Monte Carlo Approaches for Pricing Multi-Asset Options

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Abstract

This thesis uses the Monte Carlo Approach to estimate the value of American call options on the maximun of multi-assets. This methodology to price American options with finitely many exercise opportunities simulates the evolution of the underlying assets via random trees that branch at each of the possible early exercise dates. From this tree, two consistent price estimates are obtained, one biased high and the other biased low. these two estimates can be combined to provide a valid, though conservative, confidence interval for the option price.

Broadie and Glasserman had already developed several enhancements to improve the efficiency of the two estimates so that the resulting error is small. They tried to "prune" the tree by eliminating the branching whenever possible, thus shortening the simulation time and allowing for faster convergence of the estimates. In this thesis, I analyze the properties and efficiency of these approaches.

Chapter 1

Introduction

1.1 Introduction

Before 1993, there were no approaches on the use of simulation techniques to value American-style options. Tilley[1] was the first to take such an approach. Since then, a number of related articles have followed, including Boyle[2], Grant, Vora and Weeks, and Broadie and Glasserman[3]. In short, simulation approaches are particularly useful when there are multiple stochastic factors that determine an option's value.

Binomial trees that approximate a stock price distribution at future dates are commonly used to value path-independent American options. Option values are determined at the tree's terminal date, and backward induction is then used to compute values at earlier steps, taking the possibility of early exercise into account. This approach may not be useful if the option is path-dependent or based on many underlying stochastic factors, because the number of nodes becomes enormous and hard to compute. Simulation methods, however, can easily handle these difficulties and value general European options.

A problem arises with American derivatives, because the early exercise feature requires knowledge of "live" and exercise values; standard simulations do not provide estimates of live value at intermediate dates. If a poor exercise policy is employed, the estimate of the initial option value may be inaccurate.

The use of Monte Carlo simulation for pricing options was first proposed by Boyle. One advantage of this method over the binomial method of Cox, Ross, and

Rubinstein[4] is that it does not grow exponentially with the number of state variables or underlying assets. Path-dependencies can also be taken into account easily. The chief drawback of the method has been its inability to incorporate the early exercise feature of American-style derivative securities.

Broadie and Glasserman propose a way to use simulation for pricing American options with finitely many exercise opportunities. Two estimators - - - one biased high and the other biased low - - - are obtained from a simulated tree, which branches at each of the possible early exercise points. A conservative confidence interval for the option price is obtained as a consequence. These eatimators are consistent (i.e., they converge in probability) and unbiased in the limit.

1.2 Organization of This Thesis

There are five chapters in this thesis. In Chapter 1, a brief introduction is presented. In Chapter 2, we introduce some basic financial concepts. In Chapter 3, we introduce our option pricing models for American options . In Chapter 4, we show the numerical results. Finally, conclusions are in Chapter 5.

Chapter 2

Fundamental Concepts

This chapter contains several basic concepts which remind the reader of fundamental tools used in chapters to follow option basics, Black-Scholes formula, options involving several assets, and Monte Carlo simulation.

2.1 Option Basics

Options on stock were first traded on an organized exchange in 1973. Since then there has been a dramatic growth in the options markets. Now they are traded on many exchanges around the world.

There are two basic types of option contracts: *call* options and *put* options. A *call* option gives the holder the right to buy the asset at a stated price (called the *exercise price* or *strike price*). Conversely, a *put* option gives the holder the right to sell the asset at a stated price on or before a stated date.

In general, call and put options can be defined in one of two manners: *American* or *European.* A European option can only be exercised at the maturity date of the option, whereas an American option can be exercised at any time up to and including the maturity date, namely *early exercise.*

Throughout this chapter, we assume that $T - t$ denotes the time to maturity, X denotes the strike price, and S represents the current stock price.

Positions

There are two sides to each option contract. On one side is the investor who has taken the long position (i.e., has bought the option). On the other side is the investor who has taken a short position (i.e., has sold or written the option). The writer of an option receives cash up front but has potential liabilities later.

Payoffs

Four basic option positions are possible:

1. A long position in a call option.
2. A long position in a put option.
3. A short position in a call option.
4. A short position in a put option.

It is often useful to characterize European option positions in terms of the payoff to the investor at maturity. The initial cost of the option is then included in the calculation. If X is the strike price and S_T is the final price of the underlying asset, the payoff from a long position in a European call option is

$$\max(S_T - X, 0)$$

This reflects the fact that the option will be exercised if $S_T > X$ and will not be exercised if $S_T \leq X$. The payoff to the holder of a short position in the European call option is

$$-\max(S_T - X, 0)$$

or

$$\min(X - S_T, 0)$$

The payoff to the holder of a long position in a European put option is

$$\max(X - S_T, 0)$$

and the payoff from a short position in a European put option is

$$-\max(X - S_T, 0)$$

or

$$\min(S_T - X)$$

Figure 2.1 illustrate these payoffs graphically.

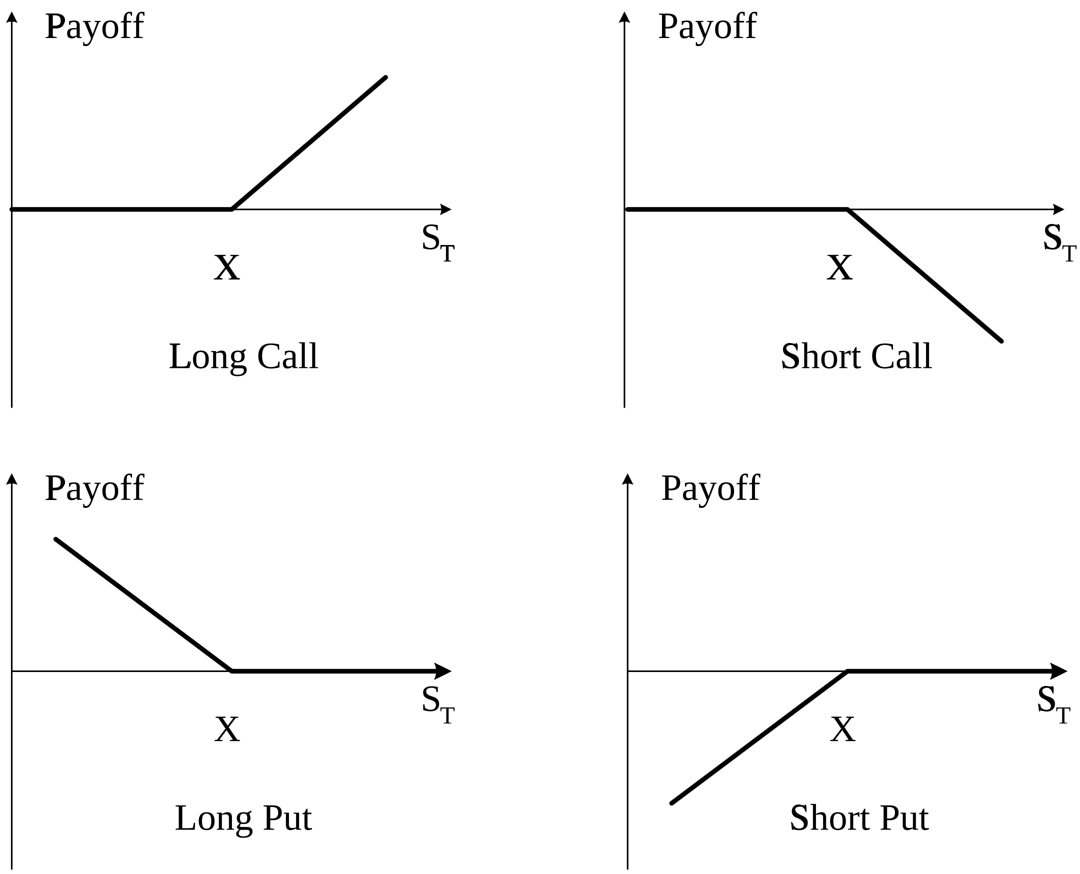

Figure 2.1: Payoffs from Positions in European Options.
X=Strike price
S_T=Price of asset at maturity.

2.2 The Black-Scholes Formula

In the early 1970s, Fischer Black and Myron Scholes made a major breakthrough by deriving a differential equation that must be satisfied by the price of any derivative security dependent on a non-dividend-paying stock. They solved this equation and obtained the values for European call and put options on stock. This formula, which has become known as the Black-Shcoles formula, is one of the most significant results in pricing financial instruments. In this section we show how a similar analysis can be used to derive the formula.

The expected value of a European call option at maturity in a risk-neutral world is

$$\widehat{E}[max(S_T - X, 0)]$$

where $\widehat{E}$ denote expected value in a risk-neutral world. From the risk-neutral valuation argument the European call option price, c, is the value of this discounted at the risk-free rate of interest, that is,

$$c = e^{-r(T-t)}\widehat{E}[\max(S_T - X, 0)] \tag{2.1}$$

We know that if a stock price S follows geometric Brownian motion,

$$dS = rSdt + \sigma S dz$$

then

$$d\ln S = (r - \frac{\sigma^2}{2})dt + \sigma dz$$

From this equation we see that variable $\ln S$ follow a generalized Wiener process. The change in $\ln S$ between time t and T is normally distributed:

$$\ln S_T - \ln S \sim \phi[(r - \frac{\sigma^2}{2})(T-t), \sigma\sqrt{T-t}]$$

where S_T is the stock price at a future time T, S is the stock price at the current time t, and $\phi(m, s)$ denotes a normal distribution with mean m and standard deviation s. Hence, $\ln S_T$ has the probability distribution equation that is

$$\ln S_T \sim \phi[\ln S + (r - \frac{\sigma^2}{2})(T-t), \sigma\sqrt{T-t}]$$

Evaluating the right-hand side of Equation (2.1) is an application of integral calculus. The result is

$$c = SN(d_1) - Xe^{-r(T-t)}N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S/X) = (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{\ln(S/X) = (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} \end{aligned}$$

and $N(x)$ is the cumulative probability distribution function for a variable that is normally distributed with a mean of zero and a standard deviation of 1 (i.e., it is the probability that such a variable will be less than x).

The value of a European put can be calculated in a manner similar to a European call. Alternatively, put-call parity can be used. The result is

$$p = Xe^{-r(T-t)}N(-d_2) - SN(-d_1)$$

Options on Stocks that Pay Continuous Dividend Yields

In the *continuous payoff model*, dividends are paid continuously. Such a model approximates, broad based stock market index portfolio, in which some company will pay a dividend nearly every day. Foreign currencies also pay daily dividends in the form of interest, hence well approximated by the continuous payoff model.

The payment of a *continuous dividend yield* at rate q reduces the growth rate of the stock price by q. In other words, a stock that grows from S to S_T with a continuous dividend yield of q would grow from $Se^{q\tau}$ to S_T without dividends. Hence, a European option on a stock with price S paying a continuous dividend yield of q has the same value as the corresponding European option on a stock with price $Se^{-q\tau}$ that pays no dividends. Black-Sholes formula can be used with S replaced by $Se^{-q\tau}$. Hence the following formulas hold,

$$\begin{aligned} C &= Se^{-q\tau}N(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}) \\ P &= Xe^{-r\tau}N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau}N(-x) \end{aligned}$$

where

$$x = \frac{ln(S/X) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

In the above formulas, C is the call option value, P is the put option value, and $N(x)$ is the cumulative probability distribution function for a variable that is normally distributed with a mean of zero and a standard deviation of 1 (i.e., it is the probability that such a variable will be less than x).

These formulas due to Merton, remain valid even if the dividend rate is not a constant. As long as it is predictable and q is replaced by the average annualized dividend yield during the life of the option.

2.3 Options Involving Several Assets

Options involving two or more risky assets are sometimes referred to as *rainbow options*. One example is the bond futures contract traded on the CBOT. The party with the short position is allowed to choose between a large number of different bonds when making delivery. Another example is what is known as a *basket options*. This is an option whose payoff depends on the value of a portfolio of assets. A third example is a LIBOR-Contingent FX option. This is an option whose payoff occurs only if a prespecified interest rate is within a certain range at maturity.

Options on the maximum of two risky assets enter the payoff function of some traded assets in a straightforward way. If V and H are the prices of two risky assets at exercised date, the call options has a payoff equal to $\max(\max(V, H) - F, 0)$, where F is the exercise price.

The Pricing of a Call on the Maximum of Several Assets

In European-style options, we know that Black-Sholes formula can help us to get the value of a call option on a single asset. But if the call options depend on several assets, how can we get the option value? Below, we show the formula of computing the value of European call options on several assets,

$$
\begin{aligned}
c_{max} &= S_1 N_n(d_1(S_1, X, \sigma_1^2), d_1'(S_1, S_2, \sigma_{12}^2), \cdots, d_1'(S_1, S_n, \sigma_{1n}^2), \rho_{112}, \rho_{113}, \cdots) \\
&+ S_2 N_n(d_1(S_2, X, \sigma_1^2), d_1'(S_2, S_1, \sigma_{12}^2), \cdots, d_1'(S_2, S_n, \sigma_{2n}^2), \rho_{212}, \rho_{223}, \cdots) \\
&+ \cdots \\
&+ S_n N_n(d_1(S_n, X, \sigma_n^2), d_1'(S_n, S_1, \sigma_{1n}^2), \cdots, d_1'(S_n, S_{n-1}, \sigma_{n-1n}^2), \rho_{n1n}, \rho_{n2n}, \cdots) \\
&- X e^{-rT}(1 - N_n(-d_2(S_1, X, \sigma_1^2), -d_2(S_2, X, \sigma_2^2), \cdots, -d_2(S_n, X, \sigma_n^2), \rho_{12}, \rho_{13}, \cdots))
\end{aligned}
$$

where n is the number of assets, X is the exercise price, T is the time to maturity, N_i is the i-variate standard cumulative normal distribution function, and

$$
\begin{aligned}
d_1(S_i, X, \sigma_i^2) &= \frac{\log \frac{S_i}{X} + (r + \frac{1}{2}\sigma_i^2)T}{\sigma_i T} \\
d_2(S_i, X, \sigma_i^2) &= \frac{\log \frac{S_i}{X} + (r - \frac{1}{2}\sigma_i^2)T}{\sigma_i T} \\
d_1'(S_i, S_j, \sigma_{ij}^2) &= \frac{\log \frac{S_i}{S_j} + \frac{1}{2}\sigma_{ij}^2 T}{\sigma_{ij} T} \\
\sigma_{ij}^2 &= \sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2 \\
\rho_{ijk} &= \frac{\sigma_i^2 - \rho_{ij}\sigma_i\sigma_j - \rho_{ik}\sigma_i\sigma_k + \rho_{jk}\sigma_j\sigma_k}{\sigma_{ij}\sigma_{ik}}
\end{aligned}
$$

for n=2, the expression simplifies to

$$
\begin{aligned}
c_{max} &= S_1 N_2(d_1(S_1, X, \sigma_1^2), d_1'(S_1, S_2, \sigma_{12}^2), \rho_{112}) \\
&+ S_2 N_2(d_1(S_2, X, \sigma_2^2), d_1'(S_2, S_1, \sigma_{12}^2), \rho_{212}) \\
&- X e^{-rT}(1 - N_2(-d_2(S_1, X, \sigma_1^2), -d_2(S_2, X, \sigma_2^2), \rho_{12}))
\end{aligned}
$$

If the option is on stocks that pay continuous dividends rates, the formula should be changed to

$$
\begin{aligned}
c_{max} &= S_1 e^{-qt} N_2(d_1(S_1, X, \sigma_1^2), d_1'(S_1, S_2, \sigma_{12}^2), \rho_{112}) \\
&+ S_2 e^{-qt} N_2(d_1(S_2, X, \sigma_2^2), d_1'(S_2, S_1, \sigma_{12}^2), \rho_{212}) \\
&- X e^{-rT}(1 - N_2(-d_2(S_1, X, \sigma_1^2), -d_2(S_2, X, \sigma_2^2), \rho_{12}))
\end{aligned}
$$

where

$$
d_1(S_i, X, \sigma_i^2) = \frac{\log \frac{S_i}{X} + (r - q + \frac{1}{2}\sigma_i^2)T}{\sigma_i T}
$$

$$\begin{aligned}
d_2(S_i, X, \sigma_i^2) &= \frac{\log\frac{S_i}{X} + (r - q - \frac{1}{2}\sigma_i^2)T}{\sigma_i T} \\
d_1'(S_i, S_j, \sigma_{ij}^2) &= \frac{\log\frac{S_i}{S_j} + \frac{1}{2}}{\sigma_{ij} T} \\
\sigma_{ij}^2 &= \sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2 \\
\rho_{ijk} &= \frac{\sigma_i^2 - \rho_{ij}\sigma_i\sigma_j - \rho_{ik}\sigma_i\sigma_k + \rho_{jk}\sigma_j\sigma_k}{\sigma_{ij}\sigma_{ik}}
\end{aligned}$$

and q is the dividend yield rate. About the details of this formula, consult [5].

Calculation of Cumulative Probability in Bivariate Normal Distribution

The above formula uses the bivariate normal distribution function. So we discuss the bivariate normal distribution function in this subsection.

We define $M(a, b; \rho)$ as the cumulative probability in a standardized bivariate normal distribution that the first variable is less than a and the second variable is less than b, when the coefficient of correlation between the variables is ρ. Drezner provides a way of calculating $M(a, b; \rho)$ to four-decimal-place accuracy.[1] If $a \leq 0$, $b \leq 0$, and $\rho \leq 0$,

$$M(a, b; \rho) = \frac{\sqrt{1-\rho^2}}{\pi} \sum_{i,j=1}^{4} A_i A_j f(B_i, B_j)$$

where

$$\begin{aligned}
f(x, y) &= exp[a'(2x - a') + b'(2y - b') + 2\rho(x - a')(y - b')] \\
a' &= \frac{a}{\sqrt{2(1-\rho^2)}} \qquad b' = \frac{b}{\sqrt{2(1-\rho^2)}}
\end{aligned}$$

$$\begin{array}{llll}
A_1 = 0.3253030 & A_2 = 0.4211071 & A_3 = 0.1334425 & A_4 = 0.006374323 \\
B_1 = 0.1337764 & B_2 = 0.6243247 & B_3 = 1.3425378 & B_4 = 2.2626645
\end{array}$$

In other circumstances where the product of a, b and ρ is negative or zero, one of the following identities can be used:

$$M(a, b; \rho) = N(a) - M(a, -b; -\rho)$$

[1] Z. Drezner "Computation of Bivariate Normal Integral," *Mathematics of Computation*, 32 (January 1978), 277-79. Note that the presentation here corrects a typo in Drezner's paper.

$$
\begin{aligned}
M(a,b;\rho) &= N(b) - M(-a,b;-\rho) \\
M(a,b;\rho) &= N(a) + N(b) - 1 + M(-a,-b;\rho)
\end{aligned}
$$

In circumstances where the product of a, b, and ρ is positive, the identity

$$M(a,b;\rho) = M(a,0;\rho_1) + M(b,0;\rho_2) - N(a) - \delta$$

can be used in conjunction with the previous results, where

$$
\begin{aligned}
\rho_1 &= \frac{(\rho a - b)sgn(a)}{\sqrt{a^2 - 2\rho ab + b^2}} \\
\rho_2 &= \frac{(\rho b - a)sgn(b)}{\sqrt{a^2 - 2\rho ab + b^2}} \\
\delta &= \frac{1 - sgn(a)sgn(b)}{4} \\
sgn(x) &= \begin{cases} +1, & \text{when} \quad x \geq 0 \\ -1, & \text{when} \quad x < 0 \end{cases}
\end{aligned}
$$

2.4 Monte Carlo Simulation

Monte Carlo simulation is a sampling scheme which is used for solving stochastic and even deterministic problems. In many important applications within finance and without, Monte Carlo simulation is the only viable tool. In some cases, the time evolution of a stochastic process is not easy to describe analytically, and Monte Carlo simulation may be the only strategy that succeeds.

Because of its simplicity, Monte Carlo simulation has become a popular method for valuing derivatives . It is computationally easy and more efficient than lattice or tree methods at valuing certain path-dependent process (such as Asian, barrier, forward start, and look-back options and mortgage-backed securities). This is because the time taken to carry out a Monte Carlo simulation increases approximately linearly with the number of variables, whereas the time taken for most other procedures increases exponentially with the number of variables. Monte Carlo simulation has the advantage that it provides a standard error for the estimates that are made. It is an approach that can accommodate complex payoffs depending on some function of the whole path followed by a variable, not just its terminal value.

In a risk-neutral environment, the value of any derivative security is the discounted value of its expected terminal date cash flow. Assume interest rates are not stochastic, and discounting occurs at the risk-free rate r. The current price of a derivative security is given by

$$\text{Price} = e^{-rT}E[f(S_0, \ldots, S_T)]$$

where T is the maturity date of derivative, $f(S_0, \ldots, S_T)$ is the derivative's terminal date cash flow, which may be dependent on the entire price history of the underlying asset, and S_0, ...,S_T is the history of prices for the underlying asset from $t = 0$ to T.

In its crudest form, Monte Carlo simulation approximates the expectation of the derivative's terminal date cash flows with a simple arithmetic average of the cash flows taken over a finite number of simulated price paths:

$$\text{Price} \approx e^{-rT}[\frac{1}{N}\sum_{n=1}^{N} f(S_0^n, ..., S_T^n)] \tag{2.2}$$

where S_0^n, ..., S_T^n, is the n-th $n = 1, 2, \ldots, N$ simulated price path of underlying asset over the life of the derivative, and $f(S_0^n, \ldots, S_T^n)$ is the derivative's terminal date cash flow from this path.

Random Price Paths and Monte Carlo Simulation

To simulate a price path of underlying asset, we assume that the asset pays no dividends and follows a geometric Brownian process. The n-th price path S_0^n, ..., S_s^n, for $n = 1, 2, \ldots, N$, follows recursively from

$$S_i^n = S_{i-1}^n exp[(r - \frac{\sigma^2}{2})\Delta t + \sigma\varepsilon_i^n\sqrt{\Delta t}]$$

where the time to maturity has been divided into s time intervals of length Δt ($i = 1, 2, ..., s = T/\Delta t$), r is the risk-free drift rate, σ is the underlying asset's volatility; and for each time interval, ε_i^n is a random sample independently drawn from a standardized normal distribution $N(0, 1)$. Figure 2.2 illustrates three price paths, each with ten time intervals, for an asset with $r = 10\%$, $\sigma = 20\%$, and $\Delta t = 0.1$ year.

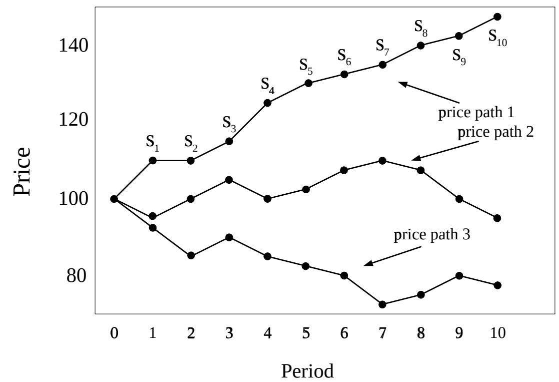

Figure 2.2: THREE RANDOM PRICE PATHS.
(Ten time intervals, $r = 10\%$, $\sigma = 20\%$)

After the entire price path $S_0^n, \ldots, S_s^n$ has been simulated, we calculate the derivative's terminal date cash flow $f(S_0^n, ..., S_s^n)$. We repeat the procedure for each additional price path. Often we will find it convenient to refer to each repetition of this procedure as "a simulation.".

After a large number of independent simulations, we compute a simple arithmetic average of the resulting terminal value. This gives us the expected terminal value of the derivative, which we then discount to the present at the risk-free rate to obtain the derivative's current price (i.e., Equation (2.2)).

Chapter 3

Model and Estimators

We use the notation following to formulate the problem of pricing American options:

- Let there be d exercise opportunities at times $0 = t_0 < t_1 < \ldots < t_{d-1} = T$, with T the time of expiration for the option initiated at time t_0 or today. Less formally, we will sometimes indicate $t = t_0, t_1, \ldots, t_{d-1}$ by $t = 0, 1, \ldots, T$.

- Consider a vector-valued Markov chain $(S_t : t = 0, 1, \ldots, T)$ consisting of all information required to determine the payoff from exercising an option. For simplicity, think of the components of S_t as lognormally distributed stock price. In practice, S_t would record all relevant information about asset prices, interest rates, exchange rates, and supplementary variables needed to eliminate path dependence. We say that state is s at time t if $S_t = s$.

- e^{-R_t} is the discount factor from $t-1$ to t. We take R_t to be a component of the vector S_t, and assume $R_t \geq 0$ for all t. In addition, let $R_{0t} = \Sigma_{i=1}^{t} R_i$.

- $h_t(s)$ is the payoff from exercise at time t in state s, $t = 0, \ldots, T-1$.

- $g_t(s) = E[e^{-R_{t+1}} f_{t+1}(S_{t+1}) | S_t = s]$ is the continuation value at time state s, $t = 0, \ldots, T-1$.

- $f_t(s) = \max[h_t(s), g_t(s)]$ is the option value at time t in state s, $t = 0, \ldots, T-1$.

- $f_T(s) = g_T(s) = h_T(s)$.

It is known that the price of American option is the solution of the optimal stopping problem:

$$f_0(S_0) = \max_{\tau} E[e^{-R_{0\tau}} h_\tau(S_\tau)] \tag{3.1}$$

where the maximum is over all stopping time τ taking values in $(0, 1, \ldots, T)$. The optimal policy stops at

$$\tau^* = \min[t = 0, \ldots, T : h_t(S_t) \geq g_t(S_t)] \tag{3.2}$$

That is, the first time the immediate exercise value is at least as great as the continuation value.

Under some restrictions, Broadie and Glasserman show that there is no unbiased estimator of Equation 3.1. As an alternative, therefore, they introduce two estimators, one biased high and the other biased low, both consistent and asymptotically unbiased.

3.1 Simulation Trees

In contrast to the Monte Carlo approach for pricing European options, the evolution of S_t is simulated using random trees rather than just sample paths. By this way, we can get better accuracy than using the sample paths that Monte Carlo generates. These random trees branch at each of the d exercise points.

Let b denote the number of branches (i.e., successor nodes) emanating from each node prior to expiration. Thus each tree consists of a total of b^{d-1} sample paths to expiration day. The nodes in these paths carry all the necessary information such as stock prices. Consequently, once a tree has been generated, we can calculate option payoffs at all nodes on the expiration day.

We then work backward through the tree, and at each node we find two estimators for the option prices by comparing the payoff from immediate exercise with the continuation value. This procedure is repeated until we reach the initial node at time 0. Two estimators for the option price are determined in this fashion.

Formally, given a value of *branching parameter* b, the evolution of tree can be described recursively as follows. From the (fixed) initial state S_0, we generate b

independent samples $S_1^1, \ldots, S_1^b$ of the state at time $t = 1$. From each of these b samples, $S_1^{i_1}$, i_1=1, ..., b, in turn, generate b new samples, $S_2^{i_1 1}, \ldots, S_2^{i_1 b}$ at time t=2. Repeat this procedure for every possible exercise time t.

To be precise from each node $S_t^{i_1 \ldots i_t}$ at time t, we generate b samples $S_{t+1}^{i_1 \ldots i_t j}$, $j = 1, \ldots, b$, conditionally independent of each other given $S_t^{i_1 \ldots i_t}$ and each having the distribution of S_{t+1} given $S_t = S_t^{i_1 \ldots i_t}$. Thus, each sequence S_0, $S_1^{i_1}$, $s_2^{i_1 i_2}, \ldots, S_t^{i_1 \ldots i_T}$ is a realization of the Markov chain, and the entire path to any node at time t, $1 \leq t \leq T$, is specified by $(i_1, i_2, \ldots, i_t)$, where $1 \leq i_1, i_2, \ldots, i_t, i_{t+1} \leq b$.

To simplify notation, we henceforth indicate the path to a node at time t by α_t; that is, for each node, α_t is a t-dimensional vector $(i_1, i_2, ..., i_t)$ that carries information about the path to that node. Thus, $S_t^{i_1 \ldots i_t}$ is written as $S_t^{\alpha_t}$. Similarly, $i_1 \ldots i_t j$ is replaced by $\alpha_t j$, so the j-th node (branch) following $S_t^{\alpha_t}$ is $S_{t+1}^{\alpha_t j}$; or, equivalently, $S_{t+1}^{\alpha_t j}$ where $\alpha_{t+1} = (i_1, i_2, \ldots, i_t, i_{t+1})$ with $i_{t+1} = j$.

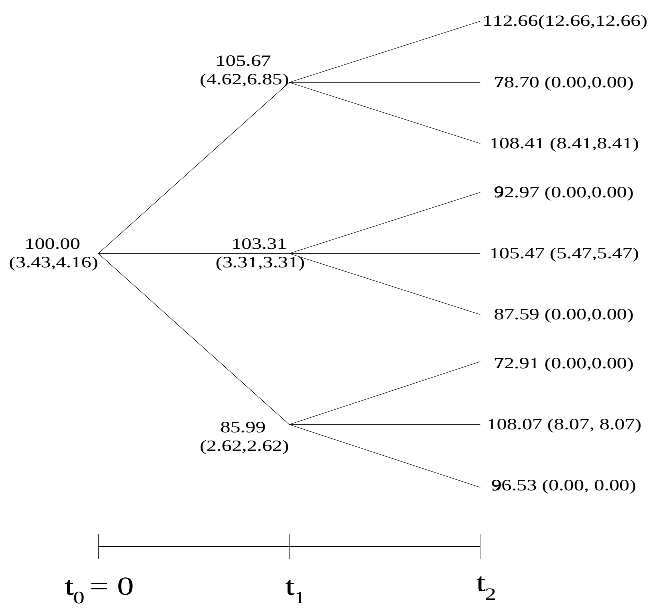

Figure 3.1: A Tree with $b = 3$ and Exercise Opportunities at t_0, t_1, and t_2 for a Call Option.

The value at each node is the stock price, and the values in parameters are the low and high estimates (q, H).
Parameter: $M = 1$, $b = 3$, $K = 100$, $r = 5\%$, $\delta = 10\%$, $T = 1.0$, $\sigma = 20\%$, and three exercise opportunities at time 0, 1/2, and one year.

Figure 3.1 illustrates a tree with parameters $b = 3$ and $M = 1$, where M denotes the dimension of the state vector. To understand the notation, consider the node at $t = T = 2$ corresponding to the stock price of 87.59. In our notation it is represented as $S_2^{2,3}$ - - - the subscript 2 indicates that we are looking at a node at time $t = 2$, and the superscripts indicate that it is the third branch among the branches emanating from the second node at $t = 1$. For this node, therefore, α_2 is (2,3). Similarly, the node corresponding to 78.70 is $S_2^{1,2}$ (α_2 for this node is [1,2]) and 85.99 is S_1^3 (α_1 for this node is [3]).

3.2 Generating Stock Prices

We assume that stock price follows geometric Brownian motion as follows:

$$\frac{dS}{S} = \mu\, dt + \sigma\, dw$$

where μ is the average return on stocks, σ is the volatility of stock price, and dw is the basic Wiener process.

Log-normal Model

Let $X = \ln S$. By Ito's lemma, we can derive the process of X as follows:

$$dX = (\mu - \frac{1}{2}\sigma^2)\, dt + \sigma\, dw \tag{3.3}$$

In the discrete time model, Equation (3.3) can be rewritten as follows:

$$\triangle X = (\mu - \frac{1}{2}\sigma^2)\, \triangle t + \sigma\, \epsilon\sqrt{\triangle t} \tag{3.4}$$

where ϵ is a random drawing from standard normal distribution $N(0, 1)$. Let S_i be the stock price at time i. An equivalent equation of Equation (3.4) is:

$$\ln S_{i+1} - \ln S_i = (\mu - \frac{1}{2}\sigma^2)\, \triangle t + \sigma\, \epsilon\sqrt{\triangle t}$$

or,

$$S_{i+1} = S_i \cdot e^{(\mu - \frac{1}{2}\sigma^2)\, \triangle t + \sigma\, \epsilon\sqrt{\triangle t}} \tag{3.5}$$

Suppose that today is time i, we can employ Equation (3.5) to generate stock price at time $i+1$. Since $\ln S_i$ is normally distributed, $\ln S_{i+1}$ is still normally distributed; thus S_{i+1} still has the log-normal property.

Getting the Price Path from Monte Carlo Model

We already know the property of the stock price. Now, we will show the equation of generating price path from Monte Carlo Model.

$$S_{t_k+1}^{(i)} = S_{t_k}^{(i)} exp[(r - \delta_i - \frac{1}{2}\sigma_i^2)(t_{k+1} - t_k) + \sqrt{t_{k+1} - t_k} W_k^{(i)}]$$
$$(i = 1, \ldots, M, andk = 0, \ldots, d-1)$$

where M is the number of the assets, d is the number of branches, r is the (constant) interest rate, δ_i are the dividend yields, and W_k^i $(i = 1, \ldots, M)$ are mean-zero normal random variates with standard deviations σ_i, and correlation ρ_{ij} for $i \neq j$.

We get the random sample from the $\phi(0, \sigma_i)$. And in this thesis, the options involving multi-assets, so we need the sample from a standardized bivariate normal distribution, an appropriate procedure is as follows. Independent samples x_1 and x_2 are obtained from $\phi(0, \sigma_i)$. The required samples ϵ_1 and ϵ_2 are then calculated as follows:

$$\begin{aligned} \epsilon_1 &= x_1 \\ \epsilon_2 &= \rho_{12} x_1 + x_2 \sqrt{1 - \rho_{12}^2} \end{aligned}$$

where ρ_{12} is the correlation between the variables in the bivariate distribution.

For an n-variate normal distribution where the coefficient of correlation between variable i and variable j is ρ_{ij}, we first sample n independent variables x_i $(1 \leq i \leq n)$ from $\phi(0, \sigma_i)$. The required samples are ϵ_i $(1 \leq i \leq n)$, where

$$\epsilon_i = \sum_{k=1}^{k=i} \alpha_{ik} x_k$$

For ϵ_i to have the correct variance and the correct correlation with the $\epsilon_j (1 \leq i < j)$, we must have

$$\sum_k \alpha_{ik}^2 = 1$$

and

$$\sum_k \alpha_{ik}\alpha_{jk} = \rho_{ij}$$

The first sample ϵ_1 is set equal to X_1. These equation for the α's can be solved, so the ϵ_2 is calculated from x_1 and x_2, ϵ_3 is calculated from x_1, x_2 and x_3, and so on.

3.3 Estimators

The *high estimator* is simply the result of applying dynamic programming to the random tree. More precisely, working backward through the tree using the recursions

$$\Theta_T^{\alpha_T} = h_T(S_T^{\alpha_T})$$

and

$$\Theta_t^{\alpha_t} = \max[h_t(S_t^{\alpha_t}), \frac{1}{b}\sum_{j=1}^{b} e^{-R_{t+1}^{\alpha_t j}}\Theta_{t+1}^{\alpha_t j}] \tag{3.6}$$

We compute the high estimator $\Theta = \Theta_0$. At any node, therefore, the high estimator is the maximum of the immediate exercise value and the average of discounted high estimates at successor nodes. The high estimator uses all branches emanating from a node to approximate both the optimal action (exercise or continue) and the value of this decision.

The *low estimator* separates the branches used to determine the action from those used to determine the continuation value. In words:

1. At each node in the tree, reserve one successor node. Average the discounted low estimator values at the other $b-1$ successor nodes.

2. If the average obtained is less than the immediate exercise value; otherwise, set the node value equal to the discounted value from the reserved node.

3. Average the resulting node value over all b ways of selecting the reserved successor node.

4. Repeat these steps backward through the tree.

For a more precise formulation, first let

$$\theta_T^{\alpha_T} = h_T(S_T^{\alpha_T}) \tag{3.7}$$

next, set

$$\eta_t^{\alpha_t j} = \begin{cases} h_t(S_t^{\alpha_t}), & \text{if } h_t(S_t^{\alpha_t}) \geq \frac{1}{b-1}\sum_{i=1 i\neq j}^{b} e^{-R_{t+1}^{\alpha_t i}} \theta_{t+1}^{\alpha_t i} \\ e^{-R_{t+1}^{\alpha_t j}} \theta_{t+1}^{\alpha_t j}, & \text{otherwise} \end{cases} \tag{3.8}$$

then let

$$\theta_t^{\alpha_t} = \frac{1}{b}\sum_{j=1}^{b} \eta_t^{\alpha_t j} \tag{3.9}$$

for $t = 0, ..., T-1$. Finally, the low estimate θ is given by θ_0.

To summarize, this method requires constructing random tree parameterized by b, the number of branches per node. Once these trees have been obtained, work backward through the tree to compute the option price. In this way, the early exercise feature is incorporated in the simulation methodology. The option price itself is determined in terms of two estimates: one is biased high, and one is biased low, but both are unbiased in the limit as b goes to infinity.

Example 1

Consider a simple call option on a single stock. The current stock price is 105, strike price is 100, time to maturity (T) is one year, annual interest rate is 5%, volatility is 10%, and the dividend rate is 10%. More important, suppose there are only two exercise opportunities, 0 and T. We exercise the option either today or at expiration.

The price of the option is therefore simply the maximum of its immediate exercise value ($105 - 100 = 5$) and the price of a European option expiring at T. Using the Black-Scholes formula, we find the price of the European option to be 3.73. Thus the option price is $\max[5.00, 3.73] = 5.00$, and the optimal decision should be to exercise immediately rather than to wait until expiration day.

Let us now attempt to use the simulation approach discussed above to find the low and high estimates for the option price. We generate five nodes on expiration day as shown in Figure 3.2. Observe that averaging the payoffs at these nodes provides an unbiased estimate for the European price, although the estimate itself may be quite inaccurate in this case since we are using only five sample paths.

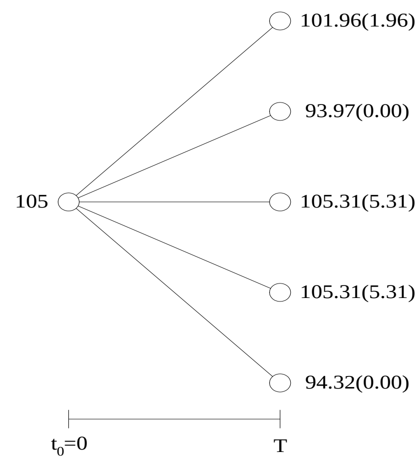

Figure 3.2: A TREE WITH $b = 5$ AND EXERCISE OPPORTUNITIES AT 0 AND T FOR A CALL OPTION.

The value at each node is the stock price, and the values in parameters is the payoff. The low and high estimates for the option price obtained from this tree are 2.38 and 5.67, respectively.
Parameter: M=1, b=5, K=100, r=5%, δ = 10%, T=1.0, σ=10%, and two exercise opportunities at time 0 and one year.

According to Equation (3.6), the high estimator is the maximum of the immediate exercise value and the discounted estimate of the continuation value. Hence, it is $\max[5, e^{-0.05}(1.96 + 22.53 + 0 + 5.31 + 0)/5] = \max[5, 5.67]$.

By erroneously choosing to continue, in this case, we have biased the estimate upward. In general, since a finite number of branches will not represent the distribution of stock prices perfectly, we will choose to continue whenever the simulated future prices are too high, even though the optimal decision is to exercise. Conversely, had the optimal decision been to continue, we will erroneously choose to exercise instead whenever the simulated future stock prices are too low. In either case, we overestimate the option price, as we are always using the large of the immediate exercise value and the estimated continuation value.

Calculation of the low estimator also requires an estimate for the continuation value. Averaging the discounted payoffs at four out of five branches provides five such estimates. By individually comparing these continuation value estimates with the immediate exercise value, we can decide whether to exercise or continue, If the decision is to continuation value – is used as one estimate of the option price. Otherwise, use the exercise value.

For example, the discounted average of the payoffs at the last four branches is $e^{-0.05}(22.53+0+5.31+0)/4 = 6.62$, which is greater than the payoff from immediate exercise. Hence a decision to continue rather than exercise is inferred, and the first low estimate is $e^{-0.05}1.96 = 1.86$(the discounted payoff of the reserved first branch).

Now reserve the second branch, and obtain the continuation value from the remaining branches as $e^{-0.05}(1.96 + 0 + 5.3 + 0)/4 = 1.73$, which is smaller than 5.00. Hence 5.00 is the second estimate. Repeating the same procedure, the remaining three estimates are 0, $e^{-0.05}5.31 = 5.05$, and 0. Finally, the low estimate is $(1.86 + 5.00 + 0 + 5.05 + 0)/5 = 2.38$.

In this example, since four of the estimates for the continuation value are greater than 5.00, we incorrectly infer continuing four out of five times. The only correct decision is due to $1.73 < 5.00$. Therefore, four of the five intermediate estimates obtained from this tree are biased. Since these four values provide an unbiased estimate for the European option price, which we already know has a value less than 5.00, the bias is downward. In general, the estimate obtained in this fashion is biased low because it is an average of some unbiased estimates (based on the correct decisions) and some other estimates that are biased low (based on the incorrect decisions).

Example 2

Suppose there are three exercise opportunities at time 0, $T/2$, and T. In particular, consider a call option on a single stock with an initial price of 100. The strike price is 100, time to maturity is one year, annual interest rate is 5%, volatility is 20%, and the dividend rate is 10%. The random tree in figure 3.1 corresponds to a single simulation run. In this figure, the low and high estimates are reported in parentheses next to the random stock price at each node.

It is easy to compute the two estimates time T(e.g., at the topmost node at time T in figure 3.1). The stock price is 112.66, so both estimates for this node are $112.66 - 100.00 = 12.66$.

At the topmost node at time $T/2$, on the other hand, since the immediate exercise value is 5.67, the high estimate is $\max[5.67, e^{-0.025}(12.66 + 0.00 + 8.41)/3] = \max[5.67, 6.85]$. The low estimate is the average of b=3 intermediate values. The first of these three values is 5.67 because $e^{-0.025}(0.00{+}8.41)/2{<}5.67$. Similarly, the second one is 0.00 because $e^{-0.025}(12.66 + 8.41)/2 > 5.67$, and the third one is $e^{-0.025}(8.41) = 8.20$ because $e^{-0.025}(12.66 + 0.00)/2 > 5.67$. Finally, the low estimate is $(5.67 + 0.00 + 8.20)/3 = 4.62$. The low and high estimates at other nodes are obtained in the same manner.

The low and high estimates for the price of the option corresponding to this simulation run are 3.43 and 4.16. Repeating the procedure many times by simulating new trees, we get refined values of the two estimators as their respective averages over all the simulation runs. More important, we can get standard errors, and hence confidence intervals, for these estimators. A conservative confidence interval for the option price itself can be obtained by taking the upper confidence limit of the high estimator and the lower confidence limit of the low estimator.

If we let d be the number of exercise opportunities and n the number of replications, the work required to carry this out grows like nb^{d-1}. In particular, increasing the branching parameter is typically far more costly than increasing the number of replications. Increasing b, however, is essential for reducing bias and thus reducing the width of the confidence interval. This difficulty motivates our investigation into techniques that help reduce the size of the confidence interval without having to increase b.

3.4 Pruning

Pricing a European option is generally easier than pricing the American option; There is no optimization involved in the European price. It is therefore natural to try to exploit information obtained from the European case in pricing the American option.

The European price provides a highly effective control variate, as example will show.

We explain how the European price can also be used for *pruning*, i.e., reducing the number of nodes in the simulated trees. Then we will introduce pruning techniques.

Pruning at the Last Step

Let $T-1$ denote the penultimate exercise opportunity. The optimal action at time $T-1$ depends on which is greater: the immediate exercise value $h_{T-1}(S_{T-1})$, or the continuation value $g_{T-1}(S_{T-1})$. The estimators Θ and θ implicitly estimate the continuation value at each node. But at time $T-1$ the continuation value is just the value of a European option initiated at time $T-1$ and maturing at time T. Computing this value directly and efficiently eliminates the need to branch at the penultimate node.

In other word, if $c_T(S_{T-1})$ denotes the price of the European option initiated at time $T-1$ with initial stock price S_{T-1} and expiring at time T. then the low and high estimators are both set to $\max[h_{T-1}(S_{T-1}), c_T(S_{T-1})]$. Since this implies that we do not need to generate successor nodes for the nodes at $T-1$, the work required is reduced to $\odot(nb^{d-2})$.

Returning to the tree in Figure 3.1, it is not necessary to branch at time t_1. As an illustration, the low and high estimates at the topmost node at time t_1 would then become $\max[5.67, 7.20] = 7.20$, where 7.20 is the price of the European call option with initial stock price 105.67 and time to maturity half a year.

Intermediate Pruning

The sole reason for branching (as opposed to simulating sample paths in the usual way) is to allow for consistent estimation of the optimal action at a node. Suppose that at time t there is a node corresponding to state s. If we knew that the immediate exercise value is smaller than the continuation value, i.e., $h_t(s) < g_t(s)$, we would know that the optimal action is to continue and there would be no need to branch; it would suffice to generate just one successor node. Of course, in general we do not know $g_t(s)$, since g_t is itself the value function of an optimal stopping problem.

But if we can find an easily computed lower bound $c(s) \leq$ (*respectively* $<$)$g_t(s)$, we can check if $h_t(s) <$ (*respectively* $\leq$)$c(s)$. If this holds, stopping is guaranteed to be suboptimal, so there is no need to branch. If there is no branching out of node $i_1 \ldots i_t$, then Equation (3.6) gets replaced by $\Theta_t^{\alpha_t} = \exp(-R_{t+1}^{\alpha_t 1})\Theta_{t+1}^{\alpha_t 1}$, and Equation (3.9) gets replaced with $\theta_t^{\alpha_t} = \exp(-R_{t+1}^{\alpha_t 1})$.

Pruning at the node in this manner considerably reduces the work required per tree because it eliminates $b - 1$ successor nodes along with their progeny. Consequently, in the same amount of time, we can simulate more trees than we could if we were not pruning. Since increasing b reduces the bias, it is a good idea to devote some of the savings in time to increasing b as well.

In virtually all practical examples, the value of an option remains strictly positive throughout its existence. Thus, a simple choice of bound is $c(s) = 0$; at any node at which the immediate exercise value is zero, there is no need to branch. This test is free because this choice of c requires no computational effort. For example, in Figure 3.1, since the immediate exercise value at time 0 is 0, it is optimal to continue, and branching is unnecessary; we should generate just one successor node at time $t = 1$ instead of three.

In the case $h_t(s) > 0$, we may decide to compare the immediate exercise value with a more refined bound. Natural choices are $c_{t+1}(s), \ldots, c_T(s)$, where $c_k(s)$ is the value of a European option initiated in state s and maturing at time k. Each of these corresponds to a particular (suboptimal) exercise policy for the American option, and thus provides a lower bound on $g_t(s)$. If $h_t(s) < c_k(s)$ for any $k = t+1, \ldots, T$, there is no need to branch. (In example later, we illustrate this approach using only $c_T(s)$.)

The idea behind pruning is pictured in Figure 3.3. Specifically, compare the price $c_T(S_{t_k})$ of the European option initiated at time t_k and expiring at T with the immediate exercise value at time t_k, $h_{t_k}(S_{t_k})$. If the latter is greater, branch in the usual fashion as shown in Figure 3.3(a). If the former is greater, generate only one successor node, i.e., only one branch emanates from this node; see Figure 3.3(b).

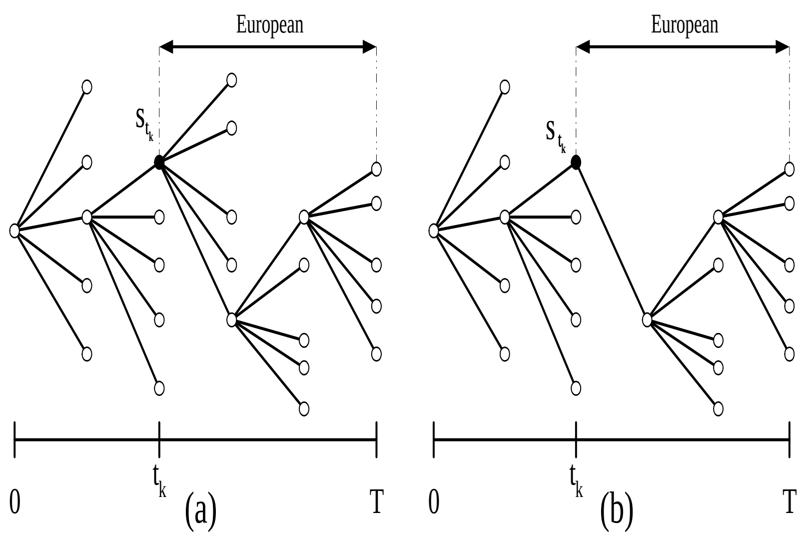

Figure 3.3: PRUNING AT TIME t_k.

Figure 3.3(a). If the European value for the option initiated at time t_k with initial stock price S_{t_k} and time to expiration T is smaller than the payoff from immediate exercise, then branch in the usual fashion. Figure 3.3(b). Otherwise, generate only one successor node at the next time step t_{k+1}.
For clarity, in this and the following figure the complete trees are not shown. They are in reality dense trees with branches emanating from every node before expiration - - - one branch from the pruned nodes and b branches from the rest.

Chapter 4

Numerical Results

We test the techniques discussed by using them here to price options. The example we consider an American option on the maximum of two assets. We use five initial prices to test them, the prices are 80, 90, 100, 110, and 120. For simplicity, we assume the two assets' initial price are identical, and the parameters that we use in every case are identical. The annualized interest rate is $r = 5\%$; Both assets have dividend yields $\delta_i = 10\%$ and volatilities $\sigma_i = 20\%$; Their correlation are $\rho_{ij} = 30\%$, $i{\neq}j$; the strike price K is 100; the time to maturity T is three years; and there are four exercise opportunities at time 0, 1, 2, and 3 years.

We implement pruning using European value at penultimate node and intermediate pruning based on first checking if the payoff from immediate exercise is positive, and then comparing it with a single European option maturing at time T . The European price is computed by the formula that we discussed in Chapter 2.3.

The price of the European call option with identical parameters is used as a control variate with both techniques. The control variate technique is applicable when there are two similar derivatives, A and B. Security A is the security under consideration; security B is a security that is similar to security A and for which an analytic solution is available. Two simulations using the same random number streams and the same Δt are carried out in parallel. The first is used to obtained an estimate, f_A^* ,of the value of A; the second is used to obtained an estimate, f_B^*, of the value of B. A better estimate of the value of A, f_A, is then obtained using the formula

$$f_A = f_A^* - f_B^* + f_B$$

where f_B is the true value of B. In this case, f_A is the value of the estimator we want, and f_B is the true value of European option. The true value we used here are obtained using the lattice approach suggested by Boyle, Evnine, and Gibbs[**?**] on the same option with four exercise opportunities.

Confidence interval for the option price are computed by taking the upper confidence limit of the high estimator and the lower confidence limit of the low estimator. Because these interval are conservative, the actual converge shall be much better.

The values labeled "Point Estimate" are the averages of the corresponding high and low estimators. Taking the midpoint as the price estimate is a fairly arbitrary way of compromising between the two. Nevertheless, the relative error "Real Error" is computed from this estimate.

The result of using original approach, We show it on the table 4.1. In this approach, we only use pruning method at penultimate node and take the European value as control variate.

Immediate pruning approach, in which we use the pruning method at every nodes of the simulation tree. And the penultimate pruning is used at the penultimate nodes, the European price is also used as control variate. The result we show it on the table 4.2.

Although the simulation number is not large, but we can see that the real error on table 4.1 is much small. This is because the approach expand the random tree completely, so we can get good point estimate. Alternatively, we use much time to compute by this approach. On the table 4.2, we can see that immediate pruning approach have worse real error than original approach in same simulation numbers. This is because immediate pruning approach do not expand the random tree completely, it delete the branches of the nodes. In the other hand, this let immediate pruning approach use less time to compute the option value.

We had shown the simulation results of two approaches respectively. Because using the immediate pruning approach, the branches of simulation tree are eliminated to

Table 4.1: Option on Maximum of Two Assets
(Original Approach)

Initial Price	Low Estimate	High Estimate	90% Confidence Interval	Point Estimate	True Value	Real Error	Simulation Num.
80	3.634	3.664	$[3.601, 3.697]$	3.649	3.643	0.17%	100
90	7.208	7.281	$[7.191, 7.350]$	7.245	7.234	0.15%	100
100	12.315	12.502	$[12.282, 12.535]$	12.409	12.412	0.02%	100
110	18.842	19.167	$[18.809, 19.200]$	19.005	19.059	0.28%	100
120	26.666	27.083	$[26.535, 27.199]$	26.875	26.875	0.00%	100
80	3.624	3.659	$[3.603, 3.680]$	3.642	3.643	0.02%	250
90	7.187	7.278	$[7.166, 7.299]$	7.232	7.234	0.01%	250
100	12.331	12.517	$[12.310, 12.538]$	12.424	12.412	0.10%	250
110	18.891	19.212	$[18.871, 19.232]$	19.051	19.059	0.03%	250
120	26.693	27.104	$[26.612, 27.179]$	26.898	26.875	0.08%	250
80	3.627	3.660	$[3.612, 3.675]$	3.644	3.643	0.02%	500
90	7.193	7.287	$[7.179, 7.302]$	7.240	7.234	0.09%	500
100	12.300	12.494	$[12.286, 12.508]$	12.397	12.412	0.11%	500
110	18.873	19.187	$[18.858, 19.202]$	19.030	19.059	0.14%	500
120	26.614	27.062	$[26.549, 27.115]$	26.838	26.875	0.13%	500

Initial price=80 means $S_0^{(1)} = S_0^{(2)} = 80$.
This technique includes pruning at last step. European price is used as control variate.
Payoff:$max[max(S^{(1)}, S^{(2)}) - K, 0]$.
Parameter:$b = 50$, $K = 100$, $r = 5\%$, $\delta = 10\%$, $T = 3.0$, $\sigma = 20\%$, $\rho = 30\%$, and four exercise opportunities at times 0, 1, 2, and 3 years.
European prices: 3.269 for $S_0 = 80$, 6.293 for $S_0 = 90$, 10.513 for $S_0 = 100$, 15.835 for $S_0 = 110$, 22.080 for $S_0 = 120$.

Table 4.2: Option on Maximum of Two Assets
(Immediate Pruning Approach)

Initial Price	Low Estimate	High Estimate	90% Confidence Interval	Point Estimate	True Value	Real Error	Simulation Num
80	3.653	3.674	$[3.619, 3.709]$	3.664	3.643	0.58%	100
90	7.188	7.272	$[7.133, 7.326]$	7.230	7.234	0.04%	100
100	12.333	12.493	$[12.248, 12.575]$	12.412	12.412	0.00%	100
110	19.014	19.274	$[18.921, 19.389]$	19.157	19.059	0.51%	100
120	26.633	27.090	$[26.428, 27.238]$	26.862	26.875	0.04%	100
80	3.636	3.658	$[3.614, 3.679]$	3.647	3.643	0.11%	250
90	7.187	7.278	$[7.166, 7.299]$	7.232	7.234	0.01%	250
100	12.331	12.517	$[12.310, 12.538]$	12.424	12.412	0.10%	250
110	18.891	19.212	$[18.871, 19.232]$	19.051	19.059	0.03%	250
120	26.693	27.104	$[26.612, 27.179]$	26.898	26.875	0.08%	250
80	3.641	3.664	$[3.625, 3.680]$	3.652	3.643	0.27%	500
90	7.214	7.288	$[7.189, 7.314]$	7.251	7.234	0.24%	500
100	12.321	12.479	$[12.284, 12.516]$	12.400	12.412	0.09%	500
110	18.963	19.215	$[18.915, 19.261]$	19.089	19.059	0.15%	500
120	26.630	27.061	$[26.550, 27.121]$	26.845	26.875	0.10%	500
80	3.630	3.655	$[3.619, 3.666]$	3.643	3.643	0.00%	1000
90	7.206	7.282	$[7.188, 7.300]$	7.244	7.234	0.14%	1000
100	12.350	12.506	$[12.324, 12.532]$	12.428	12.412	0.13%	1000
110	18.954	19.207	$[18.920, 19.240]$	19.080	19.059	0.11%	1000
120	26.645	27.076	$[26.589, 27.116]$	26.860	26.875	0.05%	1000

This technique also includes pruning at last step. European price is used as control variate.
Payoff:$max[max(S^{(1)}, S^{(2)}) - K, 0]$.
Parameter:$b = 50$, $K = 100$, $r = 5\%$, $\delta = 10\%$, $T = 3.0$, $\sigma = 20\%$, $\rho = 30\%$, and four exercise opportunities at times 0, 1, 2, and 3 years.
European prices: 3.269 for $S_0 = 80$, 6.293 for $S_0 = 90$, 10.513 for $S_0 = 100$, 15.835 for $S_0 = 110$, 22.080 for $S_0 = 120$.

Table 4.3: The Computing Efficiency
(Original Approach vs. Immediate Pruning Approach)

Initial Price	90% Confidence Interval	Point Estimate	True Value	Real Error	Simulation Num	Simulation Time (second)
			Original Approach			
80	$[3.579, 3.680]$	3.629	3.643	0.36%	97	100
90	$[7.195, 7.357]$	7.276	7.234	0.58%	96	100
100	$[12.289, 12.539]$	12.414	12.412	0.02%	99	100
110	$[18.809, 19.200]$	19.005	19.059	0.28%	100	100
120	$[26.512, 27.177]$	26.852	26.875	0.08%	98	100
			Immediate Pruning			
80	$[3.619, 3.666]$	3.643	3.643	0.00%	1000	94
90	$[7.187, 7.307]$	7.247	7.234	0.19%	584	100
100	$[12.272, 12.525]$	12.399	12.412	0.10%	312	100
110	$[18.882, 19.279]$	19.081	19.059	0.12%	217	100
120	$[26.402, 27.208]$	26.837	26.875	0.13%	159	100

decrease the complexity of the computation. So We know that immediate pruning approach can use less time to do simulation than original approach. And then We want to compare the accuracy and computing efficiency of the two approaches we used. We show that how many simulations numbers that both approaches can make by using same time. Alternatively, the real error is also shown on the table. From this, we can know that whether the immediate pruning approach can use less time to get better real error.

Next, we show the situations of convergence of the two case, the initial price 80 (deep out-of-the money) and 120 (deep in-the-money), on the Figure 4.1, 4.2, 4.3, and 4.4. We know that we can do more simulations with equal computing time by immediate pruning approach, and then we want to compare it with the original approach to know that whether we can get good convergence property with equal computing time by the immediate pruning approach.

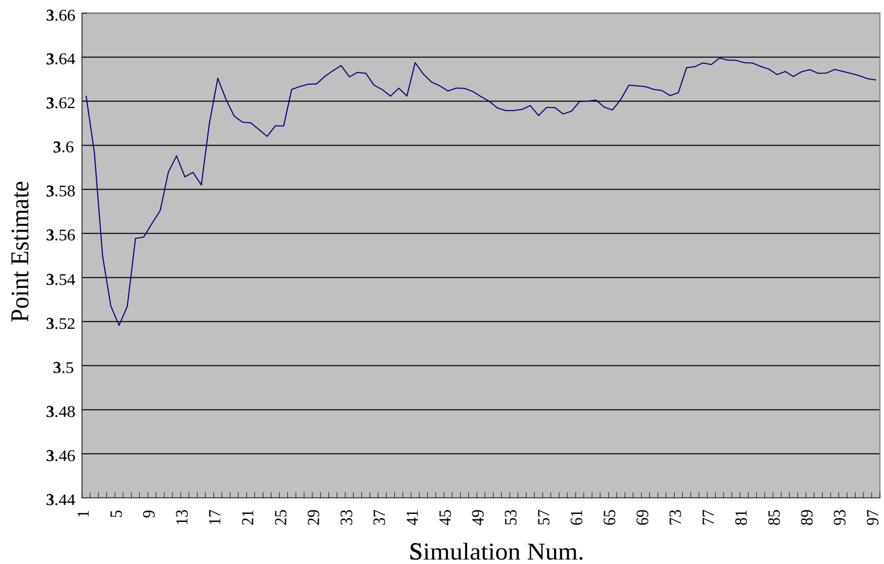

Figure 4.1: The Convergence of Original Approach ($S_0 = 80$).

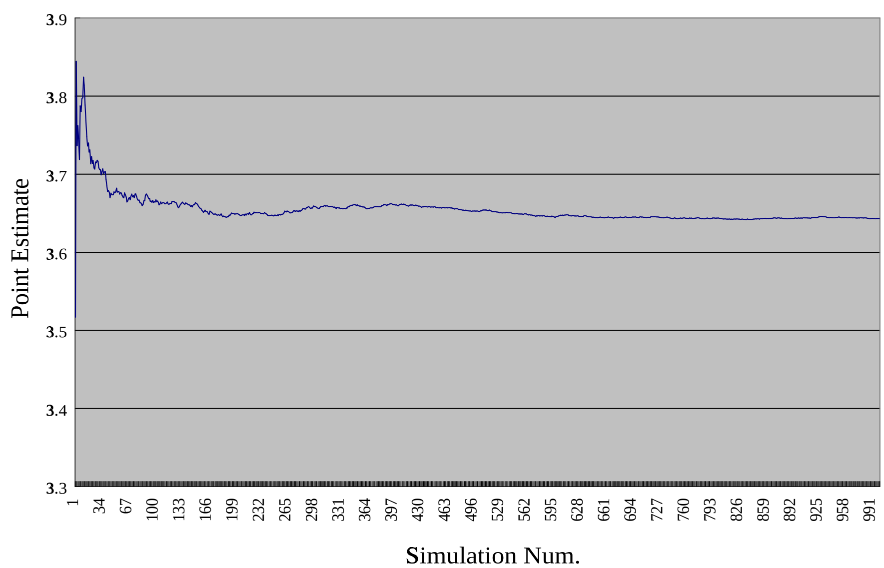

Figure 4.2: The Convergence of Immediate Pruning Approach ($S_0 = 80$).

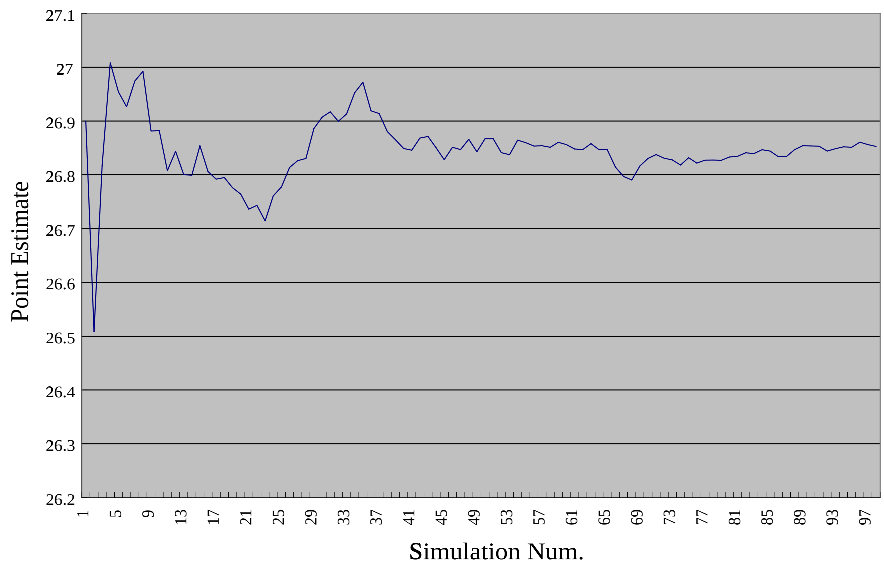

Figure 4.3: The Convergence of Original Approach ($S_0 = 120$).

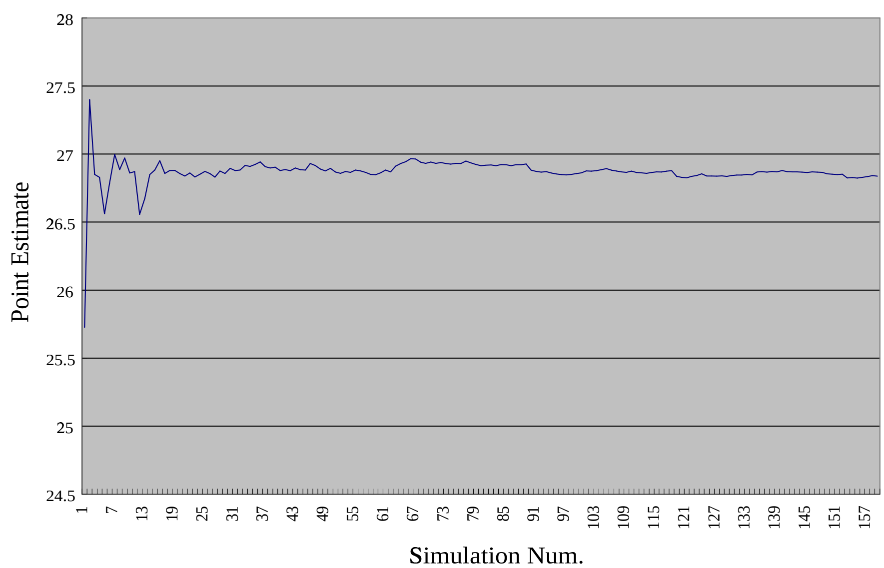

Figure 4.4: The Convergence of Immediate Pruning Approach ($S_0 = 120$).

Chapter 5

Conclusions

As the number of state variables increases, simulation becomes the only computationally feasible numerical approach for pricing options. Unlike other numerical methods, it allows for complex payoff functions and path-dependencies. Hence it becomes important to look for possible ways of using simulation for pricing American options.

Because the approach presented here relies on two estimates, one biased high and the other biased low, and a confidence interval obtained from them, it is important to reduce the bias and variance of these estimators. The computation including a control variate causes further enhancement.

The method is most promising for pricing American-style securities with multiple state variables. Although estimators were developed for option with two decisions, exercise or not, they are easily extended to a finite number of decision.

This work can be extended in several directions. Variations of the low estimator, e.g., using b_1 branches to determine the exercise decision and b_2 branches to evaluate the resulting payoff, remain to be explored. The number of branches per node does not need to be constant throughout the tree. The convergence rate of the algorithm and the effect of the choice of n and b on the error remain to be explored. Alternative variance reduction techniques, including other control variates could be tested. Quasi-Monte Carlo method, also termed low discrepancy methods that we may use to replace the original monte carlo method. Additional computational testing on other American-style securities remains to be done.

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