Pricing Moving-Average-Lookback Options

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Abstract

This thesis investigates computational methods for pricing complex path-dependent derivative securities, especially geometric- and arithmetic-moving-average-lookback options. The latter security was first issued by Polaris Securities in 1999. Our methodology can be easily modified to price similarly structured options issued by other securities firms. The moving-average-lookback option is a call option struck at the minimum moving average of underlying asset price. We consider both geometric averaging and the much harder arithmetic averaging used by Polaris. The pricing results show that our algorithms on the CRR model converge quickly to the correct value. We also find that the price difference between geometric averaging and arithmetic averaging is slight. As it takes much less time to price the geometric-moving-average version, it serves as a good approximation to the arithmetic-moving-average version. The least-squares simulation, introduced by Longstaff and Schwartz (2001), can be applied to price American-style moving-average-lookback options. Compared with our algorithms, the least-squares approach systematically undervalues the options. When applied to the two arithmetic-moving-average-lookback options issued by Polaris Securities in 1999, our algorithm prices them almost exactly. The numerical delta and gamma of the options are also investigated.

Chapter 1

Introduction

In Taiwan's warrants[1] market, some securities companies have issued warrants whose strike prices are related to the arithmetic moving average of the underlying stock price. The list of firms consists of all major players in the marketplace: Grand Cathay Securities (capital 10.5 billion NTD[2]), Yuanta Securities (capital 11.7 billion NTD), National Securities (capital 8 billion NTD), Fubon Securities (capital 10.5 billion NTD), Capital Securities (capital 9.2 billion NTD), and Polaris Securities (capital 6 billion NTD). (The capitals are based on 1999 filings.) The most prominent examples are moving-average-reset warrants and moving-average-lookback warrants. A moving-average-reset warrant is struck at a series of decreasing contract-specified prices over a monitoring window based on the moving average. A moving-average-lookback warrant is slightly more complicated. It is struck at the minimum moving average of the underlying stock price over a monitoring window. Moving-average-style warrants made up a significant portion of the warrants market in the most recent bull market. In 1999, 12 moving-average-style warrants were issued. The total premium amount stood at 2.864 billion NTD, which was more than 21% of the warrants market (see Table 1.1).

In practice, moving average is a popular technical measure for short-term trends in stock prices. Hence, it is straightforward to associate the moving average with the strike price. The advantage is, first, to avoid manipulation of the stock price as the reset date approaches and, second, to provide an objective way to determine the strike price of the options. Surprisingly, there has been scant research on the pricing of moving-average-reset and moving-average-lookback options. Lee (2001) discusses moving-average-reset options, but he considers moving average in continuous time. This thesis, in contrast, focuses on discretely sampled moving average, which is used in practice. We will concentrate on the moving-average-lookback option (MAL, hereafter), as the slightly simpler moving-average-reset option can be handled similarly.

[1] Options in Taiwan are called warrants.
[2] New Taiwan dollars.

Name	Type	Issue Date	Maturity (Year)	No. Shares (Thousands)	Premium (Millions NTD)
GC 06	Reset	28-Apr	1	10,700	216.68
GC 07	Reset	27-May	1	17,500	239.40
GC 08	Reset	09-Jun	1	10,000	248.30
GC 09	Reset	14-Jun	1	13,600	206.86
GC 10	Reset	20-Oct	1	12,000	262.44
YT 07	Reset	23-Nov	1	22,000	226.64
NS 02	Reset	16-Jun	1	10,000	200.00
FB 01	Reset	08-Jul	1	20,000	220.00
FB 02	Reset	18-Aug	1	18,000	306.00
CS 04	Reset	04-Sep	1	11,700	216.74
PL 06	Lookback	21-Aug	1	10,000	269.80
PL 07	Lookback	27-Aug	1	15,000	251.40

Table 1.1: **Moving-Average-Style Warrants Issued in 1999 on the Taipei Stock Exchange.** To put the numbers in perspective, the total amount of warrants issued in 1999 was 13,381.70 million NTD. GC: Grand Cathay Securities; YT: Yuanta Securities; NS: National Securities; FB: Fubon Securities; CS: Capital Securities; PL: Polaris Securities.

Pricing moving-average-style options is difficult. First, the moving-average process is non-Markovian. Let M_t^6 be the 6-day moving average at the tth trading day, where $t \geq 5$ and S_t denotes the stock price at day t. Then

$$M_t^6 = \frac{\sum_{i=t-5}^{t} S_i}{6}.$$

The moving average M_{t+1}^6 is related to not just M_t^6 but also $M_{t-1}^6, \ldots, M_{t-4}^6$, in other words, $S_t, S_{t-1}, \ldots, S_{t-4}$. In contrast, the average process denoted by

$$A_t = \frac{\sum_{i=0}^{t} S_i}{t+1}$$

is Markovian because A_{t+1} is a function of A_t:

$$A_{t+1} = \frac{A_t(t+1) + S_{t+1}}{t+2}.$$

It stands to reason that potentially more states are needed to record past information than, say, Asian options. Particularly in pricing MAL with *arithmetic* averages, we face not only the combinatorial explosion because of the nonnormality of the sum of lognormal distributions, but also the non-Markovian property mentioned above. These two issues combine to increase the difficulty of pricing.

There are three major ways to the pricing of derivative securities. The first is to derive closed-form (or analytical) solutions of derivatives by the partial-differential-equation (PDE) or martingale method. The second, thanks to Cox, Ross, and Rubinstein (1979), is the tree approach used especially to price American-style derivatives. The last is Monte Carlo simulation approach used most effectively to price strongly path-dependent and multifactor derivatives.

We will use the martingale approach to derive an analytical solution to the MAL with geometric averaging. We then develop an algorithm based on the CRR model to value the American-style MAL whether averaging is geometric or the more difficult arithmetic. We also adopt the Monte Carlo simulation and least-squares simulation, which can be used to value American-style options in Longstaff and Schwartz (2001), to verify the prices based on the CRR model. It will be found that our algorithms converge quickly to the correct value. The price difference between the geometric and the arithmetic MAL is very small. As it takes much less time to price the geometric MAL, it is a good approximation to the arithmetic MAL. We also find that the pricing results from the least-squares simulation are systematically undervalued; however, there is no sufficient evidence to reject the results.

The remainder of this thesis is organized as follows. Chapter 2 reviews basic concepts and pricing technologies. Chapter 3 covers the pricing of geometric MALs. Chapter 4 covers the pricing of arithmetic MALs and empirical studies of two MALs issued in Taiwan. Chapter 5 discusses numerical delta and gamma of MALs. Finally Chapter 6 summarizes results and points to future research.

Chapter 2

A Primer on Derivatives Pricing

In this chapter, we review fundamental concepts and pricing techniques used in later chapters.

2.1 The Martingale Pricing Approach

The foundation of the martingale pricing approach is laid in Harrison and Kreps (1979) and Harrison and Pliska (1981). This section relies on Pelsser (2000) in introducing this approach.

2.1.1 Basic Settings

Consider a continuous trading economy with the uncertainty modeled by the probability space $(\Omega, \mathcal{F}, P)$. In this notation, Ω denotes a sample space. $\mathcal{F}$ denotes a σ-algebra on Ω, and P denotes a probability measure on $(\Omega, \mathcal{F})$.

Throughout this thesis, we work in the Black-Scholes framework. Hence there are two basic securities in the economy. The first is a risky asset, stock, the price of which is denoted by S_t and follows geometric Browning motion,

$$dS_t = \mu S_t \, dt + \sigma S_t \, dZ_t^P. \tag{2.1}$$

Above, μ is the expected instantaneous rate of return, σ is the volatility of the instantaneous rate of return and $\{Z_t^P : t \in \Re^+\}$ is the Wiener process under measure P. The second security is the risk-free money-market account, the price of which is denoted by B_t with B_0=1. B_t follows

$$dB_t = r B_t \, dt. \tag{2.2}$$

The money-market account is assumed to earn a constant interest rate r. From Eqs. (2.1) and (2.2), we can obtain

$$S_t = S_0 \exp\left[(\mu - \frac{\sigma^2}{2})\, t + \sigma Z_t^P\right] \tag{2.3}$$

and

$$B_t = e^{rt}. \tag{2.4}$$

2.1.2 Girsanov's Theorem and Itô's Lemma

Girsanov's Theorem and Itô's Lemma are two key results in the martingale pricing approach. We state them below without proof.

Theorem 1 (Girsanov's Theorem) *For any random process $\lambda(t)$ such that*

$$\int_0^t \lambda(s)^2 ds < \infty$$

with probability one, consider the Radon-Nikodym derivative $dQ/dP = \rho(t)$ given by

$$\rho(t) = \exp\left(-\int_0^t \lambda(s)\, dZ_s^P - \frac{1}{2}\int_0^t \lambda(s)^2\, ds\right).$$

Then

$$Z_t^Q = Z_t^P + \int_0^t \lambda(s)\, ds,$$

where $t \in \Re^+$, is a Wiener process under measure Q.

Lemma 1 (Itô's Lemma) *Suppose the stochastic process x follows the stochastic differential equation $dx = \mu(t,\omega)\, dt + \sigma(t,\omega)\, dZ$ and function $f(t,x)$ is sufficiently differentiable. Then f satisfies*

$$df = \left(\frac{\partial f(t,x)}{\partial t} + \mu(t,\omega)\frac{\partial f(t,x)}{\partial x} + \frac{1}{2}\,\sigma(t,\omega)^2 \frac{\partial^2 f(t,x)}{\partial^2 x}\right) dt + \sigma(t,\omega)\, \frac{\partial f(t,x)}{\partial x}\, dZ.$$

The price of the money market account is strictly positive and can be used as a numeraire. Hence, relative price $S_t^* = S_t/B_t$ can be obtained. By Itô's lemma, the relative price follows

$$dS_t^* = (\mu - r) S_t^*\, dt + \sigma S_t^*\, dZ_t^P.$$

Apply Girsanov's theorem and let $\lambda(t) = (\mu - r)/\sigma$. The process S_t^* follows

$$\begin{aligned} dS_t^* &= (\mu - r) S_t^*\, dt + \sigma S_t^* \left(dZ_t^Q - \frac{\mu - r}{\sigma}\right) \\ &= \sigma S_t^*\, dZ_t^Q, \end{aligned} \tag{2.5}$$

which is a martingale under the new measure Q.

2.1.3 Equivalent Martingale Measure

Suppose the marketed asset V_1 is a numeraire and $V_n^* = V_n/V_1$ denote the relative price processes of another marketed asset V_1. Let $(\Omega, \mathcal{F}, P)$ be the probability space earlier. Consider the set that contains all probability measures P^* such that:

1. P^* is equivalent to P; i.e., measures P^* and P have the same null set.

2. The relative price processes V_n^* are martingales under measure P^* for all n; i.e., we have $E_t^{P^*}(V_n^*(T)) = V_n^*(t)$ for $t \leq T$.

Then we say the measures P^* are equivalent martingale measures associated with P.

Theorem 2 (The Unique Equivalent Martingale Measure) *A continuous economy is without arbitrage opportunities and every derivative security is attainable if there exists a unique equivalent martingale measure for every choice of numeraire such that relative price processes are martingales.*

Equation (2.5) implies that the discount process, the price process with the price of money-market account as the numeraire, of the stock is a martingale under the new measure Q. Therefore, the measure Q is an equivalent martingale measure associated with P. Besides, because of its uniqueness by the theorem above, we can conclude that the Black-Scholes economy is arbitrage-free and complete. Under the measure Q, the stock price process follows

$$dS_t = rS_t\,dt + \sigma S_t\,dZ_t^Q \tag{2.6}$$

or, equivalently

$$d\ln S_t = (r - \frac{1}{2}\sigma^2)\,dt + \sigma\,dZ_t^Q. \tag{2.7}$$

Different from Eq. (2.3), the solution of the above stochastic differential equation is

$$S_t = S_0 \exp\left[(r - \frac{\sigma^2}{2})\,t + \sigma Z_t^Q\right]. \tag{2.8}$$

The martingale property reflects mathematically the fact that it is impossible to outperform the market systematically in an arbitrage-free economy. From the result above, it follows that for a given numeraire M_t with unique equivalent martingale measure P^M, the relative value of a self-financing strategy $f_t^*(\delta)=f_t(\delta)/M_t$ is a P^M-martingale. Hence, for a replicating strategy δ_C that duplicates the derivative security whose payoff is C_T at expiration T, we obtain

$$E_t^M\left(\frac{C_T}{M_T}\right) = E_t^M\left(\frac{f_T(\delta_C)}{M_T}\right) = \frac{f_t(\delta_C)}{M_t}. \tag{2.9}$$

The last equality results from the definition of a martingale. Rearranging Eq. (2.9) yields

$$f_t(\delta_C) = M_t E_t^M \left(\frac{C_T}{M_T} \right). \tag{2.10}$$

Equation (2.10), the key result of the martingale pricing approach, can be used to determine the price of any derivative security C_T at time $t < T$. Replacing M with B and combining Eq. (2.4), we obtain the familiar equation

$$C_t = e^{-r(T-t)} E_t^Q(C_T). \tag{2.11}$$

2.1.4 The Change of Numeraire Theorem

The next theorem relates measure with numeraire.

Theorem 3 (Change of Numeraire) *Let P^N be the equivalent martingale measure with respect to numeraire N_t and P^M be the equivalent martingale measure with respect to numeraire M_t. The Radon-Nikodym derivative that changes the equivalent martingale measure P^M into P^N is given by*

$$\frac{dP^N}{dP^M} = \frac{N_T/N_t}{M_T/M_t}$$

Proof. From Eq. (2.10), which is feasible to any choice of numeraire and associated probability measure, it follows that

$$N_t E_t^N \left(\frac{C_T}{N_T} \right) = M_t E_t^M \left(\frac{C_T}{M_T} \right).$$

This expression can be rewritten as

$$E_t^N \left(G_T \right) = E_t^M \left(G_T \frac{N_T/N_T}{M_T/M_t} \right),$$

where $G_T = C_T/N_T$. The theorem holds for all random variable G and all numeraires N and M. $\triangle$

Applying the change of numeraire theorem in the Black-Scholes economy, we obtain

$$B_t E_t^Q \left(\frac{C_T}{B_T} \right) = S_t E_t^R \left(\frac{C_T}{S_T} \right),$$

where R is the equivalent martingale measure with respect to numeraire S_t, and

$$E_t^R(G_T) = E_t^Q(\xi_T G_T),$$

where ξ_T is the Radon-Nikodym derivative that changes the equivalent martingale measure Q into R. By the change of numeraire theorem,

$$\xi_T = \frac{S_T}{S_t}\, e^{-r(T-t)} = \exp\left[\sigma(Z_T^Q - Z_t^Q) - \frac{\sigma^2}{2}(T-t)\right]. \tag{2.12}$$

Mapping to Girsanov's theorem, one can find $\lambda(s) = -\sigma$ and hence

$$Z_t^R = Z_t^Q - \sigma t,$$

where $t \in \Re^+$, is the Wiener process under measure R. Furthermore, the dynamics of the stock price becomes

$$d \ln S_t = (r + \frac{1}{2}\,\sigma^2)\, dt + \sigma\, dZ_t^R,$$

which differ from Eq. (2.7).

The martingale pricing approach is the foundation of option pricing. We can calculate the price and other hedge parameters most efficiently via closed-form solutions resulting from this pricing approach. However, most derivatives in the markets are American-style. To price them, we will introduce the tree model in the next section.

2.2 Tree Models and Auxiliary State Variables

In this section, we review two useful pricing techniques. The first, the tree model, is mainly used to solve American-style options. The second, auxiliary state variables approach, is a general method to price path-dependent derivatives on the tree.

2.2.1 Tree Models

There are many kinds of tree models. Here we focus on the simplest but very powerful one, the CRR model, introduced in Cox, Ross, and Rubinstein (1979).

From Eq. (2.8), we can derive out, under the measure Q, or risk-neutral valuation, the expected value of the stock price after a small interval time Δt is $S_0 e^{r\Delta t}$ and the variance of the proportional stock price change after Δt time is $\sigma^2 \Delta t$. Now consider the discrete-time version of Eq. (2.6) and change the normal diffusion to a discrete random variable, ΔB. It follows that

$$\Delta S_t = r S_{t-\Delta t} \Delta t + \sigma S_{t-\Delta t} \Delta B_t.$$

Assume ΔB follows the Bernoulli distribution such that

$$S_{t+\Delta t} = \begin{cases} S_t u, & \text{with probability } p \ , \\ S_t d, & \text{with probability } (1-p) \ , \end{cases}$$

where u and d are the proportional change of S_t in the up and the down state. In order to describe the stock price process properly in discrete time, we let ΔB satisfy the mean and variance function mentioned above. This yields the following conditions,

$$\begin{aligned} e^{r\Delta t} &\approx pu + (1-p)d \\ \sigma^2 \Delta t &\approx pu^2 + (1-p)d^2 - [pu + (1-pd)]^2. \end{aligned}$$

With a third condition $ud = 1$, we obtain a possible solution:

$$\begin{aligned} p &= \frac{e^{r\Delta t} - d}{u - d}, \\ u &= e^{\sigma\sqrt{\Delta t}}, \\ d &= e^{-\sigma\sqrt{\Delta t}}. \end{aligned}$$

Thereafter, we can use p, u, and d above to describe the stock price process in discrete time. Let T denote expiration date, n denote the number of partitions, and $\Delta t = T/n$. The stock price on node $N(i,j)$ reachable from the root with j up and $i-j$ down moves is

$$S(i,j) = S_0 u^j d^{i-j},$$

and the value of derivatives C on node $N(i,j)$ can be obtained by the backward-induction formula:

$$C(i,j) = [\, pC(i+1, j+1) + (1-p)C(i+1,j)\,]\, e^{-r\Delta t}, \tag{2.13}$$

for $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, i$. When pricing American-style options, we change Eq. (2.13) into

$$C(i,j) = \max([\, pC(i+1, j+1) + (1-p)C(i+1,j)\,]\, e^{-r\Delta t}, \text{exercise value}).$$

In either case, the answer emerges in $C(0,0,0)$. We will apply the CRR model to price American-style MALs.

2.2.2 Auxiliary State Variables

This section draws on Dai (1999), which provides a general method for pricing path-dependent derivatives on CRR tree. Auxiliary state variables are memory space to record the past information needed in handling the path dependency. Let $C(i,j,k)$ denote the option value on node $N(i,j)$. In addition to i and j, which provide the information of time and the current stock price, we need an additional k to record the information arising from path dependency.

To apply backward induction without using approximation, we have to allocate enough auxiliary state variables for all the possible situations at each node. The size of auxiliary state variables therefore depends on the number of possible situations as determined by path dependency. This technique is not suitable for cases which need huge sizes of auxiliary state variables such as Asian options. However, the auxiliary state variables approach is useful in pricing "weakly" path-dependent derivatives. We remark that the alleged shortcoming of this approach no longer holds if we allow approximations.

Next we present two examples to illustrate the auxiliary-state-variables approach based on the CRR model.

Example 1 (Lookback Option) The payoff function of the lookback option at expiration date T is

$$\max(S_T - M_T, 0),$$

where

$$M_T = \min_{t=0,1,\ldots,n} S_t.$$

At time $i\Delta t$, the possible minimum stock prices belong in the set

$$\{S_0 u^{-i}, S_0 u^{-i+1}, \ldots, S_0\}.$$

Because $0 \leq i \leq n$, we have to allocate up to $n+1$ auxiliary state variables in each node, each denoting a particular minimum price. Let $C(i,j,k)$ denote the value of the lookback option on node $N(i,j)$ and k be the power index of the minimum stock price $S_0 u^k$ until time $i\Delta t$. For example, $k = -1$ means the minimum stock price until $i\Delta t$ is $S_0 u^{-1}$, and $k = -2$ means the minimal stock price until $i\Delta t$ is $S_0 u^{-2}$, and so on. The backward-induction formula is

$$C(i,j,k) = [\, pC(i+1, j+1, k_u) + (1-p)C(i+1, j, k_d)\,]\, e^{-r\Delta t},$$

where

$$\begin{aligned} k_u &= k, \\ k_d &= \min(k, 2(j+1) - (i+1)). \end{aligned}$$

After plugging in the terminal conditions, with backward induction we obtain the value of the lookback option in $C(0,0,0)$.

Example 2 (Geometric Asian Option) The payoff function of the geometric Asian option at expiration date T is

$$\max(S_T - A_T, 0),$$

where

$$A_T = (\prod_{t=0}^{n} S_t)^{1/(n+1)}.$$

At time $i\Delta t$, the possible products of the stock prices belong in the set

$$\{S_0^{i+1}u^{\frac{i(i+1)}{2}}, S_0^{i+1}u^{\frac{i(i+1)}{2}-2}, \ldots, S_0^{i+1}u^{-\frac{i(i+1)}{2}+2}, S_0^{i+1}u^{-\frac{i(i+1)}{2}}\}.$$

We have to allocate $\frac{n(n+1)}{2}+1$ auxiliary state variables at each node. Let $C(i,j,k)$ denote the value of the geometric Asian option on node $N(i,j)$ and k be the power index of the product of the stock prices until $i\Delta t$. For example, $k=1$ means the product is $S_0^{i+1}u$, and $k=3$ means the product is $S_0^{i+1}u^3$, and so on. The backward-induction formula is

$$C(i,j,k) = [\, pC(i+1,j+1,k_u) + (1-p)C(i+1,j,k_d)\,]\, e^{-r\Delta t},$$

where

$$\begin{aligned} k_u &= k + 2(j+1) - (i+1), \\ k_d &= k + 2j - (i+1). \end{aligned}$$

After plugging in the terminal conditions, with backward induction we obtain the value of the lookback option in $C(0,0,0)$.

The algorithm above uses $O(n^2)$ space and $O(n^3)$ time for the lookback option, and space $O(n^3)$ and time $O(n^4)$ for the geometric Asian option. Optimization is possible. For more efficient algorithms, we refer the reader to Lyuu (2002) for the geometric Asian option and Babbs (2000) and Cheuk et al. (1997) for the lookback option.

2.3 Least-Squares Simulation

The simulation approach, first introduced in Boyle (1977), is very effective in pricing derivatives with strong path dependency, multiple factors, or non-Markovian features. However, a serious weakness of the simulation approach is that it cannot deal with the problem of early exercise. Unfortunately, American-style options are popular in practice. To overcome this difficulty, we review a simple approach, named LSM by Longstaff and Schwartz (2001) for approximating the value of American-style options by simulation. The key feature is the application of the ordinary least squares (OLS) to estimate the conditional expected payoff in deciding whether to exercise the option or not. Below, we repeat the numerical example on pp. 115–120 of Longstaff and Schwartz (2001) in pricing American-style puts.

Consider an American-style put option struck at 1.10 and exercisable at times 1, 2, 3, where time 3 is the expiration date. The risk-free rate is 6% (the discount factor is hence 0.94176) and eight paths are used. These sample paths are generated by Eq. (2.8) and tabulated below in a matrix called the stock price matrix.

Path	$t=0$	$t=1$	$t=2$	$t=3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34

The option values with for each path at time 3 are tabulated below.

Path	$t=1$	$t=2$	$t=3$
1	-	-	0.00
2	-	-	0.00
3	-	-	0.07
4	-	-	0.18
5	-	-	0.00
6	-	-	0.20
7	-	-	0.09
8	-	-	0.00

Now consider the five paths which are in the money at time 2. Let X denote the stock prices with respect to these five paths at time 2 and Y the discounted option value of these five paths. The arrays X and Y are tabulated below.

Path	Y	X
1	0.00×0.94176	1.08
2	-	-
3	0.07×0.94176	1.07
4	0.18×0.94176	0.97
5	-	-
6	0.20×0.94176	0.77
7	0.09×0.94176	0.84
8	-	-

By regressing Y on a constant, X, and X^2, we obtain the conditional expected payoff function at time 2,

$$E(Y|X) = -1.070 + 2.983X - 1.813X^2.$$

$E(Y|X)$ means the expected option value of continuation given X. The exercise values and expected values of continuation at time 2 are as follows.

Path	Exercise Value	Continuation Value $E(Y\|X)$
1	0.02	0.0369
2	-	-
3	0.03	0.0461
4	0.13	0.1176
5	-	-
6	0.33	0.1520
7	0.26	0.1565
8	-	-

If the exercise value is greater than the expected value of continuation, the option value is set to the exercise value; otherwise, the option value is set to the discounted option value from time 3. Consequently, the option values with respect to each path at time 2 are as follows.

Path	$t = 1$	$t = 2$
1	-	0.00
2	-	0.00
3	-	0.94176×0.07
4	-	0.13
5	-	0.00
6	-	0.33
7	-	0.26
8	-	0.00

Repeat the procedure until time 0 when the option value can be determined by averaging the values of the eight paths. See Longstaff and Schwartz (2001) for the reasons only in-the-money paths are considered.

The valuing procedure of the LSM algorithm can be summarized as follows.

1: Generate the stock price matrix.
2: Determine the boundary conditions.
3: Determine the function form of $E(Y|X)$, named by basis function. Longstaff and Schwartz (2001) offer many types of basis functions with the polynomials being the simplest one.
4: Determine the order of the basis function such as quadratic, cubic or higher. If the option is path-dependent, the relevant variables such as average or minimum stock price should be included in regressors.
5: For the in-the-money paths, regress corresponded discounted option values on the regressors determined in Step 4.

6: Compare the exercise values and the expected values of continuation to determine the option values with respect to each path at that time.
7: Go to Step 5 and repeat the procedure until time 0.

We cite Proposition 1 in Longstaff and Schwartz (2001) below without proof.

Proposition 1 *For any finite choice of M, K, and vector $\theta \in \mathbb{R}^{M\times(K-1)}$ representing the coefficients for the M basis functions at each of the $K-1$ early exercise dates, let N denote the number of in-the-money paths, $V(X)$ denote the true value of the American-style option and $LSM(\omega; M, K)$ denote the discounted cash flow resulting from following the LSM rule of exercising when the immediate exercise value is positive and greater than or equal to $\widehat{E(Y|X)}$ as defined by θ. Then the following inequality holds almost surely,*

$$V(X) \geq \lim_{N\to\infty} \sum_{i=1}^{N} LSM(\omega_i; M, K).$$

Here, ω denotes an in-the-money path, and ω_i denotes the ith such path.

Longstaff and Schwartz (2001) claim that when given the order of the basis function and the number of the exercisable dates, the option value calculated by the LSM algorithm will converge to the real value when the number of in-the-money paths goes to infinity. Regarding the choice of M, we refer to Proposition 2 in Longstaff and Schwartz (2001). It claims that the order of basis function needed to obtain a desired level of accuracy need not go to infinity as $N \to \infty$. However, this proposition is limited to one-dimensional settings. Longstaff and Schwartz (2001) conjecture that similar results can be obtained for higher-dimensional problems by finding conditions under which uniform convergence occurs.

The LSM algorithm provides a simple yet powerful way to price American options by simulation. However, the LSM algorithm is very time-consuming, especially in the multifactor and non-Markovian cases which make the regression procedure become a burden of the LSM algorithm.

Chapter 3

Pricing Geometric-Moving-Average-Lookback Options

This chapter investigates the pricing of moving-average-lookback options. Because of the normality of the geometric average of stock prices modeled by geometric Browning motion, we begin by pricing geometric-moving-average-lookback options (GMALs). This serves as the benchmark against which the tree approach will be compared. Once we are satisfied with the tree model's accuracy, we will go on to price the arithmetic version in the next chapter using a similar tree method.

3.1 Defining the GMAL

Let $0 = t_0 < t_1 < t_2 < \cdots < t_n = T_s \leq T$, where n is the number of trading days before reset date T_s, and t_i be the time points when the moving average is calculated; however, our result is more general than this. The t_i are expected to coincide with trading dates as closing prices are used in moving averages. Suppose the time interval between monitor times are equal and $\Delta t = T_s/n$, i.e., $t_i = i\Delta t$. Define $S_i \equiv S_{i\Delta t}$, the stock price at time $i\Delta t$, for ease of notation. When calculating a moving average, a stock prices will be involved. The geometric moving average at time t_i equals

$$m_a(i) \equiv \left(\prod_{j=i-a+1}^{i} S_j \right)^{1/a}, \quad a-1 \leq i \leq n.$$

The minimum a-day geometric moving average as of the reset date $t_n = T_s$ is defined as

$$m_a = \min_{a-1 \le i \le n} m_a(i) = \min \left[\left(\prod_{i=0}^{a-1} S_i \right)^{1/a}, \left(\prod_{i=1}^{a} S_i \right)^{1/a}, \ldots, \left(\prod_{i=n-a+1}^{n} S_i \right)^{1/a} \right]. \quad (3.1)$$

Note that it is evaluated at discrete times. The payoff function of the GMAL at expiration date T is

$$\begin{aligned} C_T &= \max(S_T - X, 0), \\ X &= \max(\min(m_a, \mathrm{UB}), \mathrm{LB}). \end{aligned} \quad (3.2)$$

Here, the strike price of the option, X, is determined at reset date T_s. The upper bound on the strike price, UB, is initially set to S_0 by the contract. As the time goes by, if the prevailing minimum moving average is lower than the current UB, the prevailing minimum moving average becomes the new UB. The change of UB may happen at times t between t_{a-1} and T_s. LB, the lower bound of the strike price, is determined by the contract and will never be changed. Equation (3.2) means that the strike price of the option, X, is struck at the minimum a-day moving average but banded between LB and UB.

3.2 An Analytical Solution to GMAL

In this section, we apply the approach mentioned in Section 2.1 to derive an analytical solution to GMALs. For simplicity, we drop LB and UB from Eq. (3.2); hence $M = m_a$, which is defined in Eq. (3.1). By Eq. (2.11), the desired price equals

$$e^{-rT} E^Q[(S_T - m_a)\mathbf{1}_{(S_T > m_a)}],$$

where

$$m_a = \min_{a-1 \le i \le n} m_a(i).$$

With the Radon-Nikodym derivative in Eq. (2.12), we obtain price

$$S_0 \left\{ E^R \left[\mathbf{1}_{(S_T > m_a)} \right] - E^R \left[\frac{m_a}{S_T} \mathbf{1}_{(S_T > m_a)} \right] \right\} \equiv V_1 - V_2.$$

Let $Y_i \equiv \ln \frac{m_a(i)}{S_T}$. Now,

$$\begin{aligned} V_1 &= S_0 E^R[\, \mathbf{1}_{(S_T > m_a)} \,] \\ &= S_0 E^R \left\{ \mathbf{1}_{[\min(\ln \frac{m_a(a-1)}{S_T}, \ln \frac{m_a(a)}{S_T}, \ldots, \ln \frac{m_a(n)}{S_T}) < 0]} \right\} \\ &= S_0 E^R \left\{ \mathbf{1}_{[\min(Y_{a-1}, Y_a, \ldots, Y_n) < 0]} \right\} \\ &= S_0 \sum_{k=a-1}^{n} E^R \left[\mathbf{1}_{(Y_k < Y_{a-1}, Y_k < Y_a, \ldots, Y_k < Y_{k-1}, Y_k < 0, Y_k < Y_{k+1}, Y_k < Y_n)} \right]. \end{aligned}$$

Let random variable

$$X_i^k = \begin{cases} Y_k - Y_i, & \text{if } k \neq i, \\ Y_k, & \text{otherwise} \end{cases}$$

for $i = a-1, a, \ldots, n$. Define $V_{1k} \equiv S_0 E^R \left[\mathbf{1}_{(X_{a-1}^k < 0, X_a^k < 0, \ldots, X_n^k < 0)} \right]$. Then,

$$V_1 = \sum_{k=a-1}^{n} V_{1k}.$$

Similarly, by defining $V_{2k} \equiv S_0 E^R \left[e^{X_k^k} \mathbf{1}_{(X_{a-1}^k < 0, X_a^k < 0, \ldots, X_n^k < 0)} \right]$, we can show that

$$V_2 = \sum_{k=a-1}^{n} V_{2k}.$$

Apply a theorem in Chen et al. (2002) to obtain

$$\begin{aligned} V_{1k} &= S_0 N_{n-a+2} \left(d_{1,a-1}, d_{1,a}, \ldots, d_{1,n}; \rho^k \right), && (3.3) \\ V_{2k} &= S_0 e^{E^R(X_k^k) + \frac{1}{2} V^R(X_k^k)} N_{n-a+2} \left(d_{2,a-1}, d_{2,a}, \ldots, d_{2,n}; \rho^k \right), && (3.4) \end{aligned}$$

where

$$\begin{aligned} d_{1,i} &\equiv \frac{-E^R(X_i^k)}{\sqrt{V^R(X_i^k)}}, \\ d_{2,i} &\equiv \frac{-[E^R(X_i^k) + \mathrm{Cov}^R(X_{a-1}^k, X_k^k)]}{\sqrt{V^R(X_i^k)}}. \end{aligned}$$

Here, $N_d(x_1, x_2, \ldots, x_d; \rho)$ denotes the cumulative density function (CDF) of the standard d-dimensional normal distribution with correlation matrix ρ. In particular, ρ^k denotes the correlation matrix of the random vector $\mathbf{X^k} = [X_{a-1}^k, X_a^k, \ldots, X_n^k]'$. The proof of Eq. (3.3), Eq. (3.4) and the formulas of the various moments of $\mathbf{X^k}$ are in the appendix.

The price of the GMAL is associated with the CDF of the multidimensional normal distribution. To be practical, there must exist an efficient and sufficiently accurate algorithm to calculate the CDF of the d-dimensional normal distribution with $d \geq 20$ because, in practice, $n \geq 20$. Unfortunately, this seems still unsolvable, computationally. We have implemented the analytical solution with Monte Carlo simulation for the CDF; however, the standard error is greater than that generated by the crude Monte Carlo method over the stock prices.

3.3 Pricing European-Style GMALs on the CRR Model

In this section, we proceed to price the European-style GMAL (EGMAL) on the CRR model. By the law of iterated expectation, it is easy to verify that the price of the EGMAL equals

$$e^{-rT_s}E^Q[\text{BS}_{\text{call}}(S_{T_s}, M, T-T_s)].$$

Therefore, when pricing the EGMAL, we only need to build a tree up to the reset date and then plug in the Black-Scholes call formula on the terminal nodes.

The definitions of terms will be slightly different from the earlier ones because we are now dealing with discrete time points $0, 1, 2, \ldots$. A period refers to the period of time between two adjacent time points. Recall that n is the number of trading days before the reset date. Let L denote the number of periods between two adjacent monitoring time points (which will coincide with daily closing times). By making Δt a day, we make L the number of trading points per day. The number of trading points before the reset date, N, is equal to nL. We will build the binomial tree up to the reset date. The tree hence covers N periods and contains $\sim N^2/2 = O((nL)^2)$ nodes.

In order to speed up the algorithm and because moving averages involve only daily closing prices, we will simplify the N-period tree based on ideas from Ritchken and Trevor (1999). (For more general cases, one can refer to Dai and Lyuu (2002).) So although there are N periods before the reset date, we actually only care about nodes on monitoring days, i.e., at times $0, \Delta t, 2\Delta t, \ldots, n\Delta t$. And there are only n periods, each period now lasting for one full day Δt. We therefore merge every L levels of the binomial tree into one, creating an $(L+1)$-ary tree with n periods in the process. For example, when L=2, a trinomial tree is created (see Fig. 3.1). Each node $N(i,j)$ has $L+1$ successor nodes

$$N(i,j), N(i,j+1), \ldots, N(i,j+L).$$

The probability of the state moving from node $N(i,j)$ to node $N(i,j+\ell)$ equals

$$p(\ell) = \binom{L}{\ell} p^{\ell}(1-p)^{L-\ell},$$

where p is upward-moving probability in the CRR model. The stock price associated with node $N(i,j)$ is

$$S(i,j) = S_0 u^{2j-iL}$$

for $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, iL$. The new tree has $O(Ln^2)$ nodes, a reduction of order L from the original binomial tree.

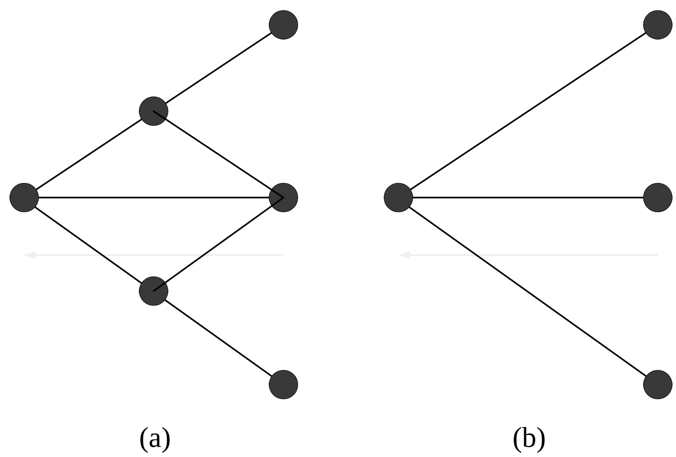

Figure 3.1: **Turning a Binomial Tree into an $(L+1)$-ary Tree.** (a) A 2-period binomial tree. (b) A 1-period trinomial tree. Both trees cover the same time period except that a period for the trinomial tree is twice as long as that for the binomial tree.

After building the $(L+1)$-ary tree, we have to determine the size of the auxiliary state variables on the nodes. Different from the simple examples in Section 2.2.2, they are no longer arrays but matrices. Define

$$\begin{aligned} k_{\mathrm{UB}} &= \left\lceil \frac{a \ln \frac{\mathrm{UB}}{S_0}}{\ln u} \right\rceil, \\ k_{\mathrm{LB}} &= \left\lfloor \frac{a \ln \frac{\mathrm{LB}}{S_0}}{\ln u} \right\rfloor. \end{aligned}$$

Note that $k_{\mathrm{UB}} = 0$ if $\mathrm{UB} = S_0$. Let $C(i,j,k,b)$ denote the option price on node $N(i,j)$. The k in $C(i,j,k,b)$ has the property that the minimum a-day moving product of the stock prices until time $i\Delta t$ is $S_0^a u^k$, where $k_{\mathrm{LB}} \leq k \leq k_{\mathrm{UB}}$. It is in effect the power index of of the product. In order to speed up the calculation, we apply forward induction to determine the minimum and maximum k, denoted by $\min k_{i,j}$ and $\max k_{i,j}$, respectively, for each node $N(i,j)$ before applying backward induction.

The b in $C(i,j,k,b)$ provides information of the current moving average of the stock prices. It is easy to see that $0 \leq b \leq b_{\max}$, where $b_{\max} = (L+1)^{a-1} - 1$. It is important to understand the role played by b. Consider the case with $a = 3$ and $L = 2$: Each node has three branches. Suppose 0, 1, and 2 mean down, flat, and up movement, respectively. Then b encodes the 2-day movements as following.

b	Price Movement
0	0 0
1	0 1
2	0 2
3	1 0
4	1 1
5	1 2
6	2 0
7	2 1
8	2 2

It basically encode them with the tertiary number system.

For example, starting from node $N(i, j)$, $b = 5$ means that the state at prior time $(i-2)\Delta t$ is node $N(i-2, j-3)$ and state at time $(i-1)\Delta t$ was node $N(i-1, j-2)$. The current moving product is

$$S_0^a u^{\mathrm{MS}} \equiv S_0^a u^{[\,6j-3L(i-1)-10\,]},$$

where MS denotes moving product's power index. The current geometric moving average is $S_0 u^{\mathrm{MS}/a}$. It is possible that $S_0 u^{\mathrm{MS}/a}$ may lie outside the tree.

The terminal conditions are given by the Black-Scholes formula for all combinations of $S(n, j)$ and $S_0 u^{k/a}$, where $\min k_{i,j} \leq k \leq \max k_{i,j}$ with $\tau = T - T_s$, volatility σ, risk-free rate r, and dividend yield q. (Options in Taiwan are dividend-protected; hence there is no need to consider dividends.) Then we use the backward-induction formula:

$$C(i, j, k, b) = e^{-r\Delta t} \sum_{\ell=0}^{L} p(\ell) C(i+1, j+\ell, k(\ell), b(\ell)),$$

where

$$\begin{aligned} k(\ell) &= \begin{cases} k, & \text{if } k \leq \mathrm{MS}(b(\ell)) \\ \mathrm{ms}(b(\ell))), & \text{if } k_{\mathrm{LB}} \leq \mathrm{ms}(b(\ell))) < k \\ k_{\mathrm{LB}}, & \text{if } \mathrm{MS}(b(\ell)) < k_{\mathrm{LB}} \end{cases} \\ b(\ell) &= (b(L+1)) \bmod (L+1)^{a-1} + \ell. \end{aligned} \tag{3.5}$$

$\mathrm{MS}(b(\ell))$ is moving product's power index with respect to the path from node $N(i, j)$ to node $N(i+1, j+\ell)$. The function $k(\ell)$ selects the index associated with the minimum moving product sum in successor nodes.

We use a simple example to illustrate how the function $b(\ell)$ works. Suppose $a = 3$ and $L = 2$. The following table shows that when the state move forward by one day, it drops the earliest movement, achieved by shifting the number to the left.

b	Past Movement	Future Moment	$b(\ell)$
0	0 0	0 ℓ	$0 \times 3 + \ell$
1	0 1	1 ℓ	$1 \times 3 + \ell$
2	0 2	2 ℓ	$2 \times 3 + \ell$
3	1 0	0 ℓ	$0 \times 3 + \ell$
4	1 1	1 ℓ	$1 \times 3 + \ell$
5	1 2	2 ℓ	$2 \times 3 + \ell$
6	2 0	0 ℓ	$0 \times 3 + \ell$
7	2 1	1 ℓ	$1 \times 3 + \ell$
8	2 2	2 ℓ	$2 \times 3 + \ell$

Comparing the second column with the definition of b, we derive the third column. The intention of Eq. (3.5) should be clear by now. The first term on the right hand side of Eq. (3.5) describes the action of keeping necessary information regarding past movements, and the second term describes the new movement. The answer appears in $C(0, 0, 0, 0)$. The algorithm for the EGMAL runs in space $O((L + 1)^{a+3})$ and time $O((L + 1)^{a+3})$

3.4 Pricing American-Style GMALs

There is one detail regarding American-style GMALs (AGMALs) that needs to be straightened out before we proceed. We may assume that the option holder can exercise the option only on or *after* the reset date (Scenario one). Note that after the reset date, when the strike price is set, the option is like an ordinary vanilla American-style option. Or we may assume that the option holder can exercise it on or after the $(a - 1)$th trading date. Our tree algorithm for the AGMAL can deal with both assumptions without loss of efficiency. However, because the least-squares simulation method becomes less efficient for Scenario two, we will focus on Scenario one.

3.4.1 The CRR Model

Under Scenario one, the tree model for pricing the AGMAL is identical to its European counterpart except for terminal conditions. All we need to to is to replace the Black-Scholes formula with an algorithm on the CRR model to price an American-style call . Let n' denote the number of periods on this tree before reset date T_s and expiration T. Under Scenario two, on the other hand, we further consider early exercise when applying backward induction on the n-period original tree.

3.4.2 The Least-Squares Simulation Method

Now, we state how to price AGMAL with least-squares simulation. As in Section 2.3, the first step is to generate the stock price matrix. But now we only preserve the stock price matrix $S(i, j)$ after T_s. Here i refers to time $T_s + i\Delta t'$, where $\Delta t' = (T - T_s)/n'$, and j signifies the jth path. The strike price vector $M(j)$ is determined by the stock price process before T_s. After giving the terminal conditions, $S_{n',j} - M_j$, we pick the following basis function:

$$e^{-r\Delta t'} C_{i+1,j} = \beta_0 + \beta_1 S_{i,j} + \beta_2 S_{i,j}^2 + \beta_3 S_{i,j} M_j + \beta_4 M_j + \beta_5 M_j^2. \tag{3.6}$$

Regressing with Eq. (3.6), we obtain this conditional expected payoff function at time $T_s + i\Delta t'$:

$$E_i(C_{i,j}|S_{i,j}, M_j) = \widehat{\beta_0} + \widehat{\beta_1} S_{i,j} + \widehat{\beta_2} S_{i,j}^2 + \widehat{\beta_3} S_{i,j} M_j + \widehat{\beta_4} M_j + \widehat{\beta_5} M_j^2,$$

where $\widehat{\beta_i}$ denotes the OLS estimator of β_i. Compare the exercise value, $S_{i,j} - M_j$, and the expected value of continuation, $E_i(C_{i,j}|S_{i,j}, M_j)$, to determine the option value with respect to each path at time $T_s + i\Delta t'$ as follows:

$$C_{i,j} = \begin{cases} S_{i,j} - M_j, & \text{if } S_{i,j} - M_j > E_i(C_{i,j}|S_{i,j}, M_j) \\ e^{-r\Delta t'} C_{i+1,j}, & \text{otherwise} \end{cases}.$$

After repeating the procedure in a backward fashion for $i = n' - 1$ to 0 (or time T_s), we can now price the option by discounting the value in $C(0, j)$ for all j with e^{-rT_s} and averaging over all paths.

We are now able to explain why we do not use the LSM algorithm for options which can be exercised before T_s (Scenario two). Equation (3.6) contains 6 regressors for just 2 variables $S_{i,j}$ and M_j). The number grows fast with the number of variables, which happens under Scenario two. The reason is the non-Markovian property of the moving average. Consider an AGMAL with 3-day moving average for example. The basis function needs a constant term, the current stock price, the stock price one period and two periods earlier, the minimum moving average until now, their squares, and the cross terms. There are at least 15 regressors in the basis function, and all this just for 3-day moving average. When T_s is less than two months and the dividend yield is not too high, the possibility of early exercise before T_s for the AGMAL is very low anyway.

3.5 Numerical Results

3.5.1 The European-Style Case: EGMAL

We use the CRR model and Monte Carlo simulation to price the 3-day and 5-day EGMAL. Assume $S_0 = \text{UB} = 50$, $r = 2\%$, $q = 4\%$, $T = 1$, and $T_s = 1/12$. Suppose

		$a = 3$			$a = 5$		
LB	σ	CRR	MC	Std.	CRR	MC	Std.
45	0.3	6.1689	6.1712	(0.0019)	6.0769	6.0745	(0.0019)
	0.4	8.1916	8.1942	(0.0029)	8.0924	8.0871	(0.0028)
	0.5	10.1367	10.1392	(0.0039)	10.0360	10.0339	(0.0038)
40	0.3	6.2694	6.2723	(0.0018)	6.1566	6.1521	(0.0018)
	0.4	8.4219	8.4242	(0.0026)	8.2832	8.2797	(0.0027)
	0.5	10.4953	10.4992	(0.0036)	10.3402	10.3332	(0.0036)
35	0.3	6.2714	6.2731	(0.0018)	6.1579	6.1551	(0.0018)
	0.4	8.4414	8.4460	(0.0026)	8.2970	8.2922	(0.0026)
	0.5	10.5581	10.5604	(0.0035)	10.3882	10.3836	(0.0035)

Table 3.1: **Pricing EGMAL.** The parameters are $S_0 = \text{UB} = 50$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$ ($n = 22$), $L = 8$ for the $a = 3$ cases, and $L = 3$ for the $a = 5$ cases. "Std." is the sample standard deviation of Monte Carlo simulations (MC).

there are 22 trading days in a month, so $n = 22$. We will vary LB, σ, and the moving-average period a in the experiments. Fix L=8 for the 3-day cases ($a = 3$) and L=3 for the 5-day cases ($a = 5$). The pricing results by Monte Carlo simulation are based on 1,000,000 paths: 500,000 plus 500,000 antithetic. The results are tabulated in Table 3.1. We make the following observations. First, the prices calculated by our algorithm are within two times the standard deviations generated by Monte Carlo simulation. Our algorithm is therefore basically correct. Second, not surprisingly, the option value decreases with the moving-average period a and LB, but increases with σ.

Next we check the convergence of the pricing. Figure 3.2 shows the price converges quickly, up to two decimal places when $L \geq 3$. The pattern of convergence oscillates. Specifically, the odd-L points and the even-L ones each converge *monotonically*. This immediately suggests Richardson's extrapolation: $2V_i - V_{i-2}$, where $i \geq 3$, and it indeed leads to tighter option prices. The pattern of overvaluation and undervaluation in Figure 3.2 also carries over to Table 3.1.

3.5.2 The American-Style Case: AGMAL

We use both the CRR model and the LSM algorithm to price the 3-day and 5-day AGMALs. Assume $S_0 = \text{UB} = 50$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$, and $n' = 50$. Let L=8 for the 3-day cases, and L=3 for the 5-day cases. The pricing results by LSM are based on 100,000 paths: 50,000 plus 50,000 antithetic. The results are shown in Table 3.2 and Figure 3.3. We make the following observations. First, compared with the tree algorithm, the prices calculated by the LSM method are systematically

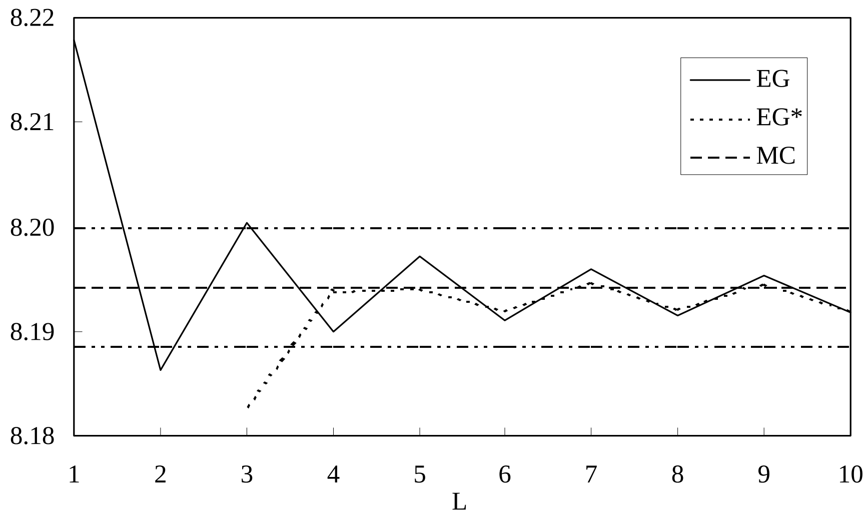

Figure 3.2: **Convergence of EGMAL.** The parameters are $S_0 = \text{UB} = 50$, $\sigma = 40\%$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$ ($n = 22$), and $a = 3$. EG* and MC represent the price obtained by Richardson's extrapolation and the Monte Carlo simulation (MC), respectively. Here, MC gives a vlaue of 8.1942 with a sample standard deviation of 0.0029. A band with a width of 2 times the standard deviation above and below MC is plotted for reference.

		$a = 3$			$a = 5$		
LB	σ	CRR	LSM	Std.	CRR	LSM	Std.
45	0.3	6.3228	6.2870	(0.0256)	6.2280	6.1555	(0.0247)
	0.4	8.3571	8.3265	(0.0360)	8.2558	8.1852	(0.0352)
	0.5	10.3149	10.2864	(0.0476)	10.2117	10.2090	(0.0474)
40	0.3	6.4256	6.3706	(0.0253)	6.3099	6.2832	(0.0255)
	0.4	8.5921	8.5353	(0.0359)	8.4505	8.4115	(0.0355)
	0.5	10.6800	10.6045	(0.0459)	10.5220	10.5057	(0.0472)
35	0.3	6.4277	6.4245	(0.0255)	6.3111	6.2814	(0.0256)
	0.4	8.6118	8.5807	(0.0357)	8.4646	8.3980	(0.0356)
	0.5	10.7436	10.7014	(0.0465)	10.5708	10.5530	(0.0476)

Table 3.2: **Pricing AGMAL.** The parameters are $S_0 = \text{UB} = 50$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$ ($n = 22$), $L = 8$ for the $a = 3$ cases, $L = 3$ for the $a = 5$ cases, and $n' = 50$. "Std." is the sample standard deviation of Monte Carlo simulations (MC).

undervalued. The reason can be traced to Proposition 1. Second, the fact that the prices calculated by the tree algorithm are within twice the sample standard deviation given by the LSM method shows that the tree algorithm is likely to converge to the correct value. Third, the early-exercise premiums are relatively stable. In the unusual case where $r = 2\%$ and $q = 4\%$, they are just slightly over 0.165. In fact, as show in Figure 3.4, they are roughly fixed when $L \geq 3$. In that Figure, we move ahead of ourselves by plotting the early-exercise premium of the arithmetic-moving-average lookback option as well.

In order to examine the option values of the AGMAL between Scenario one and Scenario two by our lattice algorithm, the most important factors q and T_s are varied. Table 3.3 shows that the prices are more sensitive to T_s than q. However, there is little difference when T_s is at most two months whatever the value of q. And even if T_s is three months long, the difference is still insignificant. These result show that the probability of early exercise for the AGMAL before T_s with a relatively short reset period is too small to be noticed.

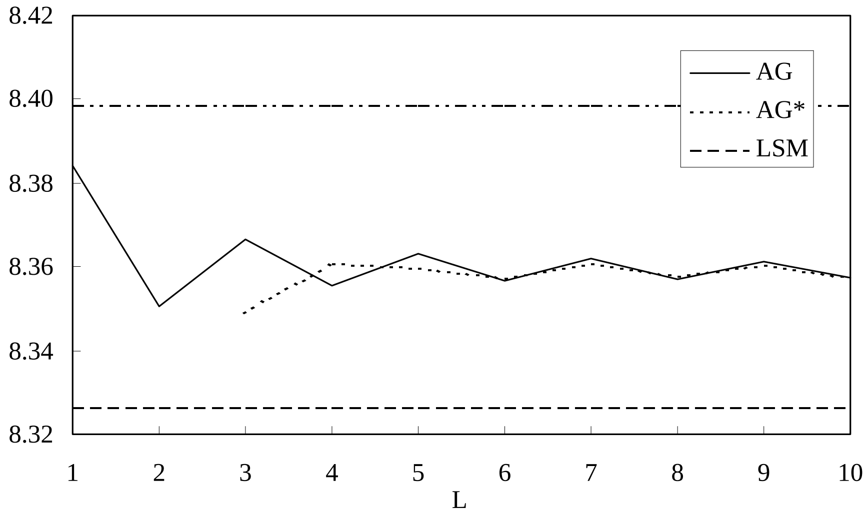

Figure 3.3: **Convergence of AGMAL.** The parameters are S_0 = UB = 50, σ = 40%, r = 2%, q = 4%, T = 1, T_s = 1/12 (n = 22), a = 3 and n' = 50. AG* and LSM represent the price obtained by Richardson's extrapolation and the least-square simulation method, respectively. LSM gives a value of 8.3265 with a sample standard deviation of 0.0360. A band with a width of 2 times the standard deviation above and below LSM is plotted for reference.

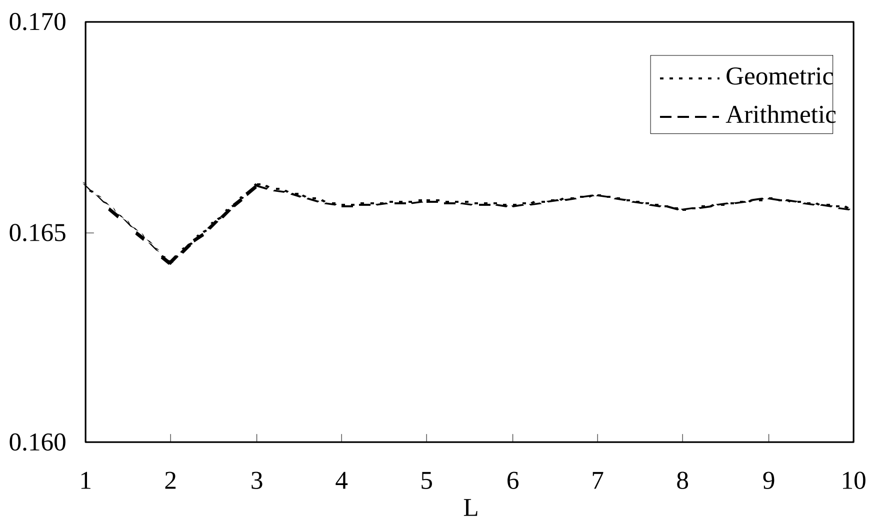

Figure 3.4: **Stability of the Early Exercise Premium.** The parameters are S_0 = UB = 50, σ = 40%, r = 2%, q = 4%, T = 1, T_s = 1/12 (n = 22), a = 3, and n' = 50. Both the geometric- and arithmetic-moving-average lookback options are plotted.

T_s	1/12		2/12		3/12	
q	Scenario 1	Scenario 2	Scenario 1	Scenario 2	Scenario 1	Scenario 2
2%	6.778140	6.778140	7.069512	7.069512	7.216221	7.216223
4%	6.322439	6.322439	6.607496	6.607502	6.750990	6.751187
6%	5.923372	5.923372	6.203751	6.203848	6.344009	6.345403

Table 3.3: **Early Exercise before T_s for AGMAL.** The parameters are $S_0 = \text{UB} = 50$, $\sigma = 30\%$, $r = 2\%$, $\text{LB} = 45$, $T = 1$, $n = 22$ for the $T_s = 1/12$ cases, $n = 45$ for the $T_s = 2/12$ cases, $n = 67$ for the $T_s = 3/12$ cases, $a = 3$, and $n' = 50$.

Chapter 4

Pricing Arithmetic-Moving-Average-Lookback Options

In this chapter, we price the arithmetic-moving-average-lookback options (AMALs). AMAL is similar to GMAL except that the geometric moving average is replaced with the arithmetic version:

$$m_a \equiv \min_{a-1 \leq t \leq n} \frac{\sum_{i=t-a+1}^{t} S_i}{a}.$$

Due to the nonnormality of the arithmetic sum of the stock prices when modeled by geometric Browning motion, we are not expected to derive the analytical solution for the AMAL. Instead, we use the CRR model.

4.1 Pricing AMAL on the CRR Model

The basic price tree remains the same as the geometric version. The difference lies in the auxiliary state variables for the nodes. Let $C(i, j, k, b)$ denote the option price on node $N(i, j)$. The meaning of b is the same as the geometric version. But that of k differs. The size of the auxiliary state variables depends on the set formed by the possible strike prices. In the geometric version, the set is

$$\{S_0 u^{k/a} : k \text{ is an integer}, k_{\text{LB}} \leq k \leq k_{\text{UB}}\}.$$

In contrast, in the arithmetic version, it is no longer so simple. We first find all the possible strike prices between LB and UB and put each in array $M(k)$, where $k = 0, 1, \ldots, k_{\max}$, softed from the smallest to the largest one such that $M(0) = \text{LB}$ and $M(k_{\max}) = \text{UB}$. (Notice that $M(0)$ may not equal LB. This happens when LB is smaller than all possible strike prices.) The possible strike prices in the arithmetic version forms the set

$$\{M(k) : k \text{ is an integer}, 0 \leq k \leq k_{\max}\}.$$

So k is not the power index but the rank of the moving average in the array.

Unfortunately, this set grows with $O(L^{a-1})$ which is exponential in a. In order to overcome this complexity, we observe that the accuracy of the strike price is rounded to 2 decimal places in the market ($x = 2$). Consequently, the size of the set is much more limited. For example, when UB $= 50$, LB $= 45$, there are at most 501 possible strike prices: $\{45.00, 45.01, \ldots, 50.00\}$.

The backward-induction formula for the AMAL is similar to that for the GMAL except for the function $k(l)$. The function $k(l)$ now becomes

$$k(l) = \begin{cases} k, & \text{if } M(k) \leq \mathrm{MA}(b(l)) \\ f(\mathrm{MA}(b(l))), & \text{if LB} \leq \mathrm{MA}(b(l)) < M(k) \\ 0, & \text{if } \mathrm{MA}(b(l)) < \mathrm{LB} \end{cases}, \tag{4.1}$$

where $\mathrm{MA}(b(l)))$ is the arithmetic moving average with respect to the path from node $N(i, j)$ to node $N(i + 1, j + l)$.

For example, suppose UB $= 50$, LB $= 45$, $\mathrm{MA}(b(l)) = 47$, and $M(k) = 48$. By function (4.1), we know that when the state moves to node $N(i + 1, j + l)$, the minimum moving average moves down to 47. Hence, we have to search the correct rank k which corresponds to a minimum moving average of 47 in the successor node $N(i + 1, j + l)$. However, searching the array can be time consuming. We now turn to a more efficient way to invert $M(k)$ to get rank k. The idea is to hard-code the correspondence once and for all. Of course a correspondence is simply an integral function, which can be coded as a table. Let $f(i)$ be the said array (called the rank-inversion table), where $\lfloor M(0)\phi \rfloor \leq i \leq \lfloor M(k_{\max})\phi \rfloor$ and $\phi \equiv 10^z$ with z being the smallest nonnegative integer such that $\lfloor M(i)\phi \rfloor \neq \lfloor M(j)\phi \rfloor$ for $i \neq j$. The table can be constructed by the following algorithm.

Algorithm 1 Construction of the Rank-Inversion Table.

```
for k = 0 to k_max do
    f(⌊M(k)φ⌋) := k;
end for
```

For example suppose $M(0) = 1$, $M(1) = 1.11$, $M(2) = 1.33$, $M(3) = 1.55$, and $M(4) = 1.7$. It is obvious that $\phi = 10$ and $f()$ works as desired. If given $M(k) = 1.33$, we can find $k = 2$ via rank-inversion table $f(\ )$ in constant time.

Location $\lfloor 10M(k) \rfloor$	Value $f(\lfloor 10M(k) \rfloor)$
10	0
11	1
12	-
13	2
14	-
15	3
16	-
17	4

Theoretically, the algorithm for the AMAL takes $O((L+1)^{a+1})$ space and $O((L+1)^{a+1})$ time. Although it may seem that the algorithm for the AMAL is faster than that for the GMAL, the numerical results show that the opposite is true in practice. The reason is that there is a big constant factor for the AMAL algorithm.

4.2 Numerical Results

We use the same parameters as in the geometric version to price both AMALs. Tables 4.1 and 4.2 show that the prices are similar to the geometric counterparts. In particular, the convergence is quite fast as shown in Fig. 4.1. Comparing AMAL and GMAL, we observe that, the price of GMAL is greater than that of AMAL because of the smaller mean of the geometric average. Second, the price difference increases with the moving-average period a and volatility σ. Still, the difference is hard to detect. Figure 4.2 shows that the difference is quite stable when $L \geq 3$. As it takes much less time to calculate the price of GMAL, it is a good approximation to the value of AMAL. Finally, Table 4.3 shows that the probability of early exercise before T_s with a short reset period is too small to be noticed.

4.3 Empirical Studies

This section conducts empirical studies of two American-style AMALs, the `PL06` and `PL07` issued by Polaris Securities in 1999. The contract specifications are tabulated in Table 4.4. The required parameters for our algorithm are listed in Table 4.5. By running our algorithms based on the parameters, the pricing results appear in Table 4.6.

We make the following observations. First, compared with the issue prices of Polaris Securities, those calculated by our algorithm are essentially identical: 26.81 vs. 26.98 and 16.65 vs. 16.67. Second, the GMAL results remain good approximations in both securities. Prices calculated by the lattice algorithms for AMAL and GMAL

		$a = 3$			$a = 5$		
LB	σ	CRR	MC	Std.	CRR	MC	Std.
45	0.3	6.1684	6.1706	(0.0019)	6.0757	6.0726	(0.0019)
	0.4	8.1909	8.1933	(0.0029)	8.0907	8.0864	(0.0028)
	0.5	10.1358	10.1380	(0.0039)	10.0340	10.0314	(0.0038)
40	0.3	6.2688	6.2715	(0.0018)	6.1552	6.1507	(0.0018)
	0.4	8.4209	8.4225	(0.0026)	8.2809	8.2775	(0.0027)
	0.5	10.4937	10.4987	(0.0036)	10.3371	10.3299	(0.0036)
35	0.3	6.2708	6.2724	(0.0018)	6.1564	6.1542	(0.0018)
	0.4	8.4404	8.4449	(0.0026)	8.2946	8.2881	(0.0026)
	0.5	10.5563	10.5573	(0.0035)	10.3847	10.3812	(0.0035)

Table 4.1: **Pricing the European-Style AMAL.** The parameters are $S_0 = \text{UB} = 50$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$ ($n = 22$), $x = 2$, $L = 8$ for the $a = 3$ cases, and $L = 3$ for the $a = 5$ cases. "Std." is the sample standard deviation of Monte Carlo simulations (MC).

		$a = 3$			$a = 5$		
LB	σ	CRR	LSM	Std.	CRR	LSM	Std.
45	0.3	6.3223	6.3000	(0.0253)	6.2268	6.1941	(0.0254)
	0.4	8.3565	8.3279	(0.0361)	8.2541	8.1816	(0.0352)
	0.5	10.3140	10.3073	(0.0478)	10.2097	10.1256	(0.0458)
40	0.3	6.4250	6.3943	(0.0256)	6.3084	6.2378	(0.0248)
	0.4	8.5911	8.5704	(0.0355)	8.4481	8.3379	(0.0349)
	0.5	10.6783	10.6431	(0.0480)	10.5188	10.4617	(0.0470)
35	0.3	6.4271	6.3854	(0.0250)	6.3096	6.2766	(0.0255)
	0.4	8.6108	8.5759	(0.0367)	8.4621	8.4348	(0.0354)
	0.5	10.7417	10.7140	(0.0472)	10.5672	10.5023	(0.0461)

Table 4.2: **Pricing the American-Style AMAL.** The parameters are $S_0 = \text{UB} = 50$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$ ($n = 22$), $x = 2$, $L = 8$ for the $a = 3$ cases, $L = 3$ for the $a = 5$ cases, and $n' = 50$. "Std." is the sample standard deviation of the least-squares simulations (LSM).

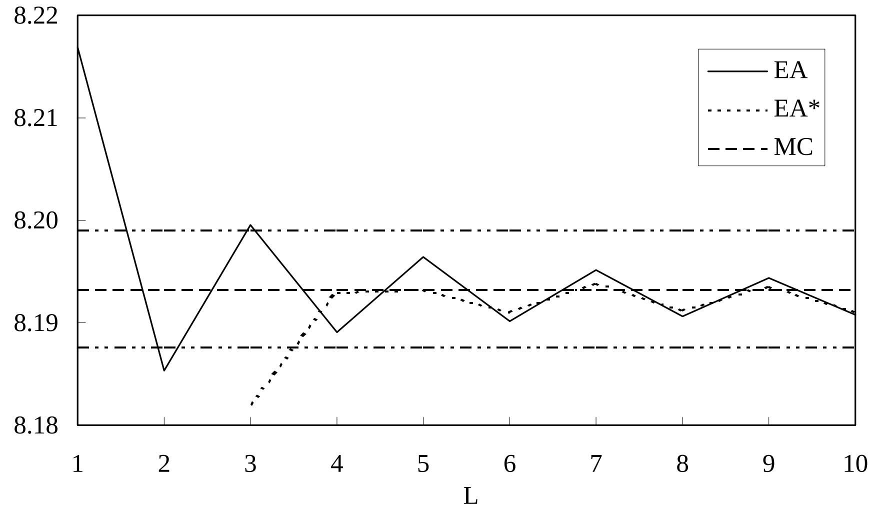

Figure 4.1: **Convergence of EAMAL.** The parameters are S_0 = UB = 50, σ = 40%, r = 2%, q = 4%, T = 1, T_s = 1/12 (n = 22), x = 3, and a = 3. EA* and MC represent the price obtained by Richardson's extrapolation and the Monte Carlo simulation, respectively. Here, MC gives a value of 8.1933 with a sample standard deviation of 0.0029. A band with a width of 2 times the standard deviation above and below MC is plotted for reference.

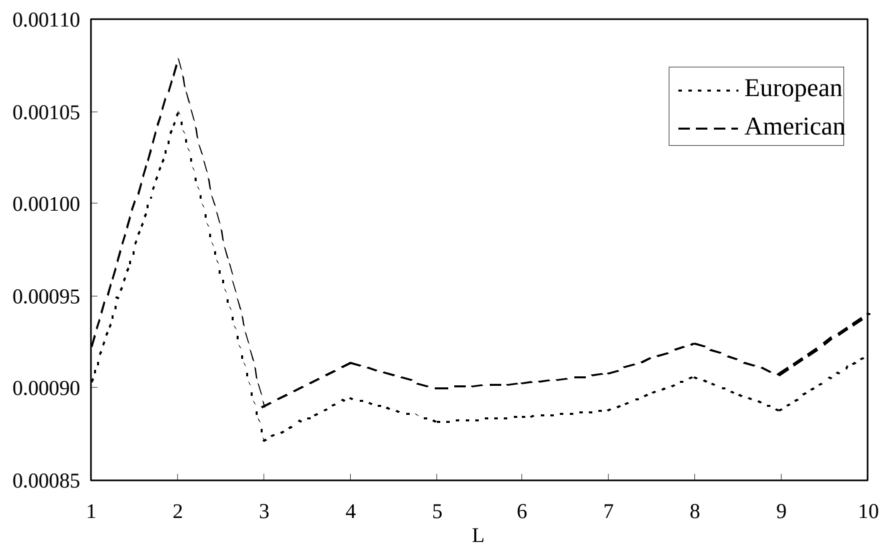

Figure 4.2: **Price Difference of Geometric and Arithmetic AMALs.** The parameters are S_0 = UB = 50, σ = 40%, r = 2%, q = 4%, T = 1, T_s = 1/12 (n = 22), dp = 3, a = 3, and n' = 50.

T_s	1/12		2/12		3/12	
q	Scenario 1	Scenario 2	Scenario 1	Scenario 2	Scenario 1	Scenario 2
2%	6.777619	6.777619	7.069056	7.069056	7.215519	7.215521
4%	6.321930	6.321930	6.607049	6.607055	6.750304	6.750486
6%	5.922871	5.922871	6.203310	6.203405	6.343335	6.344661

Table 4.3: **Early Exercise before T_s for American-Style AMAL.** The parameters are $S_0 = \text{UB} = 50$, $\sigma = 30\%$, $r = 2\%$, $\text{LB} = 45$, $T = 1$, $n = 22$ for the $T_s = 1/12$ cases, $n = 45$ for the $T_s = \frac{2}{12}$ cases, $n = 67$ for the $T_s = 3/12$ cases, $x = 2$, $n' = 50$, and $a = 3$.

	PL06	PL07
Issue date	08/21/1999	08/27/1999
First trading date	09/02/1999	09/06/1999
Reset date	09/21/1999	09/28/1999
Expiration date	09/01/2000	09/05/2000
Moving-average period (days)	6	6
UB	103.75	64.45
LB	0.9×UB	0.9×UB
Issue volatility	54.38%	54.58%
Issue price (NTD)	26.98	16.67

Table 4.4: **Contract Specifications of PL06 and PL07.**

	PL06	PL07
S_0	103.75	64.45
UB	103.75	64.45
LB	93.38	58.01
σ	54.38%	54.58%
r	5.00%	5.00%
q	0.00%	0.00%
T	378/365	376/365
T_s	31/365	31/365
n	24	23
a	6	6
L	2	2

Table 4.5: **Parameters Setup for Pricing PL06 and PL07 with the Lattice Algorithm.**

	PL06	PL07
AMAL (CRR)	26.8125	16.6495
GMAL (CRR)	26.8181	16.6528
AMAL (MC)	26.8160	16.6515
Sample standard deviation	0.0071	0.0044
Implied volatility by AMAL (CRR)	54.80%	55.04%

Table 4.6: **Pricing and Volatility of PL06 and PL07.** The parameters are based on Table 4.5.

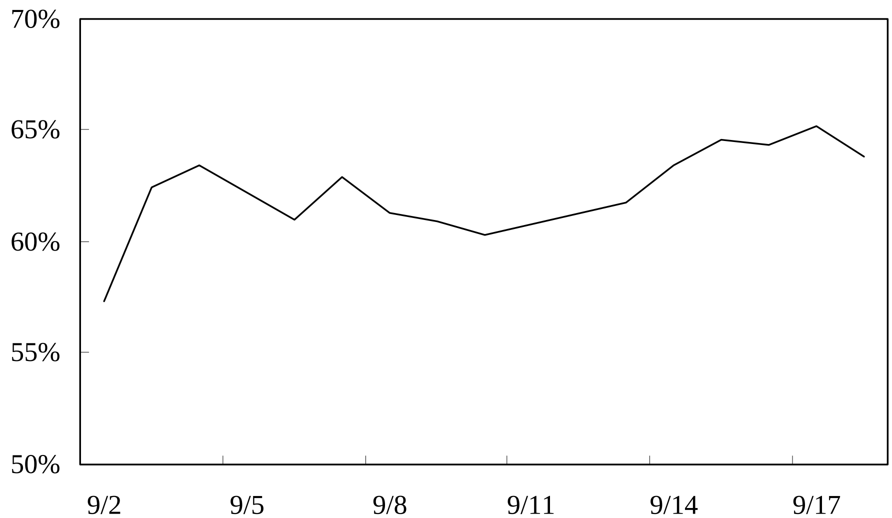

Figure 4.3: **Implied Volatilities of PL06.**

are within one sample standard deviation of the Monte Carlo simulations based on 2,000,000 paths (1,000,000 plus 1,000,000 antithetic). The tightness of the results gives us confidence in their correctness. Assured by our results, we proceed to calculate the implied volatilities of both securities in Figure 4.3 and Figure 4.4 with the AMAL algorithm. The two plots show that the volatilities lie within the band of 55% and 65%. This level of volatility is typical of stocks in Taiwan.

Figure 4.4: **Implied Volatilities of `PL07`.**

Chapter 5

Hedging

This chapter investigates the delta and gamma of MALs. Surprisingly, the delta jumps as the reset date approaches. Because of the price similarity between geometric and arithmetic MALs, we focus on EGMALs.

5.1 Delta

Let $\mathrm{C}(S_0, \Theta)$ be the value of the MAL based on our lattice algorithm and initial stock price S_0 with parameter vector $\Theta \equiv \{\, \mathrm{UB}, \mathrm{LB}, \sigma, r, q, T, T_s, n, L, a \,\}$. Numerical delta Δ is calculated as

$$\Delta = \frac{\mathrm{C}(S_0 u, \Theta) - \mathrm{C}(S_0 d, \Theta)}{S_0 u - S_0 d}$$

Figure 5.1 shows option prices with different combinations of LB and UB. The price difference between the vanilla option and the MAL vanishes as S_0 increases. On the other hand, when S_0 decreases, the reset feature makes the MAL's price decrease more slowly than the vanilla option's. In the extreme case when UB = LB, an MAL reduces to a vanilla call. One most interesting feature of the figure is the concavity of the MAL price when stock price is between LB and UB. It implies nonmonotonicity in the value of delta, unlike the case with the vanilla option.

Figure 5.2 explores the matter more closely. Indeed, the delta of MAL is not a monotonically increasing function of S_0. The delta is a decreasing function of the stock price when the stock price is roughly between LB and UB. Given the UB, the phenomenon is more prominent with lower LBs. This reflects the two forces in determining the option value. As the stock price decreases, the downward stock price will tend to lower the option value, whereas at the same time the higher probability of a low strike price will tend to raise the option value. When the stock price is between LB and UB, the strike-price effect is at its strongest. Though not strong enough to make the delta negative, it is able to counteract the decrease in the option

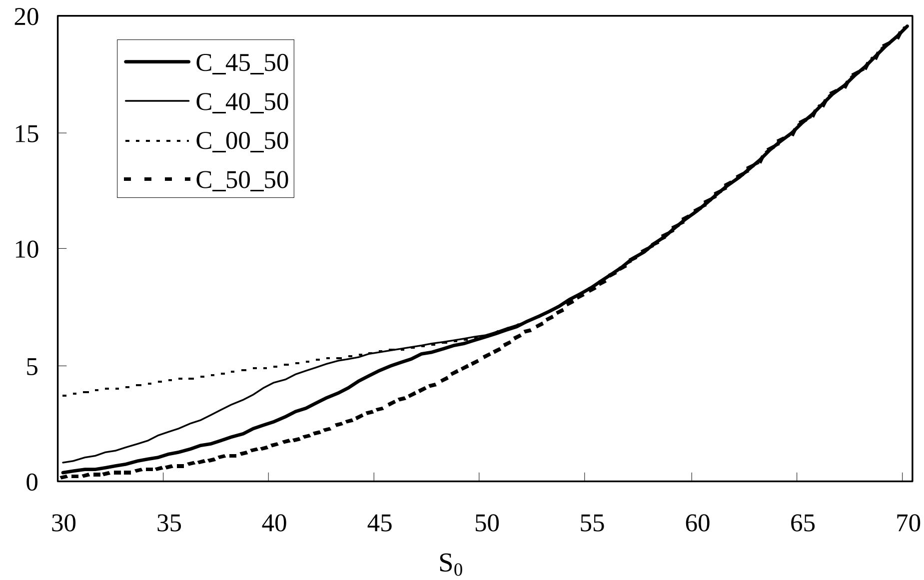

Figure 5.1: **EGMAL and the Lower and Upper Bounds.** The parameters are $\sigma = 30\%$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$ ($n = 22$), $L = 3$, and $a = 3$. C_A_B means the MAL is priced with LB = A and UB = B.

value because of the lower stock price. The option holder is therefore well protected by the downward reset feature. When stock price penetrates below LB, however, the stock-price effect will dominate the strike-price effect and the protection no longer exists.

5.2 Gamma

The numerical gamma Γ is calculated as

$$\Gamma = \frac{\frac{C(S_0u^2,\Theta)-C(S_0,\Theta)}{S_0u^2-S_0} - \frac{C(S_0,\Theta)-C(S_0d^2,\Theta)}{S_0-S_0d^2}}{(S_0u^2 - S_0d^2)/2}.$$

The complex behavior of delta suggests that gamma should behave in a complex way as well. Figure 5.3 shows that the gamma is negative roughly between LB and UB. (Because it is the moving average, not the stock price, that determines the strike price, the gamma is negative roughly, but not exactly, between LB and UB.)

For vanilla European-style calls, the gamma is positive. When a securities firm sells derivative securities with a positive gamma, the firm's position has a negative gamma. In that case, if it implements a delta-neutral hedging strategy, the negativity

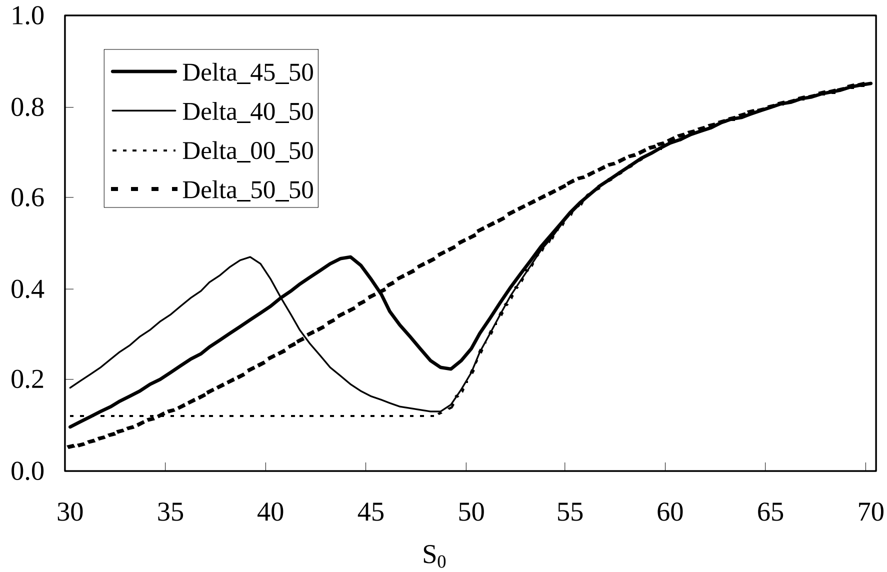

Figure 5.2: **Delta of EGMAL.** The parameters are $\sigma = 30\%$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$ ($n = 22$), $L = 3$, and $a = 3$. Delta_A_B means the delta is evaluated with LB = A, UB = B.

of gamma will result in hedging losses (the so-called gamma risk). Therefore, a delta-gamma-neutral hedging strategy should be implemented instead. This is especially relevant when the underlying stock price is volatile or the hedging frequency is low.

The opposite is true for the seller when a derivative security has a negative gamma. From the seller's print of view, a negative gamma means hedging profit when implementing the delta-neutral strategy. This is an important consideration as implementing the delta-gamma-neutral hedging requires two hedging instruments with the same underlying asset besides the money market account. This is often impossible; for example, only one option exists for a specific stock in Taiwan's options market. Apparently, the negative gamma of MAL benefits the issuing securities firm. But the issuing firm may face more gamma risk when the stock price or, more precisely, the prevailing moving average is outside the range LB and UB. A proposal will be to issue MALs without an LB. Then the gamma is zero roughly below UB, which implies that the MAL value decreases with the underlying stock price linearly because the strike price can be continuously changed downward without bounds.

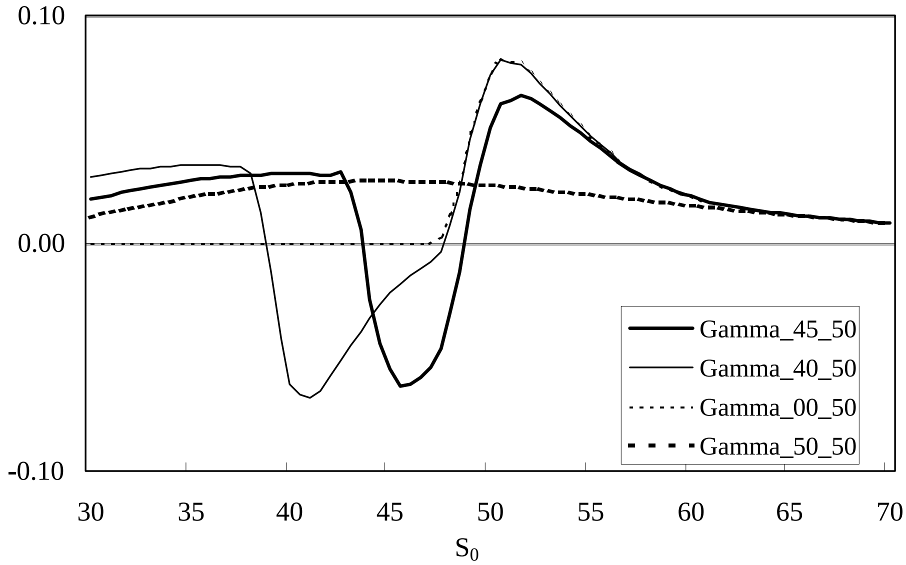

Figure 5.3: **Gamma of EGMAL.** The parameters are $\sigma = 30\%$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$ ($n = 22$), $L = 1$, and $a = 3$. Gamma_A_B means the gamma is evaluated with LB $= A$ and UB $= B$.

5.3 Delta Jumps

Like most of reset options, the MAL encounters a delta jump at the reset date. As the reset date approaches, the value of MAL is getting more and more sensitive to the moving average. Consider an MAL with $a = 3$, $S_{T_s-2} = S_{T_s-1} = 50$, and UB $= 49$. UB is the current strike price. The strike price will be revised down if $S_{T_s} < 47$, being the solution of $(50 + 50 + x)/3 = 49$. This results in a kink for the option value and a delta jump at $S = 47$ (see Figure 5.4 and Figure 5.5).

The occurrence of the delta jump relies on past $a - 1$ stock prices. It makes the stock price on the reset date, S_{T_s}, play a less critical role in MAL than in other reset options, whose strike price depend solely on the stock price directly. The probability of a delta jump at the reset date is also smaller. This is yet another advantage of MALs.

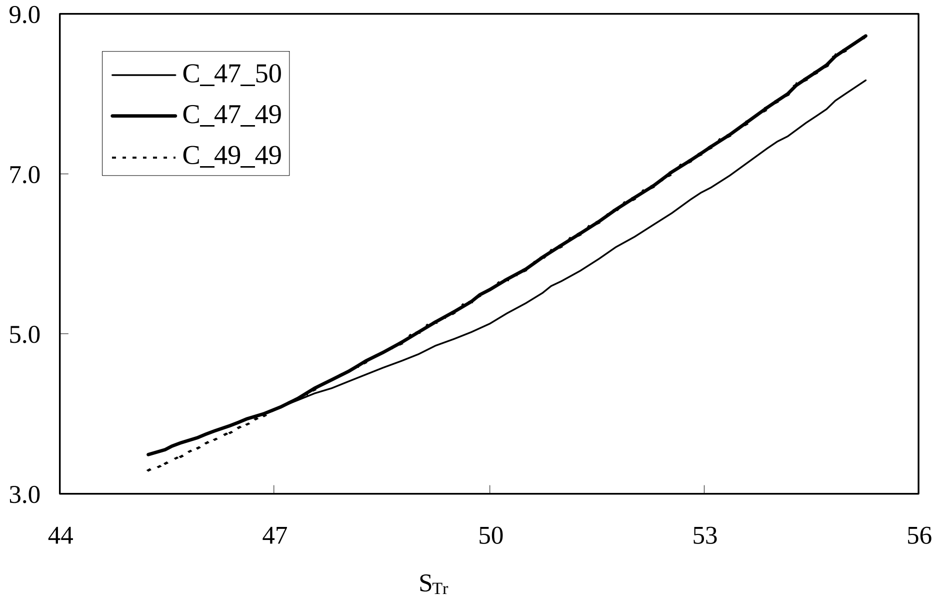

Figure 5.4: **Option Value of EGMAL.** The parameters are $S_{T_s-2} = S_{T_s-1} = 50$, $\sigma = 30\%$, $r = 2\%$, $q = 4\%$, $T = 11/12$, $T_s = 0$, and $a = 3$. C_A_B means the option price is evaluated with LB = A, UB = B.

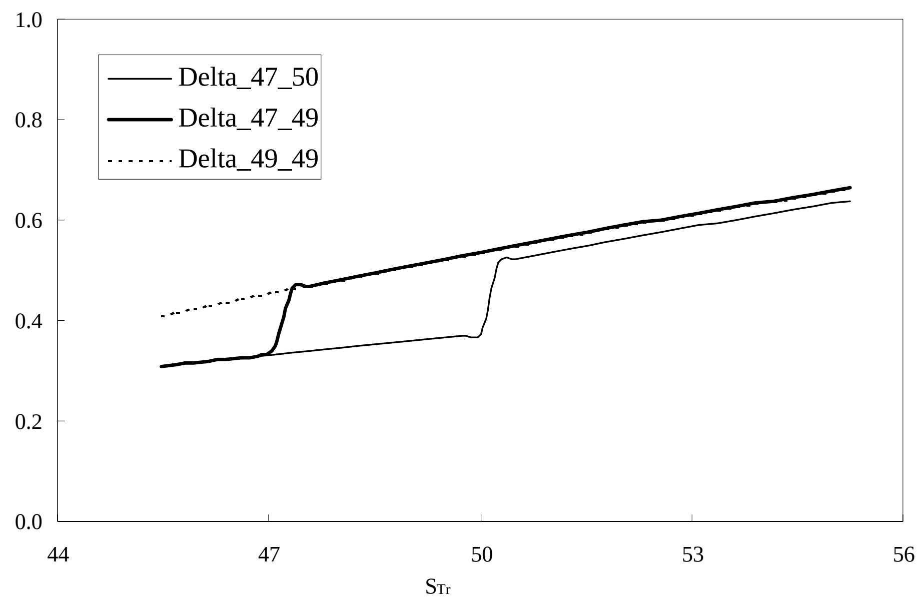

Figure 5.5: **Delta of EGMAL.** The parameters are $S_{T_s-2} = S_{T_s-1} = 50$, $\sigma = 30\%$, $r = 2\%$, $q = 4\%$, $T = 11/12$, $T_s = 0$, and $a = 3$. Delta_A_B means the delta is evaluated with LB = A, UB = B.

Chapter 6

Conclusions and Future Work

In this thesis, we investigate approaches to price derivative securities and apply them to value geometric and arithmetic moving-average-lookback options (MALs). The pricing results show that our algorithms based on the CRR model converges quickly to the correct value. We also find that the price difference between the geometric and arithmetic MALs is very small. As it takes much less time to price the geometric version, it is a good approximation to the arithmetic version. We apply the least-squares simulation to approximate the American-style MAL. Compared with the lattice algorithm, the least-squares simulation systematically undervalues the option.

This thesis highlights the need for additional research in the future. The first is to develop an efficient and accurate deterministic approximation for the CDF of the multivariate normal distribution. The reason is that the analytic solution to the geometric MAL is available as the CDF of the multivariate normal distribution. The second is to provide a more efficient way to deal with the non-Markovian problem, which is not an easy task in the framework of tree models. Finally, the LSM algorithm is a simple yet powerful method to approximate American-style derivative securities and can be applied to price complicated derivative securities which tree models cannot handle well. It is still under development. Hence, we consider it worthwhile to develop the LSM algorithm further.

Appendix A

Proofs for Analytical Solution to GMAL

A.1 Eqs. (3.3) and (3.4)

Let $\mathbf{X^k} \equiv [X_{a-1}^k, X_a^k, \ldots, X_n^k]'$, $\mu^{\mathbf{k}}$, $\mathbf{\Sigma^k}$, and $\rho^{\mathbf{k}}$ denote the mean vector, the variance-covariance matrix and the correlation matrix of $\mathbf{X^k}$ for ease of presentation. Let $\mathbf{b^k}$ be a d-row zero vector except that k^{th} element is unit, where $d = n-(a-2)$. Let $E(\ )$, $V(\ )$, and $\mathrm{Cov}(\ )$ denote $E^R(\ )$, $V^R(\ )$, and $\mathrm{Cov}^R(\ )$. Recall that R is the probability measure with the stock price as the numeraire. Define

$$
\begin{aligned}
A &= \frac{1}{2\pi^{d/2}\det(\mathbf{\Sigma^k})^{1/2}}, \\
B &= e^{\mathbf{b^k}'\mu^{\mathbf{k}} + \mathbf{b^k}'\mathbf{\Sigma^k}\mathbf{b^k}}, \\
\mathbf{Z^k} &= \mathbf{C^k}^{-1}(\mathbf{X^k} - (\mu^{\mathbf{k}} + \mathbf{\Sigma^k}\mathbf{b^k})), \\
\mathbf{C^k} &= \begin{bmatrix} \sqrt{V(X_{a-1}^k)} & 0 & \cdots & 0 \\ 0 & \sqrt{V(X_a^k)} & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \sqrt{V(X_n^k)} \end{bmatrix}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
& E(\exp\{\mathbf{b^k}'\mathbf{X^k}\}\mathbf{1}_{(\mathbf{X^k}<\mathbf{0})}) \\
= {} & A\int_{-\infty}^{\mathbf{0}} \exp\{\mathbf{b^k}'\mathbf{X^k} - \frac{1}{2}(\mathbf{X^k}-\mu^{\mathbf{k}})'\mathbf{\Sigma^k}^{-1}(\mathbf{X^k}-\mu^{\mathbf{k}})\}\, d\mathbf{X^k} \\
= {} & AB\int_{-\infty}^{\mathbf{0}} \exp\{-\frac{1}{2}(\mathbf{X^k}-(\mu^{\mathbf{k}}+\mathbf{\Sigma^k}\mathbf{b^k}))'\mathbf{\Sigma^k}^{-1}(\mathbf{X^k}-(\mu^{\mathbf{k}}+\mathbf{\Sigma^k}\mathbf{b^k}))\}\, d\mathbf{X^k}
\end{aligned}
$$

$$= \quad B \int_{-\infty}^{-\mathbf{C^{k}}^{-1}(\mu^{\mathbf{k}}+\mathbf{\Sigma^{k} b^{k}})} \frac{1}{2\pi^{d/2} \det(\rho^{\mathbf{k}})^{1/2}} \exp\{-\frac{1}{2} \mathbf{Z^{k}}' \mathbf{R^{k}}^{-1} \mathbf{Z^{k}}\} \, d\mathbf{Z^{k}}. \qquad \text{(A.1)}$$

Consequently, we can obtain Eq. (3.3) and Eq. (3.4) with Eq. (A.1).

A.2 Moments of $\mathbf{X^k}$

By definitions,

$$
\begin{aligned}
E(X_i^k) &= \begin{cases} E(Y_k) - E(Y_i) & \text{if } k \neq i \\ E(Y_k) & \text{otherwise} \end{cases} \\
V(X_i^k) &= \begin{cases} V(Y_k) + V(Y_i) - 2\mathrm{Cov}(Y_k, Y_i) & \text{if } k \neq i \\ V(Y_k) & \text{otherwise} \end{cases} \\
\mathrm{Cov}(X_i^k, X_j^k) &= \begin{cases} V(Y_k) - \mathrm{Cov}(Y_k, Y_i) - \mathrm{Cov}(Y_k, Y_j) + \mathrm{Cov}(Y_i, Y_j) & \text{if } i \neq j \\ V(Y_k) - \mathrm{Cov}(Y_k, Y_i) & \text{if } j = k \neq i \\ V(Y_k) - \mathrm{Cov}(Y_k, Y_j) & \text{if } i = k \neq j \\ V(Y_k) & \text{if } i = j = k \end{cases},
\end{aligned}
$$

where

$$
\begin{aligned}
E(Y_i) &= \left(\frac{(2i-a+1)}{2}\Delta t - T\right)\left(r + \frac{\sigma^2}{2}\right), \\
V(Y_i) &= \frac{(a-1)(2a-1)}{6a}\sigma^2 \Delta t + \sigma^2 (T - t_i), \\
\mathrm{Cov}(Y_i, Y_j) &= \left(\frac{\sigma}{a}\right)^2 \sum_{k=i-a+1}^{i} \sum_{h=j-a+1}^{j} \min(k,h)\Delta t, \\
&\quad -\sigma^2 [\,(i+j-a+1)\Delta t - T\,].
\end{aligned}
$$

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