Numerical Methods for Model Calibration under Credit Risk

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Abstract

Interest rate derivatives are instruments whose payoffs depend in some way on interest rates. To price them, it involves constructing a model to describe the probabilistic behavior of interest rates.

When valuing a derivative, it is customary to assume that there is no risk of default. However, the no-default assumption is not defensible, especially in over-the-counter markets. So dealing with credit risk issues has become more and more important.

This thesis is concerned with the above two topics: calibrating interest rate models under credit risk. We first use the yields of default-free and risky zeros to calculate the probabilities of default, then use the prices of risky zeros and options on risky bonds to calibrate a tree of possible future short rates. With the help of forward induction, the calibration process can be done efficiently. This tree can then be used to value interest-rate-sensitive securities involving credit risk.

Chapter 1

Introduction

1.1 Overview

Interest rate derivatives are instruments whose payoffs are dependent in some way on interest rates. In the 1980s and early 1990s the volume of trading in interest rate derivatives in both over-the-counter and exchange-traded markets increased rapidly. To price them, it involves constructing what is known as a yield curve model or term structure model. This is a model that describes the probabilistic behavior of interest rates.

When valuing a derivative, it is customary to assume that there is no risk of default. Unfortunately, the no-default assumption may not be defensible, especially in over-the-counter markets. In recent years, this market has become increasingly important so that dealing with credit risk issues has become an exciting and new area of research.

Our main concern, model calibration, is the inverse of the pricing problem: Given price information, it finds parameter values on the tree that generate model prices consistent with the market data. Calibration, therefore, can be perceived as a root-finding problem.

In this thesis, we try to cope with the above topics integrally. This means calibrating interest rate models under credit risk. We adopt the Jarrow-Turnbull (1995) model which concerns the credit risk in our modeling paradigm, then take the current long rates of default-free and risky bonds and the prices of options on risky bonds as inputs to construct a tree of interest rates consistent with these market data. With the help of forward induction, the calibration process can be done efficiently. This calibrated tree can then be used to value interest-rate-sensitive securities subject to default risk.

1.2 Organization of this Thesis

This thesis is organized as follows. In Chapter 2, an introduction to interest rate modeling and the Black-Derman-Toy (1990) model is presented. In Chapter 3, we introduce the concept of credit risk and the Jarrow-Turnbull (1995) model used in our thesis. In Chapter 4, we describe our numerical methods for model calibration. Forward induction is also presented in its generalities here. In Chapter 5, some experimental data are provided. Chapter 6 ends this thesis with conclusions.

Chapter 2

Introduction to Interest Rate Modeling and the Black-Derman-Toy Model

2.1 Introduction to Interest Rate Modeling

Interest rate models are usually known as term structure models or yield curve models. A term structure model describes the probabilistic behavior of interest rates. In general, term structure models are more complicated than models used to describe stock prices or exchange rates. This is not surprising because what is at issue is the movements in an entire yield curve—not with changes to a single variable. The individual interest rates in the term structure always change as time passes. Besides, the shape of the curve itself is liable to change. Interest rate models can generally be classified into equilibrium models and no-arbitrage models. In an equilibrium model the initial term structure is an output from the model; in a no-arbitrage model it is an input to the model. In the following, we will go through both kinds of models.

2.1.1 Introduction to Equilibrium Models

Equilibrium models usually derive a process for the short-term risk-free rate, r, with assumptions about economic variables. The short rate, r, at time t is the rate that applies to a very short period of time at time t. It is often referred to as the instantaneous short rate. It can be used to evaluate bond prices, option prices, and other derivative prices that depend on interest rates.

Equilibrium models include one-factor models and multi-factor models. One-factor models such as the Vasicek (1977) model and the Cox-Ingersoll-Ross (1985) model are driven by a single source of uncertainty—the short rate. A number of researchers also have investigated multi-factor models.

2.1.2 Introduction to No-Arbitrage Models

The equilibrium models mentioned above have a disadvantage that they do not fit today's yield curve. In contrast to equilibrium models that take the initial term structure as the output, no-arbitrage models take the initial term structure as input. So they are models designed to be consistent with today's term structure.

Ho and Lee proposed the first no-arbitrage model for the term structure in 1986. Thereafter, many no-arbitrage models were developed. The Hull-White (1990) model, the Black-Derman-Toy (1990) model and the Heath-Jarrow-Merton (1992) model are examples of well-known no-arbitrage models.

2.2 The Binomial Interest Rate Tree

The aim of this section is to construct an interest rate tree consistent with the observed term structure, specifically the yields and/or yield volatilities of zero-coupon bonds of all maturities. Models based on such a paradigm are arbitrage-free or no-arbitrage models which we mentioned in the previous section.

The idea behind the binomial interest rate tree is reminiscent of the binomial option pricing model. A binomial tree is constructed so that the logarithm of future short rate obeys the binomial distribution. In this way, the limiting distribution for the short rate at any time is lognormal much like the binomial option pricing model. The general idea of no-arbitrage models fitting the initial term structure is parallel to the concept of implied volatility for options. In both cases the market data are used to derive the unknown model parameters.

In the binomial interest rate process, a binomial tree of possible short rates for each future period is constructed. Each short rate is followed by two short rates for the following period. In Figure 2.1, node A coincides with the start of period j during which short rate r is in effect. At the conclusion of period j, a new short rate goes into effect for period $j+1$. This may take one of two possible values: r_l, the “low” short rate outcome (up move, for the bond price) art node B, or r_h, the ”high” short rate outcome (down move) at node C. Each of r_l and r_h has a fifty percent chance of occurring, and they are the only possibilities for period $j+1$'s short rate for node A.

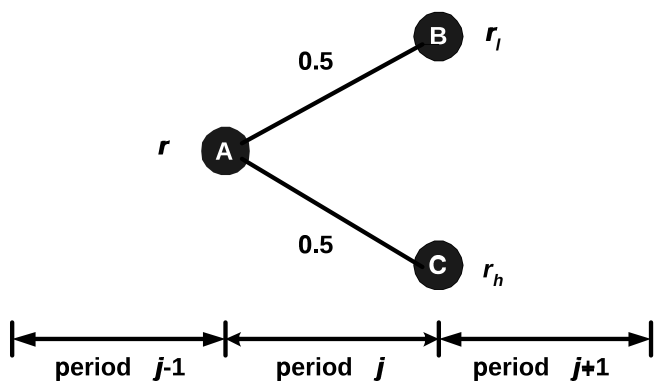

Figure 2.1: BINOMIAL INTEREST RATE PROCESS. From node A, there are two equally likely scenarios for the short rate. Rate r is applicable to node A for period j. Rate r_l is applicable to node B and rate r_h to node C, both for period $j+1$.

As the binomial process unfolds, we make sure that the paths recombine much like the binomial stock price process. Suppose the short rate r can go to r_h and r_l with equal risk-neutral probability 1/2 in a period of length Δt. Percent volatility of short rate, $\Delta r/r$, is

$$\sigma = \frac{1}{2}\frac{1}{\sqrt{\Delta t}}\ln(r_h/r_l)$$

when the short rate follows a lognormal process in the limit. By the lognormal assumption, $\ln r$ follows a Brownian motion process with the same volatility as dr/r. The variance of $\ln r$ for a time period of Δt equals

$$\sigma^2\Delta t = \frac{\ln^2 r_h + \ln^2 r_l}{2} - (\frac{\ln r_h + \ln r_l}{2})^2 = (\frac{\ln r_h - \ln r_l}{2})^2$$

by definition. So $\frac{\ln(r_h/r_l)}{2} = \sigma\sqrt{\Delta t}$. As

$$\frac{r_h}{r_l} = e^{2\sigma\sqrt{\Delta t}} \tag{2.1}$$

greater volatility, hence uncertainty, leads to greater r_h/r_l and wider ranges of possible short rates. Note that the ratio is constant across time if the volatility is constant. To nail down the values of r_h and r_l, we need information from the current term structure to establish the relationship between r and its two successors. Equation (2.1) will serve as the fundamental building block for the binomial interest tree.

As the binomial process unfolds, we make sure that the paths recombine. In general, there are j possible rates applicable for period j. They are

$$r_j, r_j v_j, r_j v_j^2, ..., r_j v_j^{j-1},$$

where

$$v_j = e^{2\sigma_j \sqrt{\Delta t}}$$

is the *multiplicative ratio* for the rates in period j. Note that the volatilities σ_j above are indexed by j because the volatility is a function of time. We shall call r_j the *baseline rate*. Figure 2.2 depicts the tree structure.

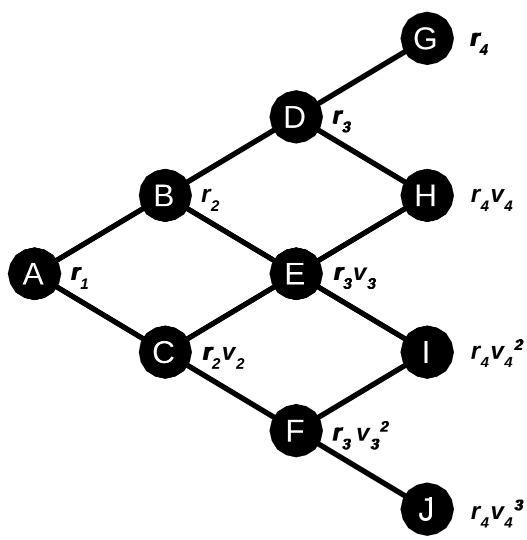

Figure 2.2: BINOMIAL INTEREST RATE TREE. The distribution at any time converges to a lognormal distribution. The baseline rates are $r_1, r_2, r_3, ..., r_n$.

2.3 The Black-Derman-Toy Model

Black, Derman and Toy brought up a no-arbitrage interest rate model in 1991. We adopt their concept and make a change in our method. In this section, we will briefly introduce this model.

2.3.1 Concept of the Model

Black, Derman and Toy assume all security prices and rates are dependent only on one factor—the short rate, then use the current structure of long rates and their estimated volatilities to construct a tree of possible future short rates before using it to value interest rate derivatives.

Take a two-year, zero-coupon bond as a example. It has a known payoff at the end of the second year, no matter what short rates take effect. Its possible prices after one year can be obtained by discounting the payoff by the possible short rates one year out. An iterative process is used to find the rates that will be consistent with current market term structure, and the price today is then determined by discounting the one-year price by the short rate.

The model has three important features:

1. Its fundamental variable is the short rate: the annualized one-period interest rate. The changes of this short rate drive all security prices.

2. The model takes long rates (yields) on zero-coupon Treasury bonds for various maturities and yield volatilities for the same bonds as inputs to form the term structure.

3. The model varies means and volatilities for the future spot rate to match the inputs.

They examined how the the model works in an world in which changes in all bond yields are perfectly correlated; expected returns on all securities over one period are equal; short rates at any time are lognormally distributed; and there are no taxes or trading costs.

2.3.2 Finding Short Rates from the Term Structure

According to the binomial interest rate model, there are j possible short rates, $r_j, r_jv_j, .., r_jv_j^{j-1}$, where r_j is the baseline rate for period j. Suppose there has a j-year zero-coupon bond, its price moves up to P_u and down to P_d one year from today. Obviously, P_u and P_d are functions of r_j and v_j. In a risk-neutral world, it must hold that

$$0.5e^{-r_1}(S_u + S_d) = 100e^{-yj} \tag{2.2}$$

where y is today's yield of the j-year zero and r is today's one-year rate (both are known).

Viewed from now, the future $(j-1)$-year yield at the beginning of year two is uncertain. Let y_u represent the $(j-1)$-year yield to maturity at the "up" node, y_d the $(j-1)$-year yield to maturity at the "down" node. Then y_u and y_d correspond to prices S_u and S_d that:

$$S_{u,d} = 100e^{-y_{u,d}(j-1)}$$

Let k^2 be the variance viewed from now of the $(j-1)$-year yield to maturity. The variance of the (j-1)-year yield is

$$k^2 = p(1-p)\ln^2(\frac{y_u}{y_d}).$$

Hence, for $p = 1/2$,

$$k = (1/2)\ln(\frac{y_u}{y_d}). \tag{2.3}$$

Equations (2.2) and (2.3) are the bases for constructing the interest rate tree.

2.3.3 A Numerical Example

Suppose we are given the market term structure in Table 2.1 and want to find the short rates one year from now by looking at the yield and the volatility for the two-year zero using the term structure of Table 2.1

From the binomial interest rate model mentioned before, we know that there are two possible short rates one year from now. Let's call the two unknown short rates r_h and r_l. Using the valuation formula—equation (2.2)—we can get

Maturity (years)	Yield (%)	Yield Volatility (%)
1	10	20
2	11	19
3	12	18
4	12.5	17

Table 2.1: A Sample Term Structure

$$0.5e^{-0.1}(e^{-r_h} + e^{-r_l}) = e^{-0.11\times 2} \tag{2.4}$$

Then consider the volatility of the two-year spot rate. From equation (2.3):

$$\sigma_2 = \frac{\ln \frac{r_h}{r_l}}{2} = 0.19. \tag{2.5}$$

By solving equations (2.4) and (2.5), we find that r_h is 14.29% and r_d is 9.77%. So an initial short rate of 10% followed by equally probable one-year short rates of 14.29% and 9.77% guarantees that our model matches the first two years of the term structures. The final short rate of the sample term structure is shown in Figure 2.3.

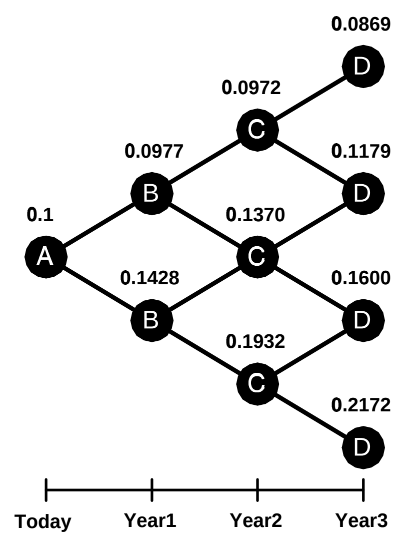

Figure 2.3: Short Rates That Match the Term Structures in Table 2.1.

To go beyond 2 years, we will need more sophisticated methods to find out the short rates. Lyuu (1995) has designed a efficient method for this.

Chapter 3

Introduction to Credit Risk and the Jarrow-Turbull Model

3.1 Introduction to Credit Risk

The pricing and hedging models are often based on the assumption that there is no risk of default. For an exchange-traded option or futures contract, this assumption might be reasonable because most exchanges have well organized trading to ensure that their contracts are always honored. However, it may be far less defensible in over-the-counter markets which have become increasingly important in recent years. Consequently, dealing with credit risk issues has become necessary.

3.1.1 Sources of Credit Risk

In general, two facets of credit risks are considered. The first is where the asset underlying the derivative security may default, paying off less than promised. For example, derivatives written on corporate bonds fit here. There is a positive probability that default may occur on the part of the corporation. The second one is where the writer of the derivative may default. An example is an over-the-counter option written on a Treasury bond. There is no default risk arising from the underlying asset—the Treasury bond. However, default risk arises from the fact that they may not be able to honor the obligation if the option is exercised. Following Johnson and Stulz (1987), we will refer to the second class as vulnerable derivative securities.

3.1.2 Development of Pricing Models

Many researchers have investigated models for pricing derivative securities involving credit risk. In general, previous models can be subdivided into three classes. The first class of models views the firm's liabilities as contingent claims on the firm's underlying assets, with the payoff in bankruptcy fully specified. Bankruptcy is determined via the evolution of the firm's assets [e.g., Chance(1990)]. The second class views risky debt as paying off an exogenously given fraction of each promised dollar when bankruptcy. Bankruptcy is determined when the value of the firm's underlying assets hits some exogenously specified boundary [e.g., Hull and White(1995)]. The third class also views risky debt as paying off a fraction of each promised dollar in the event of bankruptcy; but the time of bankruptcy is now given as an exogenous process [e.g., Jarrow and Turnbull(1995)].

3.2 The Jarrow-Turnbull Model

Jarrow and Turnbull provide a model for pricing and hedging derivatives securities involving credit risk. Two types of credit risk are both considered. They apply the foreign currency analogy to decompose the dollar payoff from a risky security into a certain payoff and a "spot exchange rate". Arbitrage-free valuation techniques are then employed. This model has an important characteristic that it avoids the needs to understand the complex structure of payoffs in bankruptcy and estimate the firms' assets value. We adopt this method in our model because of this characteristic. A brief introduction of this model will come in this section.

3.2.1 The Economy

They consider a frictionless economy with a finite horizon $[0, \tau]$. Trading can be discrete or continuous. What are traded are default-free and risky zero-coupon bonds of all maturities.

Let $p_0(t, T)$ be the time-t dollar value of the default-free zero-coupon bond paying a certain dollar at time $T \geq t$. A money money market can be constructed from this term structure by investing a dollar in the shortest maturity default-free zero coupon bond, and rolling it over at each future date. Let $B(t)$ denote the time-t value of this money market initialized with one dollar at time 0. Suppose a corporation belonging to credit class ABC issues a zero-coupon bond which is subject to default. Let $v_1(t,T)$ denote the time-t dollar value of this zero-coupon bond promising a dollar at date $T \geq t$.

Now, decompose this zero-coupon bond into the product of two hypothetical quantities: a default-free zero-coupon bond denominated in a hypothetical currency called ABC, and a price in dollars of ABCs.

First, we define

$$e_1(t) \equiv v_1(t,t). \tag{3.1}$$

The quantity $e_1(t)$ represents the time-t dollar value of one promised ABC dollar delivered immediately (at time t) and is analogous to a spot exchange rate. If ABC does not default, the exchange rate will be 1 as each promised ABC dollar is actually worth a dollar. However, if ABC does default, then each promised ABC dollar is worth δ dollar which may be worth less than one. One can view the spot exchange rate δ here as a *payoff ratio* or *recovery rate* in default.

Consider a hypothetical, ABC paying zero-coupon bond. We can deduce

$$v_1(t,T) \equiv p_1(t,T)e_1(t) \tag{3.2}$$

This quantity $p_1(t,T)$ is the time-t value, in units of XYZs, of one XYZ delivered at time T. Then we can use this decomposition concept to separately characterize the term structure of ABCs in terms of $p_1(t,T)$ and the payoff ratio $e_1(t)$.

Harrison and Pliska (1981) show that the nonexistence of arbitrage is equivalent to the existence of equivalent martingale probabilities; and that market completeness is equivalent to uniqueness of equivalent martingale probabilities. Jarrow and Turnbull make the assumption that there exists equivalent martingale probabilities making all the default-free and risky zero-coupon bond prices martingales, after normalization by the money market account. They also characterize the necessary and sufficient conditions for the existence of these equivalent martingale probabilities.

3.2.2 The Two-Period Discrete Trading Economy

To illustrate the foreign currency analogy mentioned above, we present a two-period case. Besides presenting the two-period economy, we will do formulation in such a way that readers can understand how to go beyond two periods.

Suppose there are two time periods with trading dates $t \in \{0, 1, 2\}$. We first describe the term structure for the default-free zero-coupon bonds and then the term structure for the ABC zero-coupon bonds. Face values of bonds are all assumed to be \$100.

The Default-Free Zero-Coupon Bond Term Structure

The term structure for default-free zero-coupon bonds is shown in Figure 3.1. The bond price process for default-free bond is assumed to depend only on this term structure, which gives:

$$B(0,1) = 100e^{-r_1} \tag{3.3}$$

$$B(0,2) = 0.5e^{-r_1}(100e^{-r_2} + 100e^{-r_2 v_2}); \tag{3.4}$$

where $B(t,T)$ represents the value at time t of a default-free zero-coupon bond that matures at time T.

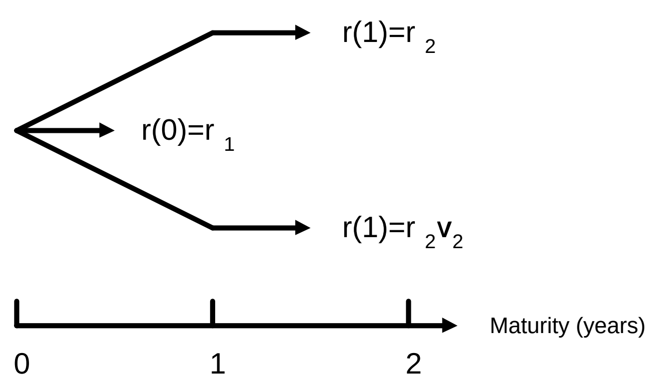

Figure 3.1: The Default-Free Short-Rate Process.

The ABC Zero-Coupon Bond Term Structure

In this section, we examine the pricing of risky bonds. Let $v(t,T;DS_t)$ denote the value at time t of a zero-coupon bond issued by ABC. The debt matures at time T and the bondholders are promised the face value of the bond at maturity.

Basis As the firm is risky, there is a positive probability that the firm might default over the life of the bond. If default occurs, the bond holder will receive less than the promised amount. The symbol DS_t is used to denote the default status of the bond at time t, that is:

$$DS_t \equiv \begin{cases} ND; \text{ default has not occurred at time } t \\ D; \text{ default has occurred at or before time } t. \end{cases}$$

As the symbol DS_t indicates, there are two possibilities: one, default does not occur before or at time t, denoted ND; and two, default does occur before or at time t, denoted D.

Defining $e(t)$ as the time-t exchange rate per ABC, we have:

$$e(t) \equiv \begin{cases} 1; \text{ probability } 1-\mu(t) \text{ if } DS_t = ND \\ \delta; \text{ probability } \mu(t) \text{ if } DS_t = D, \end{cases}$$

where $0 \leq \delta \leq 1$, and $\mu(t)$ is the martingale probability of default occurring at time t, conditional upon no default at or before time $t-1$. If default has occurred at or before time t, it is assumed that the bond remains in default and the payoff remains constant at δ dollars. This discrete-time binomial process was selected to approximate a continuous-time Poisson bankruptcy process. We can use the conditional martingale probabilities to develop pricing formulae which are arbitrage-free and the probabilities can be estimated using the observed term structure of interest rates.

To simplify the analysis, we are going to assume that the default process is independent of the level of the default-free interest rate. This implies that the movements of default-free spot rates has no effect on the probability of default. The relaxation of this assumption is discussed in Jarrow and Turnbull (1995).

Pricing and Formulation The payoff ratio process for ABC debt in the two-period economy is shown in Figure 3.2. If default has occurred at $t = 1$, the bond is assumed to remain in default. Otherwise, either default occurs or it does not at $t = 2$. At $t = 1$, with martingale probability $\mu(1)$, default occurs. Conditional upon the fact that default has not occurred at $t = 1$, the martingale probability of default occurring at $t = 2$ is $\mu(2)$. The corresponding bond price process is shown in Figure 3.3. Figure 3.3 is a combination of Figure 3.1 and Figure 3.2 under the assumption that the interest rate process and the bankruptcy process are statistically independent. The four states labelled A, B, C, and D correspond to all combinations of rate movements and bankruptcy process.

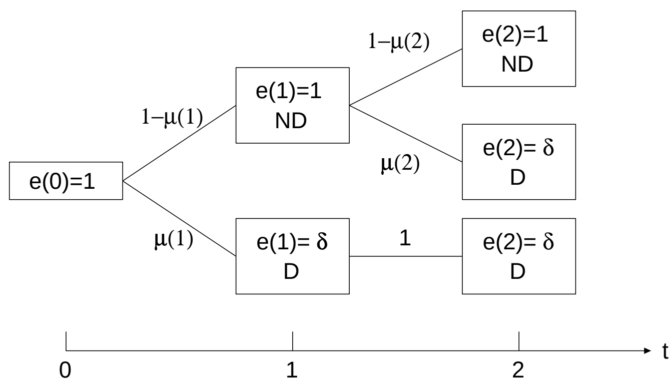

Figure 3.2: THE PAYOFF RATIO PROCESS FOR ABC DEBT IN THE TWO-PERIOD ECONOMY. $\mu(2)$ is the martingale probability that default occurs at date $t = 2$ conditional on that default has not occurred at time $t = 1$.

From Figure 3.3, let us first consider the one-year ABC zero-coupon bond. At the end of the first year, the credit risky bond's value is:

$$v(1, 1, DS_1) = 100 \begin{cases} 1; \text{ probability } 1 - \mu(1) \text{ if } DS_1 = ND \\ \delta; \text{ probability } \mu(1) \text{ if } DS_1 = D, \end{cases}$$

So we have:

$$V(0, 1; ND) = 100e^{-r_1}\{1[1 - \mu(1)] + \delta\mu(1)\} \tag{3.5}$$

we know that the conditional expected value of $e(1)$ is:

$$E_0^Q[e(1)|ND] = [1 - \mu(1)] + \delta[\mu(1)] \tag{3.6}$$

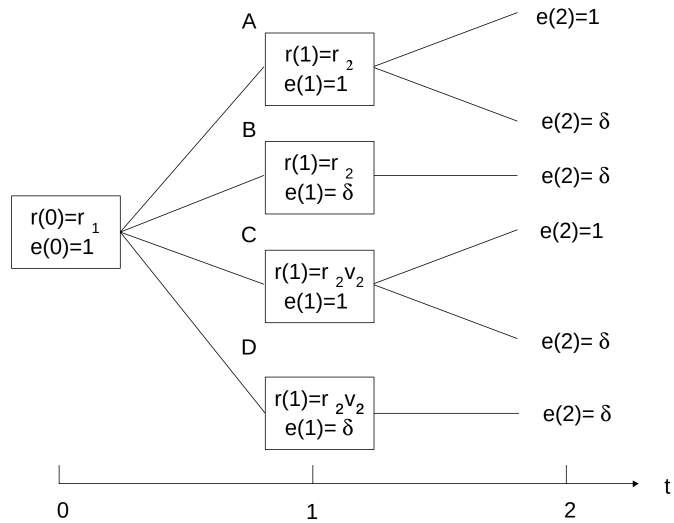

Figure 3.3: THE ABC ZERO-COUPON BOND PRICE PROCESS IN THE TWO-PERIOD ECONOMY. Martingale probabilities corresponding to State A, B, C and D are $0.5[1-\mu(1)]$, $0.5\mu(1)$, $0.5[1-\mu(1)]$ and $0.5\mu(1)$, respectively.

where $E_t^Q[e(T)|DS_t]$ denotes the time-t expected value of $e(T)$ conditional on DS_t under the martingale probabilities. Substituting equation (3.3) and equation (3.6) into equation (3.5), we have that:

$$v(0,1,ND) = B(0,1)E_0^Q[e(1)|ND] \tag{3.7}$$

The pricing of the two-period zero-coupon bond is slightly more complicated because, at the end of the first period, both interest rates and the default status of the firm are uncertain. Let us consider state A, B, C and D at $t=1$, respectively.

If default has occurred at the end of the first period, such as state B and D, the payoff at $t=2$ is:

$$v(2,2;D) = 100e(2) = 100\delta.$$

Therefore, conditional on default having occurred at $t=1$:

$$E_1^Q[e(2)|D] = \delta.$$

The values in State B and D are:

$$v_B(1,2;D) = e^{-r_2}(100\delta)$$

$$v_D(1,2;D) = e^{-r_2 v_2}(100\delta)$$

The situation that default has not occurred at the end of the first period is more interesting. In such case, the payoff at $t = 2$ will be:

$$v(2,2;DS_2) = 100e(2) = 100 \begin{cases} 1; \text{probability } 1-\mu(2) \text{ if } DS_2 = ND \\ \delta; \text{probability } \mu(2) \text{ if } DS_2 = D. \end{cases}$$

and the conditional value is:

$$E_1^Q[e(2)|ND] = [1-\mu(2)] + \delta[\mu(2)].$$

Therefore, the values in State A and C are:

$$v_A(1,2;ND) = 100e^{-r_2}\{[1-\mu(2)] + \delta\mu(2)\}$$

$$v_C(1,2;ND) = 100e^{-r_2 v_2}\{[1-\mu(2)] + \delta\mu(2)\}$$

Today at $t = 0$, default has not occurred. Therefore, when viewed from date 0:

$$E_0^Q[e(2)|ND] = [1-\mu(1)]E_1^Q[e(2)|ND] + [\mu(1)]E_1^Q[e(2)|D] \tag{3.8}$$

Referring to Figure 3.3, we have:

$$\begin{aligned} v(0,2;ND) &= 0.5e^{-r_1}\{[1-\mu(1)][v_A(1,2;ND) + v_C(1,2,ND)] + \\ &\quad \mu(1)[v_B(1,2;D) + v_D(1,2;D)]\} \end{aligned}$$

which gives:

$$\begin{aligned} v(0,2;ND) &= 0.5e^{-r_1}[e^{-r_2} + e^{-r_2 v_2}][1-\mu(1)]\{[1-\mu(2)]100 + \mu(2)100\delta\} + \\ &\quad 0.5e^{-r_1}[e^{-r_2} + e^{-r_2 v_2}]\mu(1)100\delta \end{aligned} \tag{3.9}$$

substituting equations (3.4) and (3.8) into equation (3.9) gives:

$$v(0,2;ND) = B(0,2)E_0^Q[e(2)|ND]. \tag{3.10}$$

Equations (3.7) and (3.10) are special cases of the general formulae:

$$v(0,T;ND) = B(0,T)E_0^Q[e(T)|ND] \tag{3.11}$$

and

$$v(0, T; D) = B(0, T)\delta \tag{3.12}$$

both for a risky zero-coupon bond of maturity T. Equation (3.11) gives the risky zero-coupon bond's value if the firm is not in default, and equation (3.12) gives the risky zero-coupon bond's value if the the firm is in default. These two equations are an important and intuitive result because they provide a practical way of computing the martingale probabilities of default using market data, and because they can be used for pricing derivatives on credit risky cash flows.

Chapter 4

Model Calibration under Credit Risk

4.1 Forward Induction

Forward induction proposed by Jamshidian (1991) can be used to reduce the computation time of our calibration process. This algorithm is to sweep a line across time forward and inductively figure out how much $1 at a node contributes to the total price. This number is called the *state price* since it is the price of a claim that pays $1 at that particular state (node) and zero elsewhere. We call a tree with these state prices a *state price tree*. Figure 4.1 depicts a binomial state price tree. There is no need to explicitly store the whole tree.

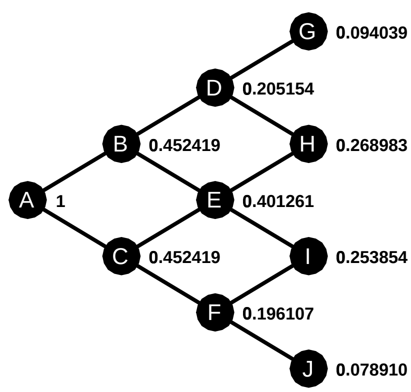

Figure 4.1: A Sample Binomial State Price Tree. The One-Year short rate at each node is the same as Figure 2.3

Let us be more precise about this mechanism. Consider a binomial interest rate tree. Suppose we are at the end of period j, so there are $j+1$ nodes. Let the baseline rate for period j be $r_j = r$, the multiplicative ratio be $v_j = v$, and P_1,...,P_j be the state prices at the nodes of the beginning of period j. One dollar j periods from now has a known market value of $1/(1+y)^j$, where y denotes the j-period spot rate. Alternatively, one dollar at the end of period j has a present value of

$$P_1 e^{-r} + P_2 e^{-rv} + P_3 e^{-rv^2} + ... + P_j e^{-rv^{j-1}}.$$

4.2 Model Calibration

This section will describe a model of interest rates that can be used to price derivatives involving credit risk. This model has four characteristics:

1. Its fundamental variable is the short rate—the annualized one period interest rate. The changes of this short rate drive all security prices.

2. This model adopts the Jarrow-Turnbull method to cope with default risk.

3. The model takes two inputs to form the term structure. One is structure of long rates (yields) of default-free and credit risky bonds for various maturities. The other are option prices on the one-period risky bond for various maturities.

4. The model varies means and volatilities for the future spot rates to match the inputs.

Because of the utilization of the Jarrow-Turnbull model to cope with default risk, the tree we use to describe the bond price process has the same structure with that in Jarrow-Turbull model, Figure 3.3. We reconstruct it in Figure 4.2 with more periods. One might imagine that the tree would not combine as it unfolds. However, the tree will combine if we take some care. As Figure 4.2 shows, the number of nodes at the end of period i is just $2(i+1)$. In fact, the bold-lined part is just a binomial interest tree which is the same as that in Figure 2.2. The light-lined part is to indicate the situations of default. Each internal node on bold lines has four branches which correspond to the movement of interests and bankruptcy. On the other hand, nodes on light lines have only two branch because once the default occurred, it would be in default in the future. So the branches just need to correspond to the movements of interests.

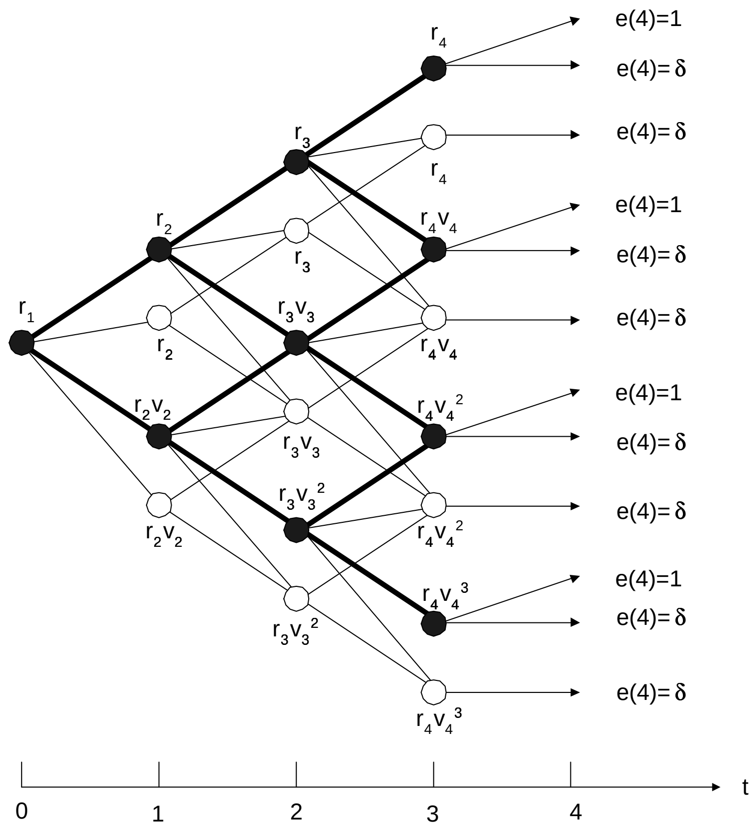

Figure 4.2: The Risky Bond Price Process. We assume the risky debt matures at time $t = 4$ and its recovery rate is δ when default occurs.

To calibrate the interest rate model, we first need to know the martingale probabilities of each state occurring. Since our tree is a combination of the default-free interest rate process and the default process under the independent assumption mentioned before, the martingale probabilities of each state are the multiplicative product of the separate probabilities of interest rates going up or down times the probabilities of default or not default. The martingale probabilities of the default-free interest rate tree that we use are 0.5. What we need to find out is the martingale probabilities of default occurring.

4.2.1 Calculating the Probabilities of Default

The martingale probabilities of default occurring are not difficult to find. Figure 4.3 describes the process of $E_0^Q[e(T)|ND]$. From this figure, we can easily derive:

$$E_0^Q[e(1)|ND] = \mu(1)\delta + [1-\mu(1)]$$
$$E_0^Q[e(2)|ND] = \mu(1)\delta + [1-\mu(1)]\mu(2)\delta + [1-\mu(1)][1-\mu(2)]$$
$$\ldots$$

Refer to (3.11) and viewing Figure 4.3, we can calculate each of these possibilities in constant time. Note that once default occurs, the bond remains in default. This means the payoff ratio will be δ with probability one hereafter.

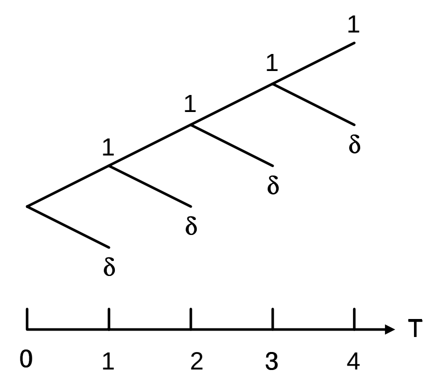

Figure 4.3: Payoff Ratio Process. The probability of default occurring at time t is $\mu(t)$ conditional upon default has not occurred at or before time $t-1$.

Let's view a two-period example. Suppose we are given the market data in Table 4.1. The second column is for default-free bonds and the third column is for bonds belonging to credit class ABC. For this credit class, if default occurs the bond holder receives $\delta = 0.32$ dollar per promised dollar.

For the one-period bond, from (3.11) and Figure 4.3, we have

$$v(0,1|ND) = B(0,1)\{\mu(1)\delta + [1-\mu(1)]\}.$$

Using the prices in Table 4.1, this implies that $\mu(1) = 0.0059$. For the two-period bond, we have

$$v(0,2|ND) = B(0,2)\{\mu(1)\delta + [1-\mu(1)]\mu(2)\delta + [1-\mu(1)][1-\mu(2)]\},$$

which implies $\mu(2) = 0.0088$.

Maturity Years	Price of Default-Free Bond	Price of Risky Bond
1	92.31	91.94
2	84.54	83.69

Table 4.1: Prices of Default-Free and Risky Bonds in the Two-Period Economy.

4.2.2 Calibration

We now present an approach to do model calibration using forward induction and the differential tree method of Lyuu (1995). Suppose we are at the end of period j, and there are $2(j+1)$ nodes. Half of them are not in default and the other half are in default. For economy of expression, we use r and v, instead of r_j and v_j, to denote the baseline rate and the multiplicative ratio for period j. Let $NSP_1, ..., NSP_j$ be the state prices of the no-default nodes at the beginning of period j and $DSP_1, ..., DSP_j$ be the state prices of the default nodes. NP and DP denote the expected payoff at the end of period j conditional on the default status at the beginning of period j. We have:

$$NP = 100\{[1-\mu(j)] + \delta[\mu(j)]\}$$
$$DP = 100\delta$$

A j-period risky zero-coupon bond now has a market value of $100e^{-yj}$, where y denotes today's yield of this risky bond. In a risk-neutral world, it must hold that:

$$\sum_{i=1}^{j}(NPe^{-rv^{i-1}})NSP_i + \sum_{i=1}^{j}(DPe^{-rv^{i-1}})DSP_i = 100e^{-yj}. \tag{4.1}$$

Viewed from now, the price of the one-period risky zero coupon at the beginning of period j is uncertain. Let C_j be the price of option whose underlying asset is one-period risky zero and maturity is $j-1$ period. Then we have:

$$\sum_{i=1}^{j} \max(X - NPe^{-rv^{i-1}}, 0) NSP_i + \sum_{i=1}^{j} \max(X - DPe^{-rv^{i-1}}, 0) DSP_i = C_j, \quad (4.2)$$

where X is the strike price. Rearranging (4.1) and (4.2) to get:

$$\begin{aligned} f(r,v) &\equiv \sum_{i=1}^{j} (NPe^{-rv^{i-1}}) NSP_i + \sum_{i=1}^{j} (DPe^{-rv^{i-1}}) DSP_i - 100e^{-yj} = 0 && (4.3) \\ g(r,v) &\equiv \sum_{i=1}^{j} \max(X - NPe^{-rv^{i-1}}, 0) NSP_i + \sum_{i=1}^{j} \max(X - DPe^{-rv^{i-1}}, 0) DSP_i \\ &\quad -C_j = 0 && (4.4) \end{aligned}$$

What we need to do now is solve $f(r,v)$ and $g(r,v)$. We use the Newton-Raphson method. Given the kth approximation $(r(k), v(k))$, the Newton-Raphson method say the $(k+1)$th approximation $(r(k+1), v(k+1))$ satisfies

$$\begin{bmatrix} \frac{\partial f(r(k),v(k))}{\partial r} & \frac{\partial f(r(k),v(k))}{\partial v} \\ \frac{\partial g(r(k),v(k))}{\partial r} & \frac{\partial g(r(k),v(k))}{\partial v} \end{bmatrix} \begin{bmatrix} \Delta r(k+1) \\ \Delta v(k+1) \end{bmatrix} = - \begin{bmatrix} f(r(k), v(k)) \\ g(r(k), v(k)) \end{bmatrix}$$

where $\Delta r(k+1) \equiv r(k+1) - r(k)$ and $\Delta v(k+1) \equiv v(k+1) - v(k)$.

4.2.3 A Two-Period Case

In this section, we provide an easy two-period case so that readers can clearly understand how our calibration process works. We are given the market data in Table 4.2. The second column is for default-free zero-coupon bonds and the third column is for bonds belonging to credit class ABC. The difference reflects the likelihood of default. The fourth column is for a 1-year option whose underlying asset is the 1-year ABC bond and the exercise price is \$90. δ for this credit class is set to 0.32. These data are taken as exogenous.

Maturity Years	Yield of Default-Free Bond	Yield of Risky Bond	Price of Option
1	0.08	0.084	
2	0.084	0.089	0.5130

Table 4.2: Initial Bond Yields and Option Prices in the Two-Period Economy.

To do calibration, first we need to determine the martingale probabilities of default. For the one-period bond, using (3.6) and (3.11), we have

$$v(0,1|ND) = B(0,1)\{\mu(1)\delta + [1-\mu(1)]\}.$$

Using the prices in Table 4.2, this implies that $\mu(1) = 0.0059$. For the two-period bond, using (3.8) and (3.11), we have

$$v(0,2|ND) = B(0,2)\{\mu(1)\delta + [1-\mu(1)]\mu(2)\delta + [1-\mu(1)][1-\mu(2)]\},$$

implying $\mu(2) = 0.0088$.

Next, we need to find the state prices at the end of period 1. Since we have known the short rate for period 1 and martingale probabilities of states occurring, we can easily find out the state prices at the end of period 1 and the payoff at the end of period 2 (Figure 4.4).

$$NSP_1 = NSP_2 = 0.5 \times (1 - 0.0059) \times e^{-0.08} = 0.4588$$

$$DSP_1 = DSP_2 = 0.5 \times 0.0059 \times e^{-0.08} = 0.0027$$

$$NP = 100 \times [(1 - 0.0088) + 0.0088 \times 0.32] = 99.4016$$

$$DP = 100 \times 0.32 = 32$$

Now we can start our calibration process. Referring to (4.3) and (4.4), we can set up two nonlinear equations:

$$f(r_2, v_2) \equiv \sum_{i=1}^{2}(NPe^{-r_2 v_2^{i-1}})NSP_i + \sum_{i=1}^{2}(DPe^{-r_2 v_2^{i-1}})DSP_i - 100e^{-2\times 0.089} = 0$$

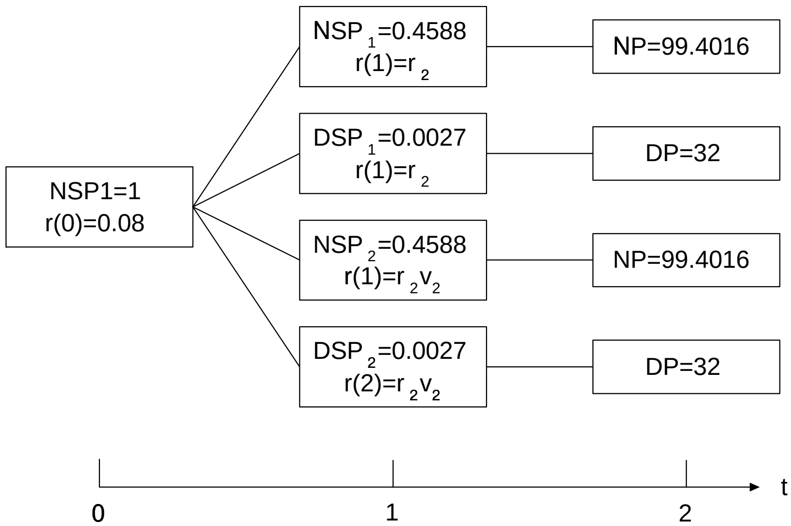

Figure 4.4: State Prices and Payoff in Two-Period Economy.

$$
\begin{aligned}
g(r_2, v_2) \quad \equiv \quad & \sum_{i=1}^{2} \max(90 - NPe^{-r_2 v_2^{i-1}}, 0) NSP_i + \sum_{i=1}^{2} \max(90 - DPe^{-r_2 v_2^{i-1}}, 0) DSP_i \\
& -0.513 = 0
\end{aligned}
$$

Solving these two equations can get short rates for period 2 and complete the calibration (Figure 4.5). It doesn't get more difficult when going beyond two periods. Once we have solved the short rates for period j, we can use them to find out the state prices at the end of period j, and we are able to go on calibrating the short rates for period $j + 1$.

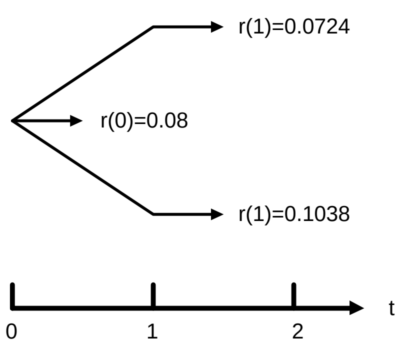

Figure 4.5: Short Rate Tree in the Two-Period Economy.

Chapter 5

Experimental Results

5.1 Experimental Process

In this chapter, we examine the performance of our calibration method. We will go through its convergence, efficiency and stability. As for the convergence and efficiency, we care about how many iterations and how much time our calibration process needs to approach the anticipated precision. To investigate the stability, we will need a benchmark tree. We use this benchmark tree to generate the price information our calibration method needs. Then we use the price information as inputs to calibrate the interest tree and see whether this tree is consistent with our benchmark tree. For convenience, we use the same price information to investigate the convergence and efficiency issues.

5.1.1 Constructing the Benchmark Tree

We use the Black-Derman-Toy (1990) model to build the benchmark tree. We know that the Black-Derman-Toy model needs bond yields and their volatilities to construct the interest rate tree. The term structure we choose is

$$
\begin{array}{ll}
0.08 + 0.005\ln(t) & \text{for } t\text{-period zero-coupon bond yield} \\
1.4(1 - e^{-0.1t})/t & \text{for } t\text{-period zero-coupon bond yield volatility}
\end{array}
$$

and we assume the bonds are default-free. This term structure is attributed by

Gagnon and Johnson (1994) to Hull and White (1990). After the baseline rates and multiplicative ratios are obtained from the term structure, we have the benchmark interest rate tree. One is referred to Lyuu (1995) and Chen (1997) for the implementation of the Black-Derman-Toy model.

5.1.2 Performing Calibration

The data our calibration process needs are yields of default-free and risky bonds and prices of options whose underlying assets are one-period risky zeros. The yields of the default-free bonds are given above and the yields of the t-period risky zeros are assumed to be $0.084 + 0.0054 \ln(t)$. The exercise price of the j-period option is set to be the price of the one-period bond whose yield is the risky zero's implied forward rate for period $j+1$. Using the prices of default-free and risky zeros, we can compute the probabilities of default. Having the benchmark interest rate tree and the probabilities of default, we can construct a tree which has the same structure with Figure 4.2. Then we can use this tree to find out the prices of risky zeros and option. We know that the prices of risky zeros from the benchmark tree won't be completely identical to the prices from the term structure $0.084 + 0.0054 \ln(t)$ due to the errors occurred in calibrating our benchmark tree. We don't want the errors to affect the result of our calibration process so that we use the prices from the benchmark tree instead of the prices from the term structure. After generating the price information of risky zeros and options, we can take them as inputs to perform our calibration process according to the steps mentioned in chapter 4. Figure 5.1 depicts the flow chart of our experimental process.

We know that in order to accurately price interest-rate-sensitive derivatives, we need a tree with finely spaced steps between today and its expiration. In the following experiments, we reduce the interval size to one day which means 365 periods per year. The termination condition is set to 10^{-11} relative error. The relative error used here is relative to the price of risky zero and the option price. In our program we define it as $\{[(rb_e - rb_t)/rb_t]^2 + [(op_e - op_t)/op_t]^2\}^{0.5}$ where rb means prices of risky zeros, op means prices of options and the subscript e and t correspond to the evaluated and

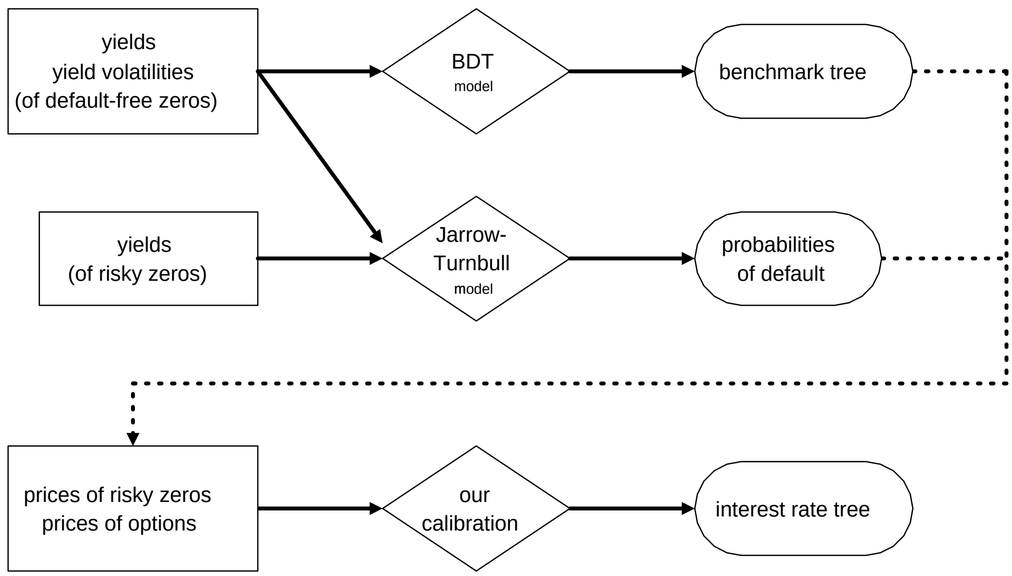

Figure 5.1: Flow Chart of Our Experimental Process.

true values. With the Newton-Raphson method, it is critical to get a suitable initial guess. Using the baseline rate and multiplicative ratio of the previous period as the initial guess for the current period has turned out to work well. We run the programs on Sun UltraSparc.

5.2 Convergence

Our calibration process converges very fast. Tabel 5.1 reveals that it takes an average of less than 3.8 iterations to achieve a relative error of 10^{-11}. The average number of iterations in Table 5.1 corresponds to $\frac{\text{total interations}}{\text{number of periods-1}}$ where -1 means we start calibration in period 2. Tighter termination condition will lead to more iterations. It might even lead to infinite loop when the termination condition is set too tight.

5.3 Efficiency

The running time of our calibration process is $O(n^2)$ because of the usage of forward induction. The running time is about $0.000016 \times n^2$ seconds. All these observations can be confirmed by the data in Table 5.2.

Number of Periods	Average Number Iterations	Number of Periods	Average Number Iterations
100	3.797980	1900	2.643497
200	3.396985	2000	2.617809
300	3.264214	2100	2.595522
400	3.253133	2200	2.574807
500	3.224449	2300	2.555024
600	3.198664	2400	2.536890
700	3.175966	2500	2.520208
800	3.156446	2600	2.505194
900	3.140156	2700	2.490923
1000	3.102102	2800	2.477671
1100	3.014559	2900	2.465333
1200	2.940784	3000	2.453484
1300	2.877598	3100	2.442723
1400	2.826305	3200	2.432323
1500	2.780520	3300	2.421946
1600	2.739212	3400	2.412180
1700	2.705121	3500	2.402115
1800	2.672040	3650	2.387503

Table 5.1: Convergence of Our Calibration Process.

5.4 Stability

Now we continue to address the stability issue. One may be interested in whether our calibrated interest rate tree is close to the benchmark interest rate tree. We use the relative error of short rates to demonstrate this issue. The average relative error in Table 5.3 is $\frac{\text{total relative errors}}{\text{number of rates}}$. The calculation begins from the rates for period 2 and the number of rates is $O(n^2)$ where n is number of periods. It is expected that the average relative error will increase as the number of periods increases, since the current baseline rate and multiplicative ratio depend on all previous ones. We can observe that the algorithm is still very accurate when the number of periods goes to 3650.

Number of Periods	Running Time (seconds)	Number of Periods	Running Times (seconds)
100	0.1	1900	57.0
200	0.5	2000	64.7
300	1.2	2100	70.9
400	2.2	2200	79.1
500	3.4	2300	85.9
600	4.8	2400	94.9
700	6.6	2500	102.0
800	8.8	2600	110.9
900	11.0	2700	119.4
1000	13.2	2800	128.2
1100	15.9	2900	138.3
1200	19.5	3000	146.8
1300	23.7	3100	157.4
1400	28.4	3200	168.1
1500	33.4	3300	178.0
1600	38.8	3400	189.3
1700	44.5	3500	200.5
1800	50.6	3650	220.1

Table 5.2: Efficiency of Our Calibration Process.

Number of Periods	Average Relative Error	Number of Periods	Average Relative Error
100	3.593265e-12	1900	4.185029e-11
200	4.642933e-12	2000	4.632705e-11
300	5.525227e-12	2100	4.984666e-11
400	6.325714e-12	2200	5.373780e-11
500	7.223715e-12	2300	5.995537e-11
600	9.194375e-12	2400	6.607607e-11
700	1.038878e-11	2500	7.297840e-11
800	1.209067e-11	2600	8.034019e-11
900	1.400110e-11	2700	9.065705e-11
1000	1.650363e-11	2800	1.045061e-10
1100	2.019819e-11	2900	1.202937e-10
1200	2.356501e-11	3000	1.397283e-10
1300	2.658802e-11	3100	1.659159e-10
1400	2.904906e-11	3200	1.948036e-10
1500	3.089136e-11	3300	2.170545e-10
1600	3.277774e-11	3400	2.425344e-10
1700	3.594489e-11	3500	2.839236e-10
1800	3.922988e-11	3650	3.882972e-10

Table 5.3: Stability of Our Calibration Process.

Chapter 6

Conclusions

This thesis presented a numerical method for model calibration under credit risk. We take advantage of the Jarrow-Turnbull (1995) model to cope with the default risk. With the technique of forward induction and differential tree, the calibration process can be done efficiently and accurately. Once the interest rate tree has been constructed, it can be used to price interest rate derivatives easily.

Appendix A

Solving Systems of Nonlinear Equations with Newton-Raphson Method

Consider the two-dimensional case. Let (x_k, y_k) be the kth approximation to the solution of the two simultaneous equations,

$$f(x, y) = 0 \text{ and } g(x, y) = 0.$$

The Newton-Raphson method leads to the following linear equations for the $(k+1)$st approximation, (x_{k+1}, y_{k+1}),

$$\begin{bmatrix} \frac{\partial f(x_k,y_k)}{\partial x} & \frac{\partial f(x_k,y_k)}{\partial y} \\ \frac{\partial g(x_k,y_k)}{\partial x} & \frac{\partial g(x_k,y_k)}{\partial y} \end{bmatrix} \begin{bmatrix} \Delta x_{k+1} \\ \Delta y_{k+1} \end{bmatrix} = - \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix}$$

where $\Delta x_{k+1} \equiv x_{k+1} - x_k$ and $\Delta y_{k+1} \equiv y_{k+1} - y_k$. The above equations have a unique solution for $(\Delta x_{k+1}, \Delta y_{k+1})$ when the *Jacobian determinant* of f and g,

$$J \equiv \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}$$

does not vanish at (x_k, y_k). The $(k+1)$st approximation is simply

$$(x_k + \Delta x_{k+1}, y_k + \Delta y_{k+1}).$$

Generalization to n dimensions is straightforward.

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