Option-Adjusted Spreads of Mortgage-Backed Securities: a Client/Server System Based on Java and C++

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Contents

1 Introduction **1**

1.1 Motivations 2

1.2 Our System 2

1.3 Option-Adjusted Spreads of Mortgage-Backed Securities 4

1.4 Organization of This Thesis 5

2 Mortgage-Backed Securities **6**

2.1 The Origination of Mortgage-Backed Securities 7

2.2 The Creation of Mortgage-Backed Securities 7

2.3 The Way Mortgage-Backed Securities Repay Investors 8

2.4 The Option-Adjusted Spread Framework 10

2.5 The Effective Duration and Convexity for 1% 12

3 Interest Rate Models **14**

3.1 Equilibrium Models 14

3.2 No-Arbitrage Models 15

3.3 Cox-Ingersoll-Ross (CIR) Model 17

3.4 Comparisons between Two Implementations of the CIR Model 22

4 Prepayment Models **27**

4.1 Measure Prepayment Speed 27

4.2 Proportional Hazard Models 29

4.3 The Proportional Hazard Model in Our System 31

4.4 The Integration of Models into a Framework 32

4.5 Properties of MBS Given OAS 35
4.6 The Way To Use Effective Duration and Convexity 38

5 A Java/C++ Client/Server System **44**
5.1 Initiate the Connection between Client and Server 44
5.2 The Connection Protocol between Client and Server 45
5.3 Representations of Objects in the String-Oriented Connection between Client and Server 47
5.4 User Guide and User Interface 49
5.5 Performance Evaluation 52

6 Conclusion and Future Work **57**

A Some Relational Data **59**

Bibliography **62**

List of Figures

1.1 Client/Server Structure Built on the World Wide Web. 3

2.1 Outstanding Volume of Agency Mortgage-Backed Securities 1980-1996. 6

2.2 The Cash Flow of Mortgage-Backed Securities. 8

2.3 An Example of an Pass-Through MBS. 9

2.4 An Overview of the OAS Framework. 10

3.1 Monte Carlo Simulation with Variance Reduction Technique. 18

3.2 Simple Binomial Process Lattice. 21

3.3 The Number of Paths and Price Difference of an MBS with PSA 100 between CIR SBP and CIR with VRT. 22

3.4 The Relation of $\triangle t$ and Price of an MBS with PSA 100 and Using CIR SBP. 23

3.5 The Relation of $\triangle t$ and Percentage Price Difference of an MBS with PSA 100 Using CIR SBP. 24

3.6 The Relation of $\triangle t$ and Percentage Price Difference of an MBS with PSA 100 between CIR with VRT and CIR SBP. 25

3.7 Comparison of Monte Carlo Simulation for CIR SBP and CIR with VRT. 26

4.1 PSA 100. 28

4.2 Pass Through MBS with PSA 100. 35

4.3 Pass Through MBS with a Proportional Hazard Model Prepayment Assumption. 36

4.4 Comparison of PSA 100 and Proportional Hazard Model Prepayment Assumptions. 37

4.5 IO under Proportional Hazard Model Prepayment Assumption. . . . 38
4.6 PO under Proportional Hazard Model Prepayment Assumption. . . . 39
4.7 A Grand Overview of Pass-Through, IO, and PO under Proportional Hazard Model Prepayment Assumption. 40
4.8 Curve of $P(r)$ and r. 41
4.9 Market Price of Those Securities in Example 1, 2, 3, and 4. 42
4.10 Effective Duration of Those Securities in Example 1, 2, 3, and 4. . . . 42
4.11 Convexity of Those Securities in Example 1, 2, 3, and 4. 43

5.1 The Structure of Our Client/Server System. 45
5.2 The Protocol of the Connection between Client and Server. 46
5.3 An Applet on Our Homepage. 49
5.4 Main Menu Window. 50
5.5 Security Tree View Dialog. 51
5.6 Pricing and Analytical Tools Dialog. 52
5.7 MBS Pass-Through Dialog. 53
5.8 The Number of Partitions in a Month Dialog. 53
5.9 Simulation Path Number Dialog. 54
5.10 CIR Interest Rate Model Dialog. 54
5.11 Proportional Hazard Prepayment Rate Model Dialog. 55
5.12 The Result. 56
5.13 The Relation between the Number of Clients and Sojourn Time. . . . 56

A.1 Data for Figure **??** 59
A.2 Data for Figure **??** 60
A.3 Data for Figure **??** 60
A.4 Data for Figure **??** 60
A.5 Data for Figure **??** 61
A.6 Data for Figure **??** 61

Abstract

The financial derivatives market has been an area of innovation over the past few years. The continuing developments of the market are a result of the growth of economies and the revolution in technology. This market offers so many opportunities that almost every investor can find a product that fits his or her needs. Unfortunately, the market also poses many new challenges to the investor.

As a matter of fact, due to the complexity of these derivatives such as MBS, every investor needs a good framework and a right tool to help himself better understand these securities and avoid the pitfalls. Although there have been many pricing frameworks and analytical tools to help investors understand these derivatives, it is not easy for investors to access them. Fortunately, today, with the rapid development of the World Wide Web, more people become users of Web browsers. To keep up with the trend and provide people with much easier access to those pricing frameworks and analytical tools, we develop the client/server system based on Java/C++. This system makes it easy for investors or researchers to access these frameworks or tools. When investors understand these securities more, the risk of investment in these securities can be reduced, and the market will hopefully develop more rapidly. Our current system handles mortgage pass-throughs, and stripped mortgage-backed securities.

Option-adjusted spreads of mortgage-backed securities are an integral part of this system. In this thesis, we will detail the structure of this system, such as option-adjusted spreads, interest rate models, and prepayment models. Beyond option-adjusted spread, effective duration, effective convexity, and the special concepts and techniques used in the development of this system are presented in the thesis. We also develop ideas that reduce the Cox-Ingersoll-Ross interest rate tree's memory requirement from quadratic to linear.

Chapter 1

Introduction

This thesis presents a Java/C++ client/server system which provides a variety of financial computing services. This system can be accessed easily by Web browsers that support Java such as Microsoft Explorer 4.0. With a system built on the World Wide Web (WWW), it becomes more convenient for investors to use because of the ubiquity of the WWW. In this system, instead of deciding which models should be used to evaluate securities, investors have a wide choice of models which they can choose. Users can choose the model which suits their market assumptions in pricing or analyzing securities.

The object-oriented programming concept is used to develop this client/server system. All securities are implemented in a class hierarchy. The object-oriented concept is the trend in computer software technology. It has a lot of benefits. In particular, making everything such as securities an object can save a lot of time while developing, improving, modifying or managing this system.

In this thesis, we focus on the option-adjusted spread of mortgage-backed securities (MBS). Mortgage-backed securities are harder to evaluate than any other derivatives in the market thanks to their complexity in structuring. Thus they are not understood by average investors. This thesis will show some properties of mortgage-backed securities with the help of the option-adjusted spread, such as the relation of price and short rates, effective duration and convexity, etc. Mortgage-backed securities such as pass-throughs, interest only (IO), and principal only (PO) are also introduced and evaluated in this thesis.

1.1 Motivations

With the rapid development of economies, investors need more financial innovations to satisfy their diverse needs. However, due to the stochastic and interdependent nature of interest rate, prepayment and other risks, pricing these securities has become one of the most difficult challenges. Although it is impossible to totally eliminate all the investment risks and completely account for all potential problems, an investor with a good framework and the right tool in hand can avoid some pitfalls of investments in these securities, while reaping the rewards.

In order to more accurately price and analyze these securities or derivatives, many models and methods have been developed. However, while using these models or methods to price and analyze these securities, there are a few things that investors must know. Note that using some models to price or analyze the same security may have very different results because models are always developed with some assumptions and these assumptions may be very different. Different assumptions leads to different results. Is there a perfect or absolutely correct model with no exogenous assumptions for securities existing in the world? Most people don't think so. This granted, investors with their different assumptions about the future may prefer different models and methods with which they want to work. Investors should have choices. Different models approach a problem from various angles. The more investors understand these securities from different angles, the less their risks are. And this will make them willing to invest in these markets.

1.2 Our System

It is a Java/C++ client/server system, supporting multiusers. Several users can log on to this system simultaneously. The supervisor subsystem keeps an eye on who has logged on and what they are doing now. The object-oriented languages (Java and C++) make it easier to maintain and update this system. With object-oriented classes, this system achieves a framework with extension capabilities. Because the client is Java-based, people can access the system via a Web browser. The perfor-

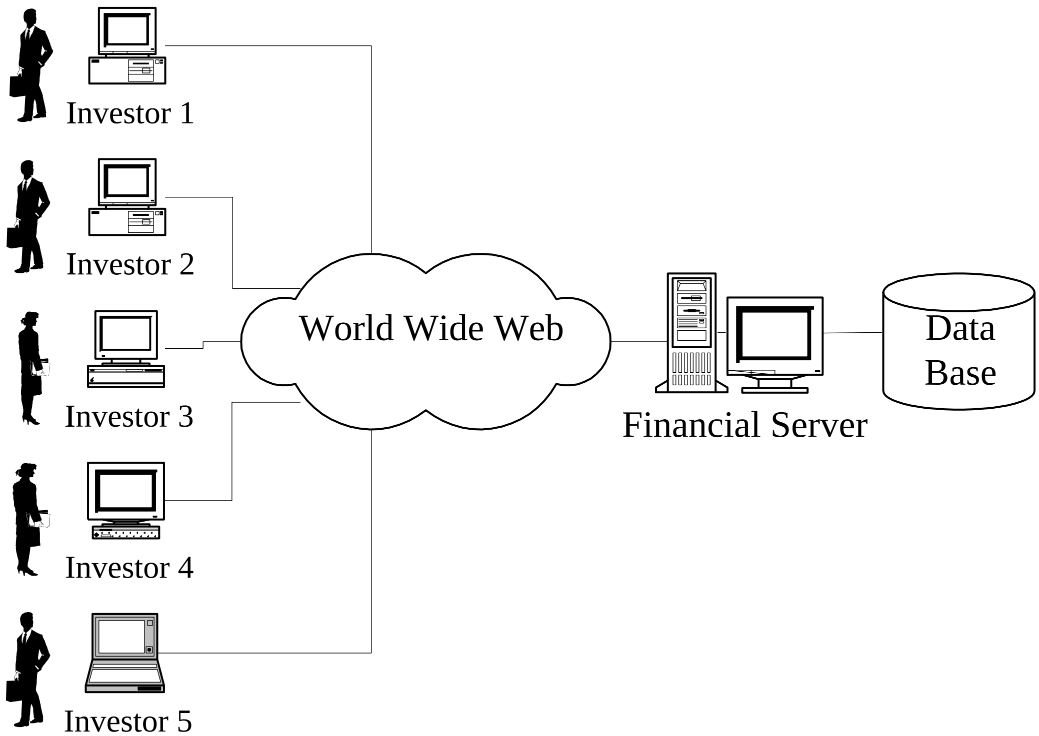

Figure 1.1: Client/Server Structure Built on the World Wide Web.

mance evaluation of the system when used by several users is done in Chapter 5. The constraints on the number of users in this system at a time depend on the system resources such as DRAM.

All securities are designed as classes in a hierarchy which can be shown in the security tree view dialog. With this security tree, it is much easier for users to have an overview of the portfolio. Interest rate models and prepayment models are handled as objects, too. With object-oriented programming methods, one can easily add a new security, a new interest rate model or a new prepayment model into this system to keep steps with the explosive development of financial innovations. And users can easily choose models which they want to work with from the friendly user interface and then combine these models to form their own frameworks. All these benefits from the Java's and C++'s object-oriented programming concept.

Two of the most object-oriented program languages, Java and VC++ 5.0 , are chosen to develope our client/server system. We pick Java to be the language to develope our client program for three reasons. The first reason is that Java is an object-oriented program language. It can fit our needs to develop an object-oriented system. The second reason is the Virtual Machine (VM) concept of Java. With the rapid development of the concept of virtual machines, the same code can work well on all kinds of platforms as long as they have virtual machine devices. By this aid, we can save a lot of development time in the future because you don't need to rewrite the code to work well on other platforms. The third reason is that a Java applet can be loaded and run by any Web browser which supports Java such as Explorer 4.0 and Navigator. Since most PCs have installed the Explorer 4.0 Web browser and more and more people have become users of the World Wide Web, it is not so difficult for investors to have the ability to access this system by the Explorer 4.0 browsers on their own personal computers, work stations or notebook computers. Thus, Java is a good choice to be the language to implement client programs.

Visual C++ 5.0 is chosen to be the platform to implement our server programs. VC 5.0 is supported by many powerful database systems and math libraries. Consider the load and complexity of financial computing. Server should be implemented with a well-developed language which can provide us with many well-defined functions and technique supports. Furthermore, a high-end PC these days can supports 2 to 4 Pentium processors at reasonable cost. This means programs written in VC++ can run on very powerful multiple-CPU computer systems. Visual C++ 5.0 is a good choice to be the language to implement server programs.

1.3 Option-Adjusted Spreads of Mortgage-Backed Securities

Mortgage-backed securities are much more complex than bonds. With prepayment and default risks, mortgage-backed securities can be viewed as coupon bonds with various options attached to them. Prepayment can be seen as a call option and default can be seen as a put option. Although MBS are almost guaranteed to avoid default risk, it is still very complicated to price them. Most street security houses use option-adjusted spread models to evaluate mortgage-backed securities. There are several steps to calculate the option-adjusted spread of a mortgage-backed security. First we need to choose an interest rate model to generate future paths of short rates. We also need to choose a prepayment model to generate cash flows along each interest rate path with security information. Then the present value can be obtained by discouting the cash flows with short rates plus a spread along the path. The average present value is obtained by averaging the present values of these paths. We work with a non-linear equation solver to find the option-adjusted spread that equates the average present value with the market price.

1.4 Organization of This Thesis

There are six chapters in this thesis. In order to understand the evaluation methodology of mortgage-backed securities, some background knowledge of MBS, interest rate models and prepayment models are provided in this thesis. In Chapter one, we introduce the motivations and objectives of this thesis. In Chapter two, we introduce the background knowledge of MBS and the framework of OAS, effective duration, and convexity. In Chapter three, we introduce interest rate models, discuss two implementations of the CIR model, and then do some comparison between them. In Chapter four, we introduce prepayment models, the proportional hazard prepayment model used in our system, the way to integrate interest rate models, prepayment models, OAS, effective duration, and convexity into a framework, and price properties with numerical examples. In Chapter five, we overview the system architecture, go over the protocols and data structure designs, show the user interface, and evaluate system performance. Finally, Chapter six is for future work and conclusion.

Chapter 2

Mortgage-Backed Securities

With high yields and assured credit quality, mortgage-backed securities appeal to many investors. In the last two decades the market for mortgage-backed securities (MBS) has rapidly grown from infancy to a multitrillion dollar industry. Such growth has changed how the mortgage market operates and has produced a keen interest in the valuation of these securities. More and more banks and other institutions such as life insurance companies that used to be the lenders now often act as the originators of these securities. In this chapter, we will review what mortgage-backed securities are, how they are created, and how they repay investors. We will discuss some problems in pricing and analysis and introduce option-adjusted spread methodology. At last, we will describe effective duration and convexity.

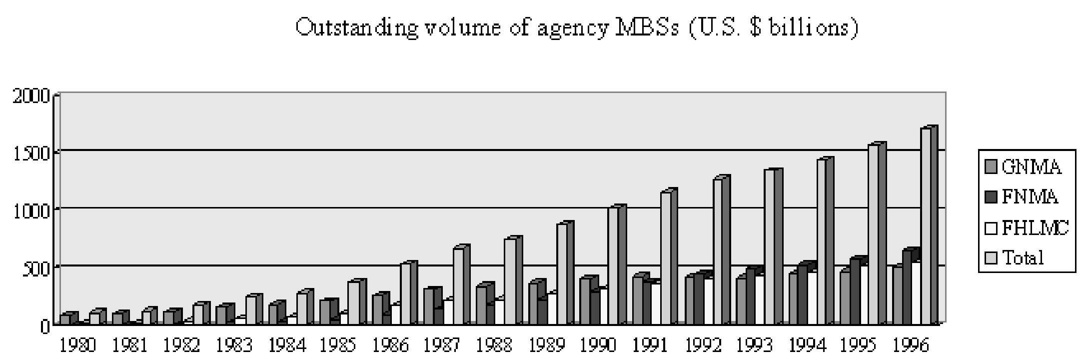

Figure 2.1: Outstanding Volume of Agency Mortgage-Backed Securities 1980-1996. Source: Public Securities Association.

2.1 The Origination of Mortgage-Backed Securities

A mortgage-backed security is an ownership interest in a pool of mortgages, where values have been added through the creation of an intermediate legal structure between investors and mortgages. Most of the time this legal structure takes the form of a pooling and servicing agreement or of a trust indenture. The issuer or trustee of the security acts as custodian to relieve investors of having to indeed hold these mortgages. There is a paying agent employed to pay investors on a monthly basis. And there is also a servicing company responsible for monthly collections from homeowners. Some kinds of guarantees or credit enhancements are usually added so that payments of full principal and interest to investors will be made even if homeowners default on their mortgages.

2.2 The Creation of Mortgage-Backed Securities

The creation of a mortgage security begins with a homeowner purchasing or refinancing a house by taking out a mortgage. Then the lender such as a bank usually sells or swaps the mortgage into a mortgage security. If the mortgage is sold, the lender will deliver the mortgage to the buyer at a specified future date and the buyer will pay a specified price in return. The buyer such as Fannie Mae and Freddie Mac usually pools together these mortgages purchased from different originators and packages them as a pass-through MBS or a CMO for resale. See Figure 2.2.

If the lender is a larger mortgage originator, he will often group his whole loans together and exchange them for a corresponding mortgage security with Ginnie Mae, Freddie Mac, or Fannie Mae. Whole loans mean unsecuritized mortgage loans. These new mortgages have all the advantages to the originator of securitization such as agency guarantees, enhanced liquidity, higher market value and the ability to be a collateral for further borrowings.

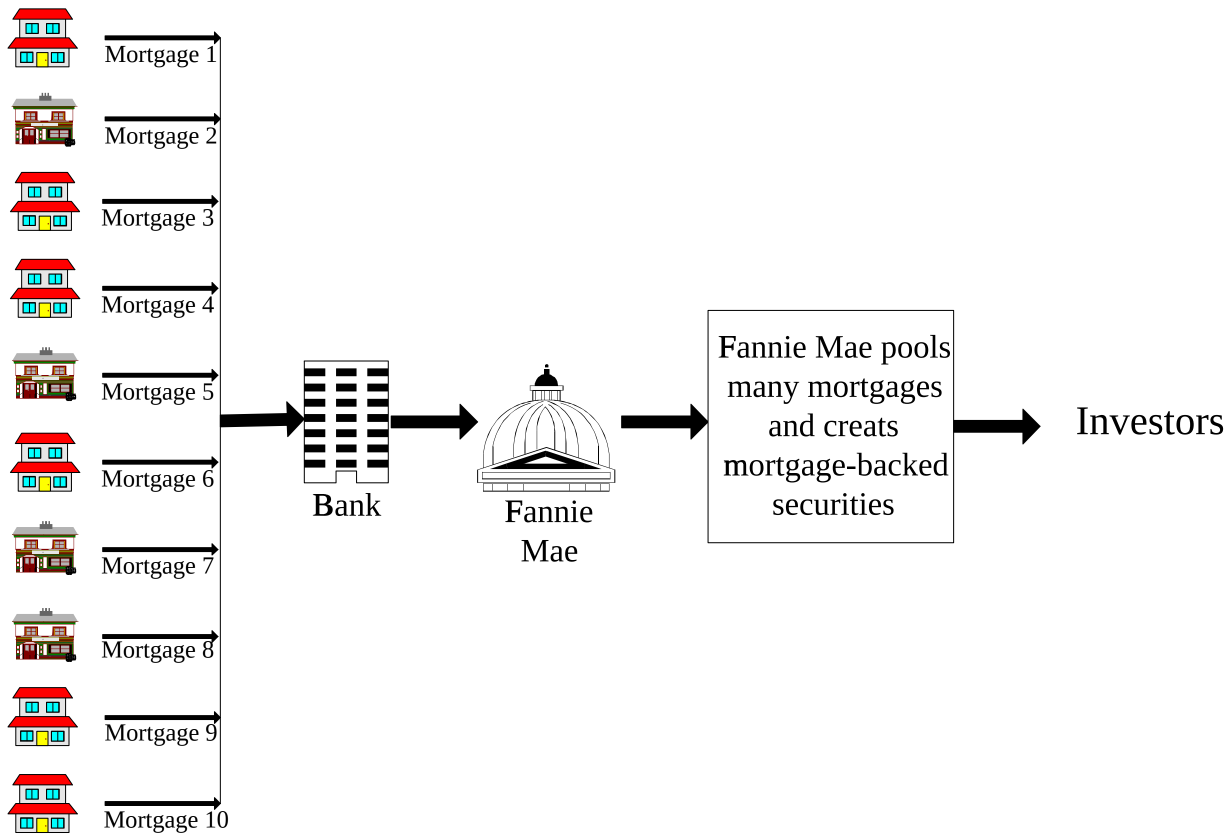

Figure 2.2: The Cash Flow of Mortgage-Backed Securities.

2.3 The Way Mortgage-Backed Securities Repay Investors

The most common type of mortgage-backed securities is a pass-through security. Payments including principal and interest from the underlying mortgages are passed through from homeowners to investors on a pro rata basis through a servicing agency after a fee is subtracted. An example of the process for a pass-through security is shown in Figure 2.3. In this example, homeowners makes the scheduled monthly payments a few days after the 1st of the month but before the 15th to avoid late penalties. The scheduled monthly payments includes principal and interest from the first day to the last day of that month. In this example, it is from January 1st to the

TRACKING THE CASH
(Fannie Mae MBS)

January 1	Mortgage interest for January begins accruing.
February 1	Homeowner makes monthly principal & interest payment.
February 11	February factor reported by agency.
February 18	Servicer remits principal & interest to agency.
February 25	Agency remits principal & interest to investors.

Figure 2.3: An Example of an Pass-Through MBS.

31st. However, mortgage interest and mortgage-backed security interest payments are both calculated on what is called a 30/360 basis. It means every month has 30 days and every year has 360 days.

Monthly payments including the scheduled monthly payments and prepayments are collected by a servicer, a company that is often but not always the originator. This company earns money by collecting a servicing fee, a small portion of the interest payment, and also profits from the collection of late payment fees such as interest earnings on its temporary possession of these monthly payments.

Each month the servicer passes through monthly mortgage payments of a pool, after a fee is subtract, to the agency or conduit on the same scheduled day. In this example, the servicer must pass through these payments to the Fannie Mae on the 18th of each month. These dates can vary with securities. Then the agency passes through these payments to investors on a pro rata basis on a specified date. It is February 25th in Figure 2.3.

Prepayments occur when some homeowners make more payments than the scheduled principal and interest. There are two forms of prepayments: *liquidation* and *curtailment*. A liquidation prepayment occurs when the homeowner pays off the entire remaining principal. If the homeowner speeds up the amortization of the mortgage by seeding in a little extra with each monthly payment, it is a curtailment prepayment. It doesn't matter which form of prepayments the homeowner takes. For a pool of mortgages, its prepayment form can be seen as a curtailment prepayment.

2.4 The Option-Adjusted Spread Framework

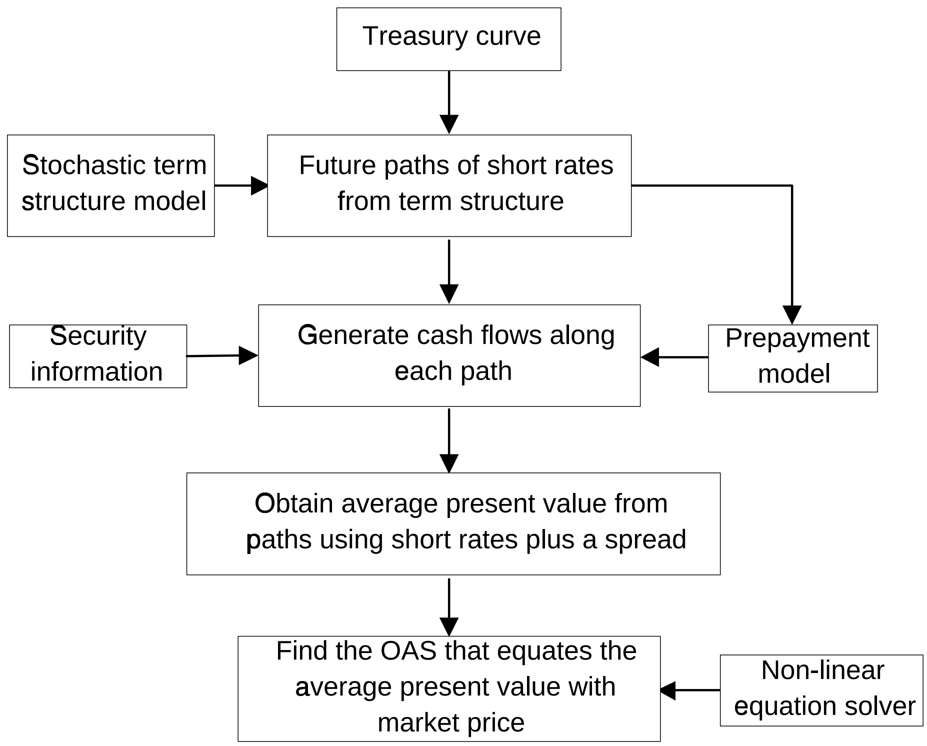

Figure 2.4: An Overview of the OAS Framework.

Pricing and analyzing MBS are known to be very complex. One of the problems is that while the required monthly payment is usually fixed, the portion that goes for principal is not. Because homeowners can pay off their mortgages at any time, cash flows of these securities are uncertain. Interest is paid first, and the remaining cash goes to pay down the principal. If there is no prepayment, the amortization term of the mortgage will be the same as the original one. When there is, the amortization of the mortgage will be irrevocably accelerated. The reason is that, there will be less interest to be paid in the next month due to less outstanding principal and more cash to go to redeem the principal, which results in even less interest the following month, and so forth until the principal is paid down. Because of this problem, prepayment models have been developed to predict the cash flow.

After deciding cash flows of a mortgage, the next problem is to find appropriate discounting functions. The present value of a mortgage is calculated by discounting its cash flows. Interest rate models are needed to derive these discounting functions. In fact, interest rates don't just play the role of discounting functions; because lower interest rates encourage homeowner to repay while higher interest rates discourage them, interest rates are one of that factors that affect prepayments.

In recent years, an option-based approach has been taken to analyze mortgage-backed securities. Models that treat mortgage-backed securities as bonds with embedded options and adjust values of these securities for the options are referred to as option-adjusted spread or OAS models.

What is option-adjusted spread (OAS)? If prepayments can be defined as a function of interest rates, then prepayment risk can be valued like a bond option. The option-adjusted spread of a mortgage security is the spread over the risk-free interest rate adjusted for the explicit value of its prepayment risk. Sometimes it is seen as the risk premium required to equate the risk-neutral probability-weighted present value of a bonds's cash flows with its market price. Sometimes it is seen as the implied prepayment hypothesis. While investors believe that OAS is the risk premium existing in the mortgage market, they should be aware of the effect of prepayment assumptions on mortgage valuation, too.

An OAS model consists of two components: an interest rate model and a prepayment model. If we describe a generic interest rate process as:

$$r_t^0 = f(r_{t-1}^0, \sigma, a_0, \ldots, a_n)$$

where r_t^0 is the risk-free interest short rate at time t, σ is the volatility of interest rate, and $a_0, \ldots, a_n$ describe the evolution of r_t^0 depending on the term structure. We describe a prepayment model as:

$$\pi_t = f(r_t^i, \ldots, r_t^k, b_0, \ldots, b_n)$$

where $r_t^i, \ldots, r_t^k$ describe the interest rate environment at time t, and parameters $b_0, \ldots, b_n$ describe the relationship between interest rates and prepayments.

These two models can be taken together to produce the relationship between price and OAS:

$$PRICE = \frac{1}{\Omega} \sum_{\varpi} P(\varpi) \sum_{i=1}^{m} PV([CF_t^{\varpi}(\pi_t^{\varpi})], [r_t^{\varpi} + OAS]) \tag{2.1}$$

where Ω describes a series of probabilistic events or paths of interest rates, $P(\varpi)$ is the probability of each such event, $[CF_t^{\varpi}(\pi_t^{\varpi})]$ is cash flow of the mortgage security at time t under event ϖ, and r_t^{ϖ} is the risk-free interest rate at time t under event ϖ. Equation (2.1) establishes a relationship between observed market price and OAS. Figure 2.4 overviews the framework used in our system. Interest rate models and prepayment models will be explored in more details in the next two chapters.

2.5 The Effective Duration and Convexity for 1%

Assume that the price of a security is a function of the current short rate, r. The price can be represented as $P(r)$. And then we have the following equation.

$$dP \approx \frac{\partial P}{\partial r} dr + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} (dr)^2 \tag{2.2}$$

If the equation above is divided by P, it becomes

$$\frac{dP}{P} \approx \frac{\partial P}{\partial r} \frac{1}{P} dr + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} \frac{1}{P} (dr)^2 \tag{2.3}$$

If the effective duration, D, and convexity, C, are defined as follows,

$$D \equiv -\frac{\partial P}{\partial r} \frac{1}{P}$$
$$C \equiv \frac{\partial^2 P}{\partial r^2} \frac{1}{P}$$

then Equation (2.3) becomes

$$\frac{dP}{P} \approx -D dr + \frac{1}{2} C (dr)^2 \tag{2.4}$$

It means, when the value of r is changed, the price percentage change can be expressed as a function of D, C, and dr. However, as the above formula stands, it is not very convenient to use in determining the price if the current short rate, r, moves to $r+1\%$. If we define D^* and C^* by

$$100 \times Ddr = D^* dr_{\%} \tag{2.5}$$

$$100 \times C(dr)^2 = C^*(dr_{\%})^2 \tag{2.6}$$

where $dr = \frac{dr_{\%}}{100}$, we can get the following

$$D^* = D \tag{2.7}$$

$$C^* = \frac{C}{100} \tag{2.8}$$

Suppose we further Equation (2.4) by 100,

$$\frac{dP}{P} \times 100 \quad \approx \quad -100 \times Ddr + \frac{1}{2} \times 100 \times C(dr)^2 \tag{2.9}$$

and substitute Equations (2.5) and (2.6) into Equation (2.9). Equation (2.9) becomes

$$\frac{dP}{P} \times 100 \quad \approx \quad -D^* dr_{\%} + \frac{1}{2} C^*(dr_{\%})^2 \tag{2.10}$$

In plain English, it means, if P is 100 at r and then r moves instantaneously into $r + 1\%$, the new price can be approximated by

$$100 - D^* \times 1 + \frac{1}{2} C^* \times 1^2 \quad = \quad 100 - D^* + \frac{C^*}{2}$$

As another example, if r changes into $r + 2\%$, the new price can be approximated by

$$100 - D^* \times 2 + \frac{1}{2} C^* \times 2^2 \quad = \quad 100 - 2D^* + 2C^*$$

So the price can be derived via simple mental calculations involving C^* and D^*. For this reason, duration and convexity will be expressed via D^* and C^* instead of D and C in our system. We will give some more examples in Chapter 4.

Chapter 3

Interest Rate Models

Interest rate models are usually known as yield curve models or term structure models. They describe the probabilistic behavior of all interest rates. Interest rate models come in one of two forms: equilibrium models and no-arbitrage models. We will go through the notion of equilibrium models and no-arbitrage models. After that, we will give more discussions on the CIR interest rate model, one of the equilibrium models. Then we will talk about techniques used to improve the performance when this interest rate model is used. In particular, we state the way to reduce the CIR tree's memory requirement from quadratic to linear. Finally, we do some comparisons with different implementations of the model.

3.1 Equilibrium Models

Equilibrium models usually specify a process for the short-term risk-free rate, r, with certain assumptions about economic variables. The short rate, r, at time t is the rate that applies to a very short period of time at time t. At times it is called the instantaneous short rate. It is used to evaluate security prices such as bond prices, MBS prices, and other derivative prices in a risk-neutral world.

If the process for r involves only one source of uncertainty, it is referred to as a one-factor model. In a risk-neutral world, r is usually described by an Ito process of the form

$$dr = m(r)dt + s(r)dz$$

where m is referred to as the instantaneous drift, and s typically as the instantaneous volatility. Both m and s are assumed to be functions of r but independent of time. In a one-factor model, all rates move in the same direction over any short time interval. However they may not all move by the same amount. As a matter of fact, it implies that the term structure may have different shapes. There are several one-factor equilibrium models. Three are sampled below.

1. $m(r) = \mu r, \quad s(r) = \sigma r$ (Rendleman and Bartter model)

2. $m(r) = a(b - r), \quad s(r) = \sigma$ (Vasicek model)

3. $m(r) = a(b - r), \quad s(r) = \sigma \sqrt[2]{r}$ (Cox, Ingersoll, and Ross model)

In a equilibrium model, the term structure of interest rates at any given time can be obtained from the value of r at that time and the risk-neutral process for r. Everything about the initial term structure can be defined once the risk-neutral process for r is defined. In fact, all the equilibrium models treat the initial term structure as an output from them.

3.2 No-Arbitrage Models

Instead of treating the initial term structure as an output in an equilibrium model, the initial term structure is an input to a no-arbitrage model. No-arbitrage models are models designed to be exactly consistent with today's term structure. Assuming the term structure depends on just one factor, the risk-neutral process for $P(t,T)$ has the form,

$$dP(t,T) = r(t)P(t,T)dt + \nu(t,T,\Omega_t)P(t,T)dz(t) \tag{3.1}$$

where $P(t,T)$ denotes the price at time t of a discount bond with one dollar principal maturing at time T, $r(t)$ denotes short-term risk-free interest rate at time t, $\nu(t,T,\Omega_t)$

denotes the volatility of $P(t,T)$, and $dz(t)$ denotes Wiener process driving term structure movements. The forward rate, $f(t,T_1,T_2)$, can be related to discount bond prices as follows:

$$f(t,T_1,T_2) = \frac{\ln P(t,T_1) - \ln p(t,T_2)}{T_2 - T_1} \tag{3.2}$$

From Equation (3.1) and Ito's lemma, we have

$$d\ln P(t,T_1) = \left[r(t) - \frac{\nu(t,T_1,\Omega_t)^2}{2}\right] dt + \nu(t,T_1,\Omega_t)dz(t) \tag{3.3}$$

and

$$d\ln\left[P(t,T_2)\right] = \left[r(t) - \frac{\nu(t,T_2,\Omega_t)^2}{2}\right] dt + \nu(t,T_2,\Omega_t)dz(t) \tag{3.4}$$

So Equation (3.2) becomes

$$df(t,T_1.T_2) = \frac{\nu(t,T_2,\Omega_t)^2 - \nu(t,T_1,\Omega_t)^2}{2(T_2-T_1)}dt + \frac{\nu(t,T_1,\Omega_t) - \nu(t,T_2,\Omega_t)}{T_2 - T_1}dz(t) \tag{3.5}$$

Equation (3.5) shows that the risk-neutral process for f depends only on the ν's. When we let $T_1 = T$ and $T_2 = T + \triangle t$ in Equation (3.5) and then take limits as $\triangle t$ tends to 0 , $f(t,T_1,T_2)$ becomes the instantaneous forward rate $F(t,T)$, the coefficient of $dz(t)$ becomes $\nu_T(t,T,\Omega_t)$, and the coefficient of dt becomes

$$\frac{1}{2}\frac{\partial[\nu(t,T,\Omega_t)^2]}{\partial T} = \nu(t,T,\Omega)\nu_T(t,T,\Omega_t)$$

Equation (3.5) now becomes

$$dF(t,T) \;=\; \nu(t,T,\Omega)\nu_T(t,T,\Omega_t)dt - \nu_T(t,T,\Omega_t)dz(t) \tag{3.6}$$

Since

$$F(t,t) \;=\; F(0,t) + \int_0^t dF(\tau,t) \tag{3.7}$$

and $r(t) = F(t,t)$, where $r(t)$ is the short rate, the equation below can be derived from Equations (3.6) and (3.7)

$$r(t) = F(0,t) + \int_0^t \nu(\tau,t,\Omega_\tau)\nu_t(\tau,t,\Omega_\tau)d\tau + \int_0^t \nu_t(\tau,t,\Omega_\tau)dz(t)$$

It means the short rate, $r(t)$, can be derived from forward rates and volatilities. Thus, no-arbitrage models can be consistent with today's term structure.

3.3 Cox-Ingersoll-Ross (CIR) Model

In 1985, Cox, Ingersoll, and Ross proposed a continuous-time first-order autoregressive process interest rate model as follows

$$dr = \lambda(\theta - r)dt + \sigma\sqrt{r}dz \tag{3.8}$$

where θ is the long-term value of short rate, λ is the speed of adjustment for short rate, $\sigma\sqrt{r}$ is the volatility of short rate, and dz is a standard Wiener process with $E[dz] = 0$, and $E[dz^2] = dt$. Both θ and σ are positive constant.

In the CIR model, the process of short rate has a standard deviation proportional to $\sqrt{r}$. That means short rate can't be negative at any time. It is more reasonable than some equilibrium models which don't have this property. Cox, Ingersoll, and Ross also showed that, in their model, the price of a zero-coupon bond paying one dollar at maturity in terms of the current short rate, $r(t)$, is

$$P(t,T) = A(t,T)\exp^{-B(t,T)r(t)}, \tag{3.9}$$

where

$$A(t,T) = \left(\frac{2\gamma\exp^{(\lambda+\gamma)(T-t)/2}}{(\gamma+\lambda)(\exp^{\gamma(T-t)}-1)+2\gamma}\right)^{2\lambda\theta/\sigma^2}$$

$$B(t,T) = \frac{2(\exp^{\gamma(T-t)}-1)}{(\gamma+\lambda)(\exp^{\gamma(T-t)}-1)+2\gamma}$$

$$\gamma = \sqrt{\lambda^2 + 2\sigma^2}$$

The CIR stochastic process can be directly simulated by the Monte Carlo method, which is a sampling scheme used for solving stochastic and even deterministic problems. In some cases, the time evolution of a stochastic process is not easy to describe analytically, and Monte Carlo simulation may be the only strategy that succeeds. Monte Carlo estimate and the true value differ owing to two reasons: sampling variance and the discreteness of the sample paths. The former can be controlled by the number of replications, while the latter can be controlled by the number of observations along the sample path.

When Monte Carlo simulation is employed, variance reduction techniques (VRT) are almost used to improve efficiency. A variance reduction technique called *antithetic variables* is used with Monte Carlo simulation here. Suppose Y_1 and Y_2 are identically distributed random variables with mean θ. We have

$$Var\left[\frac{Y_1+Y_2}{2}\right] = \frac{Var[Y_1]}{2} + \frac{Cov[Y_1,Y_2]}{2} \tag{3.10}$$

The variance would be smaller than $Var[Y_1]/2$ if Y_1 and Y_2, rather than being independent, are negatively correlated. Note that $Var[Y_1]$ is precisely the variance of the Monte Carlo method with two replications. Now consider the Ito process,

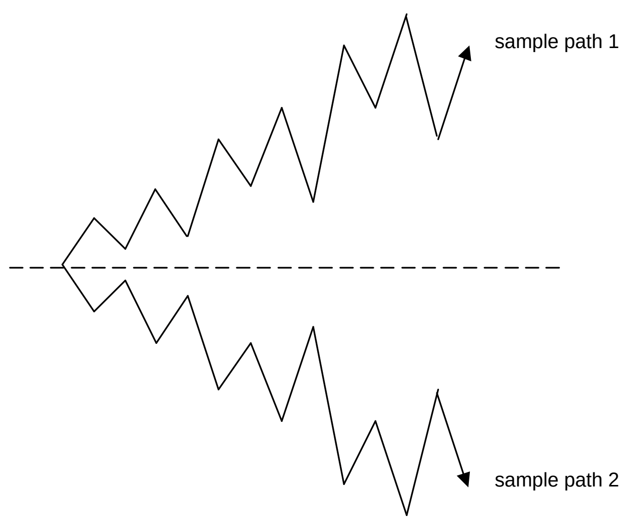

Figure 3.1: MONTE CARLO SIMULATION WITH VARIANCE REDUCTION TECHNIQUE.

$dr = \lambda(\theta - r)dt + \sigma\sqrt{r}\sqrt{dt}\xi$ where ξ is a standard normal distribution. Let g be a function of n samples from one sample path, $g(r)$, and we are interested in estimating $E[g(r)]$. Suppose one simulation run has realizations $\xi_1, \cdots, \xi_n$ for the normally distributed fluctuation term, ξ, generates samples, $r_1, \cdots, r_n$. The estimate is $g(r)$ with $r \equiv (r_1, \cdots, r_n)$. Instead of generating n more numbers for ξ from scratch, we compute another estimate, $g(r')$, from the samples $r' \equiv (r'_1, \cdots, r'_n)$ generated by $-\xi_1, \cdots$

, $-\xi_n$. The antithetic variables method amounts to outputting $(g(r)+g(r'))/2$. The estimate is based on a simple average of two averages, one based on the sample path r and the other based on r'. The fact that these two averages are negatively correlated leads to a reduction in variance. If N sample paths of r are generated, the resulting antithetic variables estimator is simply the average across all $2N$ paths. The reduction in variance comes from two sources: a doubling of the number of replications from N to $2N$ and negative correlation between the antithetic variables.

When the volatility of short rate is constant, then a model can be constructed as a binomial lattice such as Tian's simple binomial process. In the Tian's simplified binomial process, we approximates a mean-reverting diffusion process that is path independent. It means the short rate level must be the same after an upward move followed by a downward move as it is after a downward move followed by an upward move. Since the CIR model does not have a constant volatility, it is necessary to transform the short rate process in Equation (3.8) into a form that has a constant volatility in order to construct a path-independent binomial lattice tree. The following is a simple transformation that achieves this purpose.

$$\phi = \sqrt{r} \tag{3.11}$$

Thus, Equation (3.8) has the following form

$$d\phi = qdt + \nu dz \tag{3.12}$$

where

$$\begin{aligned}
q &= \lambda(\theta - r)\frac{\partial \phi}{\partial r} + \frac{\sigma^2 r \partial^2 \phi}{2 \partial r^2} \\
&= \frac{\alpha_1}{\phi} - \alpha_2 \phi \\
\alpha_1 &= \frac{4\lambda\theta - \sigma^2}{8} \\
\alpha_2 &= \frac{\lambda}{2} \\
\nu &= \frac{\sigma}{2}
\end{aligned}$$

Besides, for a mean-reverting path-independent binomial process that converges to the process in Equation (3.8), it leads to the following solution

$$\triangle\phi_{ij} = u_{ij} = d_{ij} = \nu\sqrt{\triangle t} \tag{3.13}$$

$$p_{ij} = \frac{1}{2} + \frac{q(\phi_{ij})\sqrt{\triangle t}}{2\nu} \tag{3.14}$$

However, since p is a probability, we require that $0 \leq p_{ij} \leq 1$. It implies that

$$\mid q(\phi_{ij}) \mid \leq \frac{\nu}{\sqrt{\triangle t}} \tag{3.15}$$

Thus, we can derive the following upper and lower boundaries of the lattice tree.

$$\phi_{min} \leq \phi \leq \phi_{max} \tag{3.16}$$

$$\phi_{min} = \left| -\left(\frac{\sigma}{2\lambda\sqrt{\triangle t}}\right) + \sqrt{\frac{\sigma^2(1 - \lambda \triangle t)}{4\lambda^2 \triangle t} + \theta} \right| \tag{3.17}$$

$$\phi_{max} = \left| \left(\frac{\sigma}{2\lambda\sqrt{\triangle t}}\right) + \sqrt{\frac{\sigma^2(1 - \lambda \triangle t)}{4\lambda^2 \triangle t} + \theta} \right| \tag{3.18}$$

They imply the following short rate equations,

$$r_{min} = \left[\sqrt{\frac{\sigma^2(1 - \lambda \triangle t)}{4\lambda^2 \triangle t} + \theta} - \left(\frac{\sigma}{2\lambda\sqrt{\triangle t}}\right)\right]^2 \tag{3.19}$$

$$r_{max} = \left[\sqrt{\frac{\sigma^2(1 - \lambda \triangle t)}{4\lambda^2 \triangle t} + \theta} + \left(\frac{\sigma}{2\lambda\sqrt{\triangle t}}\right)\right]^2 \tag{3.20}$$

$$r_{ij} = \left[\sqrt{r_{00}} + \left(\frac{j\sigma\sqrt{\triangle t}}{2}\right)\right]^2 \tag{3.21}$$

$$p_{ij} = \frac{1}{2} + \frac{\sqrt{\triangle t}}{\sigma} \times \left(\frac{4\lambda\theta - \sigma^2}{8\sqrt{r_{ij}}} - \frac{\lambda\sqrt{r_{ij}}}{2}\right) \tag{3.22}$$

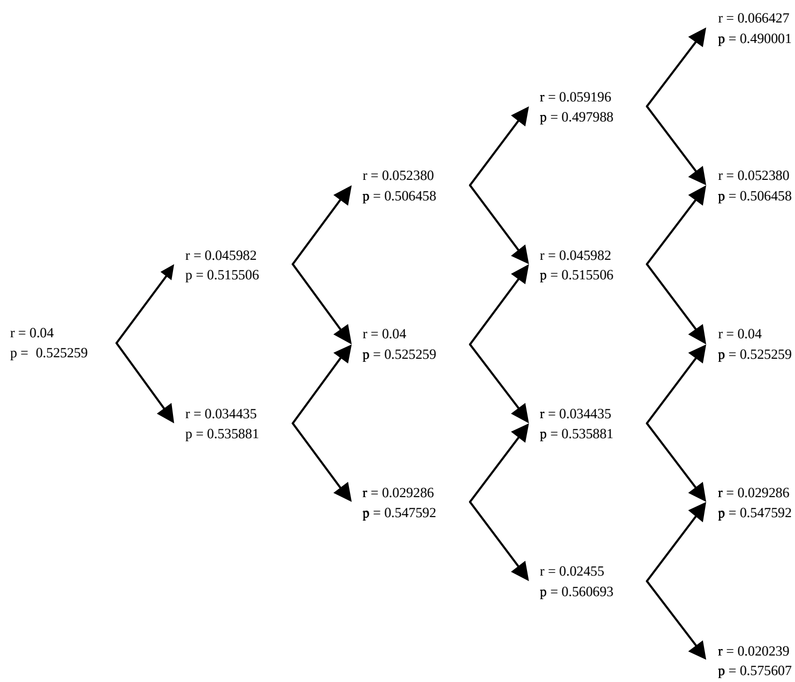

Figure 3.2: Simple Binomial Process Lattice. $r_{max} = 3.113994$ ($j_{max} = 108$) , $r_{min} = 0.001062$ ($j_{min} = -11$).

For example, if $r_{00} = 0.04$, $\lambda = 0.2$, $\theta = 0.07$, and $\sigma = 0.1$, we will have the binomial lattice in Figure 3.2.

The memory space required can be less than $O(n^2)$ if we add the following observation. Because of the constant volatility of this tree, the probability and short rate at node (i, j) are the same as the probability and the short rate at node $(i-1, j)$ or node $(i+1, j)$. It means the probabilities and short rates at nodes on the same level of j are identical. Consequently, a single node is enough to represent all the nodes on the same level of j. At most, there are $2n-1$ levels in the tree of n periods. Hence, at most, $2n-1$ nodes are needed to represent this simple binomial process tree. Therefore, the memory space required is a paltry $O(n)$. So a tree will be in reality represented by a linear array.

Assuming cash flows of a security are path-independent, we compare these following two implementations of CIR model: Monte Carlo simulation of the stochastic process and sampling paths on the simple binomial process (SBP) lattice. SBP is more efficient than Monte Carlo simulation. However, cash flows of a mortgage-backed security are not path-independent; in fact, interest rate paths and cash flows of a MBS interact strongly. It means that we can't use the simple binomial process lattice without modification. But how is it to run Monte Carlo simulation on this simple binomial process lattice? Will it be more efficient? We answer the question in the next section.

3.4 Comparisons between Two Implementations of the CIR Model

In this section, we compare the two implementations of the CIR model mentioned in Section 3.3 while working with a pass-through MBS with prepayment assumption PSA 100. First, we compare these two implementations by pricing a pass through

Figure 3.3: THE NUMBER OF PATHS AND PRICE DIFFERENCE OF AN MBS WITH PSA 100 BETWEEN CIR SBP AND CIR WITH VRT.

MBS backed by \$1,000,000 of mortgages with a 7% coupon, 360 months to maturity, a prepayment speed of PSA 100, and a 1% OAS. Cash flows of this security are interest-rate-path-independent. Then we observe the relation between the number of paths and the price difference of this security when we use these two implementations of the CIR model with the parameters: $(r_{00},\ \lambda,\ \theta,\ \sigma,\ \triangle t) = (0.07,\ 1.0,\ 0.07,\ 0.2,\ 0.00278)$.

As we see from Figure 3.3, the difference between CIR with VRT and CIR SBP is not large. If the number of paths is larger than 500, the difference is smaller than 0.1%. When the number of paths is larger than 8500, the difference is smaller than 0.01%. Of course more paths demand more time. But what if $\triangle t$ is adjusted?

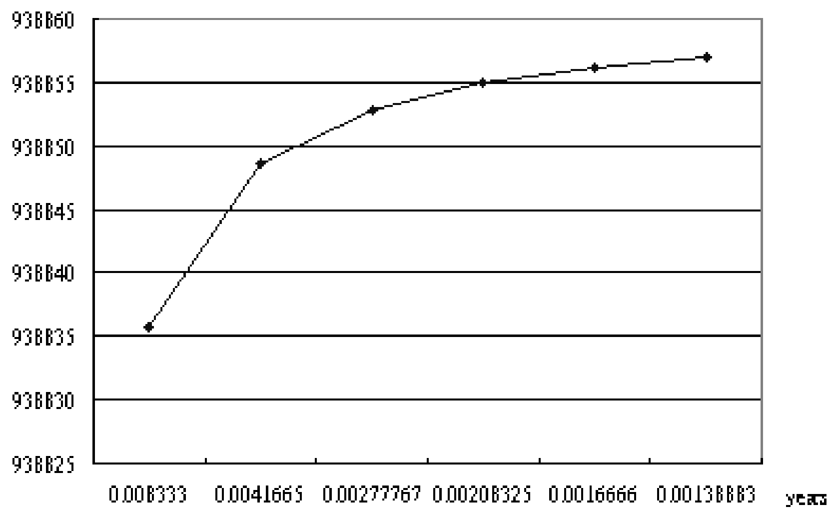

Figure 3.4: The Relation of $\triangle t$ and Price of an MBS with PSA 100 and Using CIR SBP.

What would happen to the price? Figure 3.4 tells us that the price changes with $\triangle t$ but eventually converges to a value when $\triangle t$ becomes short enough. In fact, Figure 3.5 tells us that the difference between them is smaller than 0.0025% if we take the price under $\triangle t = 0.001389$ (years), corresponding to 60 partitions in a month, as the true value. Even when $\triangle t$ is 0.08333 (years), corresponding to 1 partition in a month, their difference is not very large: only 0.06162% to the convergent value. Figure 3.6 documents what happens when we run Monte Carlo simulation with 5000

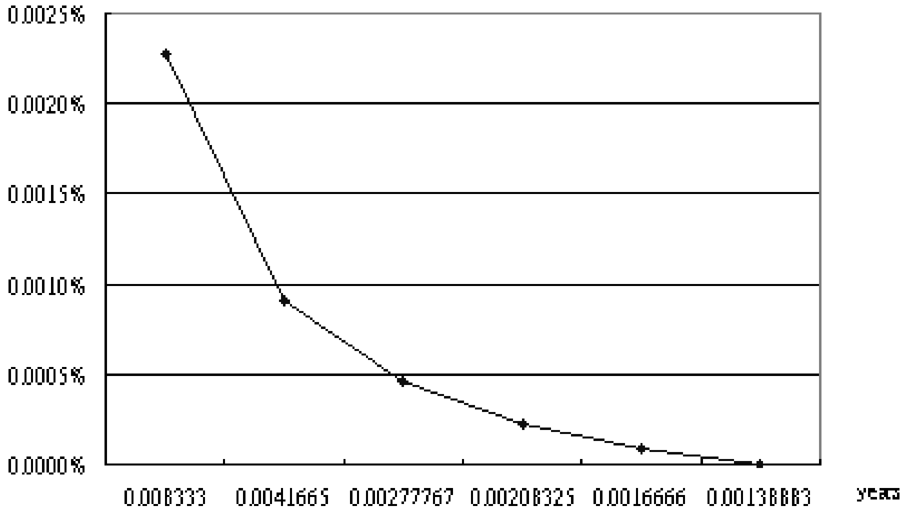

Figure 3.5: The Relation of $\triangle t$ and Percentage Price Difference of an MBS with PSA 100 Using CIR SBP.

paths for CIR with VRT and change $\triangle t$. The price calculated from CIR SBP with $\triangle t = 0.001389$ (years) is assumed to be correct. The result shows that smaller $\triangle t$ without correspondingly large numbers of simulation paths can't make the price calculated from CIR with VRT more convergent to the correct value; in fact, it will make them diverge. This result is expected because, when the space of a short rate lattice is much wider, a fixed number of paths is less likely to represent it. Even here, the percentage differences remain small enough.

Suppose we change the prepayment assumption to a proportional hazard model to be introduced shortly. We use a proportional hazard model with the parameters (γ, p, β_1, β_2, β_3, β_4) = (0.01496, 2.31217, 0.38089, 0.00333, 3.57673, 0.26570). Now we run CIR with VRT and CIR SBP as the interest rate models and then observe the relation between their prices and the path numbers. Note that the cash flows are now path-dependent.

As we see from Figure 3.7, when we run Monte Carlo simulation for CIR with SBP lattice and with VRT, the price calculated with SBP is not as stable as calculated from Monte Carlo simulation with VRT. And from Figure 3.3, we know the price calculated with VRT is very close to the price derived from SBP lattice when the

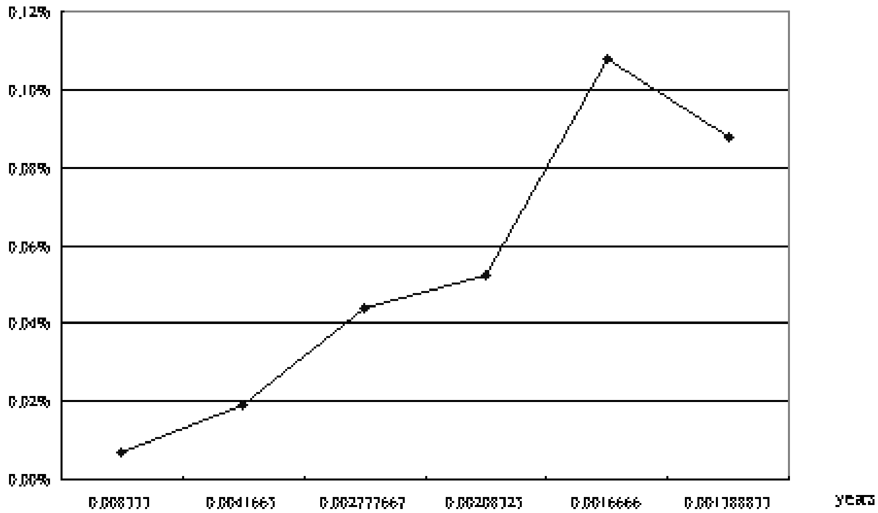

Figure 3.6: The Relation of $\triangle t$ and Percentage Price Difference of an MBS with PSA 100 between CIR with VRT and CIR SBP.

security is path-independent and the number of paths is larger than 500. Thus, running is not suitable to run Monte Carlo simulation for CIR on the SBP lattice is not recommended; it is not suitable to be used to price path-dependent securities such as MBS.

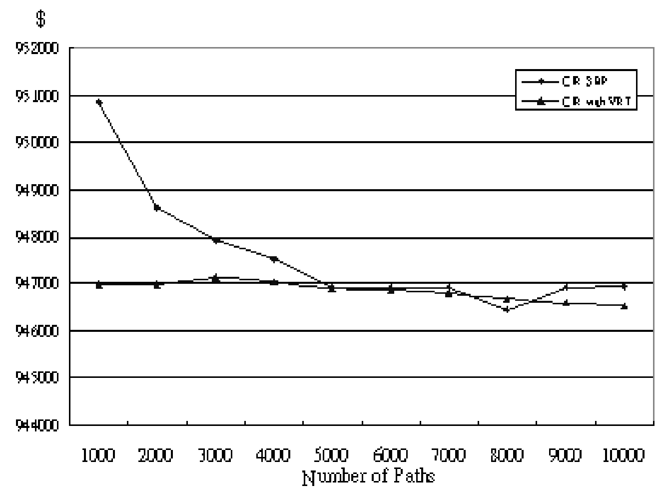

Figure 3.7: Comparison of Monte Carlo Simulation for CIR SBP and CIR with VRT. $\triangle t = 0.002777667$ (years)

Chapter 4

Prepayment Models

Homeowner prepayments of mortgages are one aspect of MBS investing that is most different from other securities. Yet changes in these prepayments are the most important risk facing the MBS investor. To predict when a mortgage will be prepaid is very difficult, but the rate of prepayment out of a million mortgages can be predicted with quite a bit of accuracy. This chapter will discuss the history of prepayment behavior, the various prepayment measurement methods used by the MBS industry, and the ways in which prepayments change with changing interest rates.

4.1 Measure Prepayment Speed

To better discuss prepayments, we need some methods for describing prepayment speeds. There are four different methods in use by the MBS industry: constant prepayment rate (CPR), FHA prepayment, Public Securities Association (PSA), and proportional hazard prepayment rate model.

The constant prepayment rate (CPR) method is the easiest to understand. It works just like it sounds: a 12% CPR means that we expect a constant 12% of the remaining mortgage pool balance to be paid down in each and every future year. While the rate of prepayments is constant at 12%, the dollar amount of the prepayments declines each year. The reason is that the pool balance declines with each prepayment and amortization. CPR is also called conditional prepayment rate, because each year's prepayments are conditional on the pool balance at the beginning of the year.

Anyway, the mathematics result is the same.

While easy to understand, the CPR method suffers from the disadvantage that prepayment rates are treated as being constant in each and every year, which is obviously inconsistent with the historical experience of actual mortgage prepayments. The second method, the FHA prepayment method, avoids such a problem. The prepayment rates of FHA are empirical rates. For this reason FHA prepayment method was in common use in the early days of the MBS market. An FHA prepayment speed is always measured relative to a particular FHA prepayment experience table. A speed of FHA 100% assumes annual prepayments exactly equal to the FHA historical experience. However, FHA prepayment speeds suffer from several disadvantages. Computations are inconvenient, possibly requiring a complete 30-year FHA prepayment model. Worse, there are a number of different FHA prepayment histories, each based on different years. Therefore, buyer and seller must make sure they are using the same tables. For these reasons, FHA speeds are rarely used today.

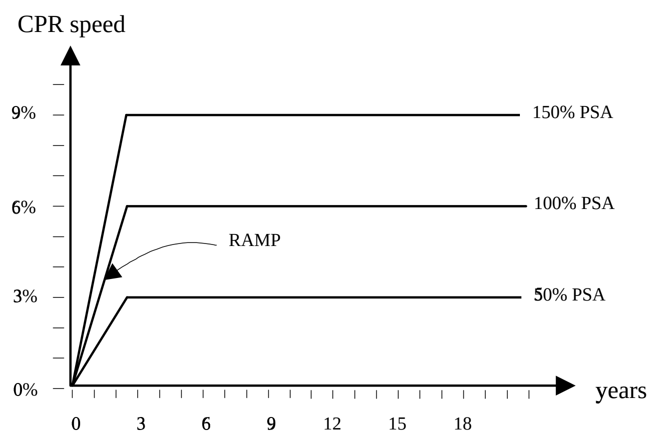

Figure 4.1: PSA 100.

The investment banking industry developed the Public Securities Association method to take the advantages of both the CPR and FHA methods, while minimizing their problems. Like the CPR method, the PSA method offers a consistent, easy-to-calculate and easy-to-understand method of measuring prepayments. Like the FHA method, it includes the gradual increase in prepayment frequency seen in the first years after mortgage origination. Beginning 30 months after origination, the PSA method assumes a level prepayment rate, which is much like the CPR method. However, the PSA method is measured in speeds that roughly approximate historical FHA speeds. The prepayment rate of a PSA 100 level is gradually grown from 0% to 6% in the initial 30 months. Once beyond the initial 30 months, the prepayment rate is exactly 6% CPR speed.

The fourth method is the proportional hazard model. We will devote the next section to it.

4.2 Proportional Hazard Models

In 1986, Green and Shoven proposed the proportional-hazard model to estimate the influence of various explanatory variables or covariates on the mortgagor's prepayment decision. Let's focus on what is a prepayment function first. A prepayment function gives the probability of a mortgagor prepaying a mortgage during a particular period, conditional on the mortgage not having been prepaid before that period. By expressing this conditional probability as a function of various explanatory variables or covariates, we can assess statistically the significance of these covariates in influencing a mortgagor's prepayment decision.

Let T be a continuous random variable representing the time until prepayment of a mortgage, and let t stands for its realization. Let $\bar{\nu}=(\nu_1,\nu_2,\cdots,\nu_n)$ be a vector of explanatory variables or covariates upon which the time until prepayment may depend, while $\bar{\theta}=(\theta_1,\theta_2,\cdots,\theta_m)$ is a vector of parameters to be estimated. The prepayment function $\pi(t;\bar{\nu},\bar{\theta})$ is defined by

$$\begin{aligned}\pi(t;\bar{\nu},\bar{\theta}) &= \lim_{\triangle t\to 0^+}\frac{P(t\leq T\leq t+\triangle t|T\geq t)}{\triangle t}\\ &= \frac{f(t;\bar{\nu},\bar{\theta})}{F(t;\bar{\nu},\bar{\theta})}\end{aligned}$$

where $F(t;\bar{\nu},\bar{\theta})$ is the survivor function

$$F(t;\bar{\nu},\bar{\theta}) = P(T\geq t|\bar{\nu},\bar{\theta})$$

and $f(t;\bar{\nu},\bar{\theta})$ is the probability-density function of T:

$$\begin{aligned} f(t;\bar{\nu},\bar{\theta}) &= \lim_{\triangle t\to 0^+} \frac{P(t \le T \le t+\triangle t)}{\triangle t} \\ &= -\frac{dF(t)}{dt} \end{aligned}$$

The prepayment function $\pi(t;\bar{\nu},\bar{\theta})$ specifies the instantaneous rate of prepayment at $T=t$ conditional upon the mortgage not having been prepaid before time t.

Now we model the prepayment function by a proportional-hazards model:

$$\pi(t;\bar{\nu},\bar{\theta}) = \pi_0(t;\gamma,p)\exp(\bar{\beta}\bar{\nu}) \tag{4.1}$$

where the base-line hazard function $\pi_0(t;\gamma,p)$ is given by the log-logistic hazard function

$$\pi_0(t;\gamma,p) = \frac{\gamma p(\gamma t)^{p-1}}{1+(\gamma t)^p}$$

and $\bar{\beta}\bar{\nu}$ is the inner product of two vectors $\bar{\beta}$ and $\bar{\nu}$. The base-line hazard function measures the probability of prepayment under homogeneous conditions, $\bar{\nu}=0$. The log-logistic hazard function admits a variety of relationships between the probability and the age of the mortgage. In fact, for $p \ge 1$, the probability of prepayment increases from zero to a maximum at

$$t^* = \frac{(p-1)^{\frac{1}{p}}}{\gamma}$$

and then decreases to zero. It means that there exists a mortgage age at which the probability of prepayment is maximum. This is consistent with the experience that, all other things being equal, conditional prepayment rates are low in the early years of a mortgage, increase as the age of the mortgage increases, and then decrease with further seasoning.

But the probability of prepayment does not only depend on a mortgage's age. The prepayment function, Equation (4.1), takes account of the fact that various explanatory variables, $\bar{\nu}$, influence the prepayment decision. Note that, according to the proportional-hazards model, these variables have an equiproportional impact at all mortgage ages. The vector of regression coefficients, $\bar{\beta}=(\beta_1,\beta_2,\cdots,\beta_n)$, measure the effect of the covariates upon the prepayment decision.

4.3 The Proportional Hazard Model in Our System

We consider four explanatory covariates in our proportional model. Since prepayment decisions are usually influenced by refinancing rates, the first explanatory covariate, $\nu_1(t)$, denotes the refinancing costs on the mortgagor's prepayment decision. It is defined as follows.

$$\nu_1(t) = c - l(t)$$

where c is the contract coupon rate, and $l(t)$ is the long-term rate seen at time t. We let $l(t)$ be the yield of a 10-year zero coupon bond seen at time t under the CIR model. $l(t)$ is defined as.

$$l(t) = \frac{-\ln P(t, t+10)}{10}$$

where $P(t, t+10)$ is the price of a zero coupon bond paying \$1 ten years from time t. The formula of $P(t, t+10)$ under the CIR model can be calculated by Equation (3.1) in Chapter 3. Note that $\nu_1(t) \geq 0$ if and only if $c \geq l(t)$. If $\nu_1 \geq 0$, there exists an incentive to prepay that is assumed to be equiproportional at all mortgage ages. Normally, the larger $\nu_1(t)$ is, the greater the incentive to prepay is. Therefore, β_1 is expected to be greater than 0.

Consider the probability that prepayments may further accelerate when refinancing rates are sufficiently lower than the contract coupon rate. We add the second explanatory covariate, $\nu_2(t)$, as follows

$$\nu_2(t) = (c - l(t))^3$$

Because transaction cost may make prepayment less profitable when refinance savings are small, for sufficiently low refinancing rates, the resultant prepayment speed is usually greater than the prepayment speed predicted by the first covariate, $\nu_1(t)$. Since for $c \geq l(t)$, there is an incentive to prepay, β_2 is also expected to be greater than 0.

The third covariate, $\nu_3(t)$, is considered for the effect called *burnout*. The effect is that, with greater past prepayment activity, mortgagors are less prone to prepay remain in the pool. $\nu_3(t)$ is defined by

$$\nu_3(t) = \ln \frac{OBM(t)}{OBM^*(t)}$$

where $OBM(t)$ is the dollar amount of the pool outstanding at time t, and $OBM^*(t)$ is the pool's principal which would prevail at t in the absence of prepayments but with amortization of the underlying mortgages. The greater the amount previously prepaid, and hence the smaller $\nu_3(t)$ is, the less incentive further prepayments are across all mortgage ages. Consequently, β_3 is expected to be greater than 0.

At last, according to experiences, seasonality may also influence prepayment activity. Because more residential real estate transactions occur in the spring and summer rather than fall and winter, prepayment rates are usually higher in the spring and summer than in the fall and winter. Hence, this covariate is represented by the dummy variable, $\nu_4(t)$, defined as follows.

$$\nu_4(t) = \begin{cases} +1 & if \quad t = \text{May–August}, \\ 0 & if \quad t = \text{September–April} \end{cases}$$

β_4 is expected to be greater than 0.

4.4 The Integration of Models into a Framework

In this section, we will show how to integrate some of the models into a framework which is used to work with a pass-through MBS. Numerical results of this framework will be shown in the next section. In the following framework, we use the CIR model as the interest rate model, a proportional hazard model as the prepayment model, and an OAS model as the pricing and analytical tool. The following describes their integration.

First, in order to do Monte Carlo simulation, we generate n interest rate paths by the CIR stochastic process. Each path is assumed to contain m periods. In order to be more efficient, a variance reduction technique called *antithetic variables* is used to reduce the variance of Monte Carlo simulation. According to this variance reduction method, we generate another n interest rate paths. Therefore, totally, the number of interest rate paths is $2n$. In reality, we generate a interest rate path and its corresponding path at the same time. When an interest rate path is generated with m standard normal distribution random variables, $\xi_1, \cdots, \xi_m$, its corresponding path can be generated with the mirror standard normal distribution random variables, $-\xi_1, \cdots, -\xi_m$.

For each path, cash flows include interests, scheduled principal payments, and prepayments. Consider a pass through MBS backed with a pool of mortgages originated now with the following security information. The principal of these mortgages is $OBM(0)$ at the beginning, the maturity date is T months from now, and the annual contract coupon is c. Then the interest paid in month t of these mortgages, $IP(t)$, can be calculated as

$$IP(t) = \frac{c}{12} \times OBM(t)$$

where $OBM(t)$ is the out-standing principal (remaining principal) in month t. The scheduled principal payment in month t, $SP(t)$, is calculated by

$$SP(t) = SMP(t) - IP(t)$$

where

$$SMP(t) = \frac{OBM(t) \times \frac{c}{12} \times (1.0 + \frac{c}{12})^{T-t}}{(1 + \frac{c}{12})^{T-t} - 1}$$

is the scheduled monthly payment in month t. $SMP(t)$ includes the interest and scheduled principal payment which occur in month t but not the prepayment. In addition, with the proportional hazard model $\pi(t; \bar{\nu}, \bar{\theta})$, the single monthly mortality in month t, $smm(t)$, can be calculated by

$$\begin{aligned} smm(t) &= 1 - \sqrt[12]{1 - \pi(t; \bar{\nu}, \bar{\theta})} \\ &= 1 - \sqrt[12]{1 - \pi_0(t; \gamma, p) \exp(\bar{\beta}\bar{\nu})} \end{aligned}$$

where $\exp(\bar{\beta}\bar{\nu}) = \exp(\beta_1 \nu_1(t) + \beta_2 \nu_2(t) + \beta_3 \nu_3(t) + \beta_4 \nu_4(t))$. So the prepayment in month t, $PreP(t)$, can be calculated by

$$PreP(t) = OBM(t) \times smm(t)$$

The total received cash in month t is

$$\begin{aligned} TCF(t) &= IP(t) + SP(t) + PreP(t) \\ &= SMP(t) + PreP(t) \end{aligned}$$

So the principal of these mortgages in next month can be calculated by

$$OBM(t+1) = OBM(t) - SMP(t) - PreP(t)$$

Thus all the cash flows on this path can been obtained by this way.

We now discount these cash flows with the interest rates on the path to get the present value of this security for this path, PV_i, as follows

$$\begin{aligned} d(t) &= \prod_{j=1}^{t} \exp\left[-(j \times \triangle t) r(j-1)\right] \\ PV_i &= \sum_{t=1}^{T} [TCF(t) \times d(t)] \end{aligned}$$

where $r(j-1)$ is the short rate in month $j-1$, and $d(t)$ is the discount factor from month t to now. Since we have $2n$ interest rate paths, we must repeat the process above $2n$ times to get $2n$ present values for each path and then average all the $2n$ present values to get an average present value,

$$APV(2n) = \frac{\sum_{i=1}^{2n} PV_i}{2n}$$

However, $APV(2n)$ may not be the same as the market price. In an OAS model, we add an OAS to all the short rates on each path to make the average present value the same as the market price. Then we have the following modified equation.

$$\begin{aligned} d^*(t) &= \prod_{j=1}^{t} \exp\left[-(j \times \triangle t)(r(j-1) + OAS)\right] \\ PV_i^* &= \sum_{t=1}^{T} [TCF(t) \times d^*(t)] \\ APV^*(2n) &= \frac{\sum_{i=1}^{2n} PV_i^*}{2n} \end{aligned}$$

Obviously, the equations above are non-linear. So we need a non-linear equation solver to find an OAS that makes $APV^*(2n)$ equal the market price. Our current system uses the *secant method* as the non-linear equation solver.

4.5 Properties of MBS Given OAS

In this section, we explore properties of MBS with a given OAS and compare MBSs different prepayment assumptions.

Example 1: This is a pass through MBS backed by \$1,000,000 of mortgages with a 40% coupon and 360 months to maturity. Assume its market price is \$900,000 with PSA 100 used as its prepayment model and a CIR model with the following parameters, the current short rate is 40%, λ is 0.1, θ is 40%, and σ is 0.1. The system calculates the OAS of this MBS to be 5.148%. Now we calculate the price with the short rate running from 1% to 100% and fixed OAS. We can observe the relation between the price and the current short rate in Figure 4.2. With PSA 100 prepayment

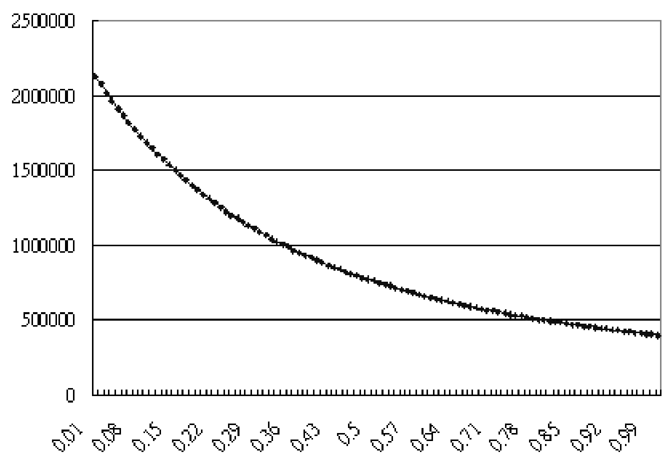

Figure 4.2: Pass Through MBS with PSA 100.

assumption, the cash flow of this MBS doesn't change when interest rates change. However, much like a bond, its price will change with interest rates. See Figure 4.2. When the current short rate becomes low, its price rapidly grows. But, when the current short rate becomes high, instead of falling down at the same speed , its price declines gradually. This is the convexity phenomenon.

Example 2: The example is the same as Example 1 except the prepayment assumption. Here, a proportional hazard model is used as its prepayment model. In this model, we let $\gamma = 0.1496$, $p = 2.31217$, $\beta_1 = 40$, $\beta_2 = 45$, $\beta_3 = 3.57673$, and $\beta_4 = 0.26570$. The system calculates the OAS of this MBS to be 4.6963%. Then we calculate the prices with the short rate running from 1% to 100% and fixed OAS. We observe the following relation between the price and the current short rate in Figure 4.3. When the current short rate becomes low, the price of this MBS doesn't

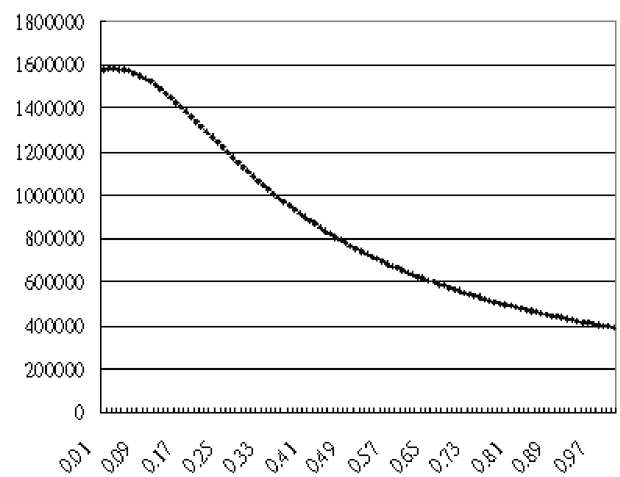

Figure 4.3: Pass Through MBS with a Proportional Hazard Model Prepayment Assumption.

always keep becoming higher. It grows at a rapid speed, then slows down, and finally declines. The reason is, with the proportional hazard model, the prepayment rate of this MBS will rapidly grows when the current short rate is low. Once the effect of accelerated prepayments which tend to decrease the price dominates the effect of low short rates which tend to increase the price, the price of this MBS declines.

In order to understand more clearly, we combine Figure 4.2 and Figure 4.3 into Figure 4.4. From Figure 4.4, the prices of MBSs with these two prepayment models aren't different from each other very much when the current short rate is relatively high. This is because when the current short rate is high, the prepayment rates are low in the proportional hazard model, as in the PSA 100 assumption. So their

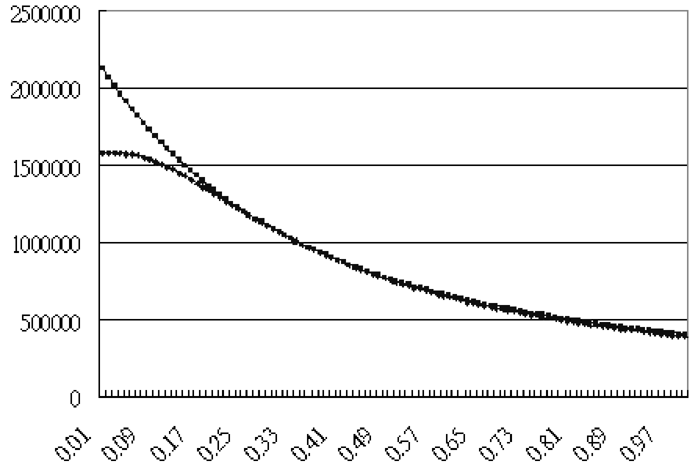

Figure 4.4: Comparison of PSA 100 and Proportional Hazard Model Prepayment Assumptions. This figure combines Figure 4.2 and Figure 4.3.

prepayments are much less than the scheduled payments.

Example 3: In order to better understand the properties of mortgage-backed securities, we now consider interest-only securities backed by the MBS in Example 2. Assume the market price of this interest-only security is $870,000. Its OAS is found to be 4.981%. Now we calculate the prices with the short rate running from 1% to 100% and fixed OAS. We observe the following relation between the price and the current short rate in Figure 4.5. As we see from Figure 4.5, the price of IO grows rapidly then declines. Because interests are the only source of IO's cash flows, when the current short rate decreases, initially the prepayment rate doesn't increase very quickly. It leaves the cash flows of IO nearly unchanged. Thus, when the cash flows are discounted at the lower rates, the price of IO increases. However, when the short rate decreases beyond some point, the prepayment rate increases quickly. As most principal of the mortgages has been prepaid, the total amount of interests received decreases sharply. Therefore, the price of IO declines sharply, too. The price of IO doesn't increase in definitely when the short rate increases because, although the amount of interests received increases, its present value is discounted at higher rates.

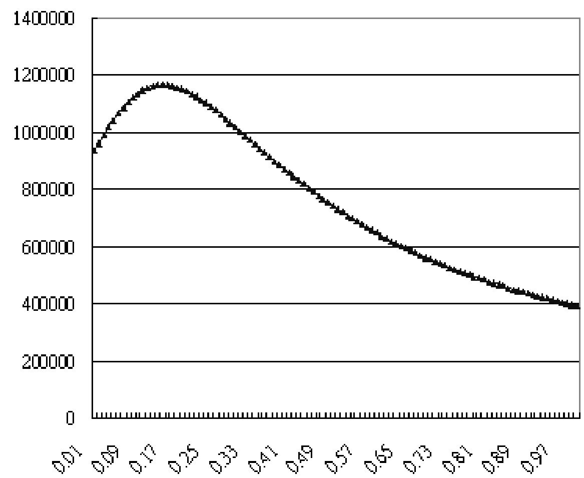

Figure 4.5: IO under Proportional Hazard Model Prepayment Assumption.

Example 4: Here we give an instance of principal-only securities backed by the MBS in Example 2. Assume the market price of this principal-only security is \$30,000. Its OAS is 0.5557%. We calculate the price with the short rate running from 1% to 100% and fixed OAS. We observe the following relation between the price and the current short rate in Figure 4.6. As we see from Figure 4.6, the price of PO increases when the short rate decreases and decreases when the short rate increases. This is because principal is the only income source of PO. Thus, when the short rate decreases, the prepayment rate increases and cash flows of PO are more early realized. Note that, although the amount of cash flows is the same, the more early they are realized the higher their present value is.

At last, we combine Figure 4.3, Figure 4.5, and Figure 4.6 into Figure 4.7 as a grand overview of these three MBSs.

4.6 The Way To Use Effective Duration and Convexity

First, let us outline the way the system calculates the effective duration and convexity. The price, $P(r)$, of a security is a function of the current short rate, r. The effective

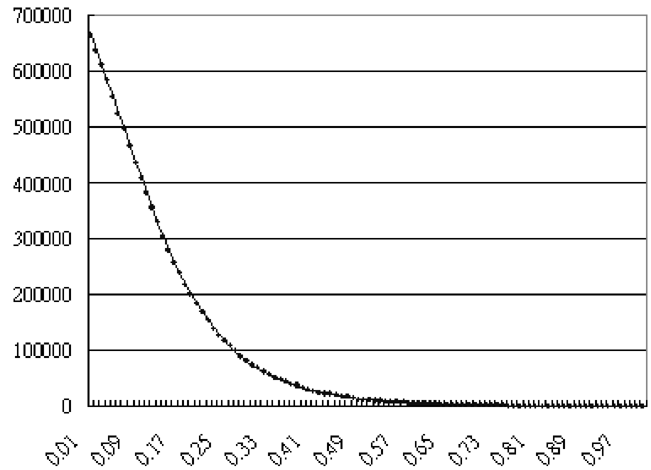

Figure 4.6: PO under Proportional Hazard Model Prepayment Assumption.

duration and convexity at $r = y$ can be calculated as follows,

$$\begin{aligned}
\frac{\partial P(r)}{\partial r} &\approx \frac{P(r_u) - P(r_d)}{r_u - r_d} \\
\frac{\partial^2 P(r)}{\partial r^2} &\approx \frac{\frac{P(r_u)-P(r)}{r_u-r} - \frac{P(r)-P(r_d)}{r-r_d}}{\frac{r_u-r_d}{2}} \\
D(r) &= \frac{\partial P(r)}{\partial r}\frac{1}{P(r)} \\
&\approx \frac{P(r_u) - P(r_d)}{r_u - r_d}\frac{1}{P(r)} \\
C(r) &= \frac{\partial^2 P(r)}{\partial r^2}\frac{1}{P(r)} \\
&\approx \frac{\frac{P(r_u)-P(r)}{r_u-r} - \frac{P(r)-P(r_d)}{r-r_d}}{\frac{r_u-r_d}{2}}\frac{1}{P(r)}
\end{aligned}$$

where r_u, r, and r_d are shown in Figure 4.8. For example, the effective duration and convexity at $r = 40\%$ in Example 2 are 1.827285 and 4.795914 respectively. If r changes into 39%, the value of the price difference, $P(39\%) - P(40\%)$, to the price at $r = 40\%$, $P(40\%)$, should satisfy

$$\frac{P(39\%) - P(40\%)}{P(40\%)} \approx -1.827285 \times (-0.01) + \frac{1}{2} \times 4.795914 \times (-0.01)^2$$

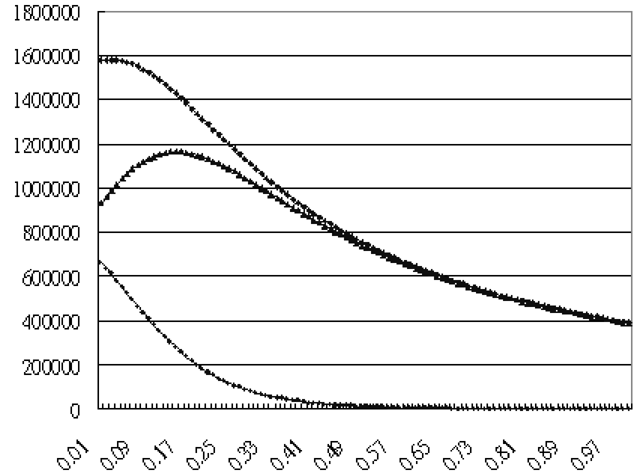

Figure 4.7: A GRAND OVERVIEW OF PASS-THROUGH, IO, AND PO UNDER PROPORTIONAL HAZARD MODEL PREPAYMENT ASSUMPTION. This figure combines Figure 4.3, 4.5, and 4.6.

where $P(40\%) = 899998.1$ and $P(39\%) = 916660.8$. Let's verify it. The value on the left-hand side is 0.0185141502 and the value on the right-hand side is 0.0185126457. Obviously, they are very close. So, if we want to get the price at $r = 39\%$, i.e., $P(39\%)$, it can be calculated as $P(39\%) = P(40\%) \times (1 + 0.0185126457) = 916659.4460$.

There is a more convenient way to get percentage price change via mental calculation. It is the method mentioned in Section 2.4 using duration and convexity for 1%. According to Equation (2.10), it can be calculated as follows

$$-D^*(40\%) \times (-1) + \frac{1}{2}C^*(40\%) \times (-1)^2 \quad = \quad 1.85126457$$

where $D^*(40\%) = 1.827285$ and $C^*(40\%) = \frac{4.795914}{100} = 0.04795914$. It means if $P(40\%)$ were 100 at $r = 40\%$, $P(39\%)$ would be $100+1.85126457 = 101.85126457$. Thus, when $P(40\%)$ is 899998.1 in this example, $P(39\%)$ can be quickly calculated as $P(40\%) \times 101.85126457\% = 916659.4460$.

For another example, with $-D^*(40\%) \times (-2) + \frac{1}{2}C^*(40\%) \times (-2)^2 = 3.75048828$, $P(38\%)$ can be calculated as $P(40\%) \times 103.75048828\% = 933752.4232607$, while the accurate value of $P(38\%)$ is 933763.184, which is very close.

Finally, we show the market prices, effective duration and convexity of those

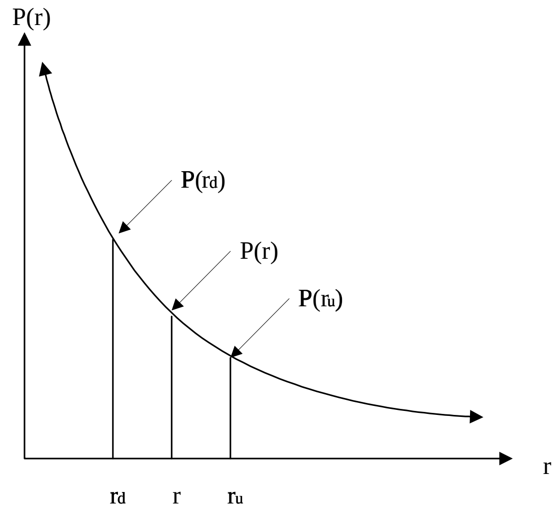

Figure 4.8: CURVE OF $P(r)$ AND r.

securities in Example 1, 2, 3, and 4 in Figure 4.10–4.11.

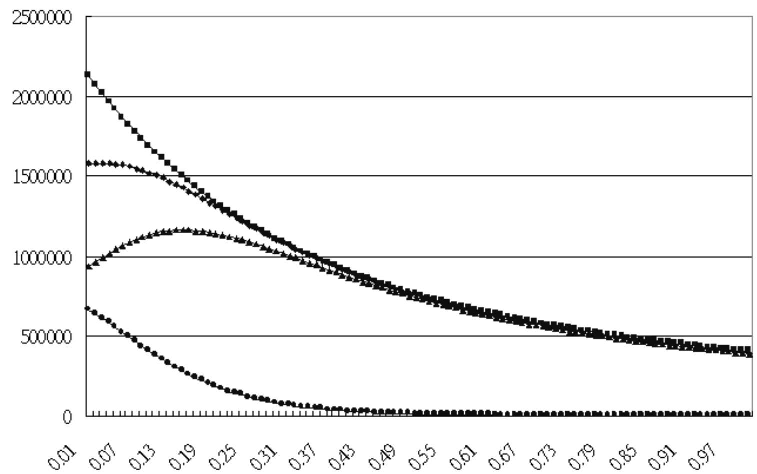

Figure 4.9: Market Price of Those Securities in Example 1, 2, 3, and 4. This graph combines Figure 4.2, 4.3, 4.5, and 4.6.

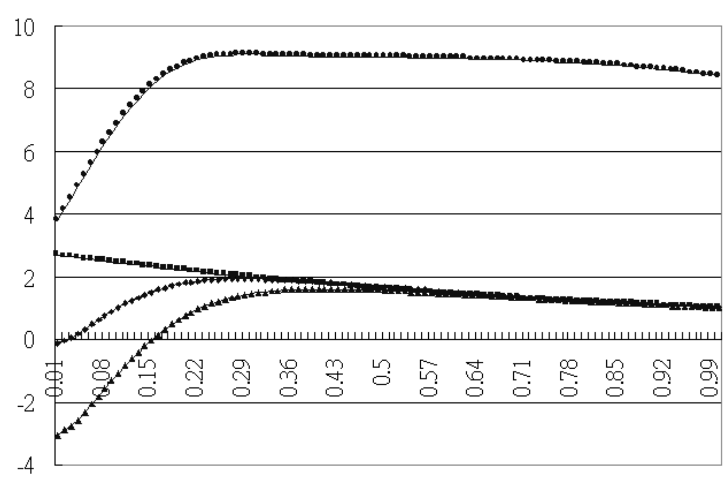

Figure 4.10: Effective Duration of Those Securities in Example 1, 2, 3, and 4.

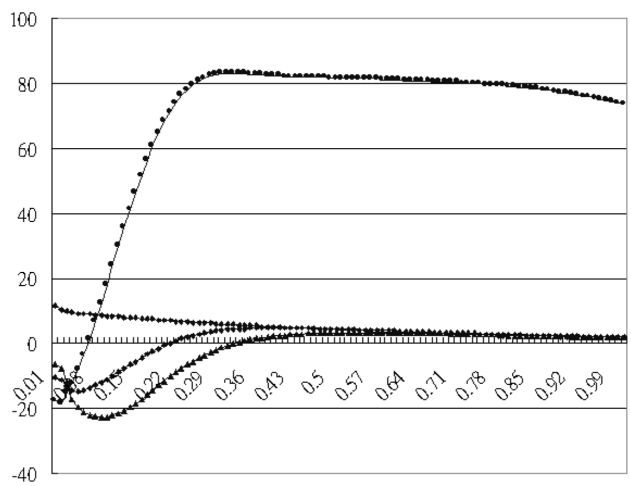

Figure 4.11: Convexity of Those Securities in Example 1, 2, 3, and 4.

Chapter 5

A Java/C++ Client/Server System

5.1 Initiate the Connection between Client and Server

This chapter presents an overview of this moderately complex system. First, let's look into how client keeps in touch with server. The user must type the server's URL in his Explorer Web browser to access our homepage. URL is the address of the server on the World Wide Web. By accessing our homepage, which is an HTML file, our client program is transported to the user's end. On our homepage, there is a Java applet working for the control of the usage of this system. In order to use this system, there are some fields you must fill first, including which kind of user you are, your login name and password. After you fill all these fields, press the OK button to send them for the server to check. As soon as you press the OK button, there is a thread generated from the applet to take care of the connection between the client and the server, which also generates a matching thread for response. If you succeed in the login process, a window titled FIRMS (Fixed Income Return Management System) is popped up right away. See Figure 5.1 for the overall protocol.

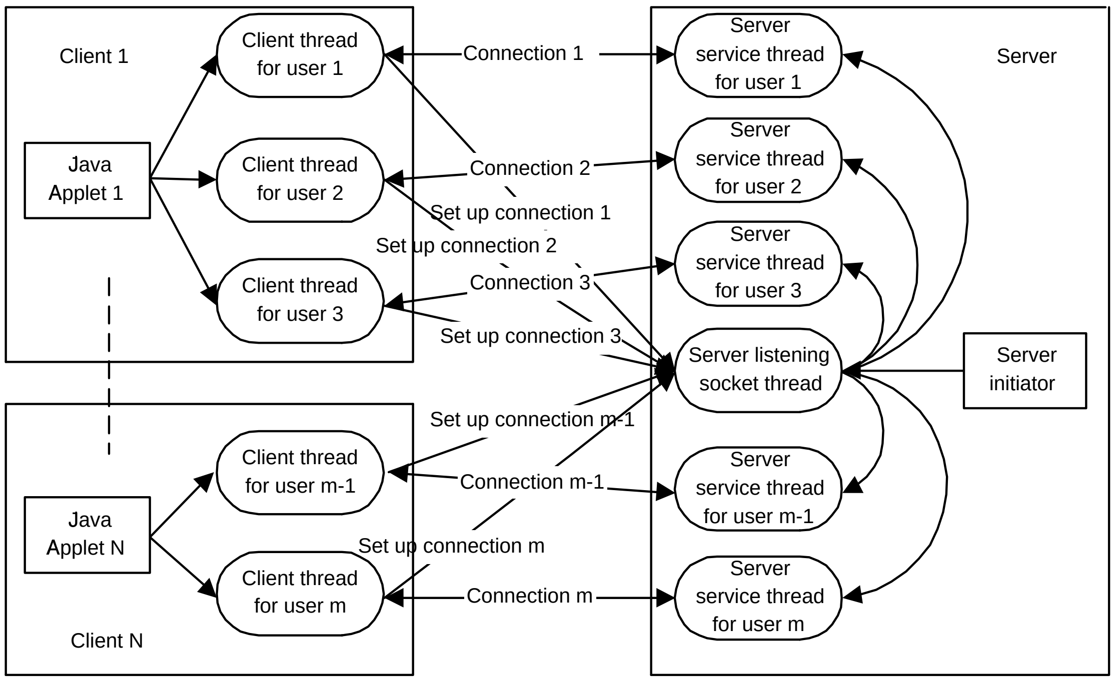

Figure 5.1: The Structure of Our Client/Server System.

5.2 The Connection Protocol between Client and Server

There are two distinct parts. The first part is used when someone wants to become a new user of this system. The second one is used when someone successfully login this system before. The protocol appears in Figure 5.2. We briefly describe each step in the following.

Step 1.a: For a new user, the client sends `NewLogin:*** Password:***` to the server.

Step 1.b: For a user registered before, the client sends `OldLogin:*** Password:***` to the server.

Step 2.a: After the server receives `NewLogin:*** Password:***`, it checks the received name with the registered users' names. If none of them are the same, the server allows the new user to register this name and sends the client a message by

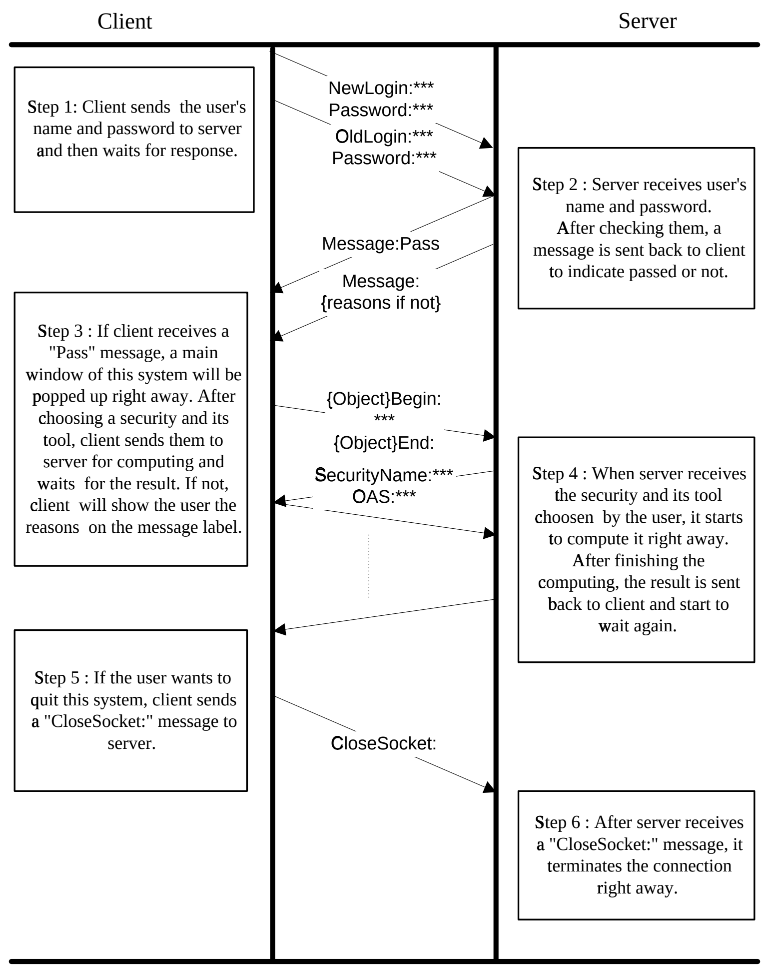

Figure 5.2: THE PROTOCOL OF THE CONNECTION BETWEEN CLIENT AND SERVER.

which the new user is allowed to use this system. Otherwise, a message is sent to the client to inform the new user to change his login name and register again.

Step 2.b: After the server receives `OldLogin:*** Password:***`, it checks the received password with the registered password according to the received name. If the received password is correct, a message is sent to the client to allow the user to use this system. If not, a message is sent to the client to inform the user. Of course, the user isn't allowed to use this system.

Step 3: After the client receives a message which means the user is allowed to use this system, the main window of this system is popped up right away and the user can use this system now. After the user chooses a security and its pricing and analytical framework, the client sends them back to the server and waits to get the result. After the client receives the result, the user can repeat Step 3.

Step 4: When the server receives the security and its pricing and analytical framework, it starts to work on them. After the job is finished, the result is sent back and the server waits for another job from the client.

Step 5: If the user wants to quit this system, the client sends `CloseSocket:` to the server and quits this system.

Step 6: When the server receives `CloseSocket:` from the client, the connection is terminated.

5.3 Representations of Objects in the String-Oriented Connection between Client and Server

Because of the different representations of classes between Java and C++, Java and C++ classes can't recognize each other. Consequently, the connection between client and server is string-oriented. Objects such as securities and their parameters are encoded as tokens to communicate with each other. The following is the general form of an object:

```
{Object}Begin:
  {Parameter  1}:{value}
  {Parameter  2}:{value}
  ⋮
  {Parameter  N}:{value}
{Object}End:
```

A parameter can be an object, too. As an instance, the following encoder is a CIR interest rate model.

```
CIR_InterestRateModelBegin:
  N:360
```

```
  DT:0.0833
  CurrentRate:0.07
  Lambda:1.0
  Theta:0.07
  Sigma:0.2
CIR_InterestRateModelEnd:
```

where `N` stands for period number, `DT` stands for period time, `Lambda` stands for long rate, `Theta` stands for mean short rate, and `Sigma` stands for the volatility of short rate. The following is a PSA prepayment rate model as another example.

```
PSA_PrepaymentRateModelBegin:
  Level:100
PSA_PrepaymentRateModelEnd:
```

To understand more deeply that a parameter can be an object, we use a MBS pass-through security as another instance. In order to price and analyze an MBS pass-through security, an interest rate model and a prepayment model must be two of the parameters of a MBS security. Below, we will show how an interest rate model and a prepayment model are encoded in a MBS pass-through security. A CIR interest rate model and a PSA prepayment model are chosen as the interest rate model and the prepayment model used in pricing and analyzing this security.

```
MBS_PassThroughBegin:
  {Parameter  1}:{value}
  ⋮
  {Parameter  M}:{value}
  CIR_InterestRateModelBegin:
    N:360
    DT:0.0833
    CurrentRate:0.07
    Lambda:1.0
```

```
    Theta:0.07
    Sigma:0.2
  CIR_InterestRateModelEnd:
  PSA_PrepaymentRateModelBegin:
    Level:100
  PSA_PrepaymentRateModelEnd:
  {Parameter  M+2}:{value}
  ⋮
  {Parameter  N}:{value}
MBS_PassThroughEnd:
```

5.4 User Guide and User Interface

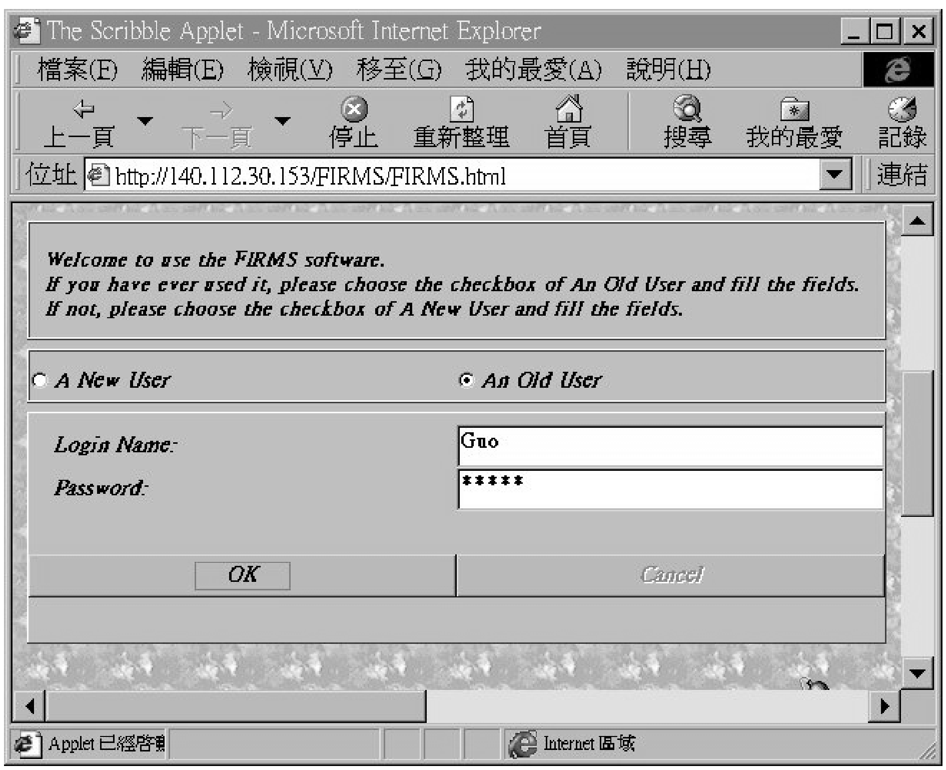

Figure 5.3: An Applet on Our Homepage.

In this section, we will show the user interface of this system and the way to use it. The following is what you will see on your Explorer browser while accessing our homepage in Figure 5.3.

After the user logins our system successfully, the main window of our system is shown right away. See Figure 5.4. Then the user should press the Security menuitem on the menu of the main window to select a security. There will be a Security Tree

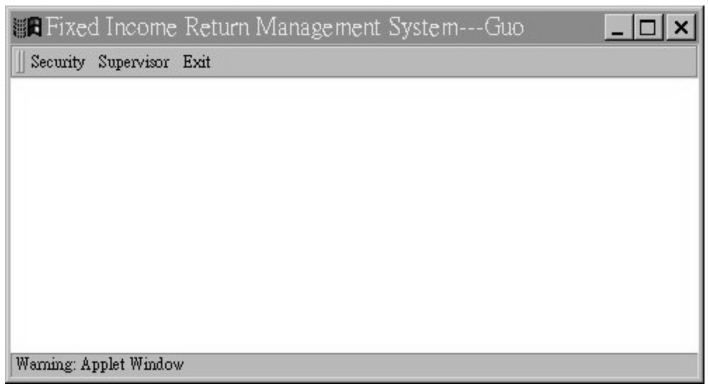

Figure 5.4: MAIN MENU WINDOW.

View window popped up. See Figure 5.5. Choose a security from the Security Tree View window such as a MBS Pass Through security, a Principal Only security based on an MBS Pass Through security or an Interest Only security based on an MBS Pass Through security. If the user wants to choose a security of MBS Pass Through, he can click on the root node, which is named Security, of the security tree, three child nodes are expanded. One of them is named Mortgage Backed Security. After it is clicked, a child node named Mortgage Pass Through Security is seen from the security tree. Both PO and IO are hidden in the node named Stripped Mortgage Backed Security. If you click on the MBS Pass Through Security node, it means you have chosen that security. Then there will be a window titled Mortgage Pass Through Pricing and Analytical Tools popped up right away. See Figure 5.6. Choose one of the tools you want to use. If you choose the Market Price Given Option-Adjusted Spread as a tool, it means you must input an option-adjusted-spread in order to get the price based on that OAS. After choosing a tool and press the OK button, a window named MBS Pass Through Dialog is showed. See Figure 5.7.

Before getting the result of pricing or analyzing that security according to the

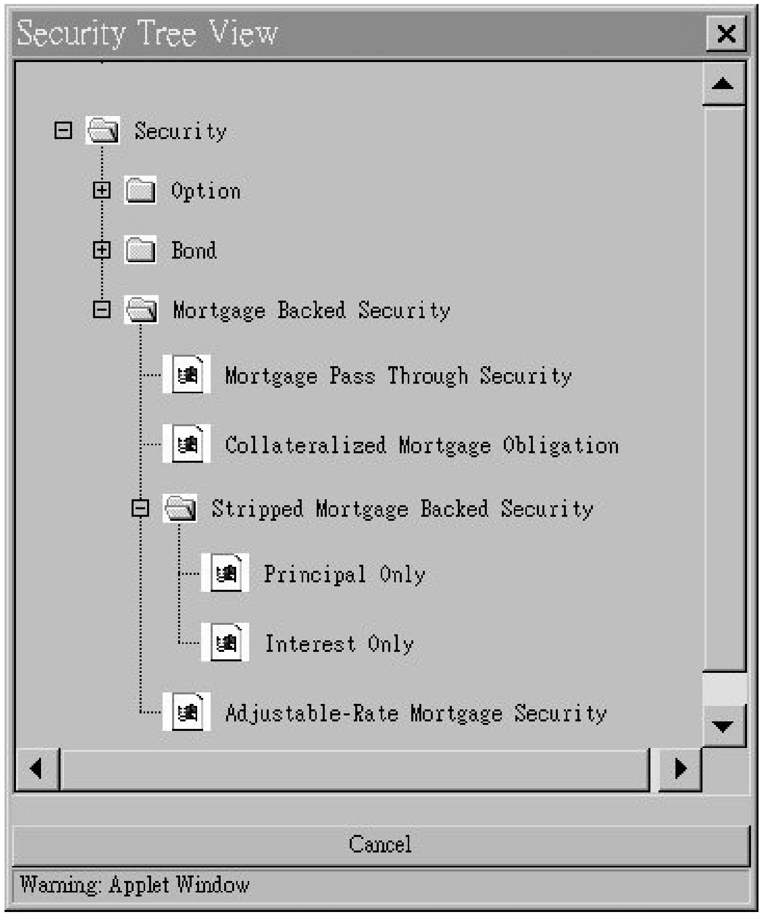

Figure 5.5: SECURITY TREE VIEW DIALOG.

framework you choose, you must fill all the required information of that security including its face value, fixed annual coupon rate, loan life, given OAS, an interest rate model and a prepayment model. You can choose one of the interest rate models from the Choice named Interest Rate Model and choose one of the prepayment models from the Choice named Prepayment Rate Model to form your own pricing and analytical framework. If you choose the CIR model as the interest rate model, then you must decide how many partitions you want to divide in one month and how many Monte Carlo simulation paths you want to work with. See Figure 5.8, Figure 5.9, Figure 5.10, and Figure 5.11.

After filling all these fields, press OK button and wait to get the result.

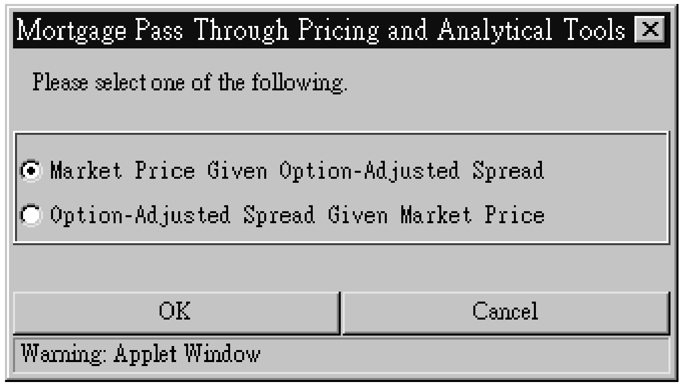

Figure 5.6: PRICING AND ANALYTICAL TOOLS DIALOG.

5.5 Performance Evaluation

This system runs on an NT server with two Pentium II 233 CPUs and 128MB memory. The performance evaluation is done by evaluating the time to calculate the market price of a pass-through security. A CIR interest rate model and a proportional hazard prepayment model are used in the calculation of a pass-through security. The security's information is as follows. Its face value is \$1000,000, and its annual coupon is 7%. Besides, the time to maturity is 30 years and its OAS is assumed to be 1%. The values of parameters used in the CIR interest rate model and proportional hazard prepayment model are: $\triangle t$ is 0.08333 (years) and the number of paths for Monte Carlo simulation is 1000. In addition, the current short rate is assumed to be 7%, λ is 100%, θ is 7%, and σ is 20%. For the proportional hazard prepayment model, $\gamma = 0.01496$, $p = 2.31217$, $\beta_1 = 0.38089$, $\beta_2 = 0.00333$, $\beta_3 = 3.57673$, and $\beta_4 = 0.26570$. The following figure shows the curve of the time to perform such a calculation with the number of users who do the same things simultaneously in this system. Figure 5.13 shows that when clients are few, the sojourn time relative to only one client is linear. However, NT doesn't seem efficient as the number of clients grows.

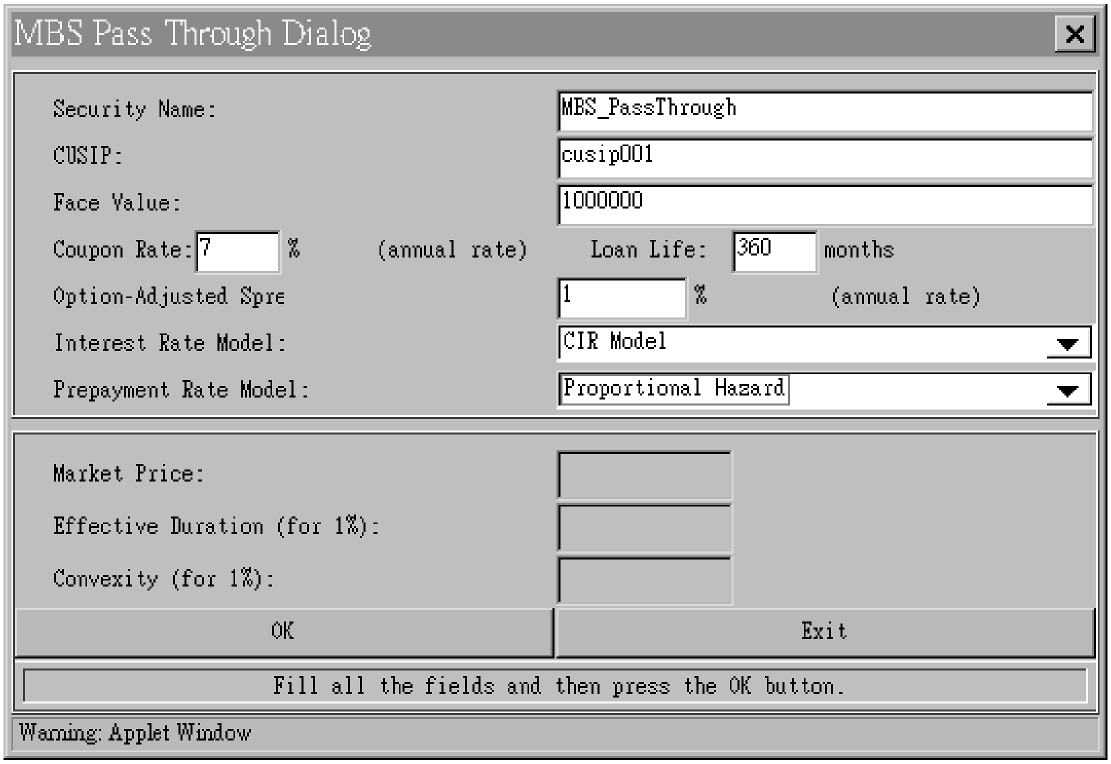

Figure 5.7: MBS Pass-Through Dialog.

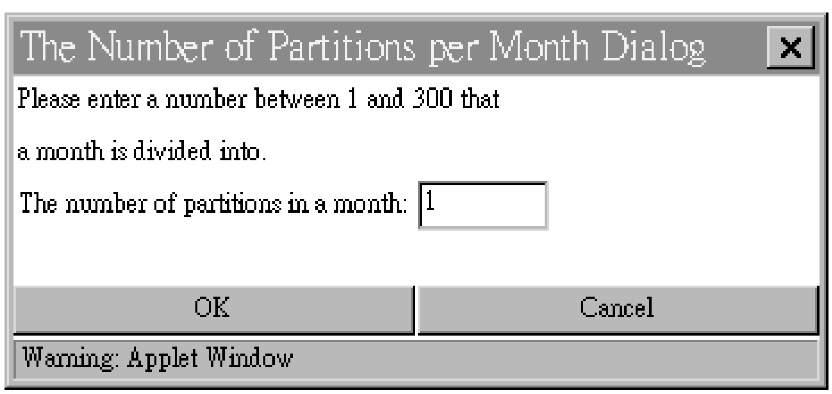

Figure 5.8: The Number of Partitions in a Month Dialog.

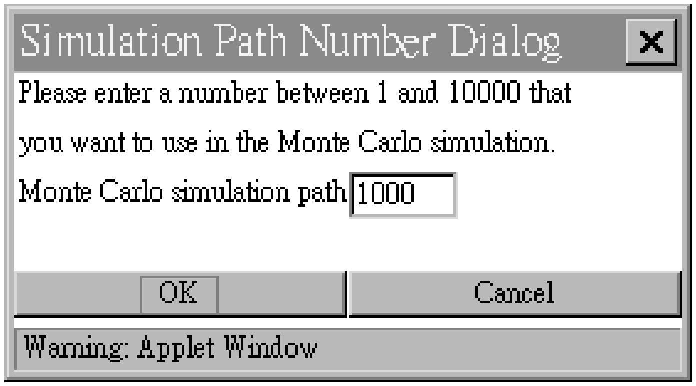

Figure 5.9: Simulation Path Number Dialog.

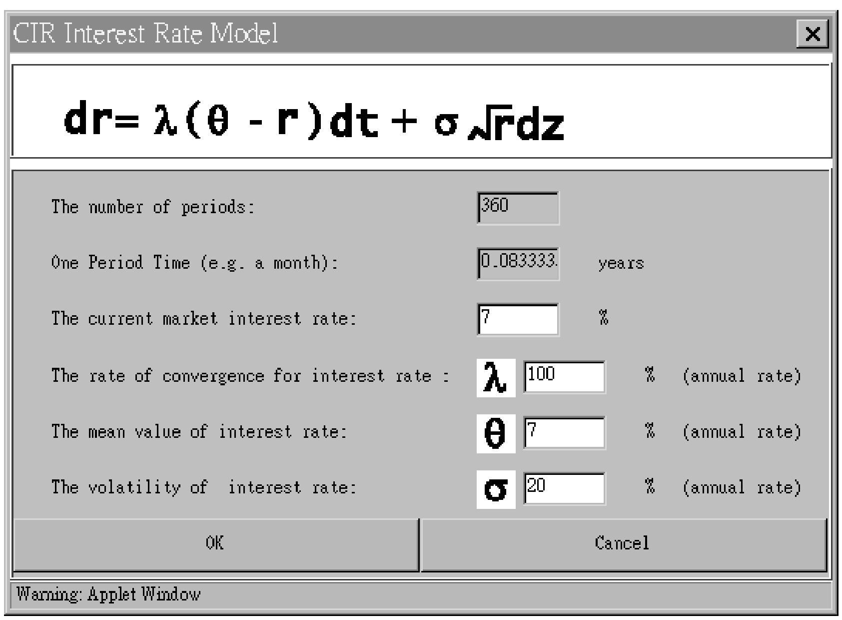

Figure 5.10: CIR Interest Rate Model Dialog.

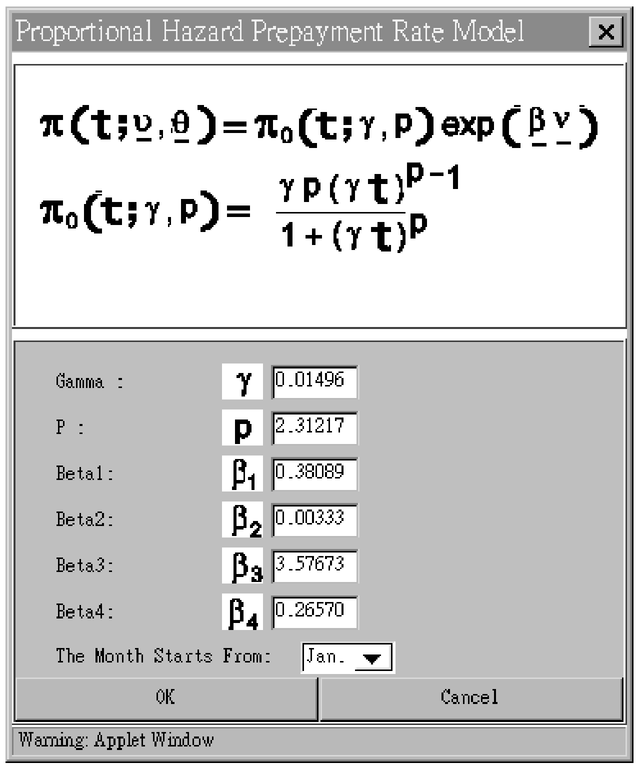

Figure 5.11: Proportional Hazard Prepayment Rate Model Dialog.

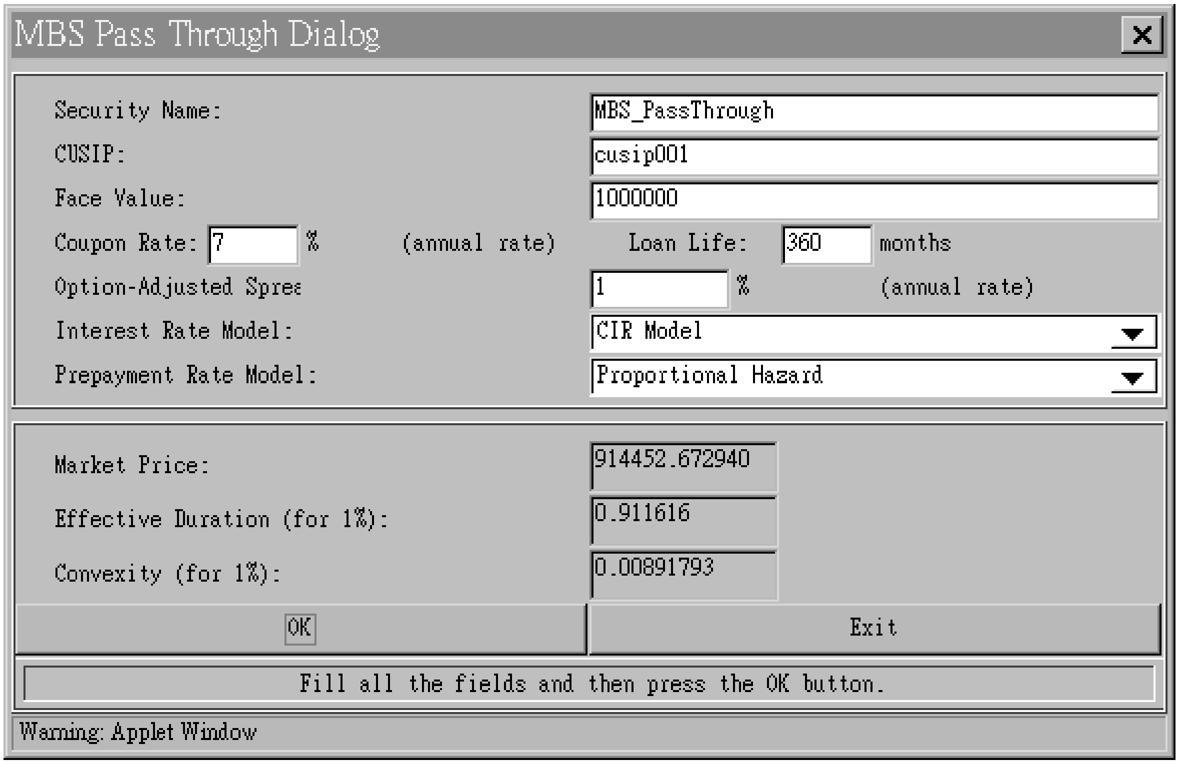

Figure 5.12: The Result.

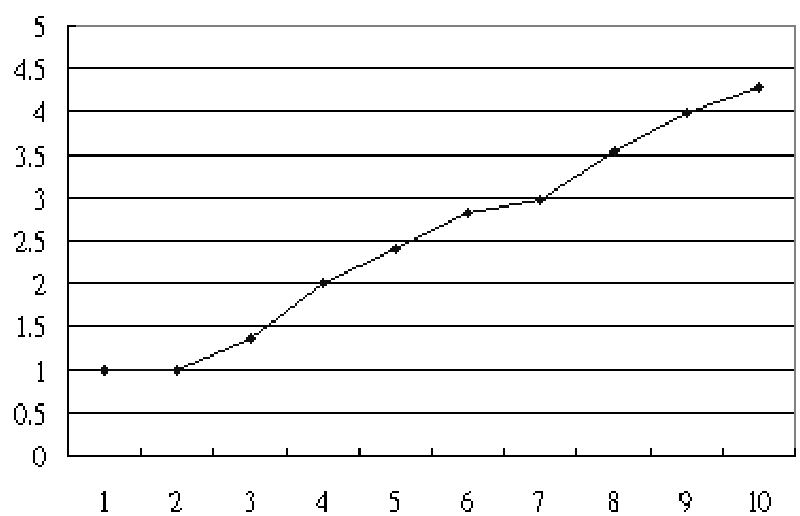

Figure 5.13: The Relation between the Number of Clients and Sojourn Time.

Chapter 6

Conclusion and Future Work

The development of derivatives is so rapid that more and more investors are attracted to this market. However, it is becoming more complex to price and analyze these derivatives. For example, to price and analyze an MBS, both the interest rates and prepayments must be taken into consideration. Without a good tool, investors may be misled when they invest in these derivatives.

This thesis constructs a framework of the C/S architecture to do financial computation based on Java and C++ with object-oriented design. It also gives methods of working with the mortgage-backed securities and their derivatives by option-adjusted spreads. With the Java-based client, this system can be easily accessed through the World Wide Web. In addition, the VM (Virtual Machine) concept of Java will make it work well on all platforms of machines. (Our system can't run on Netscape's Navigator due to software bugs in our development tools.) It means, whatever the client platform is, the user can contact to our application server and receives the services. This thesis also shows that the price of a pass through security is not like a callable bond. It doesn't keep on growing up when interest rates become low. An interest only security and a principal only security are used to make us understand it more clearly.

Future works can be pursued in two directions.

(1) System Work Financial derivatives are getting more and more complex. To correctly value them involves more and more computation time, especially working with MBS and its derivatives. Maybe we can distribute some computation to the client on users' computer to reduce the load of the server or explore some techniques to improve the efficiency of computation. To achieve a three-tier C/S architecture, we may add a data base system to our system in the future. Maybe yield curve fitting and some methods of estimation will be added then. Another work we can do in the future is to make our user interface more friendly and functional.

(2) Financial Work With a good framework, developers can easily add other types of securities, interest rate models, and prepayment models. For securities, we have in mind CMOs such as Sequential MBS, PACs, Floaters, and Inverse Floaters. Maybe ARM is another good choice. For interest rate models, no-arbitrage models such as Hull-White and Heath-Jarrow-Morton may be added. For prepayment models, we plan to add more variables such as the value of the house and the economic indicators into the proportional hazard model and add some other advanced prepayment models into our system.

Appendix A

Some Relational Data

Path Number	Market Price
500	939305.1734
1000	939398.8653
1500	939506.0342
2000	939434.3459
2500	939649.5085
3000	939588.6822
3500	939400.2819
4000	939449.4671
4500	939350.9779
5000	939266.6112
5500	939272.3533
6000	939217.6244
6500	939240.3631
7000	939165.9876
7500	939074.4048
8000	939014.9668
8500	938929.0063
9000	938903.7741
9500	938889.0176
10000	938816.2235

Figure A.1: DATA FOR FIGURE 3.3. The market price of MBS with CIR SBP is 938852.8.

Time	Market Price
0.008	938835.7496
0.004	938848.5398
0.003	938852.8023
0.002	938854.9293
0.002	938856.2124
0.001	938857.0654

Figure A.2: DATA FOR FIGURE 3.4.

Period Time	Percentage Price Difference
0.008333	0.0023%
0.004167	0.0009%
0.002778	0.0005%
0.002083	0.0002%
0.001667	0.0001%
0.001389	0.0000%

Figure A.3: DATA FOR FIGURE 3.5.

Period Time	Percentage Price Difference
0.008333	0.01%
0.004167	0.02%
0.002778	0.04%
0.002083	0.05%
0.001667	0.11%
0.001389	0.09%

Figure A.4: DATA FOR FIGURE 3.6.

Paths	CIR SBP	CIR with VRT
1000	950856.3165	946988.0411
2000	948617.4397	946989.1444
3000	947920.5434	947140.3508
4000	947517.2438	947044.3532
5000	946915.0062	946886.5798
6000	946925.2301	946852.4817
7000	946916.0315	946816.4074
8000	946455.7498	946683.5462
9000	946920.8348	946594.9702
10000	946945.8384	946528.6072

Figure A.5: Data for Figure 3.7.

Number of Clients	Time (s)	Relative Sojourn Time
1	28.062	1.000
2	28.110	1.002
3	38.453	1.370
4	55.988	1.995
5	67.075	2.390
6	79.182	2.822
7	83.558	2.978
8	99.420	3.543
9	111.622	3.978
10	120.327	4.288

Figure A.6: Data for Figure 5.13.

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