Towards Creating Taiwan's Put Market

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Abstract

An option is a financial instrument whose payoff is based upon another, more elementary financial instrument, such as stocks or bonds. It gives its owner the right to buy or sell a particular underlying asset at a stated price within a limited time. With the rapid growth and deregulation of option market in Taiwan, more and more option products have been designed to fit investors' needs. Some people, such as hedgers, use it to hedge their risk, while others, like speculators, may intend to profit from this instrument.

In Taiwan, financial options appeared on Taiwan's exchange in mid-1997. Options issuers need to hedge through delta hedge. The difficulty of delta hedge with stocks and bonds is that securities houses are not allowed to short stocks. This explains why puts are never issued, because a put's delta is negative. How to create the put market is our main concern.

Our proposal is to issue calls and puts together. Of course, the proportion has to be right, say, 4 calls for each put. This makes the overall probability of shorting stocks smaller, and we have experimental data to back up the claim. Now assume the securities house can short stock index futures. When the stock portfolio goes into negative territory because of negative option portfolio delta, the securities house shorts the index futures. Although, the result cannot be perfect hedge, it would be much better than sitting idly.

In this thesis, we derive by the optimal hedge ratio to decide how many stock index futures contracts to hold in our hedge portfolio. When the correlation between the stock price and the stock index are high, the hedging result is very encouraging. It shows with many numerical experiments that the proposal works well.

Chapter 1

Introduction

Options give their holder the right to buy or sell some underlying asset. They form one of the most important classes of financial instruments and have wide applications in finance; in fact, almost any security has option features. As far as we know, the option pricing theory is the most successful theory in finance as well as economics. The methodology developed by the theory of option pricing lays the cornerstone for the general theory of derivative pricing.

1.1 Motivations

In recent years, Taiwan's equity market has reached a point with average daily turnover in excess of US$ 6.2 billion. Active trading aroused the need for an efficient financial instrument for the market participants to manage their risk associated with the investment in Taiwan's stock market.

From the standpoint of option issuers, they can employ delta hedge, which requires either long positions or short positions in stock depending on whether the option delta is positive or negative, respectively. However, there are prohibitions against short sales of stock for securities houses in Taiwan. This explains why puts are never issued in Taiwan. To circumvent this difficulty, we propose to use stock index futures instead of stock when the delta of our option portfolio is negative. See Figure 1.1. Of course, the index should be correlated with the stock. Experiments show that the result is not satisfactory, although it is better than doing nothing. We therefore propose to lower the probability of the need for short sales by inssuing call options in tandem with the put options. Recall that a call has a positive delta. Of course, the calls and puts may not be sold to the same customer; it is the portfolio of options that we need to hedge.

The optimal hedge ratio is used to decide how many stock index futures contracts should be held in our hedge portfolio. This ratio minimizes the return variance. Experiments show that the result is very close to pure delta hedge when one is allowed to short stock. These results suggest a possible market making for puts in Taiwan.

A common misunderstanding of the proposal is that it is just a simple application of the put-call parity, which is the simplest way to create puts. Our proposal has nothing to do with the put-call parity at all. To use the put-call parity, the securities house which issues puts has to buy calls and, well, *short* stock, which it is prohibited from doing!

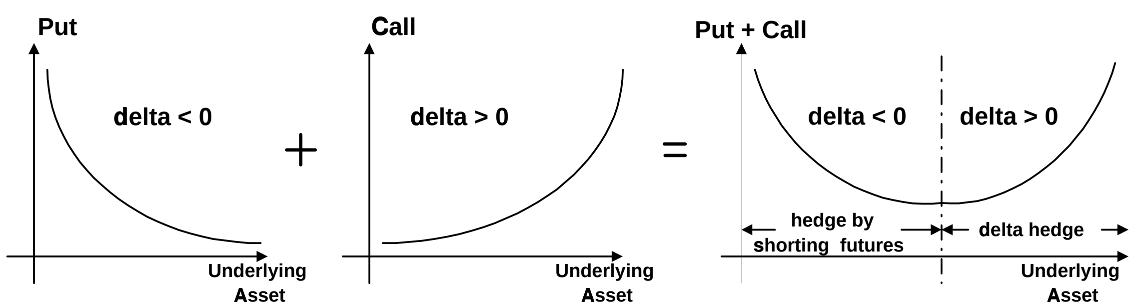

Figure 1.1: The Basic Idea.

1.2 Organization of This Thesis

There are five chapters in this thesis. Chapter 1 is a brief introduction. In Chapter 2, we introduce some basic financial concepts which will help us in the arguments later. In Chapter 3, we introduce our option pricing models and show how to generate stock prices and stock index values. In Chapter 4, we calculate the probability of negative delta for a portfolio of calls and puts. We also derive the optimal hedge ratio to determine the number of stock index futures contracts. Our proposed strategy is compared with other hedging strategies. Finally, conclusions and future work are in Chapter 5.

Chapter 2

Fundamental Concepts

This chapter reviews several basic concepts used in the following chapters. We cover option basics, futures basics, the behavior of stock prices, and delta hedge.

2.1 Option Basics

Options on stock were first traded on an organized exchange in 1973. Since then there has been a dramatic growth in options markets. Now they are traded on many exchanges around the world. In Taiwan, options were first traded in August, 1997.

There are two basic types of option contracts: *call* options and *put* options. A call option gives the holder the right to buy the asset at a stated price (called the *exercise price* or *strike price*). A put option gives the holder the right to sell the asset at the strike price.

In general, call and put options can be defined in one of two manners: *American* and *European*. A European option can only be exercised at the maturity date of the option, whereas an American option can be exercised at any time up to and including the maturity date; namely, *early exercise* is allowed.

Throughout this chapter, we assume that $T - t$ denotes the time to maturity, X denotes the strike price, and S represents the current stock price at time t.

Positions

There are two sides to each option contract. On one side is the investor who has taken the long position (i.e., has bought the option). On the other side is the investor who has taken a short position (i.e., has sold or written the option). The writer of an option receives cash up front but has potential liability later.

Payoff and profit

A call option will be exercised only if the strike price is less than the stock price. The value of a call option at its exercise date is therefore $\max(0,\ S - X)$. A put option will be exercised only if the stock price is less then the strike price. The value of a put option at its exercise date is $\max(0,\ X - S)$. A call (put) option is said to be *in the money* if $S > X$ $(S < X)$, *at the money* if $S = X$, and *out of money* if $S < X$ $(S > X)$. The profit from a long position in a European call option is

$$\max(0, S - X) - C$$

The profit from a short position in a European call option is

$$\max(0, X - S) + C$$

The profit from a long position in a European put option is

$$\max(0, X - S) - P$$

The profit from a short position in a European put option is

$$\max(0, S - X) + P$$

where C and P are the call premium and the put premium at maturity date respectively.

Figure 2.1 illustrates their profit/loss graphically.

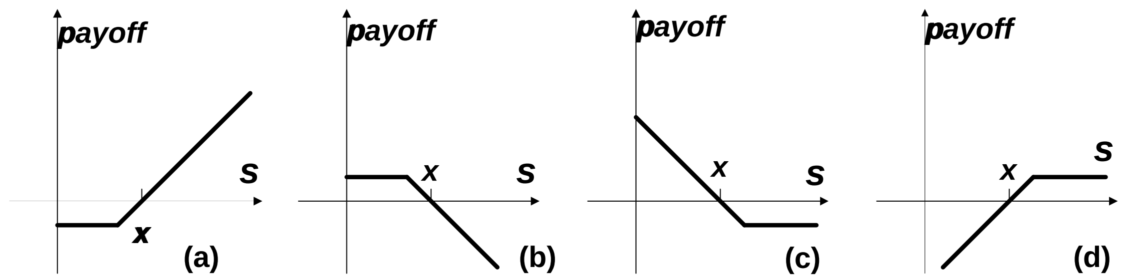

Figure 2.1: Profit/loss of options. (a) Long a call. (b) Short a call. (c) Long a put. (d) Short a put.

2.2 Stock Index Futures

A *stock index* tracks the changes in the value of a hypothetical portfolio of stocks. The weight of stock in the portfolio equals the proportion of the portfolio invested in the stock. The percentage increase in the value of a stock index over a small interval of time is usually defined so that it is equal to the percentage increase in the total value of the stocks comprising the portfolio at that time.

A *stock index futures* contract is an agreement between two parties to buy or sell portfolio of stocks at a stated time in the future for a stated price. They are normally traded on an exchange. As the two parties to the contract do not necessarily know each other, the exchange provides a mechanism which gives the two parties a guarantee that the contract will be honored. *Taiwan Stock Exchange Capital Weighted Index Futures* (TAIEX Futures) appeared on *Taiwan International Mercantile Exchange Corporation*, namely TAIMEX, in 1998. That can be used for the stock index futures of our proposed strategy.

Futures contracts on stock index are usually settled *in cash*, not by delivery of the underlying asset. All contracts are *marked to market* at the end of each trading day and the positions are deemed closed.

Futures price

Most indices can be thought of as securities that pay dividends. The security is the portfolio of stocks underlying the index, and the dividends paid by the security are the dividends that would be received by the holder of this portfolio. To a reasonable approximation, the dividends can be assumed to be paid continuously. Assume that S_t is the current value of stock index, r is the continuously compounded risk-free interest rate, and q is the dividend yield rate. Then the arbitrage-free futures price F_T is

$$F_T = S_t e^{(r-q)(T-t)}. \tag{2.1}$$

2.3 A Model of Stock Price and Stock Index

Here we introduce the Wiener process. The stock price process used for the thesis is also defined.

Wiener process

A *Wiener process* is a particular type of *Markov* stochastic process which will be used to derive our stock and stock index prices. The behavior of a variable, w, which follows a Wiener process can be understood by considering the changes in its value in small intervals of time. Consider a small interval of time of length $\triangle t$ and let $\triangle w$ be the change in w during $\triangle t$. There are two basic properties for $\triangle w$.

Property 1. $\triangle w$ must follow the equation

$$\triangle w = \epsilon\sqrt{\triangle t} \tag{2.2}$$

where ϵ is a random drawing from a standardized normal distribution $N(0,1)$.

Property 2. The values of $\triangle w$ for any two different short intervals of time $\triangle t$ are independent.

By Property 1, $\triangle w$ is a normal distribution $N(0, \sqrt{\triangle t})$. Property 2 implies that w follows a Markov process. If $\triangle t$ approaches zero, we can write the limiting case of

(2.2) as:

$$dw = \epsilon\sqrt{dt} \tag{2.3}$$

Generalized Wiener process

The basic Wiener process has a drift rate of zero and a variance rate of 1. A *generalized Wiener process*, say s, can be defined in terms of the basic Wiener process, dw, as follows:

$$ds = a\,dt + b\,dw$$

where a and b are constants. See Figure 2.2.

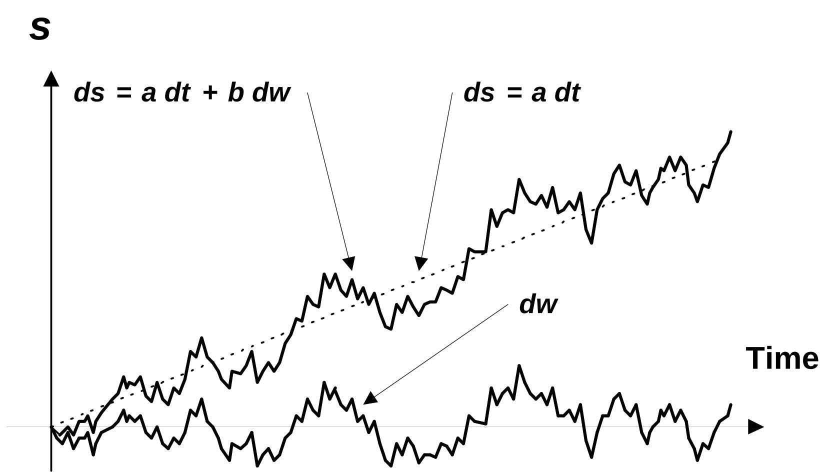

Figure 2.2: GENERALIZED WIENER PROCESS.

The process for stock prices and stock index

We assume that our stock price, which excludes the dividend payment, follows the stochastic process described as follows:

$$\frac{dS}{S} = \mu\,dt + \sigma\,dw \tag{2.4}$$

where μ is the stock's expected rate of return per unit time and σ is the volatility of the stock price. Equation (2.4) is the most widely used model of stock price and is also known as *geometric Brownian motion*.

Stock index is usually thought of as a stock providing a continuous dividend yield. We assume that the dividend yield is constant throughout this thesis. The stochastic process it followed is described as:

$$\frac{dF}{F} = (r - q)\,dt + \beta\,dz \tag{2.5}$$

where F is the stock index, r is the continuously compounded risk-free interest rate, β is the volatility of the stock index, and q is the dividend yield of the stock index.

2.4 Delta Hedge

The *delta* $\triangle$ of a derivative security is defined as the ratio of the change in its price with respect to the change in the price of the underlying asset. More formally, $\triangle = \frac{\partial f}{\partial S}$, where f is the price of the derivative security and S is the price of the underlying asset.

Consider a call option on a stock. Figure 2.3 shows the relationship between the call price and the underlying stock price. When the stock price at P and the option price at Q, the delta of the call is the slope of the line. As an approximation,

$$\triangle = \frac{\triangle C}{\triangle S}$$

where $\triangle C$ is the small change in the call price corresponding to a $\triangle S$ change in the stock price.

Suppose, for example, that the delta of the call option is 0.4. When the stock price changes by a small amount $\triangle S$, as a result, the option price will change by $0.4\triangle S$. If you hold 0.4 share of stock and a short position in the call option, the delta of this portfolio is zero. A portfolio with a delta of zero is referred to as *delta neutral.* As an example, at the end of weekdays, the stock price might drop from \$50 to \$45. As indicated by Figure 2.3, a decrease in the stock price leads to a decrease in delta. Suppose that the delta drops from 0.4 to 0.3. To remain delta neutral, we need to sell 0.1 share of stock. A strategy that maintain delta neutrality at all times is called *delta hedge.*

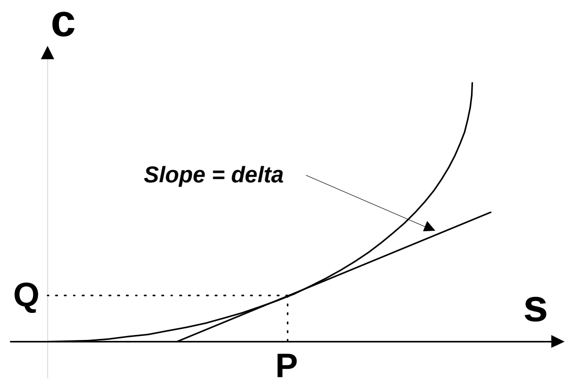

Figure 2.3: CALCULATION OF DELTA.

When the call price and the stock price fluctuate with time, the delta of this combined portfolio will change too. So, the delta neutral condition lasts only a short period of time. Even if the stock price does not change, the delta will still change. To remain delta neutrality, we need to adjust continuously the proportion between the number of stock shares and the position of call options. That is called *rebalancing*. Hedging schemes that involve frequent adjustments are known as *dynamic hedging schemes*.

In reality, however, it is impossible to hedge the risk in every seconds, which would result in a lot of transaction cost. In the real world when delta hedge is being implemented, the hedge position will be adjusted periodically.

2.5 Risk-Neutral Valuation

We will refer to a world where everyone is risk neutral as a *risk-neutral world*. In such a world investors require no compensation for risk, and the expected return on all securities is the risk-free interest rate. For example, from Figure 2.4, we assume there is a derivative security, C, today which will either move C_u with probability q or down to C_d with probability $1-q$ after time Δt. We get:

$$E(C) \quad = qC_u + (1-q)C_d = Ce^{r\Delta t}$$

where r is the risk-free interest rate. This general principle in pricing is known as *risk-neutral valuation.*

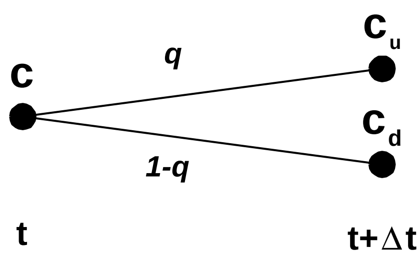

Figure 2.4: Risk-Neutral Valuation.

Chapter 3

Option Pricing

We illustrate the *binomial model* for stock price and stock index processes in this chapter. It can be proved that as the time interval approaches zero, the binomial model approaches geometric Brownian motion and the stock price distribution is log-normal. We also apply the binomial model to price American options and the Black-Scholes formula to price European options. How delta is calculated is also discussed.

3.1 Basic Assumptions

We assume that the following some assumptions we listed here can be relaxed by other methodologies.

1. The stock price follows the log-normal distribution, so does the stock index.
2. The rate of return on stock, the volatility of the stock price, and the risk-free rate of interest are constant throughout the life of the option. The same holds for the stock index.
3. The coefficient of correlation between the stock price and the stock index remains constant through the life of the option.
4. There are no transaction costs and tax. All securities are perfectly divisible.
5. There are no riskless arbitrage opportunities.

3.2 Generating Stock Prices and Stock Index Values

We assume the stock price follows the following geometric Brownian motion:

$$\frac{dS}{S} = \mu\, dt + \sigma\, dw \tag{3.1}$$

where μ is the instantaneous rate of return on stock, σ is the volatility of the stock price, and dw is the Wiener process.

Log-normal model

Let $X = \ln S$. By Ito's lemma,

$$dX = (\mu - \frac{1}{2}\sigma^2)\, dt + \sigma\, dw \tag{3.2}$$

In the discrete time model, Equation (3.2) can be rewritten as

$$\triangle X = (\mu - \frac{1}{2}\sigma^2)\, \triangle t + \sigma\, \epsilon \sqrt{\triangle t} \tag{3.3}$$

where ϵ is a random drawing from the standard normal distribution $N(0,1)$. Let S_i be the stock price at time i. An equivalent statement is

$$\ln S_{i+1} - \ln S_i = (\mu - \frac{1}{2}\sigma^2)\, \triangle t + \sigma\, \epsilon \sqrt{\triangle t}$$

or,

$$S_{i+1} = S_i\, e^{(\mu - \frac{1}{2}\sigma^2)\, \triangle t + \sigma\, \epsilon \sqrt{\triangle t}} \tag{3.4}$$

Suppose that today is time i, we can employ (3.4) to generate the stock price at time $i+1$. Since $\ln S_i$ is normally distributed, $\ln S_{i+1}$ is still normally distributed; thus S_{i+1} still has the log-normal property.

Now, we still use the same steps to derive the equation for generating the stock index values,

$$F_{i+1} = F_i\, e^{(r - q - \frac{1}{2}\beta^2)\, \triangle t + \beta\, \zeta \sqrt{\triangle t}} \tag{3.5}$$

where F is the stock index, q is the dividend yield, r is the continuously compounded risk-free interest rate, β is the volatility of the stock index, and ζ is a random drawing from the standard normal distribution $N(0,1)$ different from ϵ in (3.4).

Table 3.1: A simulation of the stock price and the stock index with ρ=0.900.

Period	Stock Price	Stock Index	ϵ	ζ
0	49.000	5000.000	−1.526360	−1.291432
1	47.541	4898.343	1.086914	1.091457
2	48.600	4985.244	1.338490	0.803586
3	49.934	5050.371	−0.009936	0.501351
4	49.939	5091.667	0.498767	−0.078926
5	50.455	5085.862	0.388009	0.092680
6	50.863	5094.030	1.335276	1.046999
7	52.255	5180.716	−1.006798	−0.112015
8	51.229	5172.071	−0.044260	0.377650
9	51.199	5204.052	−1.125502	−1.647234
10	50.075	5069.305		

Correlation between stock price and stock index

In reality, there may exits some relation between the stock price and the stock index. Of course, the closer their correlation is, the better the result would be. In the limit of a correlation of 1, the hedge is perfect.

Assume that the coefficient of correlation between ϵ in (3.4) and ζ in (3.5) is ρ. Independent x_1 and x_2 are two different random drawings from the standard normal distribution $N(0,1)$. Then we can use the following method to generate both ϵ and ζ,

$$\begin{aligned} \epsilon &= x_1 \\ \zeta &= \rho x_1 + x_2\sqrt{1-\rho^2} \end{aligned}$$

As ϵ and ζ have been calculated, we can use (3.4) and (3.5) to generate stock prices and stock indices. Table 3.1 shows a particular sample path of stock price and stock index with $\rho = 0.900$, where μ=5%, σ=20%, r=5%, β=16%, q=5%, S=\$49, F=\$5000, and $\triangle t$=0.01.

3.3 The Binomial Model

The *binomial model* is a discrete-time approximation of the continuous-time pirce model. This is a binomial tree that represents the possible paths that might be followed by the price over the life of the option.

Suppose the current stock price is S. After a small time interval Δt, it may move up to Su with probability q and down to Sd with probability $1-q$. Figure 3.1 illustrates the three periods of the binomial tree where

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta t}} \\ d &= e^{-\sigma\sqrt{\Delta t}} = \frac{1}{u} \\ q &= \frac{e^{\mu\Delta t} - d}{u - d} \end{aligned}$$

We shall call them the *CRR parameters*, because Cox, Ross and Rubinstein proposed them.

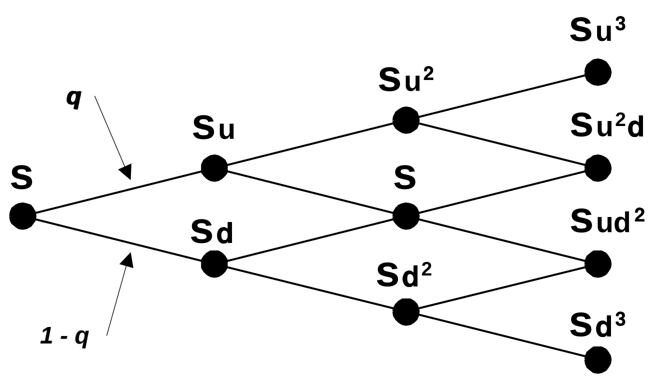

Figure 3.1: Binomial model for three periods. Stock price movements over three time periods under the binomial model.

As for the stock index, similar binomial tree structure can be developed. Figure 3.2 plots the three-period binomial tree where

$$\begin{aligned} u &= e^{\beta\sqrt{\Delta t}} \\ d &= e^{-\beta\sqrt{\Delta t}} = \frac{1}{u} \\ q &= \frac{e^{(r-q)\Delta t} - d}{u - d} \end{aligned}$$

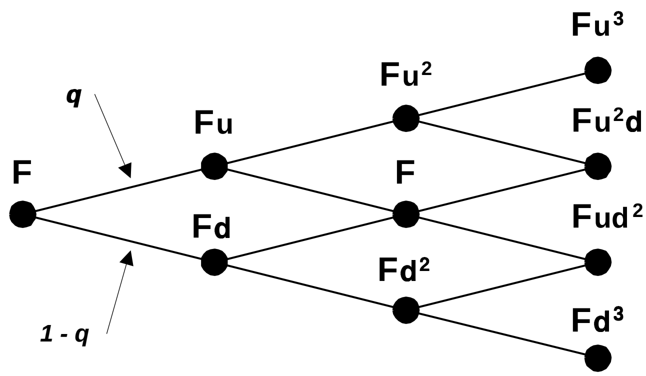

Figure 3.2: BINOMIAL MODEL FOR THREE PERIODS. Stock index movements over three time periods under the binomial model.

Once the binomial model for the stock price has been developed, it can be used to calculate option's price and option delta. It have been proved that as the time interval Δt approaches zero, the binomial model converges to geometric Brownian motion.

Pricing European stock options

One way to price current European stock options is to employ the Black-Scholes Formula. It was derived by Fischer Black and Myron Scholes in the early 1970s. Assume C is the call price and P is the put price. Then,

$$C = SN(d_1) - Xe^{-rT}N(d_2) \tag{3.6}$$

$$P = Xe^{-rT}N(-d_2) - SN(-d_1) \tag{3.7}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\ln(S/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \end{aligned}$$

$N(x)$ = cumulative normal probability

σ^2 = annualized variance of the continuously compounded return on the stock

r = continuously compounded risk-free rate
T = time to maturity

This is a continuous-time option pricing model. The assumptions are continuous security trading, short selling of securities, and no dividends during the life of the derivatives. The other assumptions are the same as those in Section 3.1. The deltas

$$\begin{aligned} Delta(C) &= N(d_1) \\ Delta(P) &= -N(-d_1) \end{aligned}$$

Pricing American stock options

Firstly, we show how to use a two-period binomial tree to price European options and then extend it to price American stock options. Take a call option for example. Figure 3.3 illustrates the case of two steps for stock and option prices.

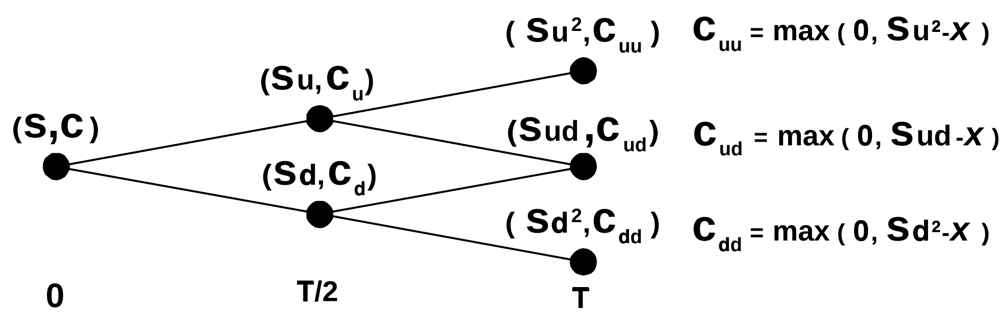

Figure 3.3: Stock and option prices in a two-period tree.

The stock price is initially S, and the strike price is X. The option provides a payoff at time T. According to the risk neutral valuation, which states that we can with complete impunity assume that the world is risk neutral when pricing options, we get obtain

$$C_u = e^{-r\Delta t}[qC_{uu} + (1-q)C_{ud}] \tag{3.8}$$

$$C_d = e^{-r\Delta t}[qC_{ud} + (1-q)C_{dd}] \tag{3.9}$$

$$C = e^{-r\Delta t}[qC_u + (1-q)C_d] \tag{3.10}$$

Substituting from equation (3.8) and (3.9) in (3.10), we get

$$C = e^{-2r\Delta t}[q^2 C_{uu} + 2q(1-q)C_{ud} + (1-q)^2 C_{dd}]$$

where the risk-free interest rate is r and the length of the period is $\Delta t = T/2$ years. The variables q^2, $2q(1-q)$, and $(1-q)^2$ are the probabilities of the upper, middle, and lower final nodes being reached from the root. The European call option's price is equal to its expected payoff in a risk-neutral world discounted at the risk-free interest rate. If the time between 0 and T is divided into n periods, we get

$$\begin{aligned} C &= e^{-rT} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} q^j (1-q)^{n-j} \max(0, Su^j d^{n-j} - X) \\ P &= e^{-rT} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} q^j (1-q)^{n-j} \max(0, X - Su^j d^{n-j}) \end{aligned}$$

Where $\Delta t = T/n$. We can also calculate the deltas of the European call and put as

$$Delta(C) = \frac{C_u - C_d}{Su - Sd}$$
$$Delta(P) = \frac{P_u - P_d}{Su - Sd}$$

We now move on to consider how American options are valued under the binomial tree such as the one in Figure 3.3. The procedure is to work back through the tree from the end to the beginning, testing at each node to see whether early exercise is optimal. The value of the option at the final nodes is the same as for European option. At earlier nodes the value of the option is the greater of

1. The value given by equation (3.10).

2. The payoff from early exercise.

That is:

$$\begin{aligned} C_u &= \max(e^{-r\Delta t}[qC_{uu} + (1-q)C_{ud}], Su - X) \\ C_d &= \max(e^{-r\Delta t}[qC_{ud} + (1-q)C_{dd}], Sd - X) \\ C &= \max(e^{-r\Delta t}[qC_u + (1-q)C_d], S - X) \end{aligned}$$

This methodology can be used to price American put options as well. We can also calculate the deltas of the American call and put as

$$Delta(C) = \frac{C_u - C_d}{Su - Sd} \tag{3.11}$$

$$Delta(P) = \frac{P_u - P_d}{Su - Sd} \tag{3.12}$$

Chapter 4

Issuing and Hedging the Put Option

For markets that prohibit the shorting of stock for securities houses, our proposal for issuing the puts is to issue multiple calls for each put and short stock index futures to hedge the option portfolio when the portfolio delta is negative. We investigate how likely the delta of the portfolio will be below zero. Then we also derive the optimal hedge ratio for the number of futures contracts and assess the performance of this hedging strategy.

4.1 The Probability of Negative Delta

We have illustrated the idea of delta hedge by using long and short stock. Short positions are mandated when the portfolio's delta is negative, we recall. This theoretical result will run into difficulties in markets like Taiwan which disallow the securities firms to short stocks. Put options have not be issued presumably for precisely this reason because the put delta is always negative. What if we short stock index futures in our delta hedge when the portfolio has negative deltas? Well, this idea is fine, but a put has a negative delta for any stock price, making this first cut an idea that may not work well. But how about issuing calls and puts simultaneously, say h calls for each put? We will show that this idea, by lowering the probability of negative delta, is precisely the scheme that will work well enough to deserve serious practical implementation.

The options' prices and deltas can be calculated from equations in chapter 3. We can also use the values of their deltas to find out the stock price at which the value of the overall portfolio's delta is zero. From this stock price, the option issuer can compute the probability of negative portfolio delta, thus using stock index futures.

Suppose the stock price is S, the expected rate of return on stock is μ, the volatility of the stock price is σ, the time to maturity is $(T-t)$, the risk-free rate of interest is r, and our portfolio consists of h calls and one put which may have different strike prices. According to (3.11) and (3.12), the delta of our overall portfolio will becomes $f(S) = h \times Delta(C) + Delta(P)$. Which S would make $f(S) = 0$?

One of the simplest and failure-free methods to find out the stock price when the delta of the portfolio becomes zero is the *bisection method*. Suppose $f(S_a)$ and $f(S_b)$ are the deltas of our portfolio, respectively at stock prices S_a and S_b. Furthermore, they are of opposite signs and $f(S_a)f(S_b) \neq 0$. Then $f(\xi) = 0$ for some ξ between S_a and S_b, written as $\xi \in [S_a, S_b]$. If we evaluate f at the min-point S_c between S_a and S_b, then either (1) $f(S_c) = 0$, (2) $f(S_a)$ and $f(S_c)$ are of opposite signs, or (3) $f(S_c)$ and $f(S_b)$ are of opposite signs. In the first case we can stop. In the second case we continue the process with the new bracket $[S_a, S_c]$, and in the third case we continue with $[S_c, S_b]$. Note that the bracket is halved in the latter two cases. After n steps, we will have nailed down ξ within a bracket of length $(S_b - S_a)/2^n$. In this way, we can quickly approximate the desired stock price.

Now that we have calculated the approximate stock price S^* which makes the delta of the portfolio zero, we continue to calculate the probability the stock price goes below that value, i.e. $S < S^*$. $\ln S$ is normally distributed with mean μ and variance σ^2, then the density function of the lognormally distributed random variable S is

$$f(s) = \begin{cases} \frac{1}{\sigma s\sqrt{2\pi}} e^{-(\ln s-\mu)^2/2\sigma^2}, & if\ y > 0 \\ 0, & if\ y \leq 0 \end{cases}$$

The distribution function of the lognormally distributed random variable S is described as

$$Prob[S \leq s] \quad = \quad N(\frac{\ln(s) - \mu}{\sigma})$$

Table 4.1: The approximated stock prices and their corresponding probabilities where the delta of the portfolio is.

Period	Stock Price	Delta	The Approximated Stock Price S^*	Probability $S < S^*$
0	49.000	0.649415	43.93	0.212983
1	49.371	0.691627	43.98	0.203800
2	49.639	0.722069	44.03	0.197793
3	49.822	0.740876	44.08	0.194286
4	48.886	0.609434	44.13	0.223109
5	49.013	0.620154	44.19	0.221264
6	48.111	0.493286	44.24	0.251625
7	50.084	0.751803	44.29	0.193588
8	49.303	0.639784	44.34	0.217553
9	49.531	0.663813	44.39	0.212459
10	49.645	0.672709	44.44	0.210749

Through the above approach, the option issuer can calculate the probability of having to use stock index futures, i.e., the probability of the portfolio's delta being below zero. It would be a useful data for understanding how likely the futures will be employed.

Table 4.1 shows a simulation of stock price movement during the life of the option. In particular, it shows the approximated stock prices and their corresponding probabilities when the delta of the portfolio is zero. In Table 4.1, μ=5%, σ=20%, S=\$49, r=5%, $T-t$=1, h=3, hedge period Δt=0.01 year, the dollar dividend which will be paid in year 0.376 is \$2, the strike price of the call option is \$57, and the strike price of the put option is \$41. Time is measured in years, and our option is an American option with a single dollar dividend.

Table 4.2 computes the probabilities under the same parameters for h=1, 2, 3, 4. Obviously, the higher h is, the less the probability is. This is what it should be, as the calls tend to make the portfolio delta higher. The strike price spread between the call option and the put option can also be employed to lower the probabilities. In Table 4.3, the strike prices for the call and the put are (\$57, \$41), (\$56, \$42), and (\$55, \$43), respectively. The probability of negative portfolio delta varies as the

Table 4.2: The probability of negative delta when h=1, 2, 3, and 4.

Period	Stock Price	Probability (h=1)	Probability (h=2)	Probability (h=3)	Prbability (h=4)
0	49.000	0.336009	0.255255	0.212983	0.186334
1	49.371	0.323509	0.244684	0.203800	0.177903
2	49.639	0.314963	0.237620	0.197793	0.172706
3	49.822	0.309216	0.233334	0.194286	0.169512
4	48.886	0.344540	0.265137	0.223109	0.196452
5	49.013	0.340927	0.262334	0.221264	0.194776
6	48.111	0.376805	0.294988	0.251625	0.223096
7	50.084	0.304416	0.231023	0.193588	0.169564
8	49.303	0.333230	0.257407	0.217553	0.191735
9	49.531	0.326021	0.251418	0.212459	0.187350
10	49.645	0.323005	0.249146	0.210749	0.186099

spread varies.

4.2 Deriving the Optimal Hedge Ratio for Correlated Assets

If the stock price and the stock index futures are deriven by different Wiener processes, what is the optimal number of stock index futures contracts to short under negative delta? We shall combine the binomial tree with the criterion of minimal variance to arrive at a solution.

There are four possible movements between the stock price and stock index futures in the two-factor binomial tree.

Case 1: Stock price moves up, and the index moves up.

Case 2: Stock price moves up, and the index moves down.

Case 3: Stock price moves down, and the index moves up.

Case 4: Stock price moves down, and the index moves down.

Table 4.3: The probability of negative delta as the strike price spread between the call and the put varies. Where h=3.

Period	Stock Price	Probability for (57,41)	Probability for (56,42)	Probability for (55,43)
0	49.000	0.212983	0.211992	0.209687
1	49.371	0.203800	0.202835	0.200912
2	49.639	0.197793	0.196847	0.194960
3	49.822	0.194286	0.193351	0.191488
4	48.886	0.223109	0.222096	0.220074
5	49.013	0.221264	0.220257	0.218248
6	48.111	0.251625	0.250545	0.248389
7	50.084	0.193588	0.192659	0.190808
8	49.303	0.217553	0.216558	0.214906
9	49.531	0.212459	0.211479	0.209851
10	49.645	0.210749	0.210099	0.208156

Table 4.4: The assignment of probabilities for up and down movements between two assets.

Probability	Stock price movements	Stock index movements
p_1	up	up
p_2	up	down
p_3	down	up
p_4	down	down

Name the four probabilities in Table 4.4. Assume that the coefficient of correlation ρ between the stock price and the stock index futures price can be found from the historical data. and it is constant throughout the life of the option. We got the following five equations to solve for p_1, p_2, p_3, and p_4.

$$\begin{aligned}
\rho &= \frac{p_1(u_s-\mu_s)(u_f-\mu_f)+p_2(u_s-\mu_s)(d_f-\mu_f)}{\sigma_s\sigma_f}+\\
&\quad \frac{p_3(d_s-\mu_s)(u_f-\mu_f)+p_4(d_s-\mu_s)(d_f-\mu_f)}{\sigma_s\sigma_f}
\end{aligned}$$

$$p_1+p_2 = \frac{\mu_s-d_s}{u_s-d_s} \qquad p_3+p_4 = \frac{u_s-\mu_s}{u_s-d_s}$$

$$p_1+p_3 = \frac{\mu_f-d_f}{u_f-d_f} \qquad p_2+p_4 = \frac{u_f-\mu_f}{u_f-d_f}$$

where u_s, μ_s, d_s, σ_s are the CRR parameters for the stock price and u_f, μ_f, d_s, σ_f are the CRR parameters for the stock index futures. The first equation is derived from the definition of correlation.

After p_1, p_2, p_3, and p_4 have been solved, we employ them to minimize the performance deviation of shorting futures in place of stock. If a person is short stocks and long λ stock index futures, the change in the value of his position for the next Δt time is

$$-\Delta S+\lambda\Delta F,$$

where λ is called the hedge ratio. The variance of the change in the value of the position is given by

$$\begin{aligned}
&p_1[(-Su_s+S\mu_s)+\lambda(Fu_f-F\mu_f)]^2+p_2[(-Su_s+S\mu_s)+\lambda(Fd_f-F\mu_f)]^2+\\
&p_3[(-Sd_s+S\mu_s)+\lambda(Fu_f-F\mu_f)]^2+p_4[(-Sd_s+S\mu_s)+\lambda(Fd_f-F\mu_f)]^2
\end{aligned}$$

or

$$A\lambda_2-2B\lambda+C,$$

where

$A=(p_1+p_3)(Fu_f-F\mu_f)^2+(p_2+p_4)(Fd_f-F\mu_f)^2$

$$B = (Fu_f - F\mu_f)[p_1(Su_s - S\mu_s) + p_3(Sd_s - S\mu_s)] + (Fd_f - F\mu f)[p_2(Su_s - S\mu_s) + p_4(Sd_s - S\mu_s)]$$
$$C = (p_1 + p_2)(Su_s - S\mu_s)^2 + (p_3 + p_4)(Sd_s - S\mu_s)^2$$

When $\lambda = A/B$, it has the minimum variance $(AC - B^2)/A$. The optimal ratio is therefore A/B.

The two-factor binomial tree

We now introduce an fast way to solve p_1, p_2, p_3, and p_4. Define $R_i \equiv \ln S_i(\Delta t)/S_i$, i=1, 2. So $R_i \sim N(\mu_i^{'}, \sigma_i^2 \Delta t)$, where $\mu_i^{'} \equiv r - \sigma_i^2/2$. Note that (R_1, R_2) has a bivariate distribution. Hence, its moment generating function is

$$\begin{aligned} E[e^{t_1 R_1 + t_2 R_2}] &= \exp[(t_1\mu_1^{'} + t_2\mu_2^{'})\Delta t + (t_1^2\sigma_1^2 + t_2^2\sigma_2^2 + 2t_1t_2\sigma_1\sigma_2\rho)\frac{\Delta t}{2}] \\ &= 1 + (t_1\mu_1^{'} + t_2\mu_2^{'})\Delta t + (t_1^2\sigma_1^2 + t_2^2\sigma_2^2 + 2t_2t_2\sigma_1\sigma_2\rho)\frac{\Delta t}{2} + o(\Delta t) \end{aligned}$$

Under the binomial model, (R_1, R_2)'s moment generating function is

$$\begin{aligned} &E[e^{t_1 R_1 + t_2 R_2}] \\ &= p_1 e^{(t_1\sigma_1 + t_2\sigma_2)\sqrt{\Delta t}} + p_2 e^{(t_1\sigma_1 - t_2\sigma_2)\sqrt{\Delta t}} + p_3 e^{(-t_1\sigma_1 + t_2\sigma_2)\sqrt{\Delta t}} + p_4 e^{(-t_1\sigma_1 - t_2\sigma_2)\sqrt{\Delta t}} \\ &= (p_1 + p_2 + p_3 + p_4) + t_1\sigma_1(p_1 + p_2 - p_3 - p_4)\sqrt{\Delta t} + t_2\sigma_2(p_1 - p_2 + p_3 - p_4) \\ &\quad \sqrt{\Delta t} + (t_1^2\sigma_1^2 + t_2^2\sigma_2^2 + 2t_1t_2\sigma_1\sigma_2(p_1 - p_2 - p_3 + p_4))\frac{\Delta t}{2} + o(\Delta t) \end{aligned}$$

Match the above two equations to obtain

$$p_1 = \frac{1}{4}(1 + \rho + \sqrt{\Delta t}(\frac{\mu_1^{'}}{\sigma_1} + \frac{\mu_2^{'}}{\sigma_2})) \quad (4.1)$$
$$p_2 = \frac{1}{4}(1 - \rho + \sqrt{\Delta t}(\frac{\mu_1^{'}}{\sigma_1} - \frac{\mu_2^{'}}{\sigma_2})) \quad (4.2)$$
$$p_3 = \frac{1}{4}(1 - \rho + \sqrt{\Delta t}(-\frac{\mu_1^{'}}{\sigma_1} + \frac{\mu_2^{'}}{\sigma_2})) \quad (4.3)$$
$$p_4 = \frac{1}{4}(1 + \rho + \sqrt{\Delta t}(-\frac{\mu_1^{'}}{\sigma_1} - \frac{\mu_2^{'}}{\sigma_2})) \quad (4.4)$$

Thus p_1, p_2, p_3, and p_4 can be calculated from those equations. For the general case of more than tow assets, consult [10] for more detailed information. Table 4.5 shows

Table 4.5: An numerical example of p_1, p_2, p_3, and p_4.

Probability	Method 1	Method 2
p_1	0.477959951665	0.477960616562
p_2	0.025966080302	0.025965062857
p_3	0.024035706520	0.024034937143
p_4	0.472038261512	0.472039383438

that the four probabilities from the above method are very close to the ones from solving the five equations on page 25. We used ρ=0.9, the expected rate of return of stock is 5% per annum, r is the continuously compounded risk-free interest rate, the dividend yield from the stock index is 5% per annum, the volatility of the stock price is 20% per annum, the volatility of the stock index is 16% per annum, Δt=1/365 year, and the time to maturity is 1 year. Method 1 means the four probabilities are calculated by solving the five equations on page 25, whereas Method 2 means the results are from (4.1) – (4.4).

4.3 Simulation Results

First, we implement delta hedge for in-the money and out-of-the-money European put options respectively in Table 4.6 and Table 4.7. The stock prices used in the two tables are generated from (3.4) with an expected rate of return of 5% per annum and a volatility of 20% per annum. The stock price starts from $49, the strike price is $50, the risk-free interest rate is 5% per annum, the time to maturity is 20 weeks, i,e, 0.3846 year, and the European put option is on 100,000 shares of non-dividend-paying stock. We rebalance the stock position weekly. The cost of hedging and the theoretical cost are listed at the bottom of each table. If we rebalance the position more frequently, the difference between the two costs should be reduced.

Now we take up the case of using futures instead of stock when the delta is negative. The option issuer issues h American call options on stock and one American put option on stock at the same time. The underlying asset of the two options is the

Table 4.6: Simulation of delta hedge: the case of in-the-money put.

Period	Price	Delta	Shares Purchased	Cost of Shares Purchased	Cumulative Cost	Interest Cost
0	49.00	−0.478398	−47839.8	−2344152	−2344152	−2255.0
1	48.39	−0.482274	−387.6	−18758	−2365164	−2275.2
2	47.37	−0.528455	−4618.1	−218758	−2586197	−2487.8
3	47.02	−0.607337	−7888.3	−370937	−2959622	−2847.0
4	47.11	−0.640208	−3287.1	−154870	−3117339	−2998.8
5	44.46	−0.642569	−236.1	−10495	−3130833	−3011.7
6	47.08	−0.828904	−18633.5	−877236	−4011081	−3858.5
7	44.49	−0.665345	16355.9	727736	−3287204	−3162.2
8	45.41	−0.852278	−18693.3	−848820	−4139186	−3981.7
9	45.08	−0.812294	3998.3	180251	−3962916	−3812.2
10	44.27	−0.847839	−3554.5	−157358	−4124087	−3967.2
11	46.09	−0.906106	−5826.7	−268550	−4396604	−4229.4
12	47.45	−0.816154	8995.2	426830	−3974003	−3822.8
13	47.95	−0.720644	9551.0	457958	−3519867	−3386.0
14	48.75	−0.690667	2997.7	146131	−3377122	−3248.7
15	46.63	−0.617999	7266.8	338857	−3041513	−2925.8
16	47.66	−0.877081	−25908.2	−1234723	−4279162	−4116.4
17	48.44	−0.819830	5725.0	277311	−4005967	−3853.6
18	47.94	−0.770478	4935.2	236610	−3773211	−3629.7
19	46.62	−0.928715	−15823.7	−737646	−4514487	−4342.8
20	45.43	−1.000000	−7128.5	−323856	−4842685	

The hedge cost is:	$157315
After discounting:	$154319
The theoretical value is:	$244815.38

Table 4.7: Simulation of delta hedge: the case of out-of-the-money put.

Period	Price	Delta	Shares Purchased	Cost of Shares Purchased	Cumulative Cost	Interest Cost
0	49.00	−0.478398	−47839.8	−2344152	−2344152	−2255.0
1	49.00	−0.482274	−387.6	−18992	−2365399	−2275.4
2	48.38	−0.486349	−407.5	−19715	−2387389	−2296.6
3	48.39	−0.534840	−4849.1	−234653	−2624339	−2524.5
4	46.33	−0.540076	−523.6	−24260	−2651123	−2550.3
5	47.12	−0.699037	−15896.1	−748973	−3402646	−3273.2
6	46.77	−0.652014	4702.3	219916	−3186004	−3064.8
7	48.43	−0.689144	−3713.0	−179823	−3368892	−3240.7
8	48.29	−0.565059	12408.5	599176	−2772956	−2667.5
9	47.82	−0.586238	−2117.9	−101269	−2876893	−2767.5
10	48.98	−0.638936	−5269.8	−258139	−3137799	−3018.4
11	48.73	−0.540229	9870.7	481001	−2659817	−2558.6
12	48.37	−0.575586	−3535.7	−171029	−2833405	−2725.6
13	49.41	−0.626585	−5099.8	−251960	−3088091	−2970.6
14	51.85	−0.522779	10380.5	538238	−2552824	−2455.7
15	51.27	−0.243655	27912.4	1431160	−1124119	−1081.4
16	52.58	−0.291019	−4736.4	−249040	−1374241	−1322.0
17	52.33	−0.128933	16208.6	848127	−527436	−507.4
18	52.41	−0.109774	1915.9	100418	−427525	−411.3
19	50.93	−0.040217	6955.7	354243	−73693	−70.9
20	52.75	0.000000	4021.7	212150	138386	

The hedge cost is:	\$138386
After discounting:	\$135750
The theoretical value is:	\$244815.38

same stock, and each option is written on 10,000 shares of stocks. As described earlier, we use delta hedge to the short option positions but with short stock replaced by short stock index futures, say *TIMEX 200*. The number of stock index future contracts used in our hedging process are determined by the optimal hedge ratio. The delta of our portfolio can be calculated by (3.11) and (3.12). The stock price and the index value are assumed to have a coefficient of correlation of ρ.

Table 4.8 shows a simulation of our proposed strategy with h=0, ρ=0.9, and Δt=1 week for in-the money American put options. For comparison, the stock price in Table 4.8 is the same as Table 4.6. The strike price of the put is $50, and the stock does not pay dividends. In Table 4.9 – 4.11, we assume the current stock price S is $49, the current stock index F is $5000, the strike price of the call option is $56, the strike price of the put option is $42, the expected rate of return on the stock μ_s is 5% per annum, r is the continuously compounded risk-free interest rate, the volatility of the stock σ_s is 20% per annum, the volatility of the stock index σ_f is 16% per annum, the dividend yield q of the stock index is 5% per annum, the time to maturity for options and futures are both 1 year, and the dollar dividend is $2 at 0.376 year from now. We adjust h, ρ, and the hedging interval Δt to gauge the performance of our proposed strategy. We also compare our data with those calculated from pure delta hedge strategy.

First, Table 4.9 tabulates the performance of four different option hedge strategies under various rebalancing time interval lengths. Each result is calculated from 500 sample paths. **Strategy 1** is delta hedge, **Strategy 2** is our proposed strategy, **Strategy 3** is the doing nothing strategy, and **Strategy 4** is the strategy of not hedging when the delta is negative. In **mean of difference**, we can tell whether the average cost of our proposed strategy approaches that of pure delta hedge . **Performance 1** assesses the fluctuation between the hedging cost of our proposed strategy and the hedging cost of pure delta hedge. **Performance 2** assesses the fluctuation between the hedging cost of our proposed strategy and the future value of the option price. The same definitions apply to Tables 4.10 and 4.11. Below, we

Table 4.8: A simulation of our proposed strategy with h=0, ρ=0.9, and Δt=1 week.

Period	Stock Price	Stock Index	Delta	Future Contracts	Cumulative Cost	Interest Cost
0	49.00	5000.00	−0.511	−2.84	0	0.0
1	48.39	4949.42	−0.560	−3.11	−28732	−27.6
2	47.37	4864.29	−0.647	−3.94	−81707	−78.6
3	47.02	4834.74	−0.681	−4.15	−105075	−101.1
4	47.11	4841.12	−0.681	−4.15	−99880	−96.1
5	44.46	4620.65	−0.904	−5.45	−282965	−272.2
6	47.08	4835.92	−0.700	−4.27	−48586	−46.7
7	44.49	4621.29	−0.924	−5.57	−231930	−223.1
8	45.41	4696.00	−0.864	−5.23	−148922	−143.3
9	45.08	4667.93	−0.903	−5.46	−178427	−171.6
10	44.27	4599.58	−0.976	−5.88	−253237	−243.6
11	46.09	4749.11	−0.849	−5.16	−77632	−74.7
12	47.45	4859.87	−0.735	−4.50	36595	35.2
13	47.95	4899.52	−0.698	−4.28	72314	69.6
14	48.75	4963.66	−0.619	−3.81	127287	122.4
15	46.63	4789.33	−0.884	−5.39	−5430	−5.2
16	47.66	4872.40	−0.804	−4.92	84111	80.9
17	48.44	4935.02	−0.735	−4.51	145808	140.3
18	47.94	4893.49	−0.861	−5.28	108490	104.4
19	46.62	4783.79	−1.000	−6.09	−7247	−7.0
20	45.43	4685.18			−127364	

The hedge cost is:	\$329636
After discounting:	\$323358
The total option value is:	\$261599

list the definitions for the above terms.

$$\begin{aligned}
\textbf{Average Cost} &= \frac{1}{500}\sum_{i=1}^{500}(\text{each hedging cost}) \\
\textbf{Mean of} & \\
\textbf{Difference} &= \frac{1}{500}\sum_{i=1}^{500}(\text{hedging cost} - \text{hedging cost from delta hedge}) \\
\textbf{Performance 1} &= \frac{1}{500}\sum_{i=1}^{500}\left(\frac{\text{hedging cost} - \text{hedging cost from delta hedge}}{\text{hedging cost from delta hedge}}\right)^2 \\
\textbf{Performance 2} &= \frac{1}{500}\sum_{i=1}^{500}\left(\frac{\text{hedging cost} - \text{future value of the option price}}{\text{future value of the option price}}\right)^2
\end{aligned}$$

We found that the size of Δt does not affect performance much. Overall, our proposed strategy works as well as pure delta hedge.

Table 4.10 tabulates the performance of the four kinds of option hedge strategies for different proportion h. Each result is calculated from 500 sample paths. As the proportion h gets larger, the performance of our proposed strategy would approach pure delta hedge very quickly. That is because larger h lowers the probability of negative delta.

Table 4.11 tabulates the performance of the four kinds of option hedge strategies for different correlations ρ between the stock price and the stock index. Each result is calculated from 500 sample paths. Our proposed strategy is affected by correlation ρ. When the ρ is lowered, our proposed strategy works less well. That is because the stock index futures would not approximate the stock for lower ρ.

Conclusions

From Table 4.9, Table 4.10, and Table 4.11, we found that the standard delta hedge is the best way to hedge the risk of issuing options among the four methods. This is expected. The second-best method is our proposed strategy. Although the results are a little inferior to standard delta hedge, they are much better than doing nothing. Our proposed strategy's performance is affected mainly by the proportion h between call and put options, the hedging period Δt, and the coefficient of correlation ρ between

Table 4.9: Our proposed strategy with Δt = 1 week, Δt = 0.5 week, Δt = 1 day, Δt = 0.5 day. The other parameters are h=3 and ρ=0.9.

Interval (days)	Average Cost of Strategy 1	Mean of Difference	Performance 1	Performance 2
7	788407.66	0.0000000	0.0000000	0.2264698
3.5	789624.02	0.0000000	0.0000000	0.2160179
1	793562.12	0.0000000	0.0000000	0.1627655
0.5	792500.01	0.0000000	0.0000000	0.1559183

Interval (days)	Average Cost of Strategy 2	Mean of Difference	Performance 1	Performance 2
7	797571.94	9164.2732	0.2158717	0.3589524
3.5	793121.96	3497.9381	0.1654964	0.2338687
1	799676.03	6113.9085	0.0228389	0.1784682
0.5	800982.86	8482.8462	0.0218281	0.1749905

Interval (days)	Average Cost of Strategy 3	Mean of Difference	Performance 1	Performance 2
7	883561.25	95153.590	2.9097463	7.2820118
3.5	940462.30	150838.28	3.6760224	8.7465032
1	767617.52	−25944.60	2.3384684	5.2822241
0.5	811672.56	19172.548	2.5424015	6.4488288

Interval (days)	Average Cost of Strategy 4	Mean of Difference	Performance 1	Performance 2
7	798835.44	10427.780	0.1739361	0.2934093
3.5	790791.02	1167.0017	0.1178412	0.2786762
1	792419.80	−1142.318	0.1654557	0.2344123
0.5	791385.97	−1114.041	0.1067655	0.2308368

The future value of the option price after 1 year is $626136.76

Table 4.10: Our proposed strategy with h =1, 2, 3, 4, and 5. The other parameters are Δt=3.5 days and ρ=0.9.

h vs 1	Average Cost of Strategy 1	Mean of Difference	Performance 1	Performance 2
1	321786.90	0.0000000	0.0000000	0.2285220
2	552458.05	0.0000000	0.0000000	0.1539293
3	801454.12	0.0000000	0.0000000	0.1950473
4	1033381.0	0.0000000	0.0000000	0.2071219
5	1255703.1	0.0000000	0.0000000	0.2035515

h vs 1	Average Cost of Strategy 2	Mean of Difference	Performance 1	Performance 2
1	329016.84	7229.9371	0.4996975	0.2285221
2	562846.09	10388.032	0.0778123	0.1962335
3	812999.08	11544.964	0.0318617	0.2167291
4	1043707.0	10325.980	0.0177118	0.2217495
5	1268421.6	12718.504	0.0165151	0.2142318

h vs 1	Average Cost of Strategy 3	Mean of Difference	Performance 1	Performance 2
1	273305.95	-48480.95	5.1184743	3.2286545
2	496088.12	-56369.94	2.1253142	4.3717438
3	702557.91	-98896.21	1.9353576	4.9602412
4	1061438.9	28057.906	2.4543647	6.6085941
5	1436033.6	180330.54	3.5546861	8.9762178

h vs 1	Average Cost of Strategy 4	Mean of Difference	Performance 1	Performance 2
1	317660.62	-4126.281	3.9684412	0.5706308
2	542227.36	-10230.70	0.3508049	0.2897741
3	819362.28	17908.160	0.1466392	0.2655268
4	1032291.9	-1089.132	0.0906594	0.2542997
5	1274042.2	18339.146	0.0763532	0.2363905

The future value of the option price for h=1 after 1 year is \$284273.40
The future value of the option price for h=2 after 1 year is \$455205.08
The future value of the option price for h=3 after 1 year is \$626136.76
The future value of the option price for h=4 after 1 year is \$797068.44
The future value of the option price for h=5 after 1 year is \$968000.12

Table 4.11: Our proposed strategy with ρ =0.95, 0.90, 0.85, 0.80, 0.75, 0.7. The other parameters are Δt=3.5 days and h=3.

ρ	Average Cost of Strategy 1	Mean of Difference	Performance 1	Performance 2
0.95	791689.08	0.0000000	0.0000000	0.1764187
0.90	802325.86	0.0000000	0.0000000	0.1899318
0.85	792091.81	0.0000000	0.0000000	0.1979056
0.80	801912.14	0.0000000	0.0000000	0.2059129
0.75	802834.14	0.0000000	0.0000000	0.1954594
0.70	802308.34	0.0000000	0.0000000	0.2067121

ρ	Average Cost of Strategy 2	Mean of Difference	Performance 1	Performance 2
0.95	805834.67	14145.594	0.0212776	0.1952652
0.90	817418.98	15093.127	0.0279559	0.2084981
0.85	807333.63	15241.822	0.0447271	0.2221676
0.80	814382.77	12470.623	0.0661255	0.2386101
0.75	810595.28	7761.1438	0.0673743	0.2432202
0.70	815362.99	13054.646	0.0814117	0.2514781

ρ	Average Cost of Strategy 3	Mean of Difference	Performance 1	Performance 2
0.95	731238.52	−60450.56	2.1668983	5.2657528
0.90	689336.46	−112989.4	2.1144717	5.1027890
0.85	883659.38	91567.577	3.0010590	8.6795287
0.80	798000.15	−3911.993	2.6615011	6.4153498
0.75	818480.48	15646.337	2.6461852	6.6823607
0.70	708997.87	−93310.48	2.2841481	5.5965457

ρ	Average Cost of Strategy 4	Mean of Difference	Performance 1	Performance 2
0.95	787242.07	−4447.010	0.1597255	0.2626587
0.90	785160.31	−17165.54	0.0918368	0.2491453
0.85	795370.27	3278.4679	0.1360571	0.2612816
0.80	808656.39	6744.2475	0.1910225	0.2793054
0.75	799100.59	−3733.552	0.0906595	0.2625522
0.70	805234.47	2926.1310	0.1147044	0.2646407

The future value of the option price for h=3 after 1 year is \$626136.76

the stock price and the stock index. Overall, the results from our proposed strategy are very close to those from standard delta hedge for reasonable parameters.

In Figure 4.1, we plot the hedge cost for each hedge strategy. The x-axis represents individual simulations, and the y-axis represents the total hedge cost for each simulation. It is obvious that doing nothing produces the highest volatility.

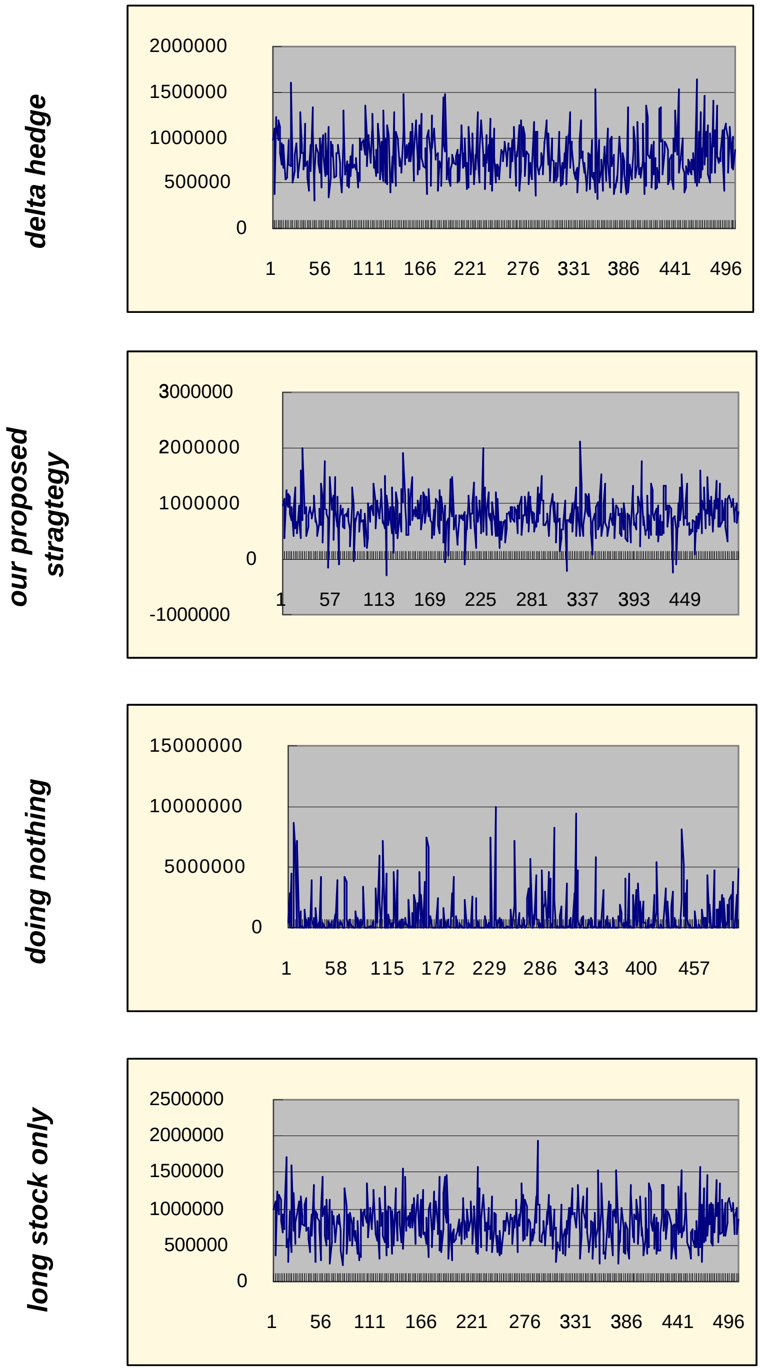

Figure 4.1: Results from four hedging strategies with Δt=1 week, ρ=0.9, and h=3. The x-axis represents individual simulations, and the y-axis represents the total hedge cost for each simulation.

Chapter 5

Conclusions

In this thesis, we propose a hedging strategy to hedge the risk of issuing put for options. This strategy consists of issuing multiple calls for each put at the same time, which reduces the chances of negative delta. The strategy replaces shorting stocks with shorting stock index futures in delta hedge that minimizes the return variance. We also calculate the probability of negative delta. This should be helpful for option issuers.

In Section 4.3, we found that the performance of our proposed strategy is affected mainly by the hedging period Δt, the coefficient of correlation ρ between the stock price and the stock index, and by the proportion h between the call option and the put option. As either h or ρ gets larger, results of our proposed strategy would get better. All our average results from simulations are better than those from doing nothing, but a little worse than the standard delta hedge. Thus, the proposed hedging strategy is an excellent strategy for put issuers to hedge their exposures.

This research points to a possible solution to the problem of puts. The impossibility of hedging puts has deterred securities firms from issuing puts. This is a loss to the market because the market loses the utility of puts in hedging risks. Our proposal shows that, with a little ingenuity, this difficulty can be circumvented.

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