Pricing Asian Options

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Abstract

With the rapid growth of economies, many financial institutions bring new financial commodities. These commodities have many different functions. Asian option is one such example.

Asian option is used to reduce the significance of the closing price at maturity of the option. It's payoff depends on the average price of the underlying asset during the start day and the maturity. Pricing these options still have not absolute solutions, and most solutions are approximate.

This thesis tests the two popular methods of valuing Asian options. We find either Hull-White or Levy's method of pricing Asian options are not good. So we introduce an Asian option put-call parity. Using the put-call parity we can get an Asian call or a Asian put option value from the other immediately.

Chapter 1

Introduction

1.1 Background

A *derivative (or derivative security)* is a financial instrument whose value depends on the values of other, more basic underlying variables. In recent years, derivatives have become increasingly important in the field of finance. Futures and options are now traded actively on many exchanges. Forward contracts, swaps, and many different types of options are regularly traded outside exchanges by financial institutions and their corporate clients. Other, more specialized derivatives often form parts of a bond or stock issue.

Very often the variables underlying the derivatives are the prices of traded securities. A stock option, for example, is a derivative whose value is contingent on the price of a stock. However, derivatives can be contingent on almost any variables.

Option is one of the outstanding examples. An option gives its owner the right to buy or sell, for a limited time, a particular good at a specified price. One can use it either to hedge the risk we face, or to speculate to profit in the market. Also, one can earn a riskless profit by simultaneously entering into two or more markets, which is called *arbitrage*.

Asian options are options whose payoff depends on the average price of the underlying asset during at least some part of the option. If the binomial tree approach is used to price the option, it is necessary to keep track of 2^n possible paths, where n is the number of periods. So it is difficult to value an Asian option.

1.2 Contributions

This thesis tests the two popular methods of valuing Asian options, the Hull-White's mothod and Levy's approximation formula. Comparing these two methods' results, we find some faults in both. We conclude that these methods are not good for valuing Asian options in all cases. Finally we introduce an Asian option put-call parity. Using the put-call parity we can get a call or a put option value from the other immediately.

1.3 Structure of the Thesis

There are four chapters in this thesis. In Chapter 1, we give a brief introduction. In Chapter 2, we describe some fundamental concepts. In Chapter 3, the two methods of valuing options are described, and their results are examined. We also introduce an Asian option put-call parity in Chapter 3. Finally, conclusion are in Chapter 4.

Chapter 2

Fundamental Concepts

Options give their holder the right to buy or sell some *underlying asset.* They form one of the most important classes of financial instruments and have wide applications to finance; in fact, almost any security has option features. As far as explaining empirical data goes, option pricing theory is the most successful theory in finance as well as economics. This chapter begins with some option basics.

A useful and very popular technique for pricing an option or other derivatives involves constructing what is known as a *binomial tree.* This is a tree that represents possible paths that might be followed by the underlying asset's price over the life of the derivative. In this chapter we will briefly discuss the *binomial model.*

Fischer Black and Myron Scholes derived a differential equation that must be satisfied by the price of any derivative security dependent on a non-dividend-paying stock. They used the equation to obtain values for European call and put options on the stock. This is the so-called *Black-Scholes formula.* This chapter will also discuss it we shall follow the exposition in [4].

2.1 Option Basics

Two basic types of options are *calls* and *puts.* More complex option-type instruments can usually be decomposed into packages of the two. Examples include *interest rate floors* and *caps, embedded options* in many fixed-income securities, notably callable bonds and mortgage-backed securities. As the value of an option depends on the price of its underlying asset, options are *contigent claims* or *derivative securities.*

A call option gives its holder the right to buy a specified number of some underlying asset by paying a specified *exercise* or *strike* price, at or before *expiration.* A put option gives its holder the right to sell a specified number of some underlying asset by paying a specified exercise or strike price, at or before expiration. The underlying asset may be stocks, stock index, foreign currencies, futures contracts, interest rates, fixed-income securities, prices of some fixed-income instruments, options, and countless others.

The individual who issues the option is the *writer.* To acquire the option, the holder pays the writer a *premium.* When a call option is *exercised,* the holder pays the writer the strike price in exchange for the stock, and the call option ceases to exist. When a put option is *exercised,* the holder receives from the writer the strike price in exchange for the stock, and the put option ceases to exist. *Early exercise* refers to the act of exercising an option prior to expiration. Besides exercising the option, at any trading date before expiration, the holder can either do nothing or sell the option. American options and European options differ in when the holder can exercise them. American options can be exercised at any time up to the expiration date, while European options can only be exercised at expiration.

An option does not oblige the holder to exercise the right. In other words, options can be allowed to expire worthless. Hence, options will be exercised only when it is in the best interest of the holder to do so. Clearly, a call option will be exercised only if the stock price is higher than the strike price. Similarly, a put option will be exercised only if the stock price is less than the strike price. The value of a call at its expiration date is therefore $\max(0, S - X)$, and that of a put at its expiration date is $\max(0, X - S)$. A call option is said to be *in the money* if $S > X$, *at the money* if $S = X$ and *out of the money* if $S < X$. Similarly, a put option is said to be *in the money* if $S > X$, *at the money* if $S = X$, and *out of the money* if $S < X$. Finding an option's value at any time before expiration, however, is much more difficult. See Figure 2.1 for the plots of values of puts and calls prior to expiration.

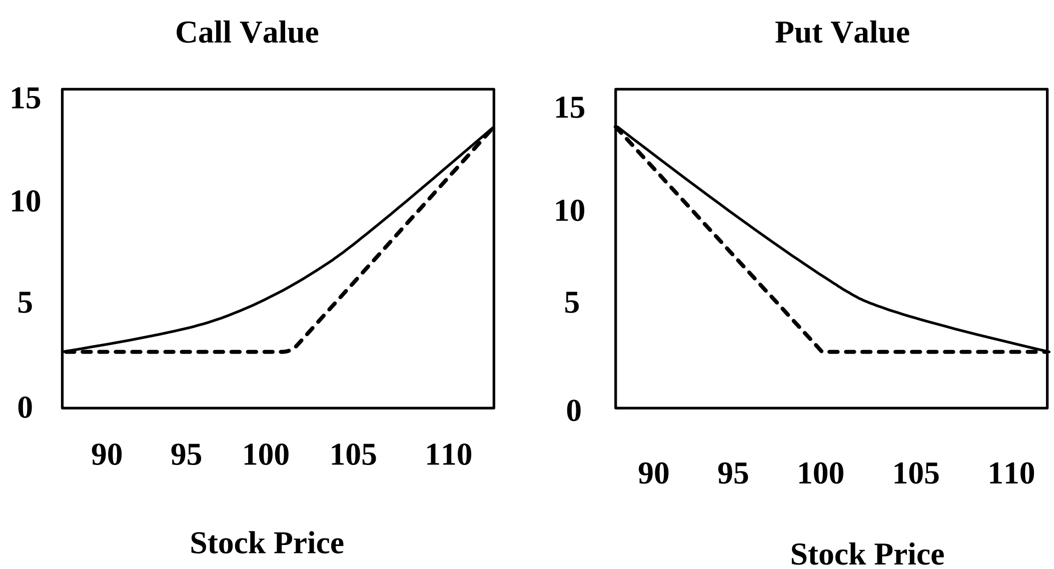

Figure 2.1: Value of Option Prior To Expiration
Plotted are the general shapes of option values as functions of the stock price before expiration. Dashed lines are the familiar option value diagrams at expiration, plotted for comparison.

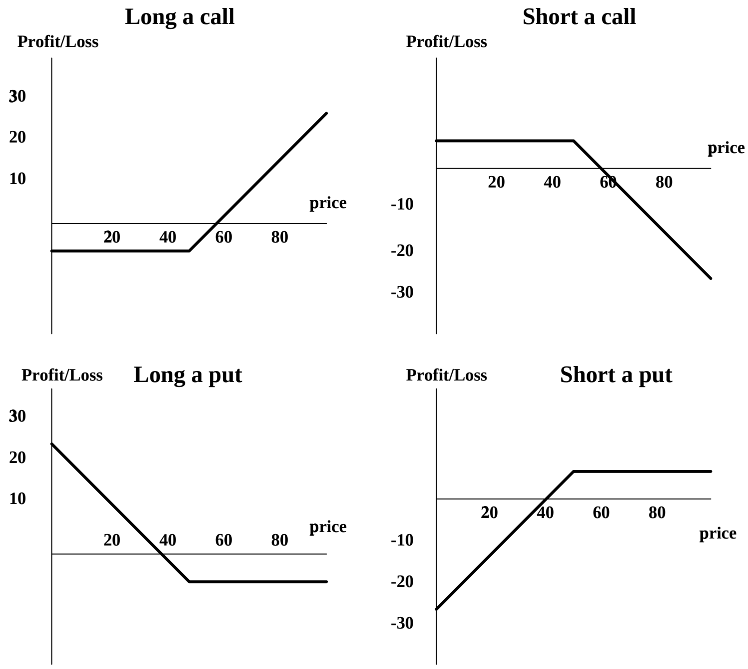

Figure 2.2: Profit/Loss of Option at expiration, where X=50 and C=P=5.

The payoff of owning a call at expiration is

$$\max(0, S - X)$$

and that of a put is

$$\max(0, X - S)$$

where C is the call premium and P the put premium. See Figure 2.2 for illustrations of the options Profit/Loss diagrams. Figure 2.3 shows the Profit/Loss diagrams of a long and short position in stock.

We call $\max(0, S - X)$ the *intrinsic* value of a call option, and $\max(0, X - S)$ the intrinsic value of a put. The intrinsic value is, in other words, the value of a American option when it is exercised immediately. The part of an American option's value above its intrinsic value is called its *time value* or *time premium.* It represents the possibility to become more valuable before the option expires. The option premium therefore consists of the intrinsic value and the time value, neither of which can be negative.

2.2 The Binomial Option Pricing Model

In this model, time is discrete, measured in periods. The central idea of the Black-Scholes analysis says five pieces of information (the current stock price, the two possible prices in the next period, the option's strike price, and the riskless interest rate)

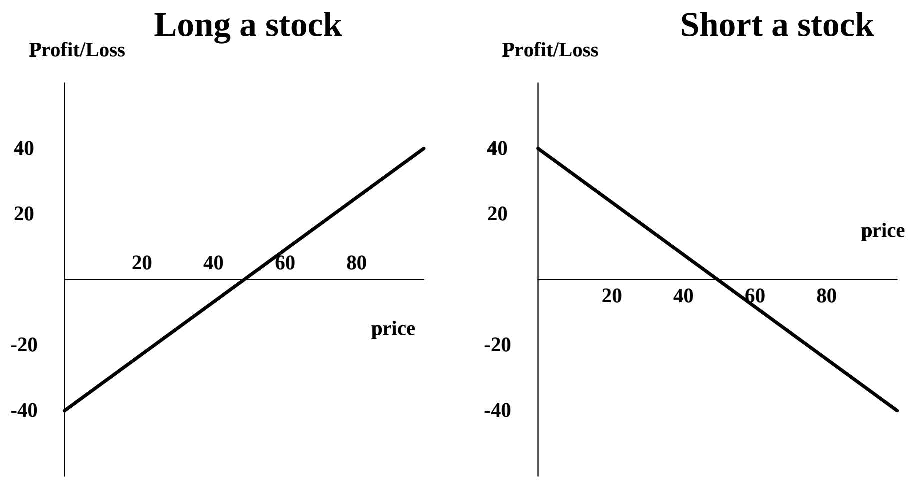

Figure 2.3: Profit/Loss of Stock

are sufficient to determine the value of an option lasting for a single period based on arbitrage considerations. The way to prove it is truly ingenious: Replicate the option by a portfolio of stocks and riskless bonds. To extend this idea to multi-period options, simply apply it recursively, from expiration to the current period. What may seem surprising is that we need to know neither the probability that the stock price will rise or fall in the next period nor the expected growth rate of the stock price [4].

Let $r > 0$ denote the constant, continuously compound riskless interest rate per period and R the gross return,

$$R = e^r$$

Denote the *binomial distribution* with parameters n and p by

$$b(j; n, p) = \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j}$$

Recall that $n! = n(n-1)\cdots 1$. (The convention for $0!$ is $0!{=}1$.) Hence, $b(j; , n, p)$ is the probability of getting j heads when tossing a coin n times. The *complementary binomial distribution function* with parameters n and p is defined as

$$\Phi(k; n, p) = \sum_{j=k}^{n} b(j; k, p)$$

$\Phi(k; n, p)$ therefore is the probability of getting at least k heads when tossing a coin n times with p being the probability of getting a head. It is not hard to see that

$$1 - \Phi(k; , p, p) = \Phi(n - k + 1; n, 1 - p)$$

Under the binomial option pricing model, if the current stock is S, it can go to Su with probability q and Sd with probability $1 - q$, where $1 > q > 0$ and $u > d$. See the illustration in Figure 2.4.

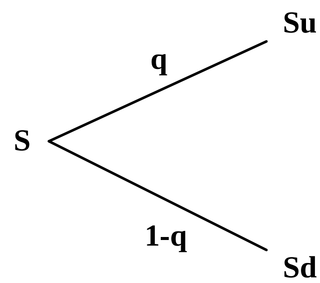

Figure 2.4: Binomial Model for Stock Price

As the first step, assume the expiration date is one period from now. Let C be the current call price, C_u be the price one period from now if the stock price moves to S_u, C_d be the price one period from now if the stock price move to S_d. Clearly,

$$C_u = \max(0, S_u - X) \text{ and } C_d = \max(0, S_d - X)$$

See Figure 8.2 for illustration.

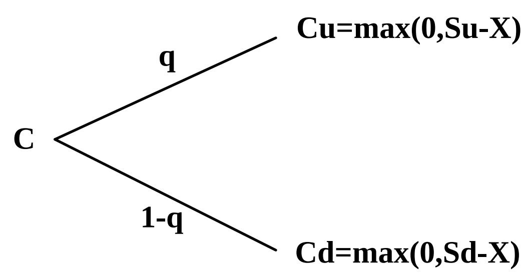

Figure 2.5: Value of Call In Binomial Option Pricing Model

Now, set up a portfolio of h shares of stock and B in riskless bonds. This costs $hS + B$. The value of this portfolio in the next period is depicted in Figure 2.6. Now we take the key step in choosing h and B such that the portfolio has the same payoff as the call option, that is,

$$hSu + RB = C_u \text{ and } hSd + RB = C_d$$

Solve the above equations to get

$$h = \frac{C_u - C_d}{(u-d)S} \geq 0 \tag{2.1}$$

$$B = \frac{uC_d - dC_u}{(u-d)R} \tag{2.2}$$

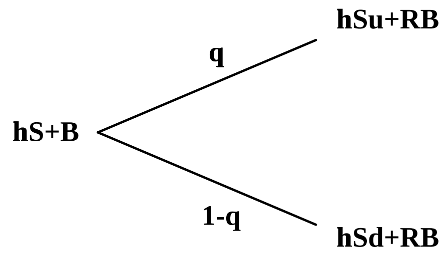

Figure 2.6: Value of Portfolio In One Period

Hence, an equivalent portfolio that replicates the call's payoff next period has been created. Note that q is not involved at all. It is not necessary to specify the underlying asset's expected return, $qSu+(1-q)Sd$. Instead, we employ the equivalent portfolio to price the option relative to the price of the underlying asset. The expected return therefore has only indirect influence on the option value through S, u and d .

By the arbitrage principle, the equivalent portfolio should cost the same as the call if the call is not exercised immediately. Since

$$uC_d - dC_u = \max(0, Sdu - Xu) - \max(0, Sud - Xd) < 0$$

the portfolio is a levered long position in stocks. We sometimes call h the *hedge ratio* or the *delta* of the option.

After substitution and rearrange, we have

$$hS + B = \frac{(\frac{R-d}{u-d})C_u + (\frac{u-R}{u-d})C_d}{R} \tag{2.3}$$

So, clearly, $hS + B \geq 0$. Eq (2.3) can be further simplified as

$$hS + B = \frac{pC_u + (1-p)C_d}{R}$$

where

$$p = (R-d)/(u-d) \text{ and } 1-p = (u-R)/(u-d)$$

We have replicated the call option as a levered *long* position in stocks–with one exception. That is, a call option, if it is American, can be exercised immediately. In contrast, the equivalent portfolio mirrors the call's payoff if the option is not exercised now. If $hS + B \geq S - X$, then the call will not be exercised immediately; thus $C = hS + B$ due to our construction. On the other hand, if $hS + B < S - X$, then the option should be exercised immediately, for we can take the proceeds $S - X$ to buy the equivalent portfolio plus some more bonds. Hence the call option is worth $S - X$. We thus have shown that

$$C = \max(hS + B, S - X) \tag{2.4}$$

In the case of European options, early exercise is not possible, hence

$$C = hS + B$$

In the case of American calls on stocks that do not pay dividends, have been proven that early exercise is not optimal; hence $C = hS + B$ should hold as well. As a result, eq (2.4) is simplified to

$$C = hS + B \tag{2.5}$$

for both European calls and American calls on stocks that pay no dividends.

We now proceed to consider a call with two periods remaining before expiration. We shall move *backward* in time in order to derive the call value. Under the binomial model, the stock can take on three possible price after two periods, Suu, Sud, Sdd. See Figure 2.7.

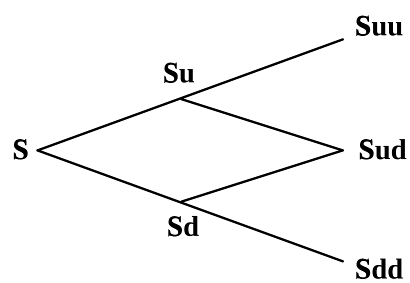

Figure 2.7: Stock Prices In Two Periods

Here, we pause to emphasize one salient feature of the stock prices. At any moment in time, the next two stock prices only depend on the current price, not prices of earlier times. This *Markovian* property is typically taken for granted for most work in this area and is the key feature of an efficient market, an original idea due to Bachelier. In the terminology of probability, we may say the stock price is taking a random walk.

Let C_{uu} be the call's value two periods from now if the stock price moves to Suu,

$$C_{uu} = \max(0, Suu - X)$$

C_{ud} and C_{dd} can be defined analogously, as

$$C_{us} = \max(0, Sud - X) \text{ and } C_{dd} = \max(0, Sdd - X)$$

See Figure 2.8 for illustration. Applying the same logic as lead to eq (2.5), we obtain the call value at the end of the current periods as

$$C_u = \frac{pC_{uu} + (1-p)C_{ud}}{R}$$

$$C_u = \frac{pC_{ud} + (1-p)C_{dd}}{R}$$

Denote the hedge ratios as h_u and h_d if the current stock price is Su and Sd, respectively. Such ratios can be derived from (2.1)–(2.2).

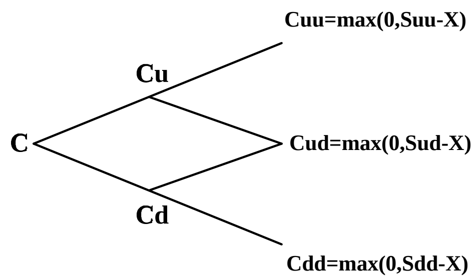

Figure 2.8: Value of Call Prior To Expiration

We now reach the current period. An equivalent portfolio of h shares of stock and B in riskless bonds can be set up for the call that costs C_u if the stock price goes to Su and C_d if the stock price goes to Sd. The values of h and B can be derived from (2.1)–(2.2). Since the hedge ratio in the current period h may not be the same as the hedge ratio in the following period, h_u or h_d, the maintenance of an equivalent portfolio is a dynamic process. By the construction, the value of the portfolio at the end of the current period, C_u or C_d, is exactly the amount needed to set up the next portfolio; it is the proportion in risky stocks that changes. This trading strategy is *self-financing* as there is neither injection nor withdrawal of funds over the time horizon. In other word, changes in portfolio values are due entirely to capital gains.

Since the option will not be exercised one period from now, $C_u > Su - X$ and $C_d > Sd - X$, and therefore

$$hs + B = \frac{pC_u + (1-p)C_d}{R} > \frac{(pu + (1-p)d)S - X}{R} = S - (X/R) > S - X$$

Hence, the call will also not be exercised in the current period even if it is American, and

$$C = hS + B = \frac{pC_u + (1-p)C_d}{R} \tag{2.6}$$

From eq (2.6) above and the formula for C_u and C_d

$$C = [p^2 C_{uu} + 2p(1-p)C_{ud} + (1-p)^2 C_{dd}]/R^2$$

$$= [p^2 \max(0, Su^2 - X) + 2p(1-p)\max(0, Sud - X) + (1-p)^2 \max(0, Sd^2 - X)]/R^2$$

The above formula can be extended to the general case with n periods to expiration, as

$$C = \frac{\sum_{j=0}^{n} b(j;n,p)\max(0, Su^j d^{n-j} - X)}{R^n}$$

which says the value of a call on a non-dividend-paying stock is the expectation of the discounted value of the payoff at expiration in a risk-neutral economy. This is the only option value consistent with there being no arbitrage opportunities in the future. Option values thus derived are often called *arbitrage values.* A similar argument can be employed to show that the value of a European put is

$$P = \frac{\sum_{j=0}^{n} b(j;n,p) max(0, X - Su^j d^{n-j})}{R^n}$$

2.3 The Black-Scholes Formula

The binomial option pricing model on the surface suffers from two unrealistic assumptions: that the tock price only takes on two possible values in a period and that trading takes place at discrete intervals. Such objections are more apparent than real because we can shorten the elapsed time of a period. As the number of periods from now to the expiration date increases, the stock price ranges over larger numbers of possible values during any fixed time interval, and trading takes place almost continuously. What remains to be done is to achieve it with proper calibration of the various parameters in the binomial option pricing model so that the result makes sense as a period takes ever shorter time. In the end, the celebrated Black-Scholes formula emerges. The derivation of this formula is quite complicated and tedious; therefore, we will omit the proof.

2.3.1 Assumptions

The assumptions used to derived the Black-Scholes differential equation are as follows:

1.The stock price follows the *log-normal* distribution. Log-normal distribution is a convenient and realistic characterization of stock prices, because it reflects stockholders' limited liability.

2.The rate of return on stock, μ, and the volatility of stock price, σ, are constant throughout the option's life.

3.The short selling of securities with full use of proceeds is permitted.

4.There are no transaction costs or taxes. All securities are perfectly divisible.

5.There are no dividends during the life of the derivative securities.

6.There are no riskless arbitrage opportunities.

7.Security trading is continuous.

8.The risk-free rate of interest, r, is constant and the same for all maturities.

2.3.2 The Black-Scholes Differential Equation

The derivation of the differential equation is quite complex. Thus, we omit the math here. We just present the final result as follow:

$$\frac{\partial f}{\partial t} + r_f S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = r_f f$$

where f is the price of a derivative security, S is the stock price, σ is the volatility on stock price, and r_f is the continuously compounded risk-free rate.

2.3.3 The Black-Scholes Formula

In their pathbreaking paper, Black and Scholes succeeded in solving their differential equation to obtain exact formulas for the prices of European call and put options. These formulas are presented below,

$$C = SN(d_1) - Xe^{-r_f T} N(d_2)$$

$$P = Xe^{-r_f T} N(-d_2) - SN(-d_1)$$

where

$$d_1 = \frac{\ln(S/X) + (r_f + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S/X) + (r_f - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

$N(x)$ = cumulative normal probability.

σ^2 = annualized variance of the continuously compounded return on the stock.

r_f = continuously compounded risk-free rate.

T = the term to maturity.

Chapter 3

Asian Option Pricing

In this chapter, we introduce two general pricing methods for Asian options, and we use Monte-carlo simulation as a benchmark to examine these two methods. First, we describe what Asian option is. In the second section, we introduce the Hull-White method for Asian options and show this model has some defects. After the Hull-White method, we introduce Levy's approximation formula, and show that it usually over-prices. At the end of this chapter, we introduce a European Asian option put-call parity. Using this put-call parity we can immediate get the call or put value if we know the other option's value.

3.1 What are Asian Options

Asian options are options whose the payoff depends on the average price of the underlying asset during at least some part of the option. The payoff from an *average price call* is $\max(0, S_{ave} - X)$ and that from an *average price put* is $\max(0, X - S_{ave})$, where S_{ave} is the average value of the underlying asset calculated over a predetermined averaging period. Average price options are less expensive than regular options and are arguably more appropriate than regular options for meeting some of the needs of corporate treasurers. Average options, by their design, reduce the significance of the closing price at the maturity of the option. This reduces the effects of any possible abnormal price movements at the maturity of the option.

Another type of Asian option is an average strike option. An *average strike call* pays off $\max(0, S - S_{ave})$, while an *average strike put* pays off $\max(0, S_{ave} - S)$. Average strike options can guarantee that the average price paid for an asset in frequent trading over a period of time is not greater than the final price. Alternatively, it can guarantee that the average price received for an asset in frequent trading over a period of time is not less than the final price.

Consider a call option of maturity T, written on the average of the past n stock prices. It is initially assumed that the maturity of the option is greater than the averaging period ($T > n$).

Let $A(t)$ denote the average at time t, defined by

$$A(t) \equiv \frac{[S(t) + S(t-1) + \cdots + S(t-(n-1))]}{n}$$

where $S(t)$ is the stock price at time t. Let $C(A(t); T, X, n)$ denote the value of a European call option written on the average. The option matures at time T; X is the exercise price; n is the number of prices included in the average. At maturity, the value of the Asian call option is

$$C(A(t); T, X, n) = \max(A(T) - X, 0)$$

So, we can conclude that the value of the Asian call option is

$$C(A(t); T, X, n) = e^{-r(T-t)} E[A(T) - X | A(T) > X]$$

3.2 The Hull-White method

3.2.1 The First Extension of the CRR model

The principle of risk-neutral valuation shows that the value of the derivative security is independent of the risk preferences of investors. This means that we may, with impunity, assume that the world is risk-neutral. We suppose that the process followed by S in a risk-neutral world is geometric Brownian motion:

$$dS = \mu S dt + \sigma S dz$$

where μ, the drift rate, and σ, the volatility, are constants. This process can be represented in the form of a Cox, Ross, and Rubinstein (1979) binomial tree, where the life of the option is divided into n time steps of length $\Delta t = T/n$). In time Δt the asset price moves up by a proportional amount u with probability p and down by a proportional amount d with probability $1 - p$, where

$$u = e^{\sigma\sqrt{\Delta t}}, d = \frac{1}{u}$$

$$a = e^{\mu \Delta t}, p = \frac{a-d}{u-d}$$

In general there are $i + 1$ nodes at time $i\Delta t$ in a tree such as Figure 3.1. We will denote the lowest node at time $i\Delta t$ by $(i, 0)$, the second lowest by $(i, 1)$, and so on. The value of S at node (i, j) is $S(0)u^j d^{i-j} (j = 0, 1, \cdots, i)$. If we were valuing a regular option, we would work back from the end of the tree to the beginning, calculating a single option value at each node. To value a path-dependent option, one approach is to value the option at each node for all alternative values of the path function $F(t, S)$

that can occur. The path function is the function of the path followed by S between time zero and time t that underlies the price of the derivative security. There are two requirements for the model to be feasible:

1. It must be possible to compute $F(t+\Delta t, S)$ from $F(t, S)$ and $S(t+\Delta t)$. This means that the path function is Markov.

2. The number of alternative values of $F(t, S)$ must not grow too fast with the size of the tree.

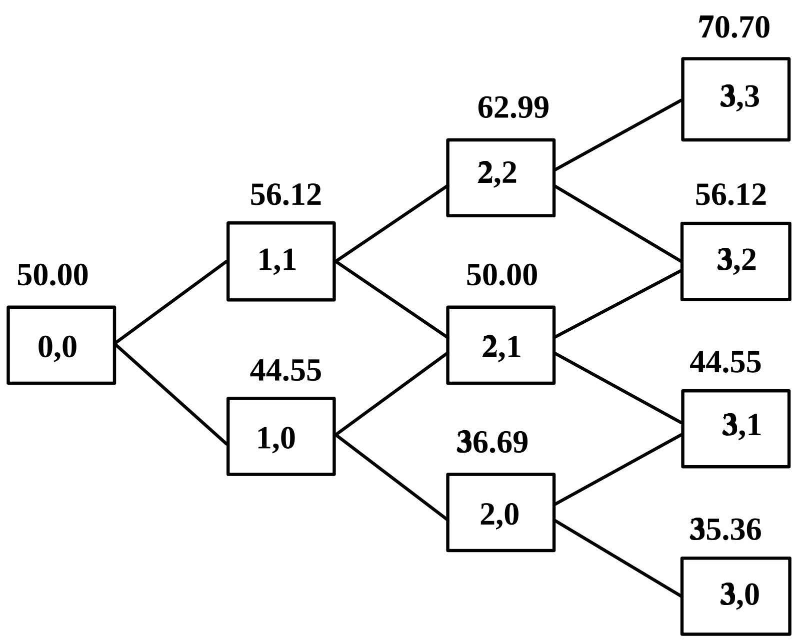

Figure 3.1: The CRR binomial tree for stock price movements

We will denote the kth values of F at node (i, j) by $F_{i,j,k}$ and define $v_{i,j,k}$ as the value of the security at node (i, j) when F has this value. The value of the derivative security at its maturity, $v_{n,j,k}$, is known for all j and all k. To calculate its value at node (i, j) where $i < n$, we note that the stock price has a probability p of moving up to node $(i+1, j+1)$ and a probability $1-p$ of moving down to node $(i+1, j)$.

We suppose that the kth value of F at node (i, j) leads to the k_uth value of F at node $(i+1, j+1)$ when there is an up-movement in the stock price and to the k_dth value of F at node $(i+1, j)$ when there is a down-movement in stock price. For a European-style derivative security this means that

$$v_{i,j,k} = e^{-r\Delta t}[pv_{i+1,j+1,k_u} + (1-p)v_{i+1,j,k_d}] \tag{3.1}$$

If the derivative can be exercised at node (i, j), the value in equation (3.1) must be compared with the early exercised value and $v_{i,j,k}$ must be set equal to the greater of the two.

3.2.2 The Hull-White Extension of CRR

The approach described above is computationally feasible when the number of alternative F-values at each nodes does not grow too fast as n, the number of time steps, is increased. An option on the arithmetic average would be very difficult to value using this approach, because the number of alternative arithmetic average that can be realized at a node grows very fast with n.

An extension to the approach that places no constraints on the number of F-values involves computing $v(S, F, t)$ at a node only for certain predetermined values of F, not all of those that can occur. The value of $v(S, F, t)$ for other values of F is computed from the known values by interpolation as required.

We illustrate this approach by using it to calculate the prices of European and American options on the arithmetic average of the stock price. In this case, F at node is defined as the arithmetic average of the asset prices from time zero to the node.

Hull and White provide two methods to choose the value of F. In his book *Options, Futures, and Other Derivatives* , They use the values of F that are equally spaced between the maximum and the minimum at each node. The other is to choose the values of F which has the form $S(0)e^{mh}$, where h is a constant and m is a positive or negative integer. The value of F considered at time $i\Delta t$ must span the full range of possible F's at that time. This is determined by inspection, using forward induction.

3.2.3 Examples of Calculation

Equally spaced F

Figure 3.2 shows the calculations that would be carried out in one small part of tree which uses equally spaced F. The stock price has a 0.5 probability of moving from a node X where the stock price is 50.00 to node Y where it is 54.46, and a 0.5 chance of moving from node X to node Z where the stock price is 45.72 . Node X is the central node at time 0.2 year (at the end of the fourth time step). Nodes Y and Z are the two nodes at time 0.25 year that can be reached from node X. The stock price at node X is 50. Forward induction shows that the maximum average stock price that is achievable in reaching node X is 53.83. The minimum is 46.65. From node X we branch to one of the two nodes, Y and Z. At node Y the stock price is 54.46 and the bounds for the average are 47.99 and 57.39. At node Z the stock price is 45.72 and the bounds for the average stock price are 43.88 and 52.48.

We have chosen the representative values of the average to be four equally spaced values at each node. We assume that backward induction has already been used to calculate the values of the option for each of the alternative values of the average at nodes Y and Z. Consider the calculations at node X for the case when average is

51.44. If the stock price moves up to node Y, the new average will be

$$\frac{5 \times 51.44 + 54.46}{6} = 51.98$$

The value of the derivative at node Y for this average can be found by interpolating between the values when the average is 51.12 and when it is 54.26. It is

$$\frac{(51.98 - 51.12) \times 8.635 + (54.26 - 51.98) \times 8.101}{54.26 - 51.12} = 8.247$$

Similarly, if the stock price moves down to node Z, the new average will be

$$\frac{5 \times 51.44 + 45.72}{6} = 50.49$$

and by interpolation the value of the derivative is 4.182. The value of the derivative at node X when the average is 51.44 is therefore

$$(0.5 \times 8.247 + 0.5 \times 4.182)e^{-0.1 \times 0.005} = 6.206$$

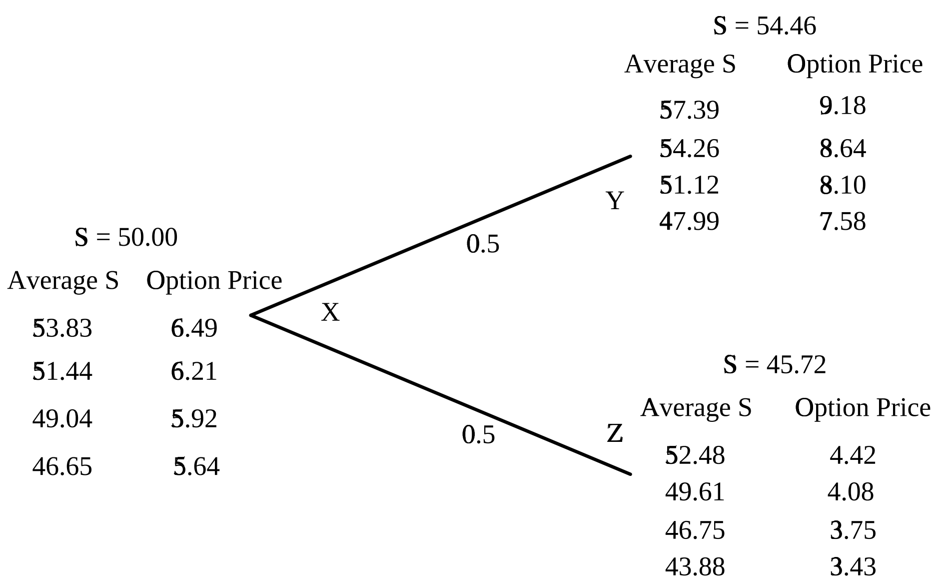

Figure 3.2: The Hull-White method with equally spaced F

F has the $S(0)e^{mh}$ Form

Figure 3.3 illustrates the calculation by supposing that F has the $S(0)e^{mh}$ form. The stock price has a 0.5 probability of moving from a node X where the stock price is 40 to node Y where it is 44, and a 0.5 chance of moving from node X to node Z where the stock price is 36.36 . In this example $h = 0.08$; the values of F considered at node X are 36.92, 40.00, and 43.33; and the values of F considered at nodes Y and Z are 34.09, 36.92, 40.00, 43.33,and 46.94.

We suppose that the values of v corresponding to these values of F are 0.10, 0.90, 1.80, 3.00, and 4.60 at node Y, and 0.01, 0.50, 1.10, 1.80, and 2.80 at node Z. We also assume that the average at node X are calculated over two time steps, that each time step is three months, and that the risk-free interest rate is 0.1 per annum.

At each node we consider certain predetermined values of the average. The upper number at each node shows the stock price; the middle numbers are the values of the average considered; the lower numbers are the values of the option. Node X is assumed to be at time $2\Delta t$; each time step is 3 months; and the probability of an up or down movement is 0.5.

Example of calculation: Consider node X when the average is 43.33. There is a 0.5 probability of moving up to node Y, where the average becomes 43.50. Using linear interpolation, the value of the option is then 3.08. There is a 0.5 probability of moving down to node Z, where the average becomes 41.59, and using linear interpolation the value of the option is 1.43. The value of of the option at node X when the average is 43.33 is therefore $(0.5 \times 3.08 + 0.5 \times 1.43)e^{-0.25\times 0.1} = 2.20$

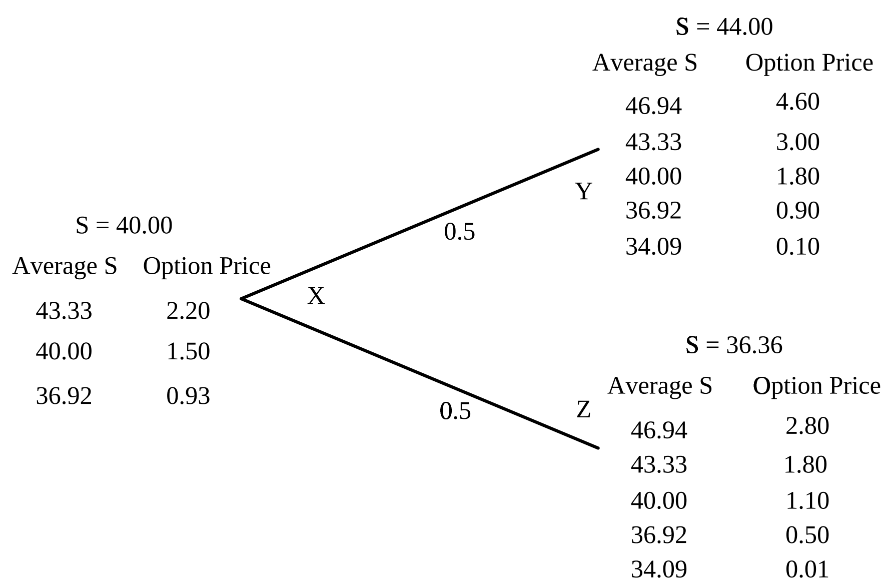

Figure 3.3: The Hull-White Method with F in the $S(0)e^{mh}$ Form

3.2.4 Numerical Investigation

First, we test the Hull-White algorithm of equally spaced F for n from 30 to 200. We compare the results with Monte-Carlo simulation, and find that Hull-White method with equally spaced F will not converge as n increases, see Table 3.1 and Figure 3.4.

Then we test the Hull-White's algorithm where F have the $S(0)e^{em}$ form. Comparing the results with Monte-Carlo simulation, we find that this Hull-White method will converge to a value that is larger than the correct value as n increase, see Table 3.2 and Figure3.5. This Hull-White method does not produce a result in the 95% confidence interval when $n < 40$, and this results first go within the confidence interval at $n = 40$ and then go out of the confidence interval again when $n > 150$. This means the method converges to a value that may has a bias from the correct value.

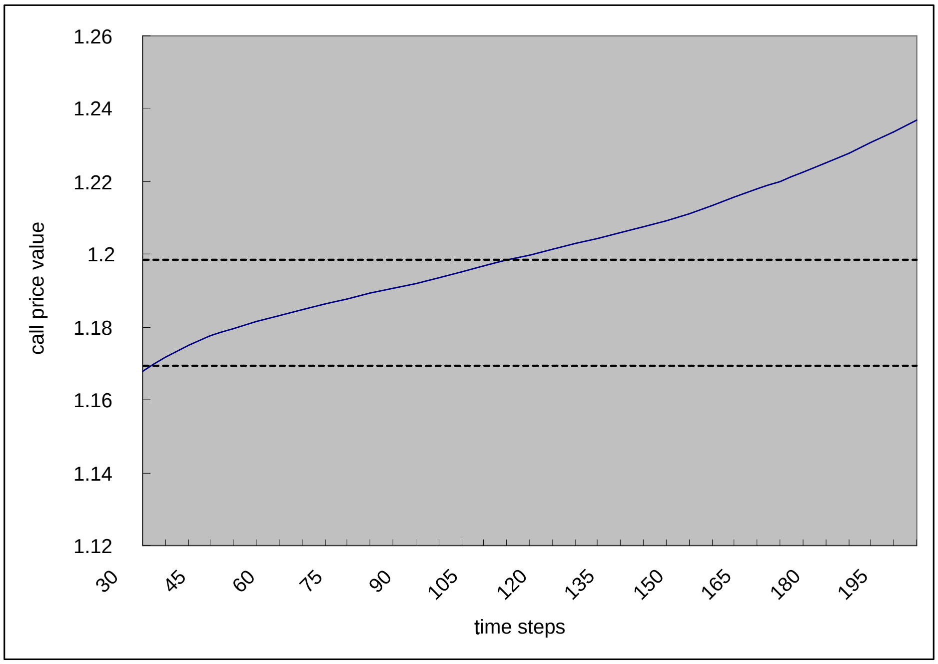

Figure 3.4: Results of the Hull-White method with equally spaced F.
Dashed lines are confidence interval.

3.3 Levy's Approximation Formula

3.3.1 Sum of Log-normals

Sum of log-normal distributions is not a log-normal distribution, but it can be approximated by an alternative distribution assumed to be log-normal [6]. If $f(x)$ denotes the true probability function and $a(x)$ the approximating distribution, where $a(x)$ is a log-normal probability density function, then

$$f(x) = a(x) + \frac{c_2 d^2 a(x)}{2! dx^2} - \frac{c_3 d^3 a(x)}{3! dx^3} + \frac{c_4 d^4 a(x)}{4! dy^4} + e(x)$$

Table 3.1: The Hull-White method with equally spaced F

time steps	option value
$n = 30$	call price=1.167806
$n = 35$	call price=1.171866
$n = 40$	call price=1.174938
$n = 45$	call price=1.177552
$n = 50$	call price=1.179633
$n = 55$	call price=1.181603
$n = 60$	call price=1.183307
$n = 65$	call price=1.184875
$n = 70$	call price=1.186455
$n = 75$	call price=1.187850
$n = 80$	call price=1.189318
$n = 85$	call price=1.190633
$n = 90$	call price=1.192105
$n = 95$	call price=1.193432
$n = 100$	call price=1.195172
$n = 105$	call price=1.196724
$n = 110$	call price=1.198359
$n = 115$	call price=1.199647
$n = 120$	call price=1.201345
$n = 125$	call price=1.202894
$n = 130$	call price=1.204331
$n = 135$	call price=1.205828
$n = 140$	call price=1.207445
$n = 145$	call price=1.209276
$n = 150$	call price=1.211139
$n = 155$	call price=1.213283
$n = 160$	call price=1.215661
$n = 165$	call price=1.217866
$n = 170$	call price=1.219982
$n = 175$	call price=1.222634
$n = 180$	call price=1.225237
$n = 185$	call price=1.227873
$n = 190$	call price=1.230668
$n = 195$	call price=1.233476
$n = 200$	call price=1.236791

stock price = 50, strike price = 60
maturity = 1.0 year, interest rate= 10% per year
volatility = 30% per year
Monte Carlo simulation = 1.185, standard error = 0.007
(number of partitions=50, number of replications=100000)

Table 3.2: The Hull-White method with F in the S(0)e^{mh} form

time steps	option value
$n = 30$	call price=0.312517
$n = 35$	call price=0.315245
$n = 40$	call price=0.317380
$n = 45$	call price=0.319077
$n = 50$	call price=0.320486
$n = 55$	call price=0.321711
$n = 60$	call price=0.322773
$n = 65$	call price=0.323709
$n = 70$	call price=0.324556
$n = 75$	call price=0.325318
$n = 80$	call price=0.326003
$n = 85$	call price=0.326623
$n = 90$	call price=0.327188
$n = 95$	call price=0.327704
$n = 100$	call price=0.328172
$n = 105$	call price=0.328597
$n = 110$	call price=0.328998
$n = 115$	call price=0.329347
$n = 120$	call price=0.329678
$n = 125$	call price=0.329983
$n = 130$	call price=0.330317
$n = 135$	call price=0.330525
$n = 140$	call price=0.330767
$n = 145$	call price=0.330995
$n = 150$	call price=0.331207
$n = 155$	call price=0.331404
$n = 160$	call price=0.331589
$n = 165$	call price=0.331764
$n = 170$	call price=0.331928
$n = 175$	call price=0.332084
$n = 180$	call price=0.332231
$n = 185$	call price=0.332369
$n = 190$	call price=0.332499
$n = 195$	call price=0.332624
$n = 200$	call price=0.332741

stock price = 50, strike price = 60
maturity = 0.5 year, interest rate= 10% per year
volatility = 30% per year
Monte Carlo simulation = 0.324, standard error = 0.003
(number of partitions=50, number of replications=100000)

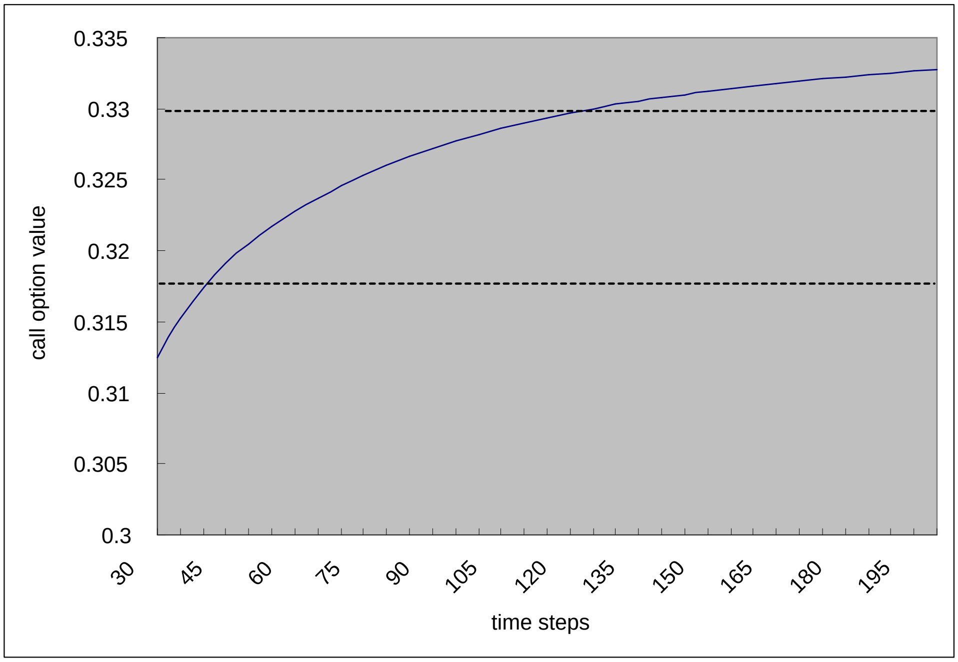

Figure 3.5: Results of the Hull-White method with F in the S(0)e^{mh} form. Dashed lines are confidence interval.

where $c_2 \equiv \chi_2(F)-\chi_2(A)$; $c_3 \equiv \chi_3(F)-\chi_3(A)$; $c_4 \equiv \chi_4(F)-\chi_4(A)+3c_2^2$; $\chi_j(F)[\chi_j(A)]$ is the jth cumulant of the exact [approximating] distribution [1]; and $e(x)$ is a residual error term. If a random variable X has a cumulative distribution function F, the first four cumulants are

$$\chi_1(F) = E(X)$$
$$\chi_2(F) = E[X - E(X)]^2$$
$$\chi_3(F) = E[X - E(X)]^3$$
$$\chi_4(F) = E[X - E(X)]^4 - 3E[X - E(X)]^2$$

where all expectations are with respect to the distribution F. The first two moments of the approximating distribution have been set equal to the first two moments of the exact distribution. The moments of a random variable X with respect to the $a(x)$ distribution are given by

$$E(X^m) = e^{\mu m+\frac{\sigma^2}{2}m^2}, m = 1, 2, \ldots$$

3.3.2 Pricing Formula

In Levy's approximation formula, we use an approximation to sum of log-normal distributions, because a large body of evidence suggests that distribution of such sums is well-approximated by another log-normal distribution. So, we use the approximating log-normal distribution to calculate the option value. We can get the mean and variance of the approximating log-normal distribution. By the mean and variance, and using Black-Scholes formula we can calculate the approximating value of the European Asian call and put option values. The approximating formula follows [2]:

notation:

S : spot price
S_a: past arithmetic price average
X : strike price
T : original time to maturity
T_2: remaining time to maturity
r : risk-free interest rate
b : cost of carry
σ: standard deviation

Then an European average call option price C is:

$$C = SeN(d_1) - Xe^{-rT2}N(d_2)$$

where

$$S_e = \frac{S}{Tb}e^{(b-r)T_2} - e^{-rT_2}$$

$$d_1 = \frac{\frac{\ln(d)}{2} - \ln(x)}{\sqrt{\sigma}}, d_2 = d_1 - \sqrt{v}$$

$$v = \ln(d) - 2(rT_2 + \ln(S_e))$$

$$x = X - \frac{T - T_2}{TS_a}, d = \frac{m}{T^2}$$

$$m = \frac{2S^2}{b + \sigma^2} \times \frac{e^{2b+\sigma^2)T_2} - 1}{2b + \sigma^2 - \frac{e^{bT_2 - 1}}{b}}$$

3.3.3 Numerical Investigation

We can find many cases calculations by Levy's formula are not accurate, see Table 3.3. If the strike price decreases or the maturity increases, the value of the European call will be over-priced, and it will go out of the confidence interval. For example, when strike price = 100 and maturity = 1.0 year, Levy's formula gets a call option value of 4.557 that is out out the confidence interval ($4.557 > 4.515 + 2 \times (0.010) = 4.535$). Although Levy's approximation formula is very fast and can value an Asian option in $O(1)$ time, but it's error is not tolerable.

Table 3.3: Levy's model

maturity	**strike price**=90		**strike price**=100		**strike price**=110	
	MC	Levy	MC	Levy	MC	Levy
0.5	6.359	6.386	2.998	3.024	1.112	1.106
	(0.005)		(0.007)		(0.005)	
1.0	7.606	7.662	4.515	4.557	2.401	2.431
	(0.008)		(0.010)		(0.009)	
1.5	8.671	8.738	5.734	5.801	3.577	3.619
	(0.010)		(0.012)		(0.012)	

stock price=100, volatility = 20% per year
interest rate= 10% per year, Levy is Levy's approximation
MC is Monte Carlo simulation:
number of partitions=50, number of replications=100000
The standard errors of the Monte-Carlo simulation are shown in paretheses

3.4 Asian Option Put-Call Parity

By the binomial tree model, each node at maturity have $\binom{n}{i}$ paths, where n is time steps and i is the ith node at maturity. Assume Z_i is the collection of paths that end ith node at maturity. We define $z \in Z_i, S_{iz} = \frac{1}{n+1}\sum_{j=0}^{n} S(j,z)$ where $S(j,z)$ means path z's jth stock price. Then $Fsum_i$ can be defined follow:

$$Fsum_i = \sum_{\forall z \in Z_i} \frac{S_{iz}}{\binom{n}{i}}$$

The expected value of an Asian call option C at maturity is

$$C = E[Fsum_i - X | Fsum_i > X]$$

similarly, the expected value of a put P at maturity is

$$P = E[X - Fsum_i | Fsum_i < X]$$

where X is strike price. So we can conclude follow:

$$C - P = E[Fsum_i - X] = E[Fsum_i] - X$$

Now, we should calculate the expected value of $Fsum_i$. Because $E[Fsum_i] = \frac{1}{n+1}E[S(0)+ S(1) + S(2) + \cdots + S(n)]$, where $S(j)$ is the price at time j, by the principle of risk-neutrality we conclude:

$$E[Fsum_i] = \frac{1}{n+1}(E[S(0)] + E[S(1)] + E[S(2)] + \cdots + Exp[S(n)])$$

$$= \frac{S(0)}{n+1} \times \frac{\Delta r^{n+1} - 1}{\Delta r - 1}$$

Where r is the risk-free rate, , n is the number of time steps, and $\Delta(r) = \frac{r}{n}$. By this put-call parity, we can immediately get a call or a put value if we have known the other option.

Chapter 4

Conclusion

This thesis describes the popular Hull-White method and Levy's approximation formula. We also examine these methods' efficiency and introduce put-call parity for the Asian options.

Examining these two methods' result, we find some problems with both. The results of Hull-White method will not converge when each node uses equally spaced values F or converge to a value that is out of the confidence interval when each node use the $S(0)e^{mh}$ form F values. The results of Levy's approximation formula are out of the confidence interval in many cases.

Finally, by the Asian option put-call parity we can immediately get an Asian call or put value, if we know the other. We can also use it to see if the call or put value calculated by some methods satisfies this put-call parity.

Bibliography

[1] J. Hull, and A. White, "Efficient Procedures for Valuing European and American Path-Dependent Options." *Journal of Derivatives*, Fall 1993, pp. 21-31.

[2] E. Levy, "Pricing European Average Rate Currency Options." *Journal of International Money and Finance*, 11 (1992), pp. 474-491.

[3] S.M. Turnbull, and L.M. Wakeman, "A Quick Algorithm for Pricing European Average Options." *Journal of Financial and Quantitative Analysis*, 26 (September 1991), pp. 377-389.

[4] Yun-Dauh Lyuu, *Introduction to Financial Computation: Principles, Mathematics, Algorithms.*, Manuscipts, 1995-1999.

[5] John C. Hull, *Options, Futures, and Other Derivative Securities.* 3rd ed. Englewood Cliffs, New Jersey: Prentice-Hall.

[6] R.L. Mitchell, "Permanence of the Log-Normal Distribution." *Journal of the Optical Society of American*, America, 58 (Sept. 1968), pp. 1267-1272.

[7] C. G. Rogers. and Z. Shi. "The Value of an Asian Option." *J. Appl. Prob.*, 32 (1995), pp 1077-1088.

[8] A.G.Z Kemna and A.C.F Vorst, "A Pricing Method for Options Based on Average Asset Values." *Journal of Banking and Finance*, 14(1990), pp. 113-129.

[9] Peter Ritchken, L. Sankarasubramanian and Anand M. Vijh. "The Valuation of Path Dependent Contracts on Average." *Financial Management*, Autumn 1991, pp. 35-41.