Efficient Algorithms for Geometric-Average-Trigger Reset Options

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Abstract

A *derivative*(or *derivative security*) is a financial instrument whose value depends on the value of other, more basic underlying variables, such as bonds or stocks. A stock option is a right to buy or sell a stock by a certain date for a certain price. The price in the contract is known as the *exercise price* or *strike price*; the date in the contract is known as the *expiration date* or *maturity*.

Stock options with reset properties are traded actively on many exchanges throughout the world. A reset option is a path-dependent option whose strike price can be reset based on certain criteria. The geometric-average-trigger reset option resets the strike price based on the geometric average of the underlying asset's price over a certain time period, the so-called monitoring interval. If there are multiple monitoring intervals, multiple resets result. Similar contracts have been traded on exchanges in Asia. For example, Grand Cathay, a securities firm in Taiwan, issued two reset options(Bloomberg 0517TT and 0522TT) in the Taiwan Stock Exchange in 1999.

This thesis suggests two numerical approaches for pricing geometric-average-trigger reset options with multiple monitoring intervals. For American-style reset puts, an $O(n^4h^2)$-time algorithm on an n-period binomial lattice is presented, where h is the length (in number of periods) of each monitoring interval. A more efficient $O(n^3hm)$-time algorithm, where m denotes the number of monitoring intervals, prices European-style reset options. It is also shown that an American-style reset call will not be exercised early if its underlying asset does not pay dividends. This makes the second approach applicable to American-style reset calls. It can be proved that the price of a geometric-average-trigger reset call is higher than that of an arithmetic-one, and vice versa for a put. Monte Carlo simulations suggest that both European-style geometric- and arithmetic-average-trigger reset options have similar values. This suggests that our approaches give very approximate price for the difficult arithmetic-average-trigger reset options. Experimental data confirm the correctness of the results above.

Chapter 1

Introduction

A *derivative*(or *derivative security*) is a financial instrument whose value depends on the value of other, more basic underlying variables, such as bonds or stocks. With the rapid development of many economies, more and more derivatives have been designed and issued by financial institutions in order to satisfy their clients. All the financial innovations make the market grow fast and become more efficient. But on the other hand, those sophisticated derivatives are getting more complicated and difficult to understand. They give rise to new problems in pricing and hedging. These problems have attracted the attention of both the industry and the academia.

A new science, named *financial engineering*, is founded under such circumstances. It is the result of the interaction between three disciplines: financial economics, computer science, and mathematics. This science involves the design, development, and implementation of innovative financial instruments and processes, through which we can meet the requirement of risk management. The field of financial engineering has developed very quickly for last 3 decades.

A Review of Reset Options

A stock option is a right to buy or sell a stock by a certain date for a certain price. The price in the contract is known as the *exercise price* or *strike price*; the date in the contract is known as the *expiration date* or *maturity*. A *call* option gives the holder the right to buy the underlying asset, while a *put* option gives the holder the right to sell the underlying asset.

Path-dependent derivatives are derivative securities whose payoff depends nontrivially on the price history of the underlying asset. Some path-dependent derivatives such as barrier options can be efficiently priced. Others, however, are known to be difficult to price in terms of speed and/or accuracy, like the (arithmetic) Asian option (see Lyuu [2002].) A reset option is a path-dependent option whose strike price can be reset based on certain criteria. To prevent price manipulation, many contracts use the average price of the underlying asset during a certain time period, the so-called monitoring interval, as a reset trigger instead of the underlying asset's price.

This reset criteria makes the options hard to price. The averaging can be either over the prices in the monitoring interval or based on a moving window sliding over the monitoring interval. The geometric-average-trigger reset option uses the geometric price average in the monitoring interval to set the strike price at the reset date. If there are multiple monitoring intervals, the end of each interval is a reset date. For example, Grand Cathay, a securities firm in Taiwan, issued two reset options(Bloomberg `0517TT` and `0522TT`) in the Taiwan Stock Exchange in 1999. We briefly describe the two options below.

- `0517TT`: The strike price was 57.25 New Taiwan dollars (TWD) initially. It would be reset to 52.65 TWD if the six-day moving average of its underlying asset's prices, the CMC Magnetics Corporation, fell below 52.65 TWD during the first three months after the option was issued.

- `0522TT`: The strike price would be reset to 98%, 96%, 94%, 92%, and 90% of the initial strike price of 81 TWD if the six-day moving average of the CMC Magnetic Corporation stock prices would fall to 98%, 96%, 94%, 92%, and 90%, respectively, of the initial strike price during the first three months after the option was issued.

Gray and Whaley [1999] derive an analytic solution for single-reset reset options. Heynen and Kat [1995] discuss the discrete lookback options, which are closely related to reset options. Cheng and Zhang [2000] derives an analytic formula for the geometric-average-trigger reset option with a single monitoring interval. Their formula can only be applied to options with only a single monitoring interval. Their formula is erroneous, however (see Fang[200].)

In this thesis two numerical approaches are introduced for pricing geometric-average-trigger reset options with multiple monitoring intervals. For American-style reset puts, it is shown that if each monitoring interval has the same duration, say h periods, then an $O(n^4h^2)$-time algorithm can be derived. A much more efficient $O(n^3hm)$ algorithm exists for European-style reset options, where m denotes the number of reset dates. Besides, Monte Carlo simulations suggest that both European-style geometric- and arithmetic-average-trigger reset options have similar values.

Structures of the Thesis

This thesis is organized as follows. In Chapter two, some background knowledge about financial derivatives is introduced, including the properties of derivatives, pricing models and methods. The definition of geometric-average-trigger reset option can also be found in Chapter 2. The lattice approach that can handle American-style options is shown in chapter 3. In chapter 4, a much more efficient combinatorial approach is introduced. Experimental results and some properties of average reset options are provided in Chapter 5.

Chapter 2

Option Pricing Basics

2.1 Basic Assumptions

Some basic assumptions in finance is presented here. Survey on mathematical tools is also given in this section.

2.1.1 Basic Assumptions in Finance

The following statements must hold for all the models in this thesis.

Rational Behavior

People in this ideal market all behave *rationally*. That is, they try to maximize their benefit. They want to gain more and avert risk. This is also a basic assumption used in most economic models.

Efficient Market

All derivatives are *priced correctly*. You can trade at the market price. This assumption implies that there is no liquidity problem in this ideal market.

Complete competitive market

All people behave like *price takers* in this market. Trading activities do not influence the prices in the market. So traders in this market do not care about the $side\ effects$ of their activities, such as price movements caused by their trading.

No Arbitrage Opportunity

Arbitrage is any trading strategy that requires no cash investment and has some probability of making profits without any risk of loss. In our *ideal* environment, there should be no arbitrage opportunity for any trading strategy. That is, you can not make excess return without taking any risk. This important assumption implies that the return of any riskless portfolio is the risk-free rate.

No Transaction Cost

No tax and shoes leather cost need to be taken into consideration. This assumption will make our models become simpler.

2.1.2 Survey of Mathematical Tools

2.1.2.1 Stochastic Process

Any variable whose value changes over time in an uncertain way is called a *stochastic process*. Stochastic processes can be classified as $discrete-time$ processes or $continuous-time$ ones. A discrete-time stochastic process is one in which the value of the variable can change only at some certain time, whereas a continuous-time stochastic process allows changes take place at any time.

Formally, a stochastic process $X = \{X(t)\}$ is a time series of random variables. In other words, $X(t)$ is a random variable for time t, and it is usually called the *process state* at time t. We often write $X(t)$ as X_t in shorthand. If the time t comes from a countable set, we call X_t a $discrete - time$ stochastic process. If the time t forms a continuum, we call it a $continuous - time$ stochastic process. Any *realization* of X is called a *sample path* or *trajectory*. Note that a sample path is but an ordinary function of t. Figure 2.1 plots a sample realization of a *Brownian motion process*.

Wiener Process A *Wiener process*, sometimes referred to as a *Brownian motion* in physics, is a particular type of Markov stochastic process. It is often used for simulating stochastic variables in physics and finance.

Assume the behavior of Z_t follows a standard Wiener process. Consider the change of its value in a small duration of $\triangle t$. Let $\triangle z$ be the change in z during $\triangle t$, then the following properties must hold :
Property 1

$$\triangle z = \epsilon\sqrt{\triangle t}$$

where ϵ is a random drawing from the standard normal distribution. [1]

Property 2

[1] A standard normal distribution is a normal distribution with mean zero and standard deviation 1.

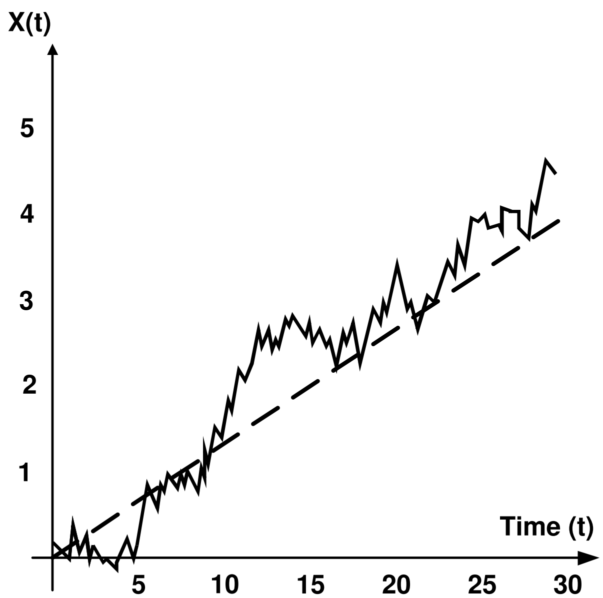

Figure 2.1: **A Sample Path of a Brownian Motion Process.** The stochastic process with volatility is testified by the jittery of the path. The related deterministic process with randomness been taken out is also plotted for reference.

The value of $\triangle z$ for any two disjoint time intervals are independent.

Thus $\triangle z$ is a normal distribution with zero mean and its standard deviation is equal to $\sqrt{\triangle t}$ by property 1. Property 2 implies that z follows a Markov process.

Generalized Wiener Process The standard Wiener process is a stochastic process with mean zero and variance 1. A *generalized process* can be defined in terms of a standard Wiener process dz as follows:

$$dx = a\,dt + b\,dz \tag{2.1}$$

where a and b are constants.

2.1.2.2 Ito Process

In this subsection, I will introduce a powerful tool, developed by Ito[3], to handle stochastic processes. An *Ito process* is a stochastic process $X = \{X_t, t \geq 0\}$ satisfying

$$X_t = X_0 + \int_0^t a_s + \int_0^t b_s dW_s ds, t \geq 0, \tag{2.2}$$

where X_0 is the "*starting point*," and a_t and b_t are two stochastic processes satisfying $\int_0^t |a_s|ds < \infty$ and $\int_0^t |b_s|ds < \infty$, respectively, almost surely for all $t \geq 0$. A shorthand for (2.2) is the following Ito differential,

$$dX_t = a_t dt + b_t \sqrt{d_t}\xi \tag{2.3}$$

where ξ is again a random variable from the standard normal distribution. From (2.3), it is easy to find that dW in (2.2) is a normal distribution with mean zero and variance dt. It is easy to see that (2.3) reduces to (2.1) when a_t and b_t are all constants.

Ito's Lemma The central tool in the Ito integral is Ito's lemma. It says that a smooth function of an Ito process is also an Ito process. Assume X_t is an Ito process of (2.1), and f is a smooth function, then the following equation follows from Ito's lemma:

$$df(X) = f'(x)adt + f'(x)bdW + \frac{1}{2}f''(x)b^2dt \tag{2.4}$$

The Ito's process can be generalized to higher dimensions for handling multi-dependent or independent Wiener processes. Consult [3] for more information.

2.2 Option Pricing Models

The basic options theory in finance and mathematical models are introduced in this section.

2.2.1 Option Basics

Option on stocks were first traded on an organized exchange in 1973. Since then there has been a dramatic growth in options markets. Huge volumes of options are also traded over the counter by banks and other financial instititions. The underlying assets include stocks, stock indices, foreign currencies, debt instruments, commodities, and futures contracts.

There are two basic types of options. A *call option* gives the holder the right to buy the underlying asset by a certain date for a certain price. A *put option* gives the holder the right to sell the underlying asset by a certain date for a certain price. The price in the contract is known as the *exercise price* or *strike price*; the date in the contract is known as the *expiration date* or *maturity*. *American options* can be exercised at any time up to the expiration date. *European options* can be exercised only on the expiration date itself.[2] Most of the options that are traded on

[2]Note that the terms *American* and *European* do not refer to the location of the option or the exchange. Some options trading on North American exchanges are European.

exchanges are American, and one contract is usually an agreement to buy or sell 100 shares(1,000 shares in the Taiwan Stock Exchange). European options are generally easier to analyze than American options, and some of the properties of an American option are frequently deduced from those of its European counterpart.

2.2.1.1 Payoffs on Standard Options

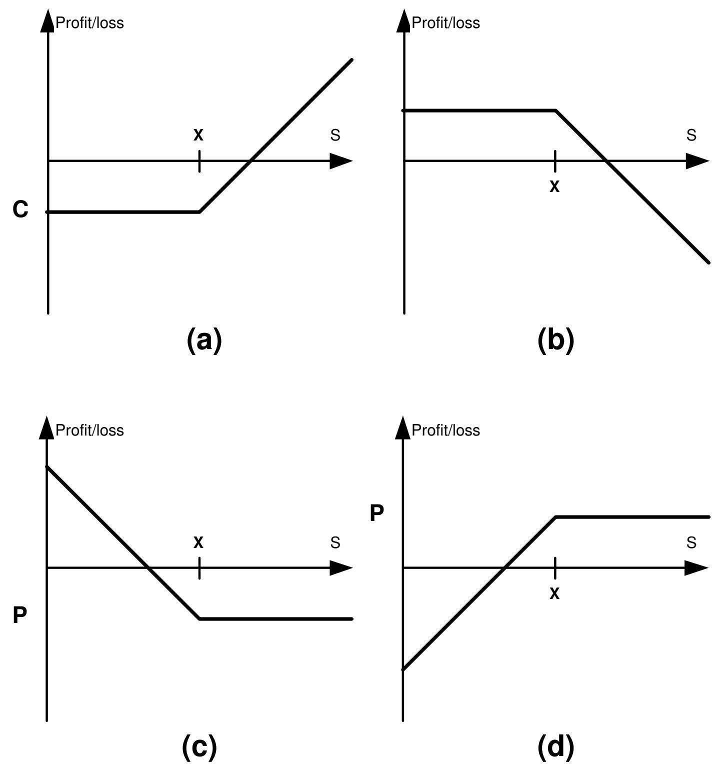

Figure 2.2: **Profit/Loss of Options.** (a) Long a call. (b) Short a call. (c) Long a put. (d) Short a put.

An option gives the holder the right to do something. The holder does not have to exercise the right. For example, consider the standard European option. Assume the value of the underlying asset is S, the strike price is X, and the premium of call/put option is represented by C and P, respectively. Then the payoff for the long position at maturity is $\max(0, S - X)$ for call options; $\max(0, X - S)$ for put options. So the profit for a long position in call options at maturity is

$$\max(S - X, 0) - C$$

The profit for a long position in put options is

$$\max(X - S, 0) - P$$

So the profit for a short position in call options is

$$-[max(S-X,0)-C]=min(X-S,0)+C$$

while the profit for a short position in put options is

$$-[max(X-S,0)-P]=min(S-X,0)+P$$

Equations above are illustrated graphically in Figure 2.2. Note that the calculations above ignore the time value of money.

2.2.2 The Black-Scholes Formula

In the early 1970s, Fischer Black, Myron Scholes, and Robert Merton made a major breakthrough by deriving a differential equation that must be satisfied by any derivative security whose underlying asset is a non-dividend-paying stock. They solved this equation and obtained the closed-form solution for European call and put options on stock. This formula, known as the Black-Scholes formula, is one of the most significant tools for pricing financial instruments.

2.2.2.1 The Log-normal Model for Stock Price

A log-normal distribution for the stock price is the standard model used in financial economics. This is because its properties can satisfy reasonable assumptions about the random behavior of stock prices. The stochastic log-normal model for the non-dividend-paying stock is

$$\frac{dS}{S}=\mu dt+\sigma dz \tag{2.5}$$

Equation (2.5) is also known as *geometric Brownian motion* where S is the stock price. The variables μ and σ are referred to as the expected return and volatility, respectively.

Clearly, the rate of return[3] on stock is a random variable with a normal distribution. That is why we call it *log-normal*. The stock price realized by this model will never be negative, and the percentage changes of S are independent and identically distributed. These nice properties make it a good model for simulating the stock price movement.

2.2.2.2 Assumptions

The assumptions used to derive the Black-Scholes differential equation are listed below:

[3]Using the continuous compounding formula.

1. The value of the underlying assets follows the log-normal distribution.
2. The rate of return on stock, μ, and the volatility of stock price, σ, are constant throughout the option's life.
3. The short selling of securities with full use of proceeds is permitted.
4. The are no transaction costs or taxes. All securities are perfectly divisible.
5. No dividends are paid during the life of the derivative security.
6. No arbitrage opportunity.
7. Security trading is continuous.
8. The risk-free interest rate, r, is constant and unchanged during the life of the security.

2.2.2.3 The Black-Scholes Differential Equation

By eliminating the random source of the underlying stochastic process [6], the final equations emerges as

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{2.6}$$

where f is the price of a derivative security, S is the stock price, σ is the volatility of the stock price, and r is the continuously compounded risk-free rate.

2.2.2.4 The Closed Form Solution for Black-Scholes Formula

The closed form solutions for the price of European calls and puts by solving (2.6) can be described as below,[4]

$$C = SN(d_1) - Xe^{-rT}N(d_2)$$

$$P = Xe^{-rT}N(-d_2) - SN(-d_1)$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\ln(S/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \end{aligned}$$

[4] C denotes the call price, P denotes the put price.

The notations for the above equation are described as below.
$N(x)$ = probability distribution function for standard normal distribution,
σ^2 = annualized variance of the continuously compounded return on stocks,
r = continuously compounded risk-free rate,
T = time to maturity.

2.2.3 The CRR Binomial Lattice

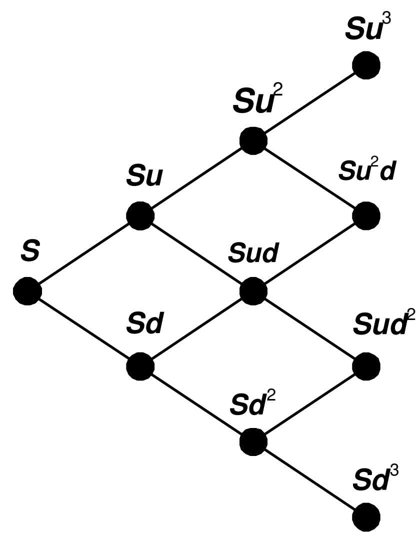

Figure 2.3: **A Three-Period Binomial Model .** Stock price moves over three time periods on the binomial model. S is the stock price at period 0 and u and d are constants indicating the upward and downward ratios of stock price movements.

The approach to build a binomial lattice for the log-normal distribution is introduced here. First of all, we should calibrate the first and the second moments. The generalized three-period binomial model is illustrated in Figure 2.3.

Let P_u and P_d indicate the probability of an up move and a down move, respectively, then the equations can be described as follows,

$$R_f = \ln u P_u + \ln d P_d$$

$$V = (\ln u - R_f)^2 P_u + (\ln d - R_f)^2 P_d$$

$$P_u + P_d = 1$$

where R_f denotes the risk-free rate and V denotes the variance of the stock return. Certainly, these four unknown variables can not be determined by the above three equations. For the convenience of pricing in lattice models, proper constraints are selected to achieve some good properties. Some examples are: the constraints we add in the CRR model is $ud = 1$, and $P_u = P_d = 0.5$ in the Jarrow's model.

2.3 Geometric-Average-Trigger Reset Options

An option starts at time 0 and matures at T. Let r denote the risk-free interest rate. $S(t)$ denotes the underlying asset's price at time t, and σ denotes the volatility of underlying asset. We assume S follows

$$dS = rS\,dt + \sigma S\,dW$$

in a risk-neutral economy.

Geometric-average-trigger reset options are reset options whose strike price can be reset to the geometric average of the underlying asset's prices over a monitoring interval. Consider a general reset option with m reset dates: $0 \leq t_1 < t_2 < \cdots < t_{m-1} < t_m \leq T$. Assume the m monitoring intervals are $[t_1 - \ell_1, t_1]$, $[t_2 - \ell_2, t_2]$, $\ldots$, $[t_m - \ell_m, t_m]$, where ℓ_i denotes the length of the ith monitoring interval. Define $\text{avg}(t_i)$ as the geometric average price of the underlying asset during the ith monitoring interval. Let $K(t_i)$ be the strike price prevailing at time t_i with $K(t_0) = K$, the original strike price. The reset procedure at time t_i is:

$$K(t_i) = \begin{cases} K(t_{i-1}), & \text{if } \ \text{avg}(t_i) \geq K(t_{i-1}) \\ \text{avg}(t_i), & \text{if } \ \text{avg}(t_i) < K(t_{i-1}) \end{cases}.$$

We will assume that $\ell_1 = \ell_2 = \cdots = \ell_m = \ell$ and that the monitoring interval are disjoint to simplify the presentation. The payoff of the call is $(S(T) - K(t_m))^+$. Similarly, the reset procedure for the put is

$$K(t_i) = \begin{cases} K(t_{i-1}), & \text{if } \ \text{avg}(t_i) \leq K(t_{i-1}) \\ \text{avg}(t_i), & \text{if } \ \text{avg}(t_i) > K(t_{i-1}) \end{cases},$$

and the payoff of the put is $(K(t_m) - S(T))^+$.

Chapter 3

The Lattice Approach to Pricing

An algorithm based on the binomial lattice model is introduced in this chapter. This algorithm can accurately price general American-style geometric-average-trigger reset call/put options. The backward-induction technique is applied in this algorithm.

3.1 The Binomial Model

The standard CRR binomial lattice will be adopted for the underlying asset's price dynamics. The binomial model starts at (discrete) time 0 and ends at time n. Because the life span of an option is T, the length of each period, denoted as Δt, equals T/n. The number of periods for a monitoring interval, h, equals $\ell/\Delta t$. Let node (i, j), $0 \leq j \leq i \leq n$, stands for the node at time i with j cumulative down moves and $S(i, j)$ denotes the underlying asset's price at node (i, j). Initially, $S(0, 0) = S(0)$. The current underlying asset's price at node (i, j), $S(i, j)$, becomes $S(i, j)u$ at node $(i + 1, j)$ (the up move) with probability p_u and $S(i, j)d$ at node $(i + 1, j + 1)$ (the down move) with probability $p_d \equiv 1 - p_u$, where $u = e^{\sigma\sqrt{\Delta t}}$ and

$$ud = 1.$$

The above identity, though not necessary for theoretical purposes, is crucial in the development of our algorithm. Clearly, $S(i, j)$ is equal to $S(0)u^{i-2j}$, and node (i, j) is reached with probability $\binom{i}{j}p_u^{i-j}p_d^j$. We assume that the underlying asset does not pay dividends for simplicity. For pricing purposes, the probability p_u is set to $(e^{r\Delta t} - d)/(u - d)$, where r denotes the continuously compounding risk-free interest rate per annum.

3.2 Basic Ideas

To price a geometric-average-trigger reset option, the geometric sum of the underlying asset's prices in a monitoring interval is a key number because it sets the strike price.

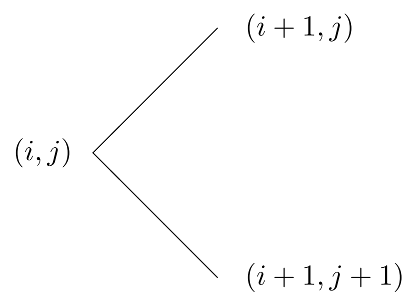

Figure 3.1: **The Binomial Model.** Node (i, j) is the result of j down moves and $i - j$ up moves, where $0 \leq j \leq i$.

Our algorithm keeps track of the prevailing strike price and the partial geometric sums of the underlying asset's prices during a monitoring interval at each node. These numbers constitute the states for a node. The question is, what is the size of the states at each node?

We first define a few key terms. Let $(S_0, S_1, \ldots, S_i)$ be a sequence of prices, where $i \leq h$. Define $S_0 S_1 \cdots S_i$ as the (partial) geometric price sum. When $i = h$, we call $(S_0 S_1 \cdots S_h)^{1/(h+1)}$ a geometric price average. For example, the geometric price sum of a price sequence $(S, Su, Su^2, \ldots, Su^i)$ is $S^{i+1} u^{i(i+1)/2}$. When $i = h$, its geometric price average is $Su^{h/2}$. For a price of form $S(0)^i u^g$, we call g the price index.

The underlying asset's prices over an n-period lattice form the set

$$\{S(0)u^{-n}, S(0)u^{-n+1}, \ldots, S(0)u^{n-1}, S(0)u^n\}.$$

The set of possible partial geometric price sums for the ith ($i \leq h$) node in a monitoring interval is some subset of

$$\{S(0)^i u^{-n(i+1)}, S(0)^i u^{-n(i+1)+1}, \ldots, S(0)^i u^{n(i+1)-1}, S(0)^i u^{n(i+1)}\},$$

which has $2n(i+1)+1$ elements. This is because the maximum geometric price sum for ith node in a monitoring interval is smaller than $\overbrace{S(0)u^n \times \cdots S(0)u^n}^{i} = S(0)u^{n(i+1)}$. Similarly, the minimum geometric price sum is larger than $S(0)u^{-n(i+1)}$. Hence, all possible geometric price sums for this node must has a integral price index between $-n(i+1)$ and $n(i+1)$. Therefore, the number of possible geometric price sums for a node falling under a monitoring interval is bounded by $O(nh)$. The number of possible strike prices is also bounded by $O(nh)$ because the strike price will only be reset to a prevailing geometric price average. As a result, the maximum number of states for each node is bounded by $O(n^2h^2)$. This estimate is pessimistic as some nodes need much fewer states.

3.3 Four Types of Nodes

Depending on where a node is located on the lattice relative to the monitoring intervals, different backward-induction formulas result. In Fig. 3.2, the lattice is divided into four areas. Area A includes the lattice nodes before the first monitoring interval. Each node in area A needs only one state to store the option value whose corresponding strike price is K. Note that partial geometric sums of the underlying asset's prices are not required since no nodes in area A fall under any monitoring interval. We use $M^A_{i,j}$ to denote the option value at node (i, j) in area A.

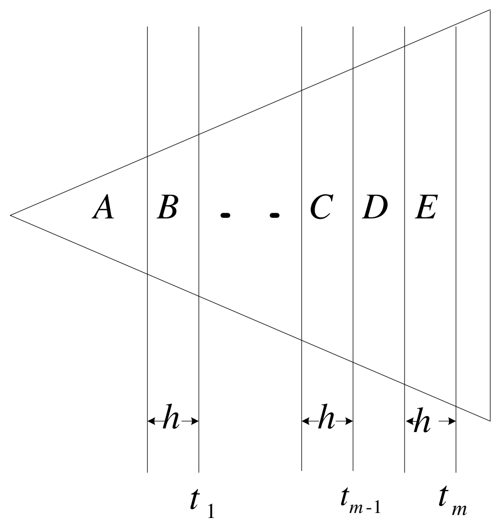

Figure 3.2: **Overview of the Lattice.**

The second area, area B, includes all the nodes falling under the first monitoring interval. Each node requires $2n(h+1)+1$ states for the $2n(h+1)+1$ different possible geometric price sums. We use $M^B_{i,j}(y)$ to denote the option value when the price index of the partial geometric price sum is $y - n(h + 1)$ at node (i, j) in area B.

The third area, area D, includes all the nodes between two adjacent monitoring intervals. Each node needs $2n(h + 1) + 1$ states since up to $2n(h + 1) + 1$ strike prices need to be recorded. The geometric price sums are not required since nodes in area D do not fall under any monitoring interval. We use $M^D_{i,j}(x)$ to denote the option value whose corresponding current strike price is $S(0)u^{\frac{x}{h+1}-n}$ at node (i, j) in area D.

The original strike price K is not necessarily equal to one of the possible geometric price averages. Define z which satisfies

$$S(0)u^{\frac{z-1}{h+1}-n} < K \leq S(0)u^{\frac{z}{h+1}-n}. \tag{3.1}$$

Then $M^D_{i,j}(z)$ as the option value whose corresponding current strike price is K instead of $S(0)u^{\frac{z}{h+1}-n}$. Note that the calculations for the states whose price indexes are greater than z can be skipped because the strike price can only be reset to some geometric price averages lower than K.

The third area, area C, includes all the nodes falling under the monitoring intervals (excluding the first monitoring interval.) $[2n(h+1)+1]^2$ states are needed at each node in area C. This is because both the geometric price sums and the prevailing strike prices are relevant. We use $M^C_{i,j}(x,y)$ to denote the option value when the current strike price and the price index of the partial geometric price sum are $S(0)u^{\frac{x}{h+1}-n}$ and $y-n(h+1)$, respectively, at node (i,j) in area C.

3.4 Pricing with Backward Induction

To evaluate an American-style option, we traverse the lattice in a backward fashion. Prices for the states at any node in area D can be evaluated by

$$M^D_{i,j}(x) = e^{-r\Delta t}[p_u M^D_{i+1,j}(x) + p_d M^D_{i+1,j+1}(x)].$$

Next, we focus on the states for a node that is located on a reset date between areas C and D. Resetting of the strike price may happen here. If the current strike price is lower than or equal to the prevailing geometric price average (i.e., $x \leq y$ in the following formula), the strike price will not be reset; otherwise, the strike price should be reset to the prevailing geometric price average. Therefore,

$$M^C_{i,j}(x,y) = \begin{cases} e^{-r\Delta t}[p_u M^D_{i+1,j}(x) + p_d M^D_{i+1,j+1}(x)], & \text{if } \ y \geq x, \\ e^{-r\Delta t}[p_u M^D_{i+1,j}(y) + p_d M^D_{i+1,j+1}(y)], & \text{if } \ y < x. \end{cases}$$

Now we focus on the states for nodes in area C. We should keep track of the current strike prices and the price indices, which represent the geometric price sums over the monitoring interval. We use the following formula:

$$M^C_{i,j}(x,y) = e^{-r\Delta t}[p_u M^C_{i+1,j}(x, y+(i+1)-2j) + p_d M^C_{i+1,j+1}(x, y+(i+1)-2(j+1))].$$

Consider the states for a node (i,j) which is one period before the beginning of a monitoring interval (but excluding the first monitoring interval), that is, located at the boundary between areas D and C. We can evaluate them by the following formula:

$$M^D_{i,j}(x) = e^{-r\Delta t}[p_u M^C_{i+1,j}(x, (i+1)-2j) + p_d M^C_{i+1,j+1}(x, (i+1)-2(j+1))].$$

For the states at node (i,j) on the first reset date, we apply the following formula:

$$M^B_{i,j}(y) = \begin{cases} e^{-r\Delta t}[p_u M^D_{i+1,j}(y) + p_d M^D_{i+1,j+1}(y)], & \text{if } \ y < z, \\ e^{-r\Delta t}[p_u M^D_{i+1,j}(z) + p_d M^D_{i+1,j+1}(z)], & \text{if } \ y \geq z. \end{cases}$$

For the states at a node in the first monitoring interval (area B), we apply the following formula:

$$M^B_{i,j}(y) = e^{-r\Delta t}[p_u M^B_{i+1,j}(y + (i+1) - 2j) + p_d M^B_{i+1,j+1}(y + (i+1) - 2(j+1))].$$

For the states at a node one period before the beginning of the first monitoring interval, we apply the following formula:

$$M^A_{i,j} = e^{-r\Delta t}[p_u M^B_{i+1,j}((i+1) - 2j) + p_d M^B_{i+1,j+1}((i+1) - 2(j+1))].$$

Finally, the backward-induction formula in area A is

$$M^A_{i,j} = e^{-r\Delta t}(p_u M^A_{i+1,j} + p_d M^A_{i+1,j+1}).$$

The final pricing result is obtained at $M^A_{0,0}$.

Since the maximum number of states for each node is bounded by $O(n^2h^2)$ and there are $O(n^2)$ nodes in the lattice, we conclude that the algorithm runs in $O(n^4h^2)$ time.

Chapter 4

The Combinatorial Approach

In the former algorithm, $O(n^2h^2)$ states are needed for every node in a monitoring interval (but excluding the first monitoring interval). This results in an $O(n^4h^2)$ algorithm since there are $O(n^2)$ nodes. A faster $O(n^3hm)$ approach using combinatorics will be described below where m denotes the number of monitoring intervals. This approach, still based on the CRR lattice, provides the same results as the former one, just faster. Forward-induction technique is employed by this approach. In other words, we assume that the probability at root node is 1, and the probabilities propagate forward in time. The option value is computed by taking the expected value of the final payoff. We will devise an efficient way to propagate the probabilities using combinatorics. See Lyuu [2002] for more information on the combinatorial methods for pricing.

4.1 Monitoring Intervals

In Fig. 4.1, each array has 4 states which are corresponding to different current strike prices K_1, K_2, K_3, and K_4, respectively. Assume $K_1 > K_2 > K_3 > K_4$. The variable a_i $(1 \leq i \leq 4)$ listed in left most array denotes the transition probability for the state whose corresponding strike price is K_i at an arbitrary node α located at the beginning of a monitoring interval. The ith term in right most array denotes the probability for the state whose corresponding strike price is K_i at another arbitrary node β located at the end of a monitoring interval. The variable b_i in the central array denotes the transition probability for the price paths that start from α and end in β with geometric price average equal to K_i. Our goal is to evaluate the probabilities for each state variable at node β efficiently. This can be divide into two different subgoals as follows.

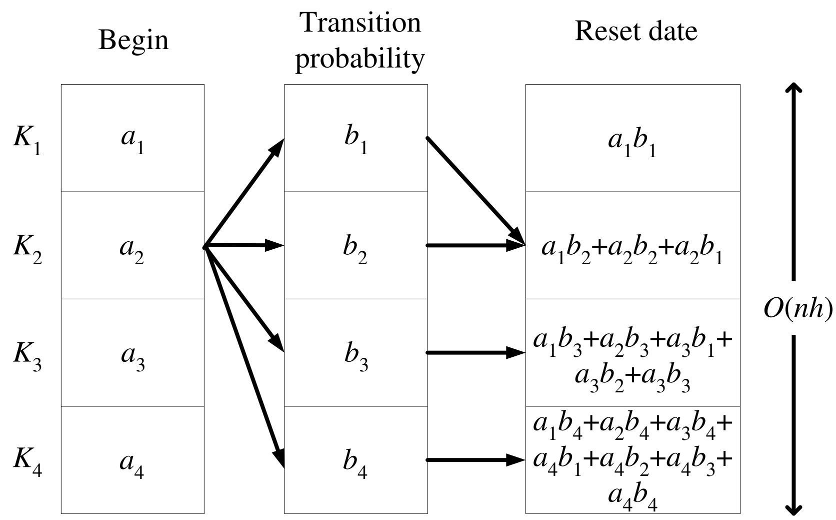

Figure 4.1: **Probability Propagation in a Monitoring Interval.**

Probabilities at Reset Dates

Recall that the strike price will be reset if the new geometric price average is lower than the original strike price. Take the second state variable in the left most array for example. The probability for the price paths reaching this state is a_2, and the current strike price for the price paths is K_2. Obviously, the probability for the price paths that pass through this state and have geometric price average K_i is $a_2 b_i$. However, the strike price will only be reset when the geometric price average is lower than the current strike price. So the probability $a_2 b_1$ should be added in the second state variable in the rightmost array. The probabilities in the right most array are derived similarly.

The computational complexity can be reduced by writing the formula for the ith state variable in the right most array as

$$\left(\sum_{k=1}^{i-1} a_k \right) b_i + \left(\sum_{k=1}^{i} b_k \right) a_i.$$

Since $\sum_{k=1}^{i-1} a_k$ and $\sum_{k=1}^{i} b_k$ for $1 \leq i \leq 2nh$ can all be evaluated in $O(nh)$ time, the transition probabilities in the right most array can be evaluated in $O(nh)$ time.

Transition Probabilities

The transition probabilities in the central array can be evaluated by the following generating function:

$$(x^{-h} + x^h y)(x^{-h+1} + x^{h-1} y) \cdots (x^{-2} + x^2 y)(x^{-1} + xy). \tag{4.1}$$

Similar generating functions can be found in Lyuu [2002] for solving geometric average Asian options. The coefficient of $x^\alpha y^\beta$ denotes the number of the paths that start from a node (i, j) and go through the node $(i + h, j + h - \beta)$ with the geometric price sum equals to $S(i, j)^{h+1} u^\alpha$. The same probabilities need for all (i, j) pairs. It is obvious that formula (4.1) can be computed in time $O(h^4)$. So the transition probabilities in the central array can be obtained in $O(h^4)$ time.

In summary, it takes $O(n^3 h)$ time to propagate the probabilities through the monitoring intervals. Assuming that there are m monitoring intervals in the lattice, the algorithm would take $O(n^3 hm)$ time.

4.2 The Algorithm

Each node needs the same amount of the states as the previous approach described to store the transition probability instead of option value.[1] We use $P^{\bullet}_{i,j}(\bullet)$ to denote the transition probability at a state in (i, j). In area A, we set $P^A_{0,0} = 1$. If node (i, j) is the first period in the monitoring interval, then it is obvious that

$$P^B_{i,j}(x) = \begin{cases} \binom{i}{j} p_u^{i-j} (1 - p_u)^j, & \text{if } \ x = (i - 2j)(h + 1) + n(h + 1), \\ 0, & \text{otherwise.} \end{cases}$$

The probabilities propagated through area B can be determined by the methods provided in the last subsection. The probability propagation between areas B and D, that is, for the state variables at node (i, j) located at node which are immediately after the first reset date, is determined by propagating the probabilities from the state variables located at first reset date by the following formula:

$$P^D_{i,j}(x) = \begin{cases} P^B_{i-1,j}(x) p_u + P^B_{i-1,j-1}(x) p_d, & \text{if } x < z, \\ \sum_{y \geq z} P^B_{i-1,j}(y) p_u + \sum_{y \geq z} P^B_{i-1,j-1}(y) p_d, & \text{if } x = z, \\ 0, & \text{if } x > z, \end{cases}$$

where z is defined in Eq. (3.1). Probability propagated through area D can be determined by $P^D_{i,j}(x) = \sum_{q=0}^{p} P^D_{p,q}(x) \binom{i-p}{j-q} p_u^{(i-p)-(j-q)} p_d^{j-q}$ for any arbitrary node (i, j) located at last period before the next monitoring interval and node (p, q) located at first period after the former monitoring interval (i.e; the boundary between area D and C). To propagate the probabilities into area C, the following recurrence is used:

$$P^C_{i,j}(x, y) = \begin{cases} P^D_{i-1,j}(x) p_u + P^D_{i-1,j-1}(x) p_d, & \text{if } y = (i - 2j)(h + 1) + n(h + 1), \\ 0, & \text{otherwise.} \end{cases}$$

[1] Only the states in the boundary nodes, the nodes between two different areas as described above, are needed in our algorithm. States at some other nodes are not required.

Obviously, to propagate probabilities through area C can be done by the methods determined in the last subsection, so we only need to consider propagating the probabilities from nodes located at the reset date to any arbitrary node (i, j) located at next period. This can be done by first setting all state variables located at node (i, j) to 0, and then propagate the probabilities from nodes located at the last period by the following formulas:

$$\begin{aligned} P^D_{i,j}(x) &= \sum_{k \geq x} P^C_{i-1,j}(x,k)p_u + \sum_{k \geq x} P^C_{i-1,j-1}(x,k)p_d \\ &+ \sum_{k > x} P^C_{i-1,j}(k,x)p_u + \sum_{k > x} P^C_{i-1,j-1}(k,x)p_d \end{aligned}$$

Finally, the option value is simply the discounted expected value of the final payoff.

Chapter 5

Behavior Analysis and Experimental Results

5.1 Behavior Analysis

Some special properties of reset options are discussed in this section. First, an American-style reset call is proved not to be exercised early if the underlying asset does not pay dividends. Second, the relations between the time span of monitoring intervals and the option value are also discussed.

Theorem 5.1.1 *An American-style reset call option will not be exercised early if the underlying asset does not pay dividends.*

Proof. It is well-known that an American vanilla call option will not be exercised early if the underlying asset does not pay dividends. That is, at any arbitrary node N of the lattice, the following inequality should always be satisfied:

$$E_v^Q[V_N] \geq S_N - K$$

where $E_v^Q[V_N]$ denotes the option value at node N if a vanilla option is not exercised immediately, and S_N denotes the underlying asset's value at node N. Since the strike price of a reset call could only be reset to a lower level, the value to hold a reset call should be at least as high as the value to hold a vanilla call . It is observed that

$$E_r^Q[V_N] \geq E_v^Q[V_N] \geq S_N - K$$

where $E_r^Q[V_N]$ denotes the option value at node N if a reset option is not exercised immediately. As a result, a reset call won't be exercised early. □

Most average reset options in real market are triggered by the arithmetic price average instead of geometric one. Pricing arithmetic-average-trigger reset options is not easy, but it is surprisingly easy to explore the relationship between these two kinds of options.

Theorem 5.1.2 *The price of the geometric-average-trigger reset call option is higher than the price of the arithmetic-average-trigger reset call option.*

Proof. Let $\text{avg}^G(t_i)$ and $\text{avg}^A(t_i)$ denote the geometric and the arithmetic price average of the i-th monitoring interval, respectively. Since $\text{avg}^G(t_i) \leq \text{Avg}^A(t_i)$, we have

$$\min\{K, \text{avg}^G(t_1), \text{avg}^G(t_2), \ldots, \text{avg}^G(t_m)\} \leq \min\{K, \text{avg}^A(t_1), \text{avg}^A(t_2), \ldots, \text{avg}^A(t_m)\}.$$

Consequently,

$$\begin{aligned} & E(S(T) - \min\{K, \text{avg}^G(t_1), \text{avg}^G(t_2), \ldots, \text{avg}^G(t_m)\})^+ \\ \geq\ & E(S(T) - \min\{K, \text{avg}^A(t_1), \text{avg}^A(t_2), \ldots, \text{avg}^A(t_m)\})^+. \end{aligned}$$

□

Theorem 5.1.3 *The price of the geometric-average-trigger reset put option is lower than the price of the arithmetic-average-trigger reset put option.*

Proof. Obviously,

$$\max\{K, \text{avg}^G(t_1), \text{avg}^G(t_2), \ldots, \text{avg}^G(t_m)\} \leq \max\{K, \text{avg}^A(t_1), \text{avg}^A(t_2), \ldots, \text{avg}^A(t_m)\}.$$

Consequently,

$$\begin{aligned} & E(\max\{K, \text{avg}^G(t_1), \text{avg}^G(t_2), \ldots, \text{avg}^G(t_m)\} - S(T))^+ \\ \leq\ & E(\max\{K, \text{avg}^A(t_1), \text{avg}^A(t_2), \ldots, \text{avg}^A(t_m)\} - S(T))^+. \end{aligned}$$

□

In real market, We compute the average price of some representative prices, like closing prices, instead of computing the continuous geometric price average during the monitoring intervals. The relationship between the sampling frequencies and the option value can be explored by taking advantages of Brownian bridge. We sample n points, including the beginning and the end points, in each monitoring interval; these n points divide the monitoring interval into $n-1$ equal sub-intervals. The average price of n sampling points located at i-th monitoring interval is denoted as $\text{avg}_n^G(t_i)$. Some required properties of Brownian bridge are explored in the following lemma.

5.2 Experimental Results

The data listed in Table 5.1 show how the the number of reset dates influence the option value. Suppose that it is a put option, the initial stock price is 100, the initial strike price is 95, the interest rate 5%, the volatility is 30%, the length of the monitoring intervals is 5 periods, and the number of periods in the lattice is 50. Assume the lifespan of the option is one year. Since the options are more likely being reset when the number of reset dates increase, the option value also increases as the reset dates increase. Monte Carlo simulations results are also listed in the same table to show the accuracy of our algorithms.

The data listed in Table 5.2 show how the different lengths of monitoring intervals influence the option value. Suppose that the initial stock price is 100, the initial strike price is 90, the interest rate 6%, the volatility is 30%, and the number of periods in the lattice is 65. The reset dates are at period 10,20,30,40,50,60. Assume the lifespan of the option is one year. We can see from the result that the option value decreases as the length of the intervals increases.

Table 5.3, Table 5.4, and Table 5.5 show that the relative difference between the value of geometric- and arithmetic-average-trigger reset options is insignificant. Fig 5.1 demonstrates the oscillation of option value with respect to n, the number of periods in the lattice. We can see that the option value converges very quickly when $n \geq 50$. Table 5.6 compares the running time of the two algorithms. Obviously the combinatorial approach is far more efficient then the lattice approach.

Reset Dates (Year)	Lattice	Lattice(Euro)	MC
1	8.73217	8.3018	8.3020
1,0.8	10.8541	10.4507	10.3667
1,0.8,0.6	12.4521	11.9824	11.9054
1,0.8,0.6,0.4	13.7323	13.1883	13.1036
1,0.8,0.6,0.4,0.2	14.735	14.1174	14.0317

Table 5.1: **Option Values with Respect to Different Reset Dates.** The option value increases with the number of reset dates. After 1,000,000 sample paths with Monte Carlo simulation with 1,000 time steps, the option values are relatively close to the values calculated by the lattice approach.

Length(year)	Lattice	MC
1/65	22.8105	22.795
2/65	22.7031	22.699
3/65	22.6586	22.625
4/65	22.5909	22.571
5/65	22.5191	22.490

Table 5.2: **Option Values w.r.t. Different Lengths of Monitoring Intervals** Option values increase as the length of monitoring intervals decrease. The 1,000,000 times sampling from Monte Carlo Simulation agrees with the values calculated by the lattice approach.

# of periods	vol=0.5	vol=0.8	vol=1.0	vol=1.2	vol=1.5
10	26.04028647	37.70778614	44.85940238	51.51328682	60.57598176
100	26.1147983	37.73757635	44.95541639	51.63071751	60.76180327
1000	26.13326108	37.76117174	45.18583957	52.01828236	61.19064968
10000	26.16468427	37.78172752	44.97135585	51.56229747	60.4602232

Table 5.3: **Evaluate Geometric-Average-Trigger Reset Calls with Monte Carlo simulations.** $T = 1$, S_0=100, K=95, r=0.05, h=0.2, and reset date=0.5.

# of periods	vol=0.5	vol=0.8	vol=1.0	vol=1.2	vol=1.5
10	26.00424102	37.46460782	44.73355475	51.56247684	60.40588585
100	26.06763879	37.68977758	44.79836599	51.49789968	60.31990169
1000	26.10137742	37.69141599	44.71671987	51.51986505	61.07059802
10000	26.10477572	37.65250356	44.78363601	51.42352387	60.16191354

Table 5.4: **Evaluate Arithmetic-Average-Trigger Reset Calls with Monte Carlo simulations.** $T = 1$, S_0=100, K=95, r=0.05, h=0.2, and reset date=0.5.

# of periods	vol=0.5	vol=0.8	vol=1.0	vol=1.2	vol=1.5
10	0.138421878	0.644902152	0.280537915	-0.095489974	0.280797615
100	0.180585399	0.126660946	0.349346999	0.257245769	0.72726871
1000	0.122004154	0.184728774	1.038200693	0.958157949	0.196192818
10000	0.228967233	0.342027665	0.417420891	0.269137735	0.493398217

Table 5.5: **Difference Percentage.** The Relative difference is insignificant.

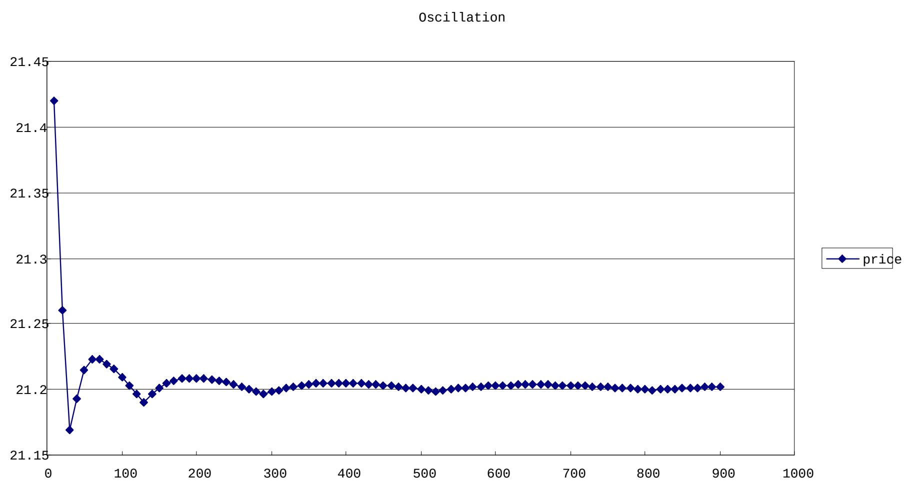

Figure 5.1: **Oscillation of Option Value.** This figure demonstrate the oscillation of option value with respect to n, the number of periods in the lattice. The option value converges very quickly when $n \geq 50$.

	$n = 50$	$n = 100$	$n = 200$	$n = 400$
Lattice	$1'33''$	N.A.	N.A.	N.A.
Combinatorial	$0.14''$	$0.48''$	$21.4''$	$22'18''$

Table 5.6: **Running-Time Comparison.** The combinatorial approach is far more efficient then the lattice approach. Assume $T = 1$, the number of monitoring interval is 2, the interval length is 0.2, and the reset dates are at 0.4 and 0.8 . n denotes the number of periods. The experiment is done on an IBM PC with an Intel Pentium III 866 MHz processor and 1 GB DRAM.

Chapter 6

Conclusion and Future Work

The geometric-average-trigger reset option resets the strike price based on the geometric average of the underlying asset's prices over a monitoring interval. Similar contracts have been traded on exchanges in Asia. An $O(n^4h^2)$-time algorithm for general American-style reset options is presented in the thesis. A more efficient $O(n^3hm)$-time algorithms is derived to price European-style options. Besides, it is proved that an American reset call option won't be exercised early if the underlying assets won't pay the dividends. Numerical results are given to suggest the correctness of these two approaches. Besides, numerical evidence suggests that our pricing approaches give very tight lower (upper) bounds on arithmetic-average-trigger reset calls (puts, respectively). Research on arithmetic-average-trigger reset options is under way.

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