Numerical Methods for Pricing Path Dependent Options

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Contents

List of Figures

Abstract

With the rapid growth of the financial market, an increasingly large number of sophisticated options are traded in the over-the-counter market to meet clients' needs. *Path-dependent* options are such sophisticated options. A *reset* option is a kind of path-dependent option that allows the exercise price to be reset when the price of the underlying asset ever hits the reset barrier during its life. A *lookback* option is another kind of path-dependent option whose payoff depends on the extreme of the underlying asset's price over a certain period of time.

In this thesis, we propose a combinatorial method to value *European*-style reset and lookback options by the use of the *reflection principle*. Under this method, we derive a linear-time algorithm to value reset options and a quadratic-time algorithm to value lookback options. Traditional methods take quadratic time to value reset options such as Ritchken's trinomial tree algorithm and cubic time to value lookback options using backward induction.

Although the combinatorial method is highly efficient in pricing European lookback options, it converges slowly. It also underestimates the analytical value. We propose an *interpolation method* to improve its convergence. We also price the *American*-style lookback options by the use of the interpolation method. The interpolation technique is found to work well for price approximations and is efficient.

In this thesis, all programs run on a PC with Intel Pentium-2 266 CPU, 64 MB DRAM, and Windows 98 platform.

Chapter 1

Introduction

1.1 Option Basics

Options were first traded on an organized exchange in 1973. Since then there has been a dramatic growth in options markets. There are two basic types of option contracts: *call* options and *put* options. A call option gives the holder the right to buy the asset at a specific price called the *exercise price* or *strike price*. A put option gives the holder the right to sell the asset at the strike price. Call and put options can be classified as: *American* or *European*. A European option can only be exercised at the maturity date of the option, whereas an American option can be exercised at any time up to and including the maturity date; namely, *early exercise* is allowed.

Let X be the strike price and S_T be the price of the underlying asset at the maturity date. The payoff from a long position in a European call option is

$$\max(S_T - X, 0)$$

This is because, the holder has the right but not the obligation to exercise it. The holder will exercise only if $S_T > X$ and then receive $S_T - X$ in effect. See Figure 1.1. Similarly, the payoff from a long position in a European put option is

$$\max(X - S_T, 0)$$

The holder will exercise only if $X > S_T$ and then receive $X - S_T$ in effect. See Figure 1.2.

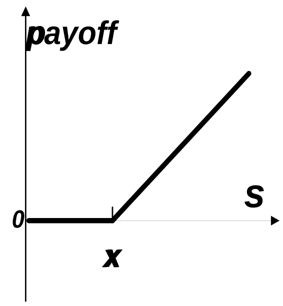

Figure 1.1: CALL OPTION.

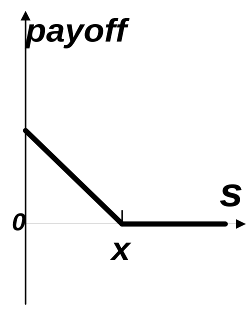

Figure 1.2: PUT OPTION.

1.2 Path-Dependent Options

A *path-dependent* option is an option whose payoff depends on the path followed by the price of the underlying asset. There are many kinds of path-dependent options, such as *lookback* and *Asian* options. Their payoffs do not merely depend on the final value of the underlying asset, but also on the way that the price was reached. This thesis concentrates on "barrier-like" path-dependent options such as reset and lookback options.

Reset Options

A *reset* option provides insurance for its holder by resetting the strike price if the price of the underlying asset is deep out of money. There are many versions of reset option. This thesis considers only single-barrier and fully-monitored reset options. As an illustration, in Figure 1.3 the price path crosses the reset barrier H, and the option is reset.

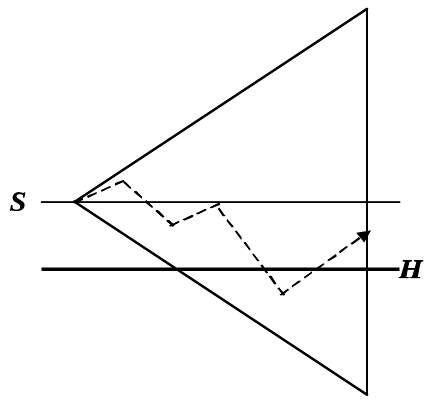

Figure 1.3: RESET OPTION ON A BINOMIAL TREE.

We assume that X denotes the strike price, H denotes the reset level, K denotes the new strike price, and S_T represents the price of the underlying asset at maturity. Thus the payoff of a European reset call is

$$\begin{cases} max(S_T - K, 0) & if\ the\ price\ ever\ hits\ the\ barrier\ H \\ max(S_T - X, 0) & otherwise \end{cases}$$

Similarly, the payoff of a European reset put is

$$\begin{cases} max(K - S_T, 0) & if\ the\ price\ ever\ hits\ the\ barrier\ H \\ max(X - S_T, 0) & otherwise \end{cases}$$

Lookback Options

Among path-dependent options *lookback* options are popular because they allow investors to buy the underlying asset at the lowest price or to sell it at the highest price over a certain period. Miscellaneous lookback specifications include *floating-strike* lookbacks and *fixed-strike* lookbacks with monitoring of the asset price over the whole period. The lookback option discussed in this thesis is floating-strike. Therefore, we abbreviate the "floating-strike lookback" by "lookback" in this thesis. A lookback call gives its holder the right to buy at the historically lowest price over a certain period. That is, the exercise price is equal to $\min_{0 \le \tau \le T} S(\tau)$. A lookback put gives its holder the right to sell at the historically highest price over a certain period. The exercise price of a lookback put is equal to $\max_{0\leq\tau\leq T} S(\tau)$ where T is the term of the option and S is the stock price. For a long position in a European lookback call, the payoff is

$$S_T - \min_{0\leq\tau\leq T} S(\tau)$$

Similarly, for a long position in a European lookback put, the payoff is

$$\max_{0\leq\tau\leq T} S(\tau) - S_T$$

Though lookback options take full advantage of a significant upward or downward trend in the price of the underlying asset, they are much more expensive than plain vanilla (classic) options. For an n-period lookback option, we can imagine that there exist $n+1$ reset barriers. It is intuitive that the premium of a lookback option increases with n. This is because when the number of time partitions increases, the number of reset barriers increases, and the extreme of asset price might lie on the added barrier.

1.3 Organization of This Thesis

There are six chapters in this thesis. We give a brief introduction in Chapter 1. In Chapter 2, some concepts in finance, mathematics, and computer science are introduced. We devote a whole chapter to pricing reset options in Chapter 3. In Chapter 4, we price lookback options and describe their behavior. Chapter 5 contains experimental results. Finally, we discuss future work and conclude in chapter 6.

Chapter 2

Fundamental Concepts

This chapter covers basic concepts used in the book. We cover the well-known Black-Scholes model, the binomial model, and the reflection principle.

2.1 The Black-Scholes Option Pricing Model

Options theory has played an important role in the modern theory of finance. In 1973, Fischer Black and Myron Scholes published the well-known option pricing model, called the Black-Scholes option pricing model, in the *Journal of Political Economy*. This formula has opened a new window into the modern theory of finance, and has been one of the most significant breakthroughs in finance. For the derivation of this formula, the mathematics is quite complex, so we omit it here. See [2] for more detailed information. We review the assumptions given in the model below.

1. The stock price follows the log-normal distribution.
2. There are no taxes or transaction costs.
3. There are no dividends during the life of the option.
4. There are no risk-less arbitrage opportunities.
5. The risk-free rate of interest, r, is constant.
6. The trading is continuous.

7. The options are European.

We assume that C denotes the call price, and P denotes the put price. The Black and Scholes formula follows:

$$\begin{aligned} C &= SN(d_1) - Xe^{-rT}N(d_2) && (2.1) \\ P &= Xe^{-rT}N(-d_2) - SN(-d_1) && (2.2) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\ln(S/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \end{aligned}$$

$N(x)$ = the cumulative normal probability

σ^2 = annualized variance of the continuously compounded return on the stock

r = continuously compounded risk-free rate

T = time to maturity

2.2 Wiener Process

A *Wiener process* is a particular type of *Markov* stochastic process. The behavior of a variable, w, which follows a Wiener process can be understood by considering the changes in its value in small intervals of time. Consider a small interval of time of length $\triangle t$ and let $\triangle w$ be the change in w during $\triangle t$. There are two basic properties for $\triangle w$.

Property 1. $\triangle w$ must satisfy the equation

$$\triangle w = \epsilon\sqrt{\triangle t} \tag{2.3}$$

where ϵ is a random variable drawn from the standardized normal distribution $N(0, 1)$.

Property 2. The values of $\triangle w$ for any two different short intervals of time $\triangle t$ are independent.

By Property 1, $\triangle w$ is a normal distribution $N(0, \sqrt{\triangle t})$, while Property 2 implies that w follows a Markov process. We assume that our stock price follows the stochastic process described below:

$$\frac{dS}{S} = \mu\, dt + \sigma\, dw \tag{2.4}$$

where μ is the stock's expected rate of return per unit time and σ is the volatility of the stock price. Equation (2.4) is the most widely used model of stock price and is also known as the *geometric Brownian motion.* We will use it to construct the binomial model in next section.

2.3 The Binomial Model

The *binomial model* is a discrete-time approximation of the continuous-time pricing model. This is a binomial tree that represents the possible paths that might be followed by the price over the life of the option.

First, we assume that the model follows the *geometric Brownian motion*, $\frac{dS}{S} = r\, dt + \sigma\, dw$. Second, we assume that we live in a *risk-neutral* world, so $\mu = r$. In such a world, everyone is risk-averse, and the expected rate of return on all securities is the risk-free interest rate. As in Figure 2.1, we assume S denotes the current price at time t, which will either increase to Su with probability p or decrease to Sd with probability $1 - p$ after time Δt. We get:

$$p\, Su + (1-p)\, Sd = Se^{r\Delta t}$$

where r is the risk-free interest rate.

After Δt, the stock price S either moves to Su with probability p or Sd with probability $1 - p$.

The variance of the binomial stock price at Δt is given by

$$p(Su)^2 + (1-p)(Sd)^2 - (Se^{r\Delta t})^2$$

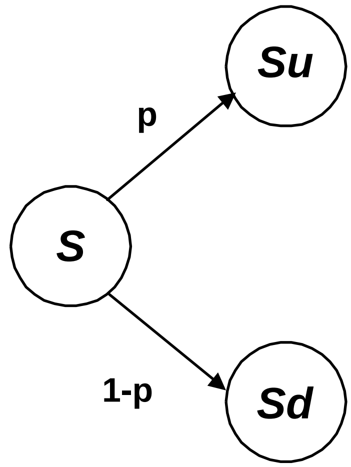

Figure 2.1: BINOMIAL MODEL.

So we obtain the following equality for variance,

$$p(Su)^2 + (1-p)(Sd)^2 - (Se^{r\Delta t})^2 = S^2\sigma^2\Delta t.$$

or

$$pu^2 + (1-p)d^2 - e^{2r\Delta t} = \sigma^2\Delta t.$$

Imposing $ud = 1$, the following equalities which satisfy the above in the limit obtain

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta t}} \\ d &= e^{-\sigma\sqrt{\Delta t}} = \frac{1}{u} \\ p &= \frac{e^{r\Delta t} - d}{u - d} \end{aligned}$$

We shall call them the *CRR* parameters, because Cox, Ross and Rubinstein proposed them. See [6] for more detailed discussion.

2.4 The Reflection Principle

Besides the methods described above, the key tool in understanding the *combinatorial method* for our algorithms is the *reflection principle* [12].

In Figure 2.2, suppose a particle starts at position $(0, -a)$, on the integral lattice and wishes to reach $(n, -b)$. Without loss of generality, assume $a, b \geq 0$. We restrict the particle to move to either $(i+1, j+1)$ or $(i+1, j-1)$ from (i, j), the way the binomial tree for the stock price is supposed to be traversed. That is,

$$\begin{cases} (i,j) \to (i+1, j+1) & \textit{can be regarded as the up move } \; S \to Su. \\ (i,j) \to (i+1, j-1) & \textit{can be regarded as the down move } \; S \to Sd. \end{cases}$$

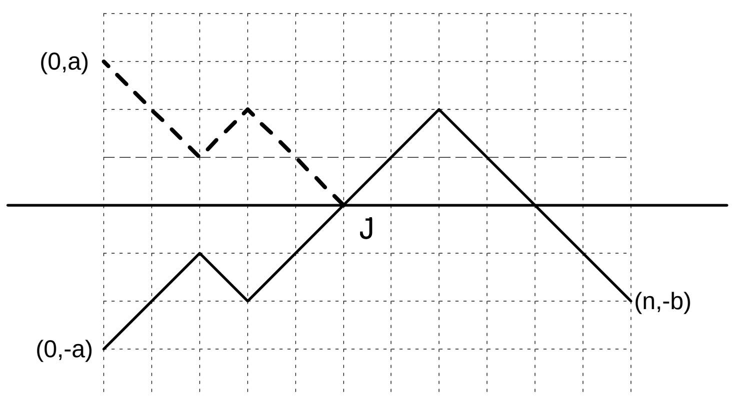

Figure 2.2: THE REFLECTION PRINCIPLE.

How many such paths the particle can take that touch or cross the x-axis? Consider any legitimate path from $(0, -a)$ to $(n, -b)$ that either touches or crosses the x-axis. Let J denote the first position it happens. By reflecting the portion of the path from $(0, -a)$ to J, a path from $(0, a)$ to $(n, -b)$ is constructed. A moment's reflection leads to the conclusion that the number of paths from $(0, -a)$ to $(n, -b)$ that touch the x-axis is exactly the number of paths from $(0, a)$ to $(n, -b)$. This is the celebrated reflection principle of André (1840-1917) published in 1887 [10]. The number of paths is thus equal to

$$\binom{n}{\frac{n+a+b}{2}} \text{ for even, non-negative } n+a+b$$

and zero otherwise. The negative $n+a+b$ case can be disregarded with the convention,

$$\binom{n}{k} = 0 \text{ for } k < 0 \text{ or } k > n$$

Chapter 3

Reset Options Pricing

In Chapter 2, we discussed the binomial model in general and the CRR model in particular. In this chapter, reset option pricing is based on the CRR model.

3.1 Backward Induction on Binomial Tree

The binomial tree method is widely used in option pricing. To price reset options, backward induction on the binomial tree is the standard scheme. We show how to price these options as follows.

Assume that X denotes the strike price, H denotes the reset level, K denotes the new strike price, and S_T represents the price of the underlying asset at maturity. We first adjust the reset barrier to the new barrier, called the effective barrier. We thus guarantee that the effective barrier coincides with one of the legal stock prices on the tree. Then, we evaluate the option by starting at the end of the tree (at time T). For a call option at maturity, the payoff is either $\max(S_T - X, 0)$ or $\max(S_T - K, 0)$. We use two arrays C and Q to store them respectively. In general, at state (i, j), where i denotes at time i and j denotes the stock price $Su^j d^{i-j}$, the value of $C(i, j)$ and $Q(i, j)$ are as follows:

$$C(i, j) = e^{-rt}(\ p\, C(i+1, j+1) + (1-p)C(i+1, j)\ )$$
$$Q(i, j) = e^{-rt}(\ p\, Q(i+1, j+1) + (1-p)Q(i+1, j)\ )$$

If the state (i,j) is on the effective barrier, then we move the value $Q(i,j)$ to $C(i,j)$. By working through all the nodes, the value of the option at time zero is $C(0,0)$. The running time of this algorithm is quadratic in n, where n is the number of time periods.

3.2 The Combinatorial Method

Counting the valid paths that lead to a particular terminal price is the idea of the combinatorial method. A European reset call option with strike price X and new strike price K can be disassembled as a down-and-out call with strike price X plus a down-and-in call with strike price K.

As an illustration in Figure 3.1, the number of paths from S to the terminal price $Su^j d^{n-j}$ is $\binom{n}{j}$, where each path has the same probability $p^j(1-p)^{n-j}$.

We assume that $H < K < X$, where H is the reset barrier. Let

$$h = n + \lfloor \frac{\ln(\frac{H}{S})}{\ln u} \rfloor$$

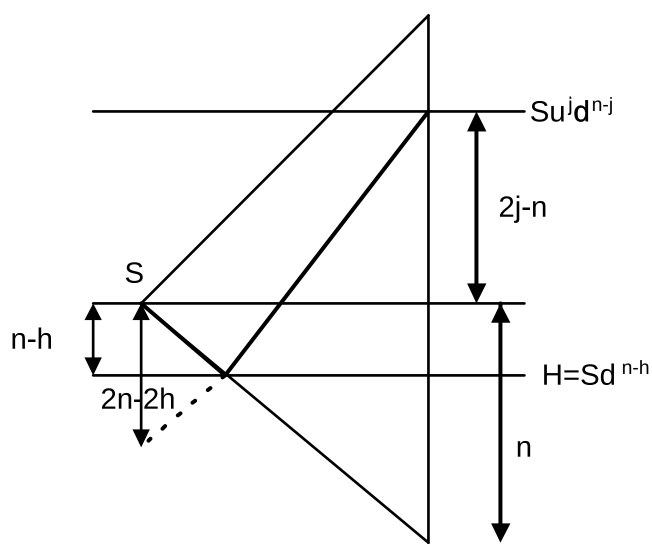

Figure 3.1: Reset call on the binomial tree.

By the reflection principle, the number of paths that hit the reset level Sd^{n-h} is:

$$\binom{n}{n+j-h}$$

Thus, the option value is

$$e^{-rT}\left(\sum_{j=\lfloor \frac{h}{2} \rfloor+1}^{h}(A+B)+\sum_{j=h+1}^{n}C\right), where$$

$$\begin{aligned}
A &= \binom{n}{n+j-h}\max(Su^jd^{n-j}-K,0)\\
B &= \left(\binom{n}{j}-\binom{n}{n+j-h}\right)\max(Su^jd^{n-j}-X,0)\\
C &= \binom{n}{j}\max(Su^jd^{n-j}-X,0)
\end{aligned}$$

The running time of this algorithm is proportional to $n-\frac{h}{2}$. The experimental results are discussed in Chapter 5.

Chapter 4

Lookback Options Pricing

The payoff from a lookback call (put) depends on the minimum (maximum) stock price reached during the life of the option. In this chapter, we price the European- and American-style lookbacks based on the CRR model. Due to the slowness of its convergence, we adopt the interpolation method to reduce the computation time.

4.1 Backward Induction on Binomial Tree

The valuation formulas of European lookback options have been proposed in 1979; see [7] for detailed discussion. Like the Black-Scholes model, this formula assumes continuous trading. Hence, under this formula, we can imagine that there exist an infinite number of reset barriers. However, continuous trading is impossible in reality. To price discrete-time lookbacks, the continuous-time valuation formulas are no longer favorable; the prices calculated by this formula are also more expensive than discrete-time models. Thus, we adopt the binomial tree method to price discrete lookbacks. Backward induction on the binomial tree is a standard method. We show how it works in the following.

We assume that the European lookback call option is issued at time zero, and the current value of the underlying asset is S. The binomial tree for an n-period European lookback call option issued at time zero is illustrated in Figure 4.1. Hence $S_{min} = S$; whose S_{min} denotes the minimum price. There exists $n+1$ reset-barriers, H_0, H_1,. . ., H_n. We work from H_0 toward H_n one barrier at a time. For each barrier H_i, we

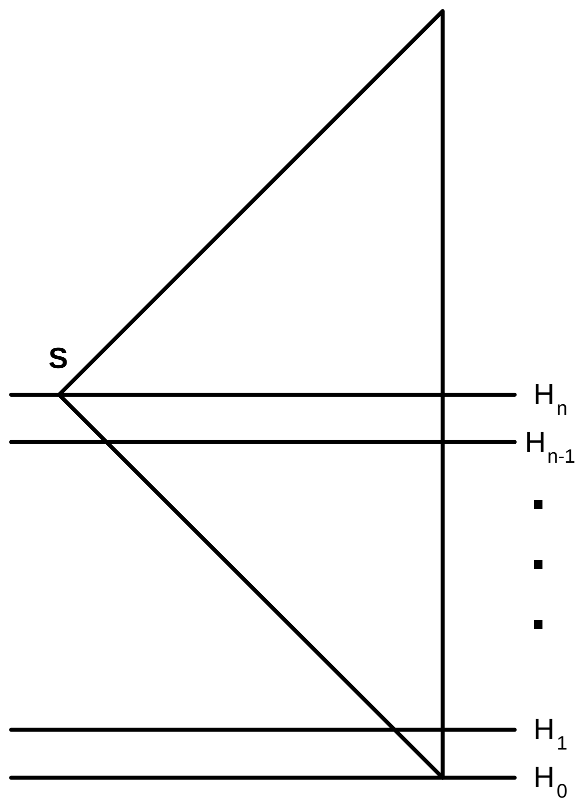

Figure 4.1: An n-period lookback call on the binomial tree.

calculate the present value of those paths with the minimum price H_i. The procedure is similar to that of pricing reset options, and it requires $O(n^2)$ time. Since there are $n+1$ barriers, this algorithm takes $O(n^3)$ time. We also observe that the price converges slowly and is below the analytical value. This Algorithm is also applicable for other types of lookback options. See Chapter 5 for the experimental results.

4.2 American Lookback Options Pricing

In this section, we concentrate on the pricing of American lookbacks. Unlike European options, it is impossible to derive closed-form expressions for the value of American options. The combinatorial method is also not applicable to the pricing of American options. If the underlying asset does not pay dividends during the life of the option, the American lookback call is equal to the European one. See [5] for more details.

Figure 4.2 illustrates a three-period American lookback put option. This binomial tree is based on the CRR model. We suppose that the initial stock price $S = S_{max} = 100$, the risk-free interest rate r is 6% per annum, the stock price volatility σ is 30% per annum, the number of time periods n is 3, and the total life of the option T is one year. Under the CRR model, other parameters are $\Delta t = 0.333$, $u = 1.189$, $d = 0.841$, $R = 1.0202$, and $p = 0.515$

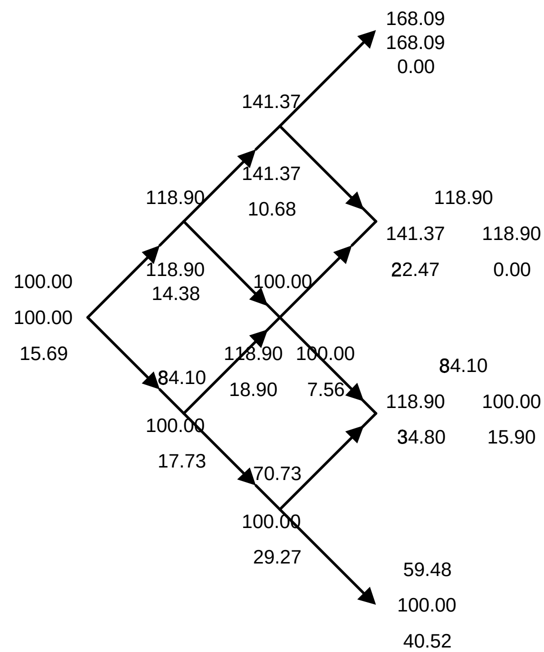

Figure 4.2: A THREE-PERIOD TREE FOR VALUING AN AMERICAN LOOKBACK PUT OPTION.

The top number at each node is the stock price. The next level of numbers at each node represents the possible maximum stock prices achievable on paths leading to the node. Whereas the final level of numbers represents the values of the option corresponding to each of the possible maximum stock prices. The values of the option at the final nodes of the tree are calculated as the maximum stock price minus the actual stock price. Rolling back through the tree, we can calculate the value of the

American lookback as \$15.69. The value of the European lookback calculated under the CRR model is \$14.69. The analytical value is \$22.75.

The algorithm for pricing American lookbacks is similar to that described in Section 4.1. Both European and American lookbacks have the characteristics of slow convergence. We will compare the convergence speeds of European and American lookbacks in Section 4.5.

4.3 An Improved Algorithm

In Section 4.1, we showed that the traditional backward induction for pricing lookback options is a cubic algorithm in n. By changing the binomial tree, we can derive a quadratic time algorithm [9]. We will briefly show how it works and the drawback of this algorithm in the following.

We use the 3-period binomial tree discussed in Figure 4.2. All the parameters we use here are the same in Section 4.2. We define $F(t)$ as the maximum stock price achieved up to time T and set

$$Y(t) = \frac{F(t)}{S(t)}$$

We use the CRR model to produce a tree for Y. Initially, $Y = 1$ because $F = S$ at time zero. If there is an up movement in S during the first time step, both F and S increase by a proportional amount u and Y stays the same. If there is a down movement during the first time step, F stays the same, then $Y = u$. We can produce the tree for Y in Figure 4.3. An up movement in Y corresponds to a down movement in the stock price, and vice versa. The probability of an up movement in Y is $1 - p$ and the probability of a down movement in Y is p. In dollars, the payoff from the option at maturity is

$$SY - S$$

In stock price units, the payoff from the option at maturity is

$$Y - 1$$

We omit the detailed procedure here, see [9] for more discussion. By rolling back through the tree, we can count the value of the American lookback put at time zero (in stock price units) as 0.1569. The dollar value of the option is therefore 0.1569×100=15.69.

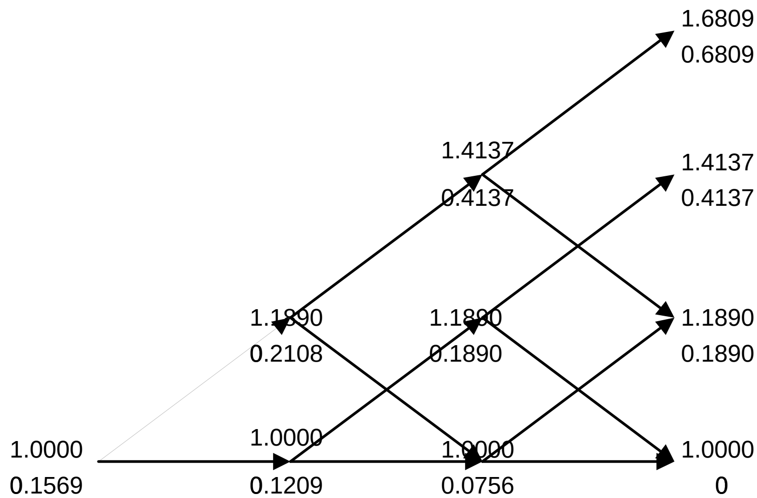

Figure 4.3: EFFICIENT PROCEDURE FOR VALUING AN AMERICAN LOOKBACK OPTION.

This algorithm is faster than that described in Section 4.1. However, when the historical extreme is not equal to the current stock price, the nodes on the lattice may not combine. To overcome this shortcoming, we then move the historical extreme to the nearest lattice node. As an illustration in Figure 4.4, S_{max} is not on the lattice nodes, we then either move it to Su ($S_{maxfloor}$) or $Su^2(S_{maxceil})$. In Figure 4.4 we move S_{max} to Su , and call it $S_{maxfloor}$. Though this procedure successfully solve the drawback, the convergence of either $S_{maxfloor}$ or $S_{maxceil}$ is biased, see Figure 4.5 as an illustration. Hence, this algorithm is still not adaptable for historical extreme values. In the next section, we will produce a quadratic time algorithm, and it can handle historical extreme values.

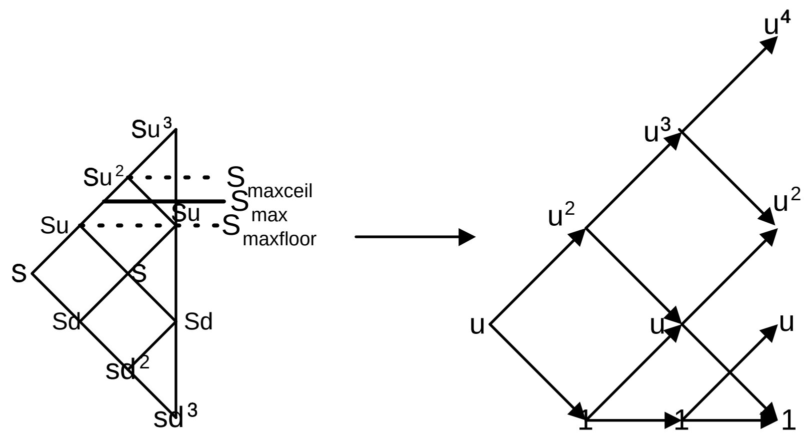

Figure 4.4: THE DRAWBACK OF THIS ALGORITHM, WHEN EXTREME VALUE IS NOT ON THE LATTICE NODES.

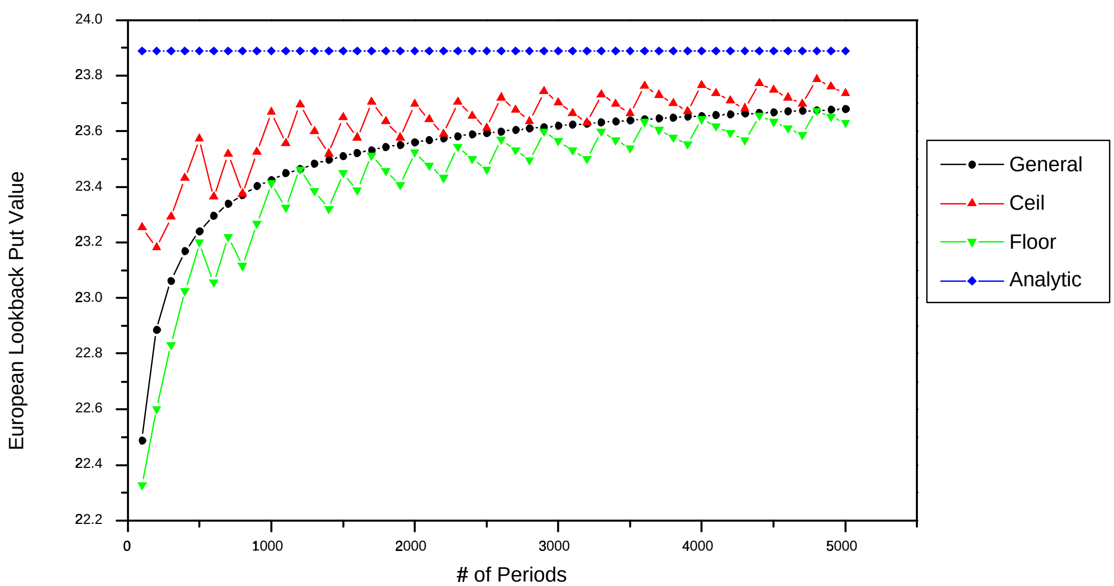

Figure 4.5: THE BIAS OF THIS ALGORITHM FOR HISTORICAL EXTREME VALUES. The parameters are $S = 100$, $S_{max} = 110$, $\sigma = 30\%$, $T = 1$ (year), $r = 6\%$, $q = 0\%$. The analytical value of the European lookback put is about 23.89.

4.4 The Combinatorial Method

In Section 4.1, we showed that the running time of the binomial tree method with backward induction is cubic in n. The improved algorithm discussed in Section 4.3 is biased when historical extreme value is not equal to the current stock price. In this section, we propose a quadratic-time algorithm to price European lookback options based on the combinatorial method. This algorithm is also applicable for historical extreme values.

For simplicity, we assume the European lookback call option is issued at time zero, and the number of time periods n is even. Then at time zero, S_{min} is equal to the value of the underlying asset, S. As illustrated in Figure 4.6, the number of paths from S to the terminal price $A_i = Su^i d^{n-i}$ is $\binom{n}{n-i}$, where each path has the same probability $p^i(1-p)^{n-i}$. There are $n+1$ reset barriers, and the reset barrier H_j is equal to Sd^{n-j}. By the reflection principle, the number of paths reaching the terminal node A_i that hit the reset level H_j is $\binom{n}{n+i-j}$.

For a terminal node A_i, where $i \leq n/2$, the minimum price reached to this node might be H_i, H_{i+1}, ... , H_{2i}. Note that A_i has the same price as H_{2i} for $0 \leq i \leq \frac{n}{2}$. We count the number of paths that hit the reset level H_i, H_{i+1},. . ., H_{2i}, and call them $n(H_i)$, $n(H_{i+1})$,. . ., $n(H_{2i})$, respectively. Thus the number of paths reaching A_i with minimum price H_j is equal to

$$\begin{cases} n(H_j) - n(H_{j-1}) & if\ j \neq i \\ 1 & if\ j = i \end{cases}$$

For a terminal node A_i, where $i > n/2$, the minimum price reached to this node might be H_i, H_{i+1}, ... , H_n, and we call them $n(H_i)$, $n(H_{i+1})$, ... , $n(H_n)$ respectively. Thus the number of paths reached to A_i with minimum price H_j is equal to:

$$\begin{cases} n(H_j) - n(H_{j-1}) & if\ j \neq i \\ 1 & if\ j = i \end{cases}$$

The option value is therefore

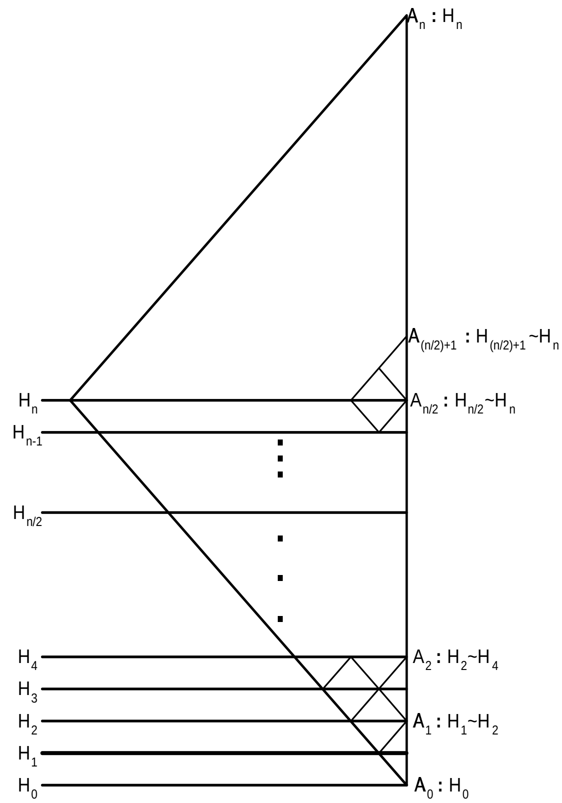

Figure 4.6: An n-period lookback call under the binomial model.

$$e^{-rT}\left(\sum_{i=0}^{\frac{n}{2}}\sum_{j=i}^{2i} A \times (A_i - H_j) + \sum_{i=\frac{n}{2}+1}^{n}\sum_{j=i}^{n} A \times (A_i - H_j)\right), where$$

$$A = \begin{cases} \binom{n}{n+i-j} - \binom{n}{n+i-(j-1)} & if\ j \neq i \\ 1 & if\ j = i \end{cases}$$

The running time of this algorithm is quadratic in n. The experimental results are in Chapter 5.

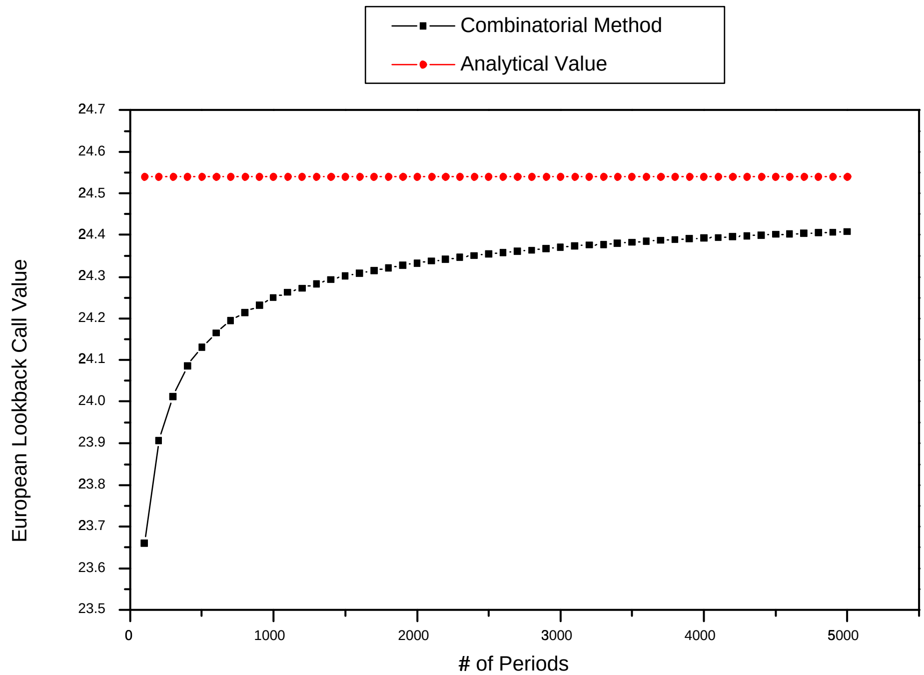

Figure 4.7: CONVERGENCE OF A EUROPEAN LOOKBACK CALL OPTION. The parameters are $S = 100$, $S_{min} = 95$, $\sigma = 30\%$, $T = 1$ (year), $r = 6\%$, $q = 0\%$. The analytical value is about 24.54.

4.5 The Convergence Speed Comparison

Figure 4.7 illustrates the convergence speed of the European lookback call option. It shows that the algorithm converges slowly to the analytical value as n increases. For a large number of time periods, e.g, $n = 3000$, the relative error is about 1%. Even under the power of the combinatorial method, it takes about 8 seconds to compute the value with 3000 periods. Hence for computing the option value with large n, the combinatorial method still takes significant running time.

Figure 4.8 illustrates the convergence speed of the American lookback put option. We can find that the American lookback option converges slower than that of the European one. It takes much time to compute the value with large n. For example, it takes about 9 hours to compute the value with $n = 2000$.

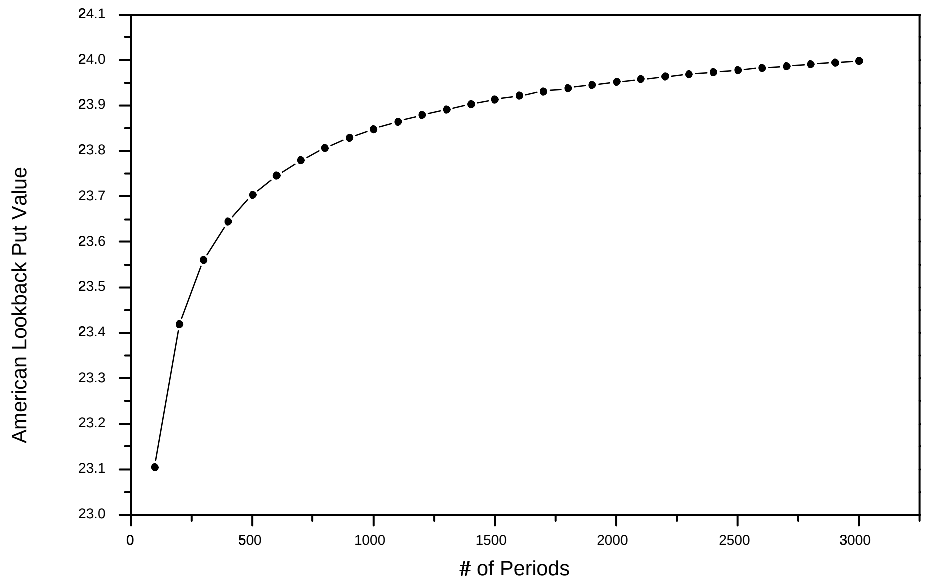

Figure 4.8: CONVERGENCE OF AN AMERICAN LOOKBACK PUT OPTION. The parameters are $S = 100$, $S_{max} = 100$, $\sigma = 30\%$, $T = 1$ (year), $r = 6\%$, $q = 0\%$. The upper bound of the put value is about 30.34.

Both European and American lookback options have the characteristics of slow convergence. And they all take much time to compute the value for large n. Due to this reason, we propose the interpolation method to reduce the computation time. We will describe how it works in the next section.

4.6 The Interpolation Method

This section proposes an interpolation method to price European and American lookback options when they are monitored discretely. There are many numerical methods for interpolation. Lagrangian polynomials and Newton's interpolations are equivalent in nature, but different in presentation. In this thesis, we concentrate on the Lagrangian polynomials.

Lagrangian polynomials

The Lagrangian polynomial is perhaps the simplest way to exhibit the existence of a polynomial for interpolation with unevenly spaced data. Data where the x-values are not equispaced often occur as the result of experimental observations or when historical data are examined.

Suppose we have a table of data with four pairs of x- and $f(x)$-values, with x_i indexed by variable i:

x	$f(x)$
x_0	f_0
x_1	f_1
x_2	f_2
x_3	f_3

Here we do not assume uniform spacing between the x-values, nor do we need the x-values arranged in a particular order. The x-values must be all distinct, however. Through these four data pairs we can pass a cubic. The Lagrangian form is

$$
\begin{aligned}
P_3(x) = {} & \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f_1 \\
& + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f_3
\end{aligned}
$$

This equation is made up of four terms, each of which is cubic in x; hence the sum is a cubic. The pattern of each term is to form the numerator as a product of linear factors of the form $(x - x_i)$, omitting one x_i in each term, the omitted value being used to form the denominator by replacing x in each of the numerator factors. In each term, we multiply by the f_i corresponding to the x_i omitted in the numerator factors. The Lagrangian polynomial for other degrees of interpolating polynomials employs the same pattern of forming a sum of polynomials all of the desired degree; it will have $n + 1$ terms when the degree is n.

It is easy to see that the Lagrangian polynomial does in fact pass through each of the points used in its construction. For example, in the preceding equation $P_3(x)$, $P_3(x_i) = f_i$ for $i = 0, 1, 2, 3$.

An interpolating polynomial, while passing through the points used in its construction, does not, in general, give exactly correct values when used for interpolation. The reason is that the underlying relationship is often not a polynomial of the same degree. Thus the error term of an interpolating polynomial with $n+1$ points is given by the expression

$$E(x) = (x - x_0)(x - x_1)\cdots(x - x_n)\frac{f^{n+1}(\xi)}{(n+1)!}$$

where ξ is in the smallest interval that contains $\{x, x_0, x_1, ..., x_n\}$, and f^{n+1} represents the $n+1$-st derivative.

The idea of interpolation and evaluation

We have pointed out that it takes lots of computation time to value either a European or an American lookback option when n is large. Trying to reduce the cost of the computation time, we adopt the Lagrangian polynomial.

The idea of the interpolation technique is heuristic; see [14] for more detailed discussion. For an interpolating equation with four points, the y-values of them are C_1, C_2, C_3, C_∞ respectively, where the subscript of C denotes the monitoring frequency. It is easy to see that C_∞ is the analytical value. The x-values are given by $x_1 = \frac{1}{1}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{1}{3}$, and $x_\infty = \frac{1}{\infty} = 0$. Hence the Lagrangian polynomial is

$$\begin{aligned} P(x) = {} & \frac{(x-x_2)(x-x_3)(x-x_\infty)}{(x_1-x_2)(x_1-x_3)(x_1-x_\infty)}C_1 + \frac{(x-x_1)(x-x_3)(x-x_\infty)}{(x_2-x_1)(x_2-x_3)(x_2-x_\infty)}C_2 \\ & + \frac{(x-x_1)(x-x_2)(x-x_\infty)}{(x_3-x_1)(x_3-x_2)(x_3-x_\infty)}C_3 + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_\infty-x_1)(x_\infty-x_2)(x_\infty-x_3)}C_\infty \end{aligned}$$

Note that $P(X_n) = P(\frac{1}{n}) = C_n$, $n = 1, 2, 3, \infty$. Then the interpolated price of C_n is $P(\frac{1}{n})$. This algorithm is easy to program and it combines speed and accuracy; see Figure 4.9 for illustration.

Other experimental results are discussed in Chapter 5.

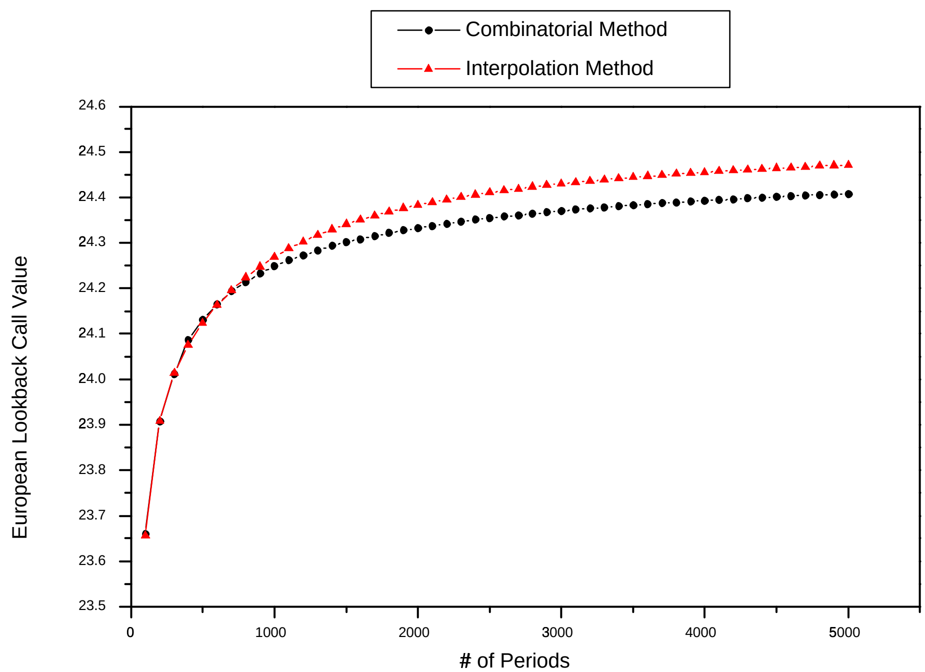

Figure 4.9: CONVERGENCE COMPARISON: COMBINATORIAL METHOD VS INTERPOLATION METHOD. Seven-point interpolation is used. The seven points are: $n = 50$, $n = 90$, $n = 130$, $n = 170$, $n = 210$, $n = 250$, and $n = \infty$.

Chapter 5

Experimental Results

5.1 Reset Options

This section concentrates on the experimental results of reset options. In Chapter 3, we described two methods, the combinatorial method and the binomial tree method with backward induction, of pricing reset options. Both should produce the same value under the same parameters, because they are both based on the CRR model. The option value oscillates as we increase the number of time periods n; see Figure 5.1.

The reason for the jittery is that the reset barrier H does not coincide with one of the $n+1$ available stock prices. To reduce this error, we need to find n, the number of time periods, that can guarantee that the barrier is almost on a layer of nodes. The n are:

$$n = \frac{m^2\sigma^2 T}{(\log \frac{S}{H})^2} \qquad m = 1, 2, 3... \tag{5.1}$$

Figure 5.2 shows the fast convergence with n from equation (5.1).

Table 5.1 tabulates their running times. From Figure 5.3, we can see clearly the dramatic difference between the linear-time and quadratic-time algorithms. The combinatorial method takes much less time for the same n.

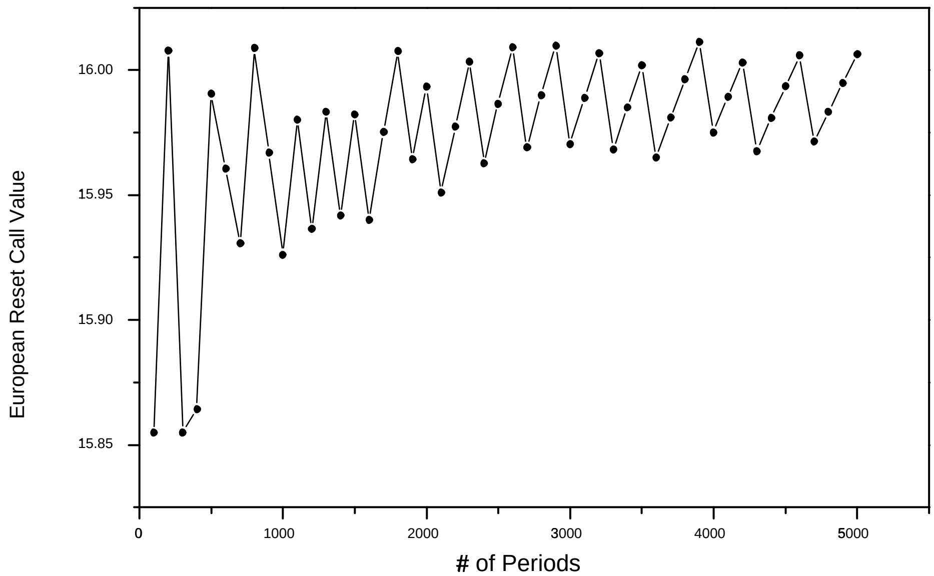

Figure 5.1: THE SAWTOOTH-LIKE CONVERGENCE BASED ON THE BINOMIAL MODEL. The parameters are $S = 100$, $X = 100$, $K = 95$, $H = 90$, $\sigma = 30\%$, $r = 6\%$, $q = 0\%$, $T = 1$ (year). The analytical value is about 16.014.

Table 5.1: Time used of a European reset call option by the two methods (combinatorial method and binomial tree method with backward induction).

Number of Periods	Combinatorial Method	Backward Induction on the Binomial Tree
72	0.2 ms	6 ms
202	0.6 ms	30 ms
656	1.8 ms	170 ms
810	2.2 ms	220 ms
1370	3.8 ms	550 ms
1589	4.4 ms	770 ms
2626	7.3 ms	2150 ms
2926	8.1 ms	2630 ms
3242	9.0 ms	3190 ms
3924	10.9 ms	4730 ms
4288	11.9 ms	5650 ms
4669	13.0 ms	6700 ms

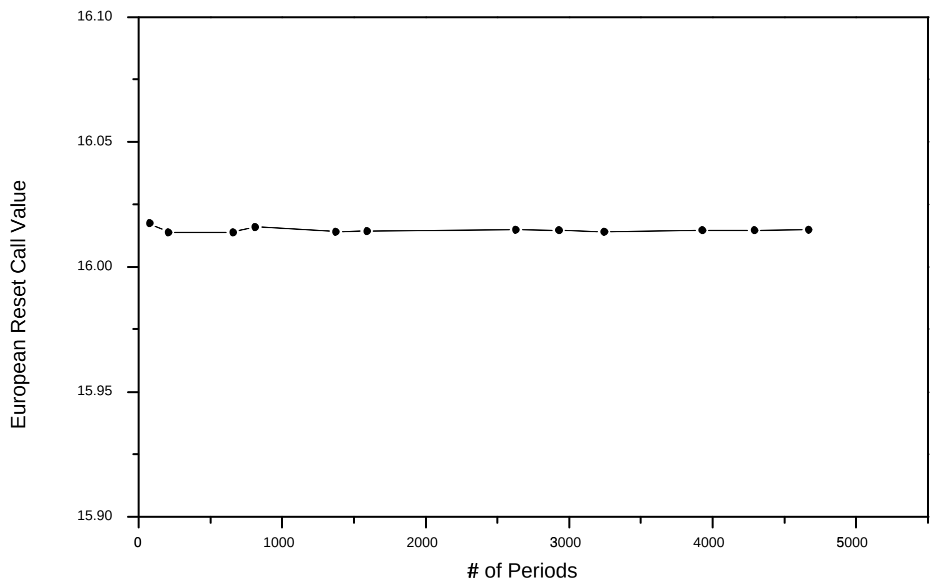

Figure 5.2: THE FAST CONVERGENCE OF A EUROPEAN RESET CALL FOR WELL CHOSEN N'S. The parameters are $S = 100$, $X = 100$, $K = 95$, $H = 90$, $\sigma = 30\%$, $r = 6\%$, $q = 0\%$, $T = 1$ (year). The n we choose are: 72, 202, 656, 810, 1370, 1589, 2626, 2926, 3242, 3924, 4288, and 4669.

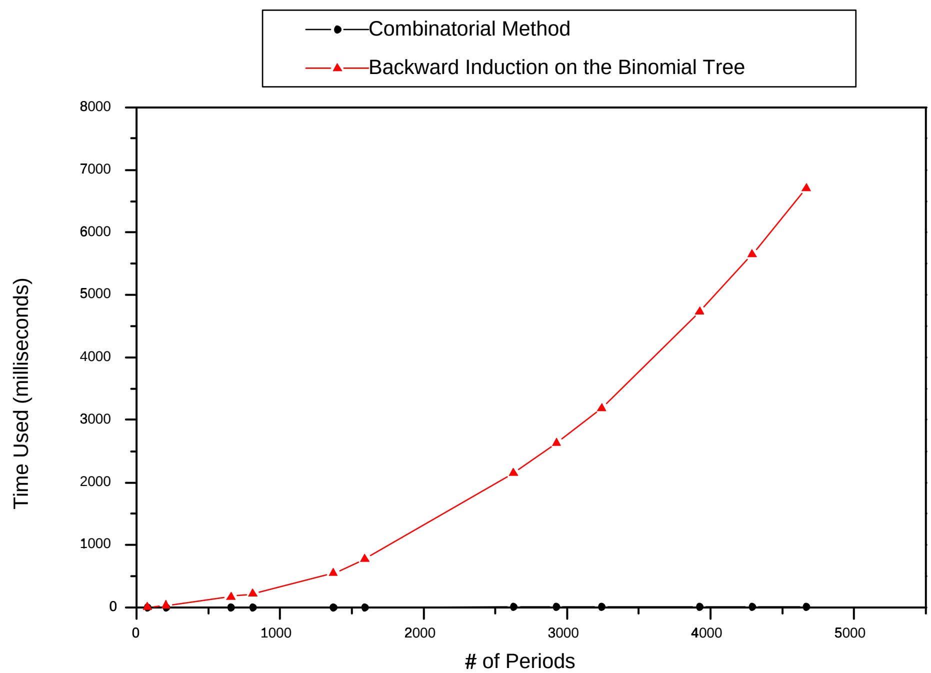

Figure 5.3: Time used of a European reset call option by the two methods (combinatorial method and binomial tree method with backward induction). The parameters are $S = 100$, $X = 100$, $K = 95$, $H = 90$, $\sigma = 30\%$, $r = 6\%$, $q = 0\%$, $T = 1$ (year).

Table 5.2: Time used of a European lookback call option by the two methods (combinatorial method and binomial tree method with backward induction).

Number of Periods	Combinatorial Method	Backward Induction on the Binomial Tree
100	0.11 s	0.77 s
300	0.16 s	5.32 s
500	0.22 s	22.44 s
1000	0.88 s	194.11 s
1500	2.03 s	657.02 s
2000	3.62 s	1566.53 s
2500	5.66 s	3063.75 s
3000	8.08 s	5300.15 s
3500	11.04 s	8418.54 s
4000	14.33 s	12568.88 s
4500	18.18 s	17900.50 s
5000	22.41 s	24574.49 s

5.2 Lookback Options

In this section, we concentrate on the experimental results of lookback options. In Chapter 4, we described two methods, the combinatorial method and the binomial tree method with backward induction, of pricing European lookback options. Both should produce the same value under the same parameters because they are both based on the CRR model. The option value converges slowly as we increase the number of time periods, and it underestimates the analytical value (see Figure 4.7).

Table 5.2 tabulates their running times. From Figure 5.4, we can see clearly the dramatic difference between the quadratic-time and cubic-time algorithms. The combinatorial method takes much less time for the same n.

For pricing American lookback options, the combinatorial method is no longer applicable. Like the European ones, the convergence of American lookback options is slow (see Figure 4.8). The pricing of American lookback options under the binomial model is time-consuming (see Table 5.3).

Figure 4.9 illustrates the convergence speed of the combinatorial method and the

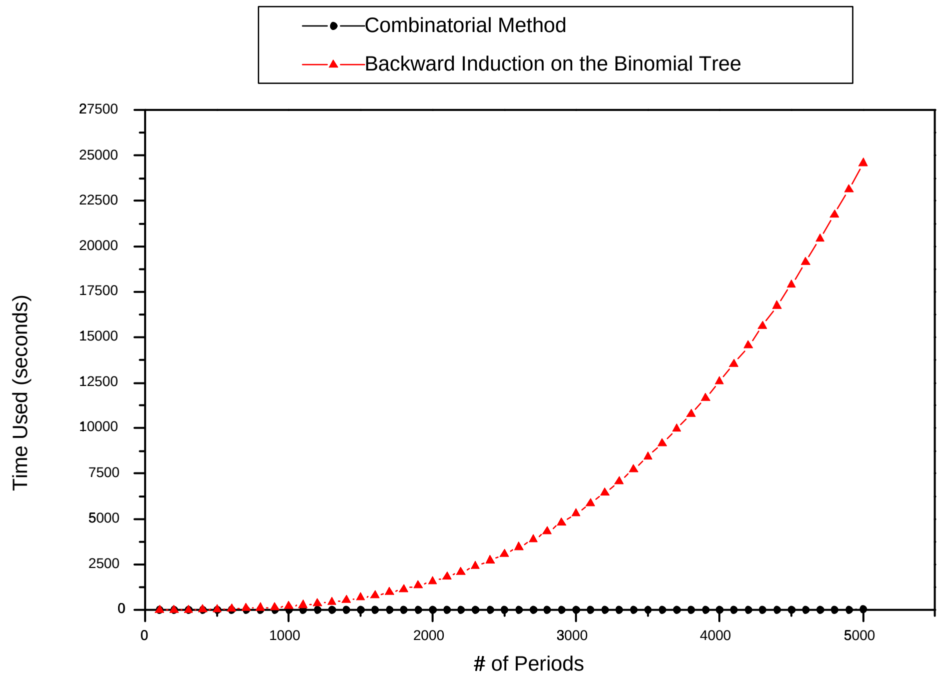

Figure 5.4: TIME USED OF A EUROPEAN LOOKBACK CALL OPTION BY THE TWO METHODS (COMBINATORIAL METHOD AND BINOMIAL TREE METHOD WITH BACKWARD INDUCTION). The parameters are $S = 100$, $S_{min} = 95$, $\sigma = 30\%$, $r = 6\%$, $q = 0\%$, $T = 1$ (year).

interpolation method for pricing a European lookback call option. We then compare their running times in Table 5.4. Note that most of the running time by the interpolation method is to evaluate the values at the interpolated points, i.e. the values at $n = 50$, $n = 90$, $n = 130$, $n = 170$, $n = 210$, $n = 250$, and $n = \infty$.

Table 5.5 shows the running times of pricing an American lookback put option by the binomial tree method with backward induction and the interpolation method. Most of the running time by the interpolation method is to evaluate the values at the interpolated points, i.e. the values at $n = 50$, $n = 70$, $n = 90$, $n = 110$, $n = 130$, $n = 150$, and $n = 170$. Figure 5.5 illustrates the convergence of the binomial tree method with backward induction and the interpolation method.

Table 5.3: Time used of an American lookback put option (backward induction on the binomial tree).

Number of Periods	Backward Induction on the Binomial Tree
100	1.59 s
300	28.45 s
500	134.62 s
700	375.36 s
900	808.56 s
1100	1493.64 s
1300	2489.99 s
1500	3855.00 s
1700	5648.81 s
1900	7927.07 s
2100	10757.76 s
2300	14195.06 s
2500	18302.44 s
3000	31892.05 s

Table 5.4: Time used of a European lookback call option by the two methods (combinatorial method and interpolation method).

Number of Periods	Combinatorial Method	Interpolation Method
100	0.11 s	0.64 s
300	0.16 s	0.64 s
500	0.22 s	0.64 s
1000	0.88 s	0.64 s
1500	2.03 s	0.64 s
2000	3.62 s	0.64 s
2500	5.66 s	0.64 s
3000	8.08 s	0.64 s
3500	11.04 s	0.64 s
4000	14.33 s	0.64 s
4500	18.18 s	0.64 s
5000	22.41 s	0.64 s

Table 5.5: Time used of an American lookback put option by the two methods (binomial tree method with backward induction and interpolation method).

Number of Periods	Backward Induction on the Binomial Tree	Interpolation Method
100	1.59 s	11.75 s
300	28.45 s	11.75 s
500	134.62 s	11.75 s
1000	1116.41 s	11.75 s
1500	3855.00 s	11.75 s
2000	9269.39 s	11.75 s
2500	18302.44 s	11.75 s
3000	31892.05 s	11.75 s

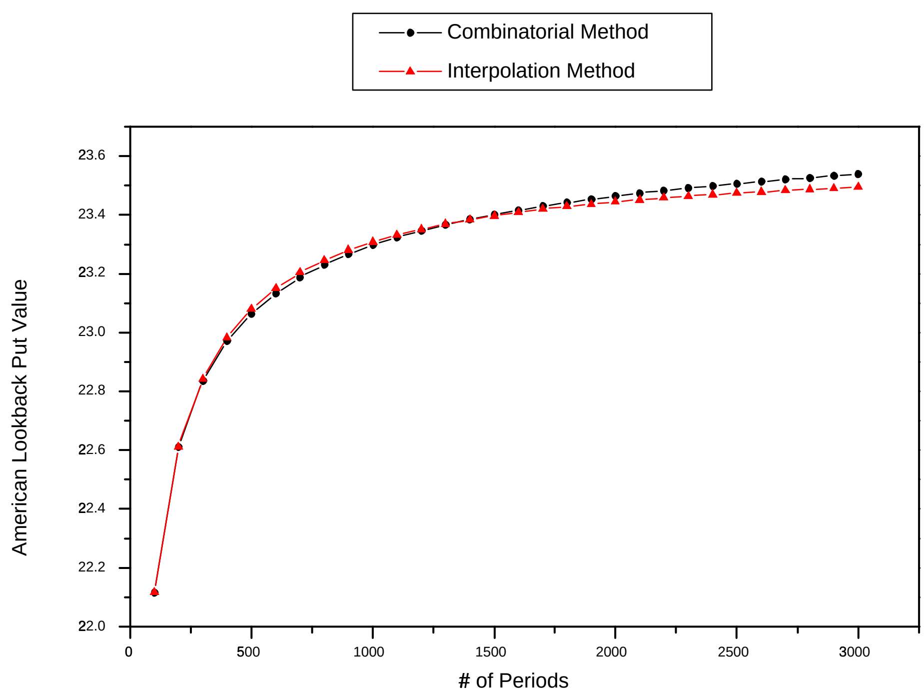

Figure 5.5: CONVERGENCE COMPARISON: BINOMIAL TREE METHOD WITH BACKWARD INDUCTION VS INTERPOLATION METHOD. Seven points interpolation. The seven points are: $n = 50$, $n = 70$, $n = 90$, $n = 110$, $n = 130$, $n = 150$, and $n = 170$.

Table 5.6: Convergence speed for pricing a European lookback call option by these methods.

Number of Periods	IP	CM	Number of Replications	MC	UMC
2500	24.411	24.355	2500	24.492	24.688
5000	24.471	24.408	5000	24.322	24.683
7500	24.493	24.432	7500	24.453	24.606
10000	24.505	24.446	10000	24.484	24.541
12500	24.511	24.456	12500	24.651	24.492
15000	24.516	24.464	15000	24.411	24.522
17500	24.519	24.469	17500	24.450	24.618
20000	24.522	24.474	20000	24.472	24.600
22500	24.524	24.478	22500	24.565	24.575
25000	24.526	24.481	25000	24.447	24.626
27500	24.527	24.484	27500	24.544	24.621
30000	24.528	24.486	30000	24.410	24.488

The parameters are: $S = 100$, $S_{min} = 95$, $\sigma = 30\%$, $T = 1$ (year), $r = 6\%$, $q = 0\%$. The analytical value is about 24.540.

Table 5.6 compares the combinatorial method (CM), the interpolation method (IP), the Monte Carlo method (MC) with $n = 1000$, and the Unbiased Monte Carlo method (UMC) [1].

5.3 More Comparisons in Convergence for Lookback Options

The payoff from a lookback call depends on the historically minimum stock price reached during the life of the option. When the option is issued today, then S_{min} is equal to S, where S denotes the current stock price. However, during the life of the option, S_{min} may not be equal to S. Table 5.7 tabulates the European lookback call

Table 5.7: The comparison of the European lookback call value for various S_{min}.

Number of Periods	S_{min} =100	95	90	70	10
2500	23.978	24.355	25.406	35.895	90.582
5000	24.044	24.408	25.449	35.906	90.582
7500	24.073	24.432	25.468	35.911	90.582
10000	24.091	24.446	25.480	35.914	90.582
12500	24.102	24.456	25.488	35.916	90.582
15000	24.111	24.464	25.493	35.917	90.582
17500	24.118	24.469	25.498	35.918	90.582
20000	24.124	24.474	25.502	35.919	90.582
22500	24.128	24.478	25.505	35.920	90.582
25000	24.132	24.481	25.507	35.921	90.582
27500	24.135	24.484	25.509	35.921	90.582
30000	24.138	24.486	25.511	35.922	90.582
Analytical	24.204	24.540	25.554	35.933	90.582

Other parameters are: $S = 100$, $\sigma = 30\%$, $T = 1$ (year), $r = 6\%$, $q = 0\%$.

price for various S_{min}, while keeping other parameters unchanged. From Table 5.7, we can see that the major the difference between S and S_{min}, the faster the convergence speed.

To price discrete lookback options using the continuous monitoring formula, Broadie, Glasserman, and Kou [1996] discover a simple correction procedure; One needs only to adjust the $n+1$ reset barriers by a factor of exp $(0.5826\times\sigma\times\sqrt{t/m})$. For a European lookback call option, we need to adjust each barrier downward by a factor calculated as exp $(0.5826\times\sigma\times\sqrt{t/m})$ by the BGK method; see Table 5.8.

When the value of the underlying asset is monitored over the whole period, the premium of the lookback options are expensive. A partial monitoring of the underlying price is one way of reducing the lookback's premium. A partial lookback option is cheaper than a classic lookback option, and the payoff of such option depends on the period monitored. We assume that the current time $t = 0$, the monitoring period of the partial lookback option starts at time T_0 and ends at time T_N prior to the

Table 5.8: The convergence of the BGK method.

Number of Periods	S_{min} =100	95	90	70	10
1000	24.268	24.589	25.588	35.943	90.582
2500	24.243	24.572	25.580	35.939	90.582
5000	24.231	24.561	25.573	35.936	90.582
7500	24.226	24.558	25.569	35.936	90.582
10000	24.223	24.555	25.566	35.936	90.582
12500	24.221	24.553	25.565	35.935	90.582
15000	24.220	24.553	25.564	35.935	90.582
17500	24.218	24.552	25.564	35.935	90.582
20000	24.217	24.551	25.563	35.935	90.582
22500	24.217	24.550	25.563	35.935	90.582
25000	24.216	24.549	25.562	35.935	90.582
27500	24.215	24.549	25.562	35.934	90.582
30000	24.215	24.549	25.562	35.934	90.582
Analytical	24.204	24.540	25.554	35.933	90.582

Other parameters are: $S = 100$, $\sigma = 30\%$, $T = 1$ (year), $r = 6\%$, $q = 0\%$.

expiration date T. Then the payoff of such a European lookback call at maturity can be written as

$$\max(S_T - \min_{T_0 \leq \tau \leq T_N} S(\tau),0)$$

Figure 5.6 illustrates the convergence of such partial lookback call option.

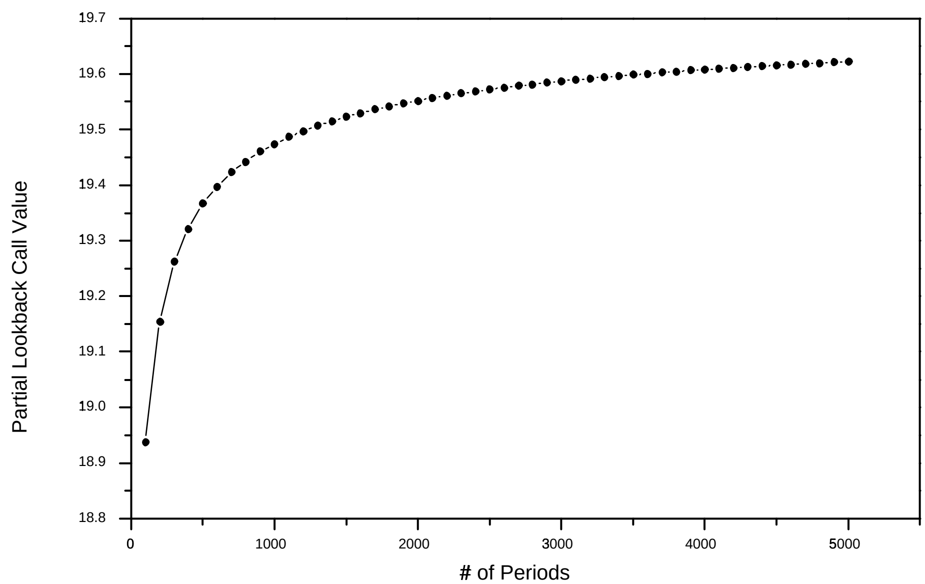

Figure 5.6: Convergence of a European partial lookback call option. The parameters are $S = 100$, $\sigma = 30\%$, $T = 12$ (month), $T_0 = 6$ (month), $T_N = 9$ (month), $r = 6\%$, $q = 0\%$.

Chapter 6

Conclusions

The combinatorial method has been widely applied in many fields. In this thesis, we extend it to pricing European-style reset and lookback options. In Chapter 5, we showed the efficiency of pricing such options by the combinatorial method. We successfully reduced the running time by an order.

We also found that the convergence of lookback options were very slow. To price a European lookback option with a large n, even under the power of the combinatorial method, it took minutes to get the result. It was clearly not efficient. We then tried the interpolation method, called the Lagrangian polynomial, to make it converge faster. We approximated the option value at a large n by interpolating with the polynomial. From our experimental results, we successfully reduced the running time and obtained well approximations. We also used this method to price American lookback options and obtained good results.

From our experimental results, the interpolation method can be applied to price other lookback-like options. We may work in the future if the interpolation method could be applicable for other complex options that have smooth curve. Second, we may want to know that how many data points are needed to obtain good approximations by the interpolation method.

Bibliography

[1] Babsiri, M.E, and G. Noel, 1998, "Simulating Path-Dependent Options: A New Approach," *Journal of Derivatives*, Winter, 65–83.

[2] Black, F., and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637–659.

[3] Boyle, P. P., and S. H. Lau, 1994, "Bumping Up Against the Barrier with the Binomial Method," *The Journal of Derivatives*, Summer, 6–14.

[4] Chalasani, P., S. Jha, and I. Saias, 1999, "Approximate Option Pricing," *Algorithmica*, 25, 2–21.

[5] Conze, A., and Viswanathan, 1991, "Path Dependent Options: the Case of Lookback Options," *Journal of Finance*, Vol XLVI, 1893–1907.

[6] Cox, J., S. Ross, and M. Rubinstein, 1979, "Option Pricing: a Simplified Approach," *Journal of Financial Economics*, 7, 229-264.

[7] Goldman, M. B., H. B. Sosin, and M. A. Gatto , 1979, "Path Dependent Options: 'Buy at the Low, Sell at the High,'" *Journal of Finance*, Vol XXXIV, No. 5, 1111-1127.

[8] Heynen, R. C., and H. M. Kat, 1995, "Lookback Options with Discrete and Partial Monitoring of the Underlying Price," *Applied Mathematical Finance*, 2, 273–284.

[9] Hull, John C., 1999, *Options, Futures, and Other Derivative Securities.*, 4th ed. Englewood Cliffs, New Jersey: Prentice-Hall.

[10] Lint, J. H. Van, and R. M. Wilson., 1994, '*A Course in Combinatorics*,' Cambridge: Cambridge University Press.

[11] Lyuu, Yuh-Dauh. *Financial Engineering and Computation: Principles, Mathematics, Algorithms.* Manuscripts, Feb. 1995–2000.

[12] Lyuu, Yuh-Dauh., 1998, "Very Fast Algorithms for Barrier Option Pricing and the Ballot Problem," *Journal of Derivatives*, Spring, 68–79.

[13] Ritchken, P., 1995, "On Pricing Barrier Options," *Journal of Derivatives*, Winter, 19–28.

[14] Wei, J. Z., 1998, "Valuation of Discrete Barrier Options by Interpolations," *Journal of Derivatives*, Fall, 51–73.