^2-2\rho\sigma_F\sigma_{S_f})T}}\right), \\ C\Sigma C^* &= \begin{bmatrix} 1 & \frac{(\sigma_F^2-\sigma_{S_f}^2)T}{\sqrt{(\sigma_F^2+\sigma_{S_f}^2-2\rho\sigma_F\sigma_{S_f})}\sigma_{S_d}T} \\ \frac{(\sigma_F^2-\sigma_{S_f}^2)T}{\sqrt{(\sigma_F^2+\sigma_{S_f}^2-2\rho\sigma_F\sigma_{S_f})}\sigma_{S_d}T} & 1 \end{bmatrix}, \end{aligned}$$

where $\sigma_{S_d}=\sqrt{\sigma_F^2+\sigma_{S_f}^2+2\rho\sigma_F\sigma_{S_f}}$. Then the pricing formula discounted by $r_d$ is

$$\begin{aligned} & e^{-qT}S_d(0)\mathrm{N}\left(\frac{\ln(S_d(0)/K)+(r_d-q+\sigma_{S_d}^2/2)T}{\sigma_{S_d}\sqrt{T}}\right) \\ - & e^{-r_dT}K\mathrm{N}\left(\frac{\ln(S_d(0)/K)+(r_d-q-\sigma_{S_d}^2/2)T}{\sigma_{S_d}\sqrt{T}}\right). \end{aligned}$$

## 4.2 Compound Options

A compound option is an option whose underlying asset is another option. Consider a call option with maturity date $T$ and strike price $K$ on the asset with volatility $\sigma$. Then a call compound option is an option on the former call option with maturity date $T_C$ and strike price $K_C$, where $T_C < T$. Assume $S(t)$ is the price of the asset at time $t$, then payoff of the call compound option can be expressed as

$$\begin{aligned} & (S(T)-K)1_{\{S(T_C)\geq S',S(T)\geq K\}}-K_Ce^{r(T-T_C)}1_{\{S(T_C)\geq S'\}} \\ = & S(T)1_{\{S(T_C)\geq S',S(T)\geq K\}}-K1_{\{S(T_C)\geq S',S(T)\geq K\}}-e^{r(T-T_C)}K_C1_{\{S(T_C)\geq S'\}}, \quad (4.2) \end{aligned}$$

where $S'$ is the price of the asset such that the compound option at the money at time $T_C$. Let $X=(X_1,X_2)$, where $X_1=\ln(S(T_C)/S(0))$, $X_2=\ln(S(T)/S(0))$. The mean and the variance of $X$ are

$$\begin{aligned}\mu = \mathrm{E}(X) &= \left((r-\sigma^2/2)T_C, (r-\sigma^2/2)T\right), \\ \Sigma = \mathrm{Var}(X) &= \begin{bmatrix} \sigma^2 T_C & \sigma^2 T_C \\ \sigma^2 T_C & \sigma^2 T \end{bmatrix}.\end{aligned}$$

Then equation (4.2) becomes

$$\begin{aligned} & S(0)e^{X_2}1_{\{X_1 \geq \ln(S'/S(0)), X_2 \geq \ln(K/S(0))\}} \\ - \;& K1_{\{X_1 \geq \ln(S'/S(0)), X_2 \geq \ln(K/S(0))\}} \\ - \;& e^{r(T-T_C)}K_C 1_{\{X_1 \geq \ln(S'/S(0))\}}.\end{aligned}$$

Let

$$\begin{aligned} A_1 &= \{X | X_1 \geq \ln(S'/S(0))\}, \\ A_2 &= \{X | X_1 \geq \ln(S'/S(0)), X_2 \geq \ln(K/S(0))\}, \\ b &= (0, 1).\end{aligned}$$

The excepted value of the above equation are

$$\begin{aligned} & S(0)e^{rT}\int_{A_2} \frac{e^{-\left\{[x-(\mu+b\Sigma)]\Sigma^{-1}[x-(\mu+b\Sigma)]^*\right\}/2}}{2\pi(\det\Sigma)^{\frac{1}{2}}}dx \\ - \;& K\int_{A_2} \frac{e^{-\left\{[x-\mu]\Sigma^{-1}[x-\mu]^*\right\}/2}}{2\pi(\det\Sigma)^{\frac{1}{2}}}dx \\ - \;& e^{(T-T_C)r}K_C\int_{A_1} \frac{e^{-\left\{[x-\mu]\Sigma^{-1}[x-\mu]^*\right\}/2}}{2\pi(\det\Sigma)^{\frac{1}{2}}}\}dx. \end{aligned} \tag{4.3}$$

Fortunately, although there are two integration areas, we only need one transformation. Suppose that

$$\begin{aligned} Y_1 &= -X_1/\sqrt{\mathrm{Var}(X_1)} = -X_1/\sqrt{\sigma^2 T_C}, \\ Y_2 &= -X_2/\sqrt{\mathrm{Var}(X_2)} = -X_2/\sqrt{\sigma^2 T}.\end{aligned}$$

Therefore, the transformation $C$ is

$$C = \begin{bmatrix} -1/\sqrt{\sigma^2 T_C} & 0 \\ 0 & -1/\sqrt{\sigma^2 T} \end{bmatrix}.$$

Then,

$$\begin{aligned} A'_1 &= \{Y | Y_1 \geq \ln(S(0)/S')/\sqrt{\sigma^2 T_C}\}, \\ A'_2 &= \{Y | Y_1 \geq \ln(S(0)/S'), Y_2 \geq \ln(S(0)/K)/\sqrt{\sigma^2 T}\}, \\ \Sigma' = C\Sigma C^* &= \begin{bmatrix} 1 & \sqrt{T_C/T} \\ \sqrt{T_C/T} & 1 \end{bmatrix}.\end{aligned}$$

Finally, the pricing formula is equation (4.3) multiplied by the discount factor:

$$\begin{aligned} & e^{-rT}\left[S(0)e^{rT}\mathrm{N}(v_1,\Sigma') - K\mathrm{N}(v_2,\Sigma') - e^{(T-T_C)r}K_C\mathrm{N}(v_3,1)\right] \\ = \; & S(0)\mathrm{N}(v_1,\Sigma') - e^{-rT}K\mathrm{N}(v_2,\Sigma') - e^{-rT_C}K_C\mathrm{N}(v_3,1). \end{aligned}$$

where

$$\begin{aligned} v_1 &= d_1 - (\mu + b\Sigma)C^* = \left(\frac{\ln(S(0)/S') + (r+\sigma^2/2)T_C}{\sigma\sqrt{T_C}}, \frac{\ln(S(0)/S') + (r-\sigma^2/2)T_C}{\sigma\sqrt{T_C}}\right), \\ v_2 &= d_2 - (\mu + b\Sigma)C^* = \left(\frac{\ln(S(0)/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \frac{\ln(S(0)/K) + (r-\sigma^2/2)T}{\sigma\sqrt{T}}\right), \\ v_3 &= \frac{\ln(S(0)/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}. \end{aligned}$$

## 4.3 Rainbow Options

A rainbow Option is an option with $m$ underlying assets. It has maturity date $T$ and strike price $K$. For a rainbow call option, then its payoff can be expressed as

$$\max\{(\max\{S_1(T), S_2(T), \ldots, S_m(T)\} - K), 0\},$$

where $S_i(t)$ denotes the price of the $i$th underlying asset at time $t$. It means we can choose the favorite one among the $m$ underlying assets to exercise. Similarly, a rainbow put option can be expressed as below

$$\max\{(K - \min\{S_1(T), S_2(T), \ldots, S_m(T)\}), 0\}.$$

The premium of a rainbow option must be higher than that of the vanilla option. We can also get the pricing formula for rainbow options by our approach.

Consider a rainbow option with maturity date $T$ and strike price $K$ based on two assets. For $i = 1, 2$, assume $S_i(t)$ and $\sigma_i$ are the price and volatility of the $i$th asset, and $\rho$ is the correlation coefficient between the two assets. By the assumption of price behavior, $X = (X_1, X_2)$ is a two-dimensional normal distribution, where $X_1 = \ln(S_1(T)/S_1(0))$ and $X_2 = \ln(S_2(T)/S_2(0))$. Hence, the mean and the variance matrix of $X$ are

$$\begin{aligned} \mu &= \mathrm{E}(X) = ((r-\sigma_1^2/2)T, (r-\sigma_2^2/2)T), \\ \Sigma &= \mathrm{Var}(X) = \begin{bmatrix} \sigma_1^2 T & \rho\sigma_1\sigma_2 T \\ \rho\sigma_1\sigma_2 T & \sigma_2^2 T \end{bmatrix}. \end{aligned}$$

So the payoff of the rainbow option can be expressed as

$$\begin{aligned}
& \max\{\max\{S_1(T), S_2(T)\} - K, 0\} \\
= \ & (S_1(T) - K)1_{\{S_1(T)\geq K, S_1(T)\geq S_2(T)\}} + (S_2(T) - K)1_{\{S_2(T)\geq K, S_2(T)\geq S_1(T)\}} \\
= \ & S_1(0)e^{X_1}1_{\{X_1\geq \ln(K/S_1(0)),(X_1-X_2)\geq \ln(S_2(0)/S_1(0))\}} \\
& +S_2(0)e^{X_2}1_{\{X_2\geq \ln(K/S_2(0)),(X_2-X_1)\geq \ln(S_1(0)/S_2(0))\}} \\
& -K1_{\{X_1\geq \ln(K/S_1(0)),(X_1-X_2)\geq \ln(S_2(0)/S_1(0))\}} \\
& -K1_{\{X_2\geq \ln(K/S_2(0)),(X_2-X_1)\geq \ln(S_1(0)/S_2(0))\}}. \qquad (4.4)
\end{aligned}$$

Let

$$\begin{aligned}
b_1 &= (1, 0), \\
b_2 &= (0, 1), \\
A_1 &= \{X|X_1 \geq \ln(K/S_1(0)), (X_1 - X_2) \geq \ln(S_2(0)/S_1(0))\}, \\
A_2 &= \{X|X_2 \geq \ln(K/S_2(0)), (X_2 - X_1) \geq \ln(S_1(0)/S_2(0))\}.
\end{aligned}$$

Then the expected value of equation (4.4) becomes

$$\begin{aligned}
& S_1(0)e^{rT}\int_{A_1}\frac{e^{-\left\{[x-(\mu+b_1\Sigma)]\Sigma^{-1}[x-(\mu+b_1\Sigma)]^*\right\}/2}}{2\pi(\det\Sigma)^{\frac{1}{2}}}dx \\
+ \ & S_2(0)e^{rT}\int_{A_2}\frac{e^{-\left\{[x-(\mu+b_2\Sigma)]\Sigma^{-1}[x-(\mu+b_2\Sigma)]^*\right\}/2}}{2\pi(\det\Sigma)^{\frac{1}{2}}}dx \\
- \ & K\int_{A_1}\frac{e^{-\left[(x-\mu)\Sigma^{-1}(x-\mu)^*\right]/2}}{2\pi(\det\Sigma)^{\frac{1}{2}}}dx \\
- \ & K\int_{A_2}\frac{e^{-\left[(x-\mu)\Sigma^{-1}(x-\mu)^*\right]/2}}{2\pi(\det\Sigma)^{\frac{1}{2}}}dx.
\end{aligned}$$

We need a transformation for each area. For area $A_1$, suppose that

$$\begin{aligned}
Y_1 &= -X_1/\mathrm{Var}(X_1) = -X_1/\sqrt{\sigma_1^2 T}, \\
Y_2 &= (-X_1 + X_2)/\mathrm{Var}(-X_1 + X_2) = (-X_1 + X_2)/\sqrt{(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)T}.
\end{aligned}$$

Therefore,

$$C_1 = \begin{bmatrix} -1/\sqrt{\sigma_1^2 T} & 0 \\ -1/\sqrt{(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)T} & 1/\sqrt{(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)T} \end{bmatrix}.$$

For area $A_2$, suppose that

$$\begin{aligned}
Y_1 &= (X_1 - X_2)/\mathrm{Var}(X_1 - X_2) = (X_1 - X_2)/\sqrt{(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)T}, \\
Y_2 &= -X_2/\mathrm{Var}(X_2) = -X_2/\sqrt{\sigma_2^2 T}.
\end{aligned}$$

Therefore,

$$C_2 = \begin{bmatrix} 1/\sqrt{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)T} & -1/\sqrt{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)T} \\ 0 & -1/\sqrt{\sigma_2^2 T} \end{bmatrix}.$$

Then,

$$\begin{aligned}
A_1' &= \left\{Y|Y_1 \le \ln(S_1(T)/K)/\sqrt{\sigma_1^2 T}, Y_2 \le \ln(S_1(T)/S_2(T))/\sqrt{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)T}\right\}, \\
A_2' &= \left\{Y|Y_1 \le \ln(S_2(T)/S_1(T))/\sqrt{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)T}, Y_2 \le \ln(S_2(T)/K)/\sqrt{(\sigma_2^2)T}\right\}, \\
\Sigma_1 &= C_1\Sigma C_1^* = \begin{bmatrix} 1 & \frac{\rho\sigma_1\sigma_2-\sigma_1^2}{\sqrt{\sigma_1^2(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)}} \\ \frac{\rho\sigma_1\sigma_2-\sigma_1^2}{\sqrt{\sigma_1^2(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)}} & 1 \end{bmatrix}, \\
\Sigma_2 &= C_2\Sigma C_2^* = \begin{bmatrix} 1 & \frac{\rho\sigma_1\sigma_2-\sigma_2^2}{\sqrt{\sigma_1^2(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)}} \\ \frac{\rho\sigma_1\sigma_2-\sigma_2^2}{\sqrt{\sigma_1^2(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)}} & 1 \end{bmatrix}.
\end{aligned}$$

The pricing formula emerges as

$$e^{-rT}\left[S_1(0)\mathrm{N}(v_1,\Sigma_1)+S_2(0)\mathrm{N}(v_2,\Sigma_2)-K\mathrm{N}(v_3,\Sigma_1)-K\mathrm{N}(v_4,\Sigma_2)\right],$$

where

$$\begin{aligned}
v_1 &= \left(\frac{\ln(S_1(T)/K)+(r+\sigma_1^2/2)T}{\sigma_1\sqrt{T}}, \frac{\ln(S_1(T)/S_2(T))+(\sigma_1^2/2+\sigma_2^2/2-\rho\sigma_1\sigma_2)T)}{\sqrt{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)T}}\right), \\
v_2 &= \left(\frac{\ln(S_2(T)/S_1(T))+(\sigma_1^2/2+\sigma_2^2/2-\rho\sigma_1\sigma_2)T)}{\sqrt{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)T}}, \frac{\ln(S_2(T)/K)+(r+\sigma_2^2/2)T}{\sigma_2\sqrt{T}}\right), \\
v_3 &= \left(\frac{\ln(S_1(T)/K)+(r-\sigma_1^2/2)T}{\sigma_1\sqrt{T}}, \frac{\ln(S_1(T)/S_2(T))-(\sigma_1^2/2-\sigma_2^2/2)T)}{\sqrt{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)T}}\right), \\
v_4 &= \left(\frac{\ln(S_2(T)/S_1(T))-(\sigma_2^2/2-\sigma_1^2/2)T)}{\sqrt{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)T}}, \frac{\ln(S_2(T)/K)+(r-\sigma_2^2/2)T}{\sigma_2\sqrt{T}}\right).
\end{aligned}$$

# Chapter 5

# Conclusions

Although the variety of options are large, if the payoff of the option can be expressed as a linear combination of form $e^{bX^*}1_{\{X\in A\}}$, it can be priced by the systematic approach to get the pricing formula. Besides, the formula of Cheng and Zhang [2000] is wrong. Furthermore, we get some properties of European-style geometric average reset options, summarized below:

1. Without dividends, the European-style geometric average reset call option will not be exercised early.

2. The European-style geometric average reset put option tends to be more valuable with more reset dates. But it is not true for geometric average reset call options.

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# Appendix A

# Cheng and Zhang's Analytic Formula

The analytic pricing formula offered by Cheng and Zhang [200] for geometric average reset options with one monitoring interval is as follows:

$$\begin{aligned} & S(0)\mathrm{N}(e_1, e_2, \breve{\Sigma}) - Ke^{-rT}\mathrm{N}(\hat{e_1}, \hat{e_2}, \breve{\Sigma}) \\ + \quad & S(0)\mathrm{N}(f_1, f_2, \grave{\Sigma}) - S(0)e^{-r\left(T-t_1+\frac{\ell}{2}\right)+5\ell\sigma^2/12}\mathrm{N}(f_3, f_4, \grave{\Sigma}), \end{aligned}$$

where $e_1$, $e_2$, $\breve{\Sigma}$, $\hat{e_1}$, $\hat{e_2}$, $f_1$, $f_2$, $\grave{\Sigma}$, $f_3$, $f_4$ are defined as follows:

$$\begin{aligned} e_1 &= \frac{\ln(S(0)/K) + (r + \sigma^2/2)\,(t_1 - \ell/2)}{\sigma\sqrt{t_1 + \ell/3}}, \\ e_2 &= \frac{\ln(S(0)/K) + (r + \sigma^2/2)\,T}{\sigma\sqrt{T}}, \\ \hat{e_1} &= \frac{\ln(S(0)/K) + (r - \sigma^2/2)\,(t_1 - \ell/2)}{\sigma\sqrt{t_1 + \ell/3}}, \\ \hat{e_2} &= \frac{\ln(S(0)/K) + (r - \sigma^2/2)\,T}{\sigma\sqrt{T}}, \\ \breve{\Sigma} &= \begin{bmatrix} 1 & \frac{t_1-\ell/2}{\sqrt{T}\sqrt{t_1+\frac{\ell}{3}}} \\ \frac{t_1-\ell/2}{\sqrt{T}\sqrt{t_1+\ell/3}} & 1 \end{bmatrix}, \\ f_1 &= -e_1, \\ f_2 &= \frac{(r + \sigma^2/2)\,(T - t_1 + \ell/2)}{\sigma\sqrt{T - t_1 + 4\ell/3}}, \end{aligned}$$

$$
\begin{aligned}
f_3 &= \frac{\ln(S(0)/K) + (r+\sigma^2/2)\,(t_1-\ell/2) - \sigma^2\,(t_1+\ell/3)}{\sigma\sqrt{t_1+\ell/3}}, \\
f_4 &= \frac{(r-\sigma^2/2)\,(T-t_1+\ell/2) - \sigma^2 5\ell/6}{\sigma\sqrt{T-t_1+4\ell/3}}, \\
\grave{\Sigma} &= \begin{bmatrix} 1 & \frac{5\ell/6}{\sqrt{T-t_1+4\ell/3}\sqrt{t_1+\ell/3}} \\ \frac{5\ell/6}{\sqrt{T-t_1+4\ell/3}\sqrt{t_1+\ell/3}} & 1 \end{bmatrix}.
\end{aligned}
$$