On Hull-White Models: One and Two Factors

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Abstract

This thesis contains two main parts. In the first part, we implement a new approach for constructing no-arbitrage models of the term structure in terms of the process followed by the short rate, r. The approach, which makes use of trinomial trees, is relatively simple and computationally more efficient than previous procedures. The procedure is appropriate for models where there is some function x of the short rate r that follows a mean-reverting arithmetic process. The key element of the procedure is that it produces a tree that is symmetrical about the expected value of x. A forward induction procedure is used to find the positions of the central nodes at the end of each time step.

In the second part, the new tree-building procedure is extended to model the yield curves in two different economies simultaneously. We focus on the adjusted short rate trees, which are constructed from the viewpoint of a risk-neutral investor in the economy in which the cash flows are realized. We then increase the number of economies and discuss arbitrage pricing therein.

Chapter 1

Introduction

1.1 Introduction

The traditional approach to modeling the term structure involves starting with a plausible stochastic process for the short rate, r, in a risk-neutral world and exploring what process obtains for bond prices and option prices. It is important to emphasize that it is not the process in the real world that matters.

In a number of models, Brennan and Schwartz [1979, 1982], Courtadon [1982], Cox, Ingersoll, and Ross [1985], Dothan [1978], Langetieg [1980], Longstaff [1989], Richard [1979], and Vasicek [1977], it is assumed that there is only one underlying stochastic variable (or factor); so the risk-neutral process for r is of the form

$$dr = m(r)dt + s(r)dz$$

The instantaneous drift, m, and instantaneous standard deviation, s, are assumed to be functions of r, but independent of time. The assumption of a single factor is not as restrictive as it might at first appear to be. It does not, as is sometimes supposed, imply that the term structure always has the same shape. A fairly rich pattern of term structures can occur under a one-factor model. The essence of a one-factor model is that it implies that all rates move in the same direction in any short time interval; it does not imply that all rates move by the same amount.

The disadvantage of these term structure models is that they do not automatically fit today's term structure. Choosing the parameters judiciously provides an approximate fit to many of the term structures that are encountered in practice. But the fit is not usually an exact one and in some cases there are significant errors. Most traders find this unsatisfactory. Not unreasonably, they argue that they can have very little confidence in the price of a derivative security when the model does not even price the underlying correctly. A one percent error in the price of the underlying can lead to a 50 percent error in an option price.

In the past few years, many term structure models have been developed, that take the initial term structure as an input rather than output. These models are often referred to as no-arbitrage models.

The first no-arbitrage model was proposed by Ho an Lee [1986] in the form of a binomial tree of discount bond prices. The model is automatically consistent with any specified initial term structure. The Ho and Lee model involves one underlying factor and it assumes an arithmetic process for the short rate. Their work has been extended by a number of researchers, including Dybvig [1988], and Milne and Turnbull [1989]. Heath, Jarrow, and Morton [1992] present a general multifactor interest-rate model consistent with the existing term structure of interest rates and any specified volatility structure. Another popular no-arbitrage model was proposed by Hull and White [1990]. Hull and White refer to this as the extended Vasicek model. In fact the Hull-White model generalizes the Ho and Lee model to include mean reversion.

Black, Derman, and Toy [1990] have developed another one-factor no-arbitrage model. In their model, the short rate follows a lognormal process. Another model where the term structure is in terms of the processes followed by forward rates has been proposed by Heath, Jarrow, and Morton [1992].

There are so many no-arbitrage models. It's a difficult trade-off to make a choice from these models. The Heath, Jarrow, and Morton model may provide the most realistic description of term structure movements, but it has the disadvantage that it is non-Markov (meaning the distribution of interest rates in the next period depends on the current rate and the rates in earlier periods). It's a very serious problem, because this means Monte Carlo simulation and a non-recombining tree are the only alternatives. Computations hence become very time-consuming, making it difficult to value American-style derivatives accurately.

1.2 One-Factor No-Arbitrage Models

A number of authors have proposed one-state-variable models of the term structure in which the short rate, r, follows a mean-reverting process of the form

$$dr = a(b - r)dt + \sigma r^{\beta} dz$$

where a, b, σ, and β are positive constants and dz is a Wiener process. In these models, the interest rate, r, is pulled toward a level b at rate a. Superimposed upon this "pull" is a random term with instantaneous variance $\sigma^2 r^{2\beta}$.

The situations where $\beta = 0$ and $\beta = 0.5$ are of particular interest because they lead to models that are analytically tractable. The $\beta = 0$ case was first considered by Vasicek [1977], who derived an analytic solution for the price of a discount bond. Jamshidian [1989] showed that, for this value of β, it is also possible to derive relatively simple analytic solutions for the prices of European call and put options on both discount bonds and coupon-bearing bonds. One drawback of assuming $\beta = 0$ is that the short rate, r, can become negative. Cox, Ingersoll, and Ross considered the alternative $\beta = 0.5$. In this case, r can, in some circumstance, become zero but it can never become negative. They derives analytic solutions for the prices of both discount bonds and European call options on discount bonds.

The Ho and Lee Model

Ho and Lee proposed the first no-arbitrage model of the term structure in a paper in 1986. They presented the model in the form of a binomial tree where there were two parameters, one concerned with volatility the other with the market price of risk. It has since been shown that the continuous time limit of the model is

$$dr = \theta(t)dt + \sigma dz$$

where σ, the instantaneous standard deviation of the short rate, is constant and $\theta(t)$ is a function of time chosen to ensure that the model fits the initial term structure.

Define $F(t,T)$ as the instantaneous forward rate at time t for a contract maturing at T. The equation for $\theta(t)$ is

$$\theta(t) = F_t(0,t) + \sigma^2 t$$

where the subscript denotes partial derivative. It is interesting to note that Ho and Lee's parameter concerned with the market price of risk is redundant. This is analogous to risk preferences being irrelevant in the pricing of stock options.

In the Ho and Lee model, discount bonds and European options on discount bonds can be valued analytically. The expression for the price of a discount bond at time t in terms of the short rate is

$$P(t,T) = A(t,T)e^{-r(T-t)}$$

where

$$\ln A(t,T) = \ln \frac{P(0,T)}{P(0,t)} - (T-t)\frac{d \ln P(0,t)}{dt} - \frac{1}{2}\sigma^2 t(T-t)^2$$

European options on coupon-bearing bonds can be valued by decomposing them into a portfolio of European options on discount bonds using the approach suggested by Jamshidian [1989]. American options can be valued by drawing a tree in either the way described by Ho and Lee or by using trinomial trees as will be described later in this chapter.

The Ho and Lee model has the advantage that it is a Markov, analytically tractable model. It is easy to apply and provides an exact fit to the current term structure of interest rates. One disadvantage of the model is that it gives the user very little flexibility in choosing the volatility structure. All spot and forward rates have the same instantaneous standard deviation, σ. Another related disadvantage of the model is that it has no mean reversion. This means that regardless of how high or low interest rates are at a particular point in time, the average direction in which interest rates move over the next short period of time is always the same.

The Hull and White Model

In a paper published in 1990, Hull and White explored extensions of the Vasicek and Cox, Ingersoll, and Ross models that provide an exact fit to the initial term structure.

One version of the extended Vasicek's model is

$$dr = (\theta(t) - ar)dt + \sigma dz$$

where a and σ are constants. We will refer to this as the Hull-White model. It can be characterized as the Ho and Lee model with mean reversion at rate a. Alternatively it can be characterized as the Vasicek model with a time-dependent reversion level. The Ho and Lee model is a particular case with $a = 0$.

The model has the same amount of analytic tractability as that of Ho and Lee. The bond price at time t can be determined analytically as a function of the short rate,

$$P(t, T) = A(t, T)e^{-B(t,T)r}$$

where

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a}$$

and

$$\ln A(t, T) = \ln \frac{P(0, T)}{P(0, t)} - B(t, T)\frac{d \ln P(0, t)}{dt} - \frac{1}{4a^3}\sigma^2(e^{-aT} - e^{-at})^2(e^{2at} - 1)$$

The volatility structure in the Hull-White model is determined by both σ and a. The model can represent a wider ranger of volatility structures than Ho and Lee. The price volatility at time t of a bond maturing at time T is

$$\frac{\sigma}{a}[1 - e^{-a(T-t)}]$$

The instantaneous standard deviation at time t of the zero-coupon interest rate maturing at time T is

$$\frac{\sigma}{a(T-t)}[1 - e^{-a(T-t)}]$$

and the instantaneous standard deviation of the T-maturity instantaneous forward rate is $\sigma e^{-a(T-t)}$. These functions are shown in Figure 1.1.

The parameter σ determines the short rate's instantaneous standard deviation. The reversion rate parameter, a, determines the curvature in Figure 1.1(a) and the rate at which standard deviations decline with maturity in Figure 1.1(b) and 1.1(c).

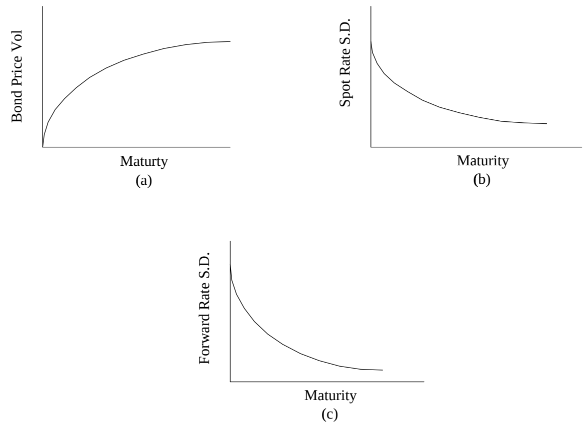


Figure 1.1: THE VOLATILITY STRUCTURE IN THE HULL-WHITE MODEL.

The higher a is, the greater the curvature and the decline are. When $a = 0$, discount bond price volatilities are a linear function of maturity, and the instantaneous standard deviations of both zeroes and forward rates are constant.

1.3 The Trinomial Tree

Hull and White [1990] had shown how trinomial trees can be used to value American bond options and other interest rate contingent claims in Vasicek's model. The value of r on the tree at time zero is the initial short rate, r_0. The values of r considered at other nodes have the form $r_0 + k\Delta r$ where k is an integer. The relationship between Δr and the time step, Δt, is chosen to be

$$\Delta r = \sigma\sqrt{3\Delta t}$$

The trinomial tree is constructed so that the change in r has the correct mean and standard deviation over each time interval Δt. The tree is more complicated than the binomial tree in three ways:

1. There are three branches emanating from each node, not two.
2. The probabilities on the branches are different in different
parts of the tree.

3. A branching process is liable to vary from node to node.

The alternative branching processes are illustrated in Figure 1.2. Figure 1.2(a) is the normal branching process. The alternative changes in r are: move up by Δr, stay the same, and move down by Δr. When r is high, it is sometimes necessary to use the branching process in Figure 1.2(c). The alternative changes in r are: stay the same, move down by Δr, and move down by $2\Delta r$. When r is low, it is sometimes necessary to use the branching process in Figure1.2(b). The alternative changes in r are then: move up by $2\Delta r$, move up by Δr, and stay the same. Other branching processes that are occasionally necessary in applications of the trinomial tree approach are indicated in Figure 1.2(d) and 1.2(e).

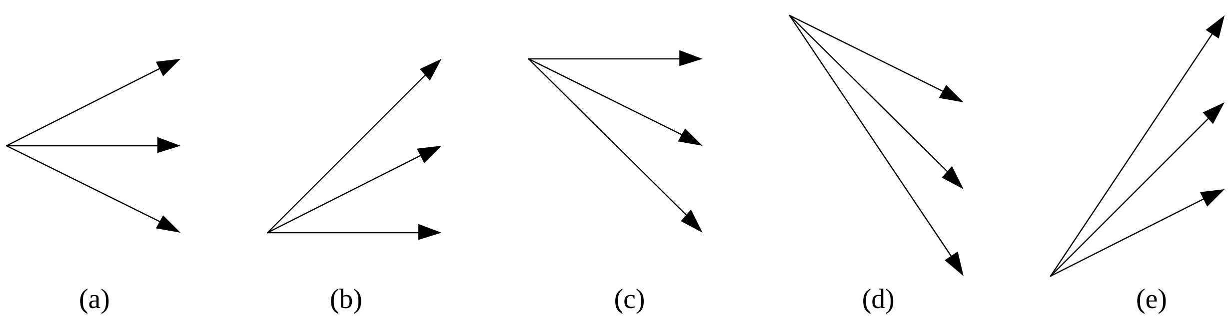


Figure 1.2: ALTERNATIVE BRANCHING PROCESSES IN TRINOMIAL TREE.

Consider the node at time $i\Delta t$ where $r = r_0 + j\Delta r$. To choose a branching process, we first calculate the expected value of r at time $(i+1)\Delta t$ given that we start at this node. We then choose the value of k which makes $r_0 + k\Delta r$ as close as possible to this expected value of r and draw the tree so that the three possible values of r that can be reached at time $(i+1)\Delta t$ are $r_0 + (k-1)\Delta r$, $r_0 + k\Delta r$, and $r_0 + (k+1)\Delta r$. If the drift in r is such that the expected change in r in time Δt is between $-\Delta r/2$ and $+\Delta r/2$, the normal branching process in Figure 1.2(a) is appropriate; if the expected change is between $\Delta r/2$ and $3\Delta r/2$, the branching process in Figure 1.2(b) is appropriate; and so on.

Bond prices are known analytically at each node of the tree. When an American bond option is being valued, it is therefore necessary for the tree to extend only to the end of the life of the option (not to the end of the life of the bond).

1.4 Organization of The Thesis

There are two main purposes in this thesis. The first one is to implement the Hull-White model and explore its stability. The second one is building the short rate trees for three different economies, and discussing the arbitrage opportunity between them. In Chapter 2 of this thesis, we discuss the single-factor Hull-White model, implement the tree building algorithm, and then test the relationship between the time period and the computing time to understand the convergence of the algorithm. Then we introduce a method to test the stability of the Hull-White model. Although the stability of the model has not been discussed before, it is an important issue for the model. To understand the stability of the Hull-White model, we implement another program named the Inverse Hull-White. The program just is essentially the inverse of the Hull-White model. By cascading the two programs together, we expect to obtain the input precisely if the Hull-White model is stable.

In Chapter 3, the major discussion is on the two-factor Hull-White model, and the main application is in modeling two correlated interest rates when each follows a process chosen from the family of one-factor models.

We introduce the swap in the beginning of Chapter 3. When the diff swaps and options on diff swaps are negotiated, the yield curves in two different countries must be modeled at the same time. Understanding the operation of the simple swaps can help us to understand the advantages of the two-factor Hull-White model. In the following section of Chapter 3, we discuss processes in two economies. In particular, the adjusting value of the DM short rate tree is calculated so that it reflects the evolution of rates from the viewpoint of a risk-neutral U.S. investor rather than a risk-neutral DM investor.

Besides the original two economies, Deutschemark (DM) and U.S. dollar (USD), a third economy, Pound Sterling, is later introduced. The Pound Sterling tree can be adjusted so that it reflects the evolution of rates from the viewpoint of a risk-neutral U.S. investor. But what interests us is the relationship between these two processes to prevent any arbitrage opportunities when the Pound Sterling tree is adjusted from the viewpoint of a DM investor as viewed from a U.S. investor.

Chapter 2

The One-Factor Hull-White Model

2.1 Introduction

Heath, Jarrow, and Morton [1992] provide the most general approach to constructing a one-factor no-arbitrage model of the term structure. Their approach involves specifying the volatilities of all instantaneous forward rates at all times. This is sometimes referred to as the volatility structure. The following equation can be used to calculate the drift of each instantaneous forward rate from which a binomial tree describing the evolution of the term structure of forward rates is constructed,

$$m(t,T) = s(t,T)\int_t^T s(t,\tau)d\tau.$$

The expected drifts of forward rates in a risk-neutral world are calculated from their volatilities, and the initial values of the forward rates are chosen to be consistent with the initial term structure.

Unfortunately, the model that results from the Heath, Jarrow, and Morton approach is usually non-Markov. It means that the HJM tree is, in general, nonrecombining in the sense that an up movement followed by a down movement does not lead to the same term structure as a down movement followed by an up movement. Generally, in the HJM tree, there are 2^n nodes after n time steps. This severely limits the number of time steps that can be used.

2.2 Tree Building for the Hull-White Model

The Hull-White model is

$$dr = [\theta(t) - ar]dt + \sigma dz$$

Hull and White [1993] construct a trinomial tree to represent movements in r by using time steps of length Δt and considering at the end of each time step r-values of the form $r_0 + k\Delta r$, where k is a integer, and r_0 is the initial value of r. The tree branching can take any of the forms shown in Figure 2.1. Here we arrange the geometry of the tree so that the central node always corresponds to the expected value of r. Doing this can lead to faster tree construction, more accurate pricing, and much more accurate values for hedge parameters.

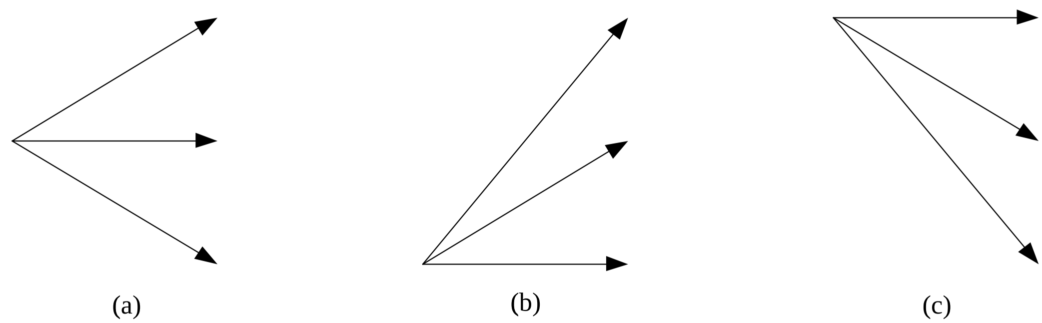


Figure 2.1: Alternative Branching Processes.

There are a few stages to build a Hull-White model tree. The first stage is to build a preliminary tree for r, setting $\theta(t) = 0$ and the initial value of $r = 0$. The process assumed for r during the first stage is therefore

$$dr = -ardt + \sigma dz$$

For this process, $r(t + \Delta t) - r(t)$ is normally distributed. For the purpose of tree construction, we define r as the continuously compounded Δt-period rate. We denote the expected value of $r(t + \Delta t) - r(t)$ by M and the variance of $r(t + \Delta t) - r(t)$ by V,

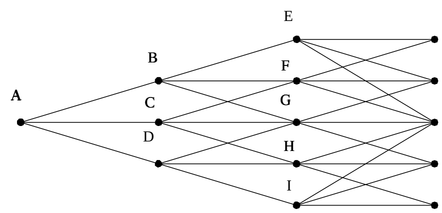
$$E[r(t + \Delta t) - r(t)] = M$$

$$var(r(t + \Delta t) - r(t)) = V$$

We first choose the size of the time step, Δt. Then set the size of the interest rate step in the tree, Δr, as

$$\Delta r = \sqrt{3V}$$

Theoretical work in numerical procedures suggests that this is a good choice of Δr from the standpoint of error minimization.



Node	A	B	C	D	E	F	G	H	I
r	0.00	1.73	0.00	-1.73	3.46	1.73	0.00	-1.73	-3.46 (%)
Pu	0.167	0.122	0.167	0.222	0.887	0.122	0.167	0.222	0.087
Pm	0.666	0.656	0.666	0.656	0.026	0.656	0.666	0.656	0.026
Pd	0.167	0.167	0.122	0.122	0.087	0.222	0.167	0.122	0.887

Figure 2.2: Simple Trinomial tree.

Our first objective is to build a tree similar to that shown in Figure 2.2, where the nodes are evenly spaced in r and t. To do this, we must resolve which of the three branching methods shown in Figure 2.1 will apply at each node. This will determine the overall shape of the tree. Once this is done, the branching probabilities must also be calculated.

Define (i, j) as the node for which $t = i\Delta t$ and $r = j\Delta r$. Define p_u, p_m, and p_d as the probabilities of the highest, middle, and lowest branches emanating from a node. The probabilities are chosen to match the expected change and variance of the change in r over the next time interval Δt. The probabilities must also sum to unity. This leads to three equations in the three probabilities. When r is at node (i, j), the expected change during the next time step of length Δt is $j\Delta rM$, and the variance of the change is V.

If the branching from node (i, j) is as in Figure 2.1(a), we can list three equations below,

$$p_u + p_m + p_d = 1$$

$$(\Delta r)p_u + 0 + (-\Delta r)p_d = j\Delta rM$$

$$(\Delta r)^2 p_u + 0 + (-\Delta r)^2 p_d = (j\Delta rM)^2 + \frac{1}{3}\Delta r^2$$

and the solution of the three equations above is

$$p_u = \frac{1}{6} + \frac{j^2M^2 + jM}{2}$$

$$p_m = \frac{2}{3} - j^2M^2$$

$$p_d = \frac{1}{6} + \frac{j^2M^2 - jM}{2}$$

respectively.

If the branching has the form shown in Figure 2.1B, the equations are:

$$p_u + p_m + p_d = 1$$

$$(2\Delta r)p_u + (\Delta r)p_m + 0 = j\Delta rM$$

$$(2\Delta r)^2 p_u + (\Delta r)^2 p_m + 0 = (j\Delta rM)^2 + \frac{1}{3}\Delta r^2$$

and the solution is

$$p_u = \frac{1}{6} + \frac{j^2M^2 - jM}{2}$$

$$p_m = -\frac{1}{3} - j^2M^2 + 2jM$$

$$p_d = \frac{7}{6} + \frac{j^2M^2 - 3jM}{2}$$

Finally, if it has the form shown in Figure 2.1C, the equations are:

$$p_u + p_m + p_d = 1$$

$$0 + (-\Delta r)p_m + (-2\Delta r)p_d = j\Delta rM$$

$$0 + (-\Delta r)^2 p_m + (-2\Delta r)^2 p_d = (j\Delta rM)^2 + \frac{1}{3}\Delta r^2$$

and the solution is

$$p_u = \frac{7}{6} + \frac{j^2M^2 + 3jM}{2}$$

$$p_m = -\frac{1}{3} - j^2M^2 - 2jM$$

$$p_d = \frac{1}{6} + \frac{j^2M^2 + jM}{2}$$

Most of the time the branching in Figure 2.1(a) is appropriate. When $a > 0$, it is necessary to switch from the branching in Figure 2.1(a) to the branching in Figure 2.1(c) when j is large. This is to ensure that the probabilities p_u, p_m, and p_d are all positive. Similarly, it is necessary to switch from the branching in Figure 2.1(a) to the branching in Figure 2.1(b) when j is small (i.e., negative and large in absolute value).

Define j_{max} as the value of j where we switch from the Figure 2.1(a) branching to the branching in Figure 2.1(c), and j_{min} as the value of j where we switch from the Figure 2.1A branching to the Figure 2.1(b) branching. It can be shown from the equations that p_u, p_m, and p_d are always positive, provided j_{max} is chosen to be an integer between $-0.184/M$ and $-0.816/M$, and j_{min} is chosen to be an integer between $0.184/M$ and $0.816/M$. That is :

$$0 \leq p_d, p_m, p_u \leq 1$$

$$\Rightarrow \frac{-0.816}{M} \leq j_{max} \leq \frac{-0.184}{M}$$

$$\frac{0.184}{M} \leq j_{min} \leq \frac{0.816}{M}$$

Note that when $a > 0, M < 0$. In practice it is most efficient to set j_{max} equal to the smallest integer greater than $-0.184/M$ and j_{min} equal to $-j_{max}$.

We illustrate the first stage of the tree construction by showing how the tree in Figure 2.2 is constructed for $\sigma = 0.01, a = 0.1$, and Δt=one year. In this example we set $M = -a\Delta t$ and $V = \sigma^2\Delta t$. This is accurate to order Δt. The steps in the construction of the tree can be described as below:

1. Calculate Δr from Δt. In this case $\Delta r = 0.01\sqrt{3} = 0.0173$.

2. Calculate the bounds for j_{max}. These are $0.184/0.1$ and $0.816/0.1$, or $1.84 < j_{max} < 8.16$. We set $j_{max} = 2$. Similarly, $j_{min} = -2$.

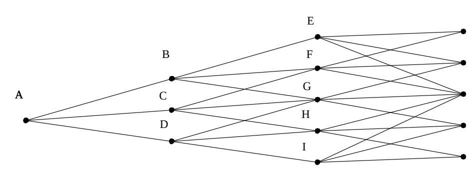
3. Using the equations for p_u, p_m, and p_d to calculate the probabilities on the branches emanating from each node.

Note that the probabilities at each node depend only on j. For example, the probabilities at node B are the same as the probabilities at node F. Furthermore, the tree is symmetrical. The probabilities at node D are the mirror image of the probabilities at node B.

This completes the tree for the simplified process. For the next stage in the tree construction, we perform the following.

4. Introduce the correct, time-varying drift.

To do this, we displace the nodes at time $i\Delta t$ by an amount α_i to produce a new tree; see Figure 2.3. The value of r at node (i,j) in the new tree equals the value of r at node (i,j) in the old tree plus α_i. The probabilities on the new tree are unchanged. The values of the $\alpha_i s$ are chosen so that the tree prices all discount bonds consistently with the initial term structure observed in the market.



Node	A	B	C	D	E	F	G	H	I
r	3.82	6.93	5.20	3.47	9.71	7.98	6.25	4.52	2.79
Pu	0.167	0.122	0.167	0.222	0.887	0.122	0.167	0.222	0.087
Pm	0.666	0.656	0.666	0.656	0.026	0.656	0.666	0.656	0.026
Pd	0.167	0.222	0.167	0.122	0.087	0.222	0.167	0.122	0.887

Figure 2.3: ADJUSTED TRINOMIAL TREE.

The effect of moving from the tree in Figure 2.2 to the tree in Figure 2.3 is to change the process being modeled from

$$dr = -ardt + \sigma dz$$

to

$$dr = [\theta(t) - ar]dt + \sigma dz$$

If we define $\hat{\theta}(t)$ as the estimate of θ given by the tree between times t and $t+\Delta t$, as the drift in r at time $i\Delta t$ at the midpoint of the tree is $\hat{\theta}(t) - a\alpha_i$, the average rate change in period Δt equals to

$$\{\theta(t+\Delta t) - a(r+\alpha_i)\} - \{\theta(t) - ar\}$$

$$= \hat{\theta}(t) - a\alpha_i$$

so that

$$[\hat{\theta}(t) - a\alpha_i]\Delta t = \alpha_i - \alpha_{i-1}$$

or

$$\hat{\theta}(t) = \frac{\alpha_i - \alpha_{i-1}}{\Delta t} + a\alpha_i$$

This equation relates the $\hat{\theta s}$ to the $\alpha_i s$. In the limit as $\Delta t \to 0$, $\hat{\theta}(t) \to \theta(t)$.

To facilitate computations, we define $Q_{i,j}$ as the present value of a security that pays off \$1 if node (i,j) is reached and zero otherwise. The α_i and $Q_{i,j}$ are calculated using forward induction. We illustrate the procedure by showing how the tree in Figure 2.3 is calculated from the tree in Figure 2.2 when the t-year continuously compounded zero-coupon rate is $0.08 - 0.05e^{-0.18t}$. (This corresponds approximately to the U.S. term structure at the beginning of 1994, with one-, two-, and three-year yields of 3.82%, 4.51%, and 5.09%, respectively.)

The value of $Q_{0,0}$ is 1. The value of α_0 is chosen to give the right price for a zero-coupon bond maturing at time Δt. That is, α_0 is set equal to the initial Δt period interest rate. Since $\Delta t = 1$ in this example. α_0= one-year-yield= 0.0382. The next step is to calculate the values of $Q_{1,1}, Q_{1,0}$, and $Q_{1,-1}$. There is a probability 0.1667 that the $(1,1)$ node is reached and the discount rate for the first time step is 3.82%. The value of $Q_{1,1}$ is therefore

$$Q_{1,1} = 0.1667e^{-0.0382} = 0.1604$$

Similarly

$$Q_{1,0} = 0.6667e^{-0.0382} = 0.6417$$

and

$$Q_{1,-1} = 0.1667e^{-0.0382} = 0.1604$$

Once $Q_{1,1}, Q_{1,0}$, and $Q_{1,-1}$ have been calculated, we are in a position to determine α_1. This is chosen to give the right price for a zero-coupon bond maturing at time $2\Delta t$. Since

$$\Delta r = \sqrt{3V} = \sigma\sqrt{3\Delta t} = 0.0173$$

and $\Delta t = 1$, the price of this bond as seen at node B is

$$e^{-(\alpha_1+\Delta r)} = e^{-(\alpha_1+0.0173)}$$

Similarly, the price as seen at node C is

$$e^{-(\alpha_1+0\Delta r)} = e^{-\alpha_1},$$

and the price as seen at node D is

$$e^{-(\alpha_1+(-\Delta r))} = e^{-(\alpha_1-0.0173)}.$$

The price as seen at the initial node A is therefore

$$P(0,2) = Q_{1,1}e^{-(\alpha_1+0.0173)} + Q_{1,0}e^{-\alpha_1} + Q_{1,-1}e^{-(\alpha_1-0.0173)}$$

where P(0,2) is the zero-coupon bind price maturing at time $2\Delta t$.

From the initial term structure, this bond price should be $e^{-0.0451*2} = 0.9137$. Substituting from the Qs in the equation above, we can obtain

$$0.1604e^{-(\alpha_1+0.0173)} + 0.6417e^{-\alpha_1} + 0.1604e^{-(\alpha_1-0.0173)} = 0.9137$$

This can be solved to give $\alpha_1 = 0.0520$.

The next step is to calculate $Q_{2,2}, Q_{2,1}, Q_{2,0}, Q_{2,-1}$, and $Q_{2,-2}$. These are found by discounting the value of a single \$1 payment at one of E-I nodes back through the tree. This can be simplified by using previously determined Q values.

Consider as an example $Q_{2,1}$. This is the value of a security that pays off \$1 if node F is reached and zero otherwise. Node F can be reached only from nodes B and C. The interest rates at these nodes are 6.93% and 5.20%, respectively. The probabilities associated with the B-F and C-F branches are 0.656 and 0.167. The value at node B of $1 received at node F is therefore $0.656e^{-0.0693}$. The value at node C is $0.167e^{-0.0520}$, and the present value is the sum of each of these weighted by the present value of $1 received at the corresponding node. This is

$$\begin{aligned}Q_{2,1} &= 0.656e^{-0.0693} \times Q_{1,1} + 0.167e^{-0.0520} \times Q_{1,0} \\ &= 0.656e^{-0.0693} \times 0.1604 + 0.167e^{-0.0520} \times 0.6417 \\ &= 0.1997\end{aligned}$$

Similarly

$$\begin{aligned}Q_{2,2} &= 0.122e^{-0.0693} \times Q_{1,1} \\ &= 0.122e^{-0.0693 \times 0.1604} \\ &= 0.0183\end{aligned}$$

and

$$\begin{aligned}Q_{2,0} &= 0.222e^{-0.0693} \times Q_{1,1} + 0.666e^{-0.0520} \times Q_{1,0} + 0.222e^{-0.0347} \times Q_{1,-1} \\ &= 0.222e^{-0.0693} \times 0.1604 + 0.666e^{-0.0520} \times 0.6417 + 0.222e^{-0.0347} \times 0.1604 \\ &= 04737\end{aligned}$$

in the same way, $Q_{2,-1} = 0.2032$, and $Q_{2,-2} = 0.0189$.

The next step is to calculate α_2.

$$P(0,3) = Q_{2,2}e^{-(\alpha_2+2\Delta r)} + Q_{2,1}e^{-(\alpha_2+\Delta r)} + Q_{2,0}e^{-\alpha_2}$$

$$+Q_{2,-1}e^{-(\alpha_2-\Delta r)} + Q_{2,2}e^{-(\alpha_2-2\Delta r)}$$

where $P(0,3) = 0.8584$ is the bond price discounted from period 3, and $\Delta r = 0.0173$. Substituting for these values in the equation above, we obtain

$$0.0183e^{-(\alpha_2+0.0346)} + 0.1997e^{-(\alpha_2+0.0173)} + 0.4737e^{-\alpha_2}$$

$$+0.2032e^{-(\alpha_2-0.0173)} + 0.0189e^{-(\alpha_2-0.0346)} = 0.8584$$

This can be solved to give α_2. After that the $Q_{3,j}s$ can then be calculated. Then α_3 can be computed; and so on.

To express the approach more formally, suppose the $Q_{i,j}$s have been determined for $i \leq m$ $(m \geq 0)$. The next step is to determine α_m so that at time 0 the tree correctly prices a discount bond maturing at $(m+1)\Delta t$. The interest rate at node (m,j) is $\alpha_m + j\Delta r$ so that the price of a discount bond maturing at time $(m+1)\Delta t$ is given by

$$P(0,m+1) = \sum_{j=-n_m}^{n_m} Q_{m,j} exp[-(\alpha_m + j\Delta r)\Delta t]$$

where n_m is the number of nodes on each side of the central node at time $m\Delta t$. The solution to this equation is

$$\alpha_m = \frac{\ln(\sum_{j=-n_m}^{n_m} Q_{m,j} e^{-j\Delta r \Delta t}) - \ln P(0,m+1)}{\Delta t}$$

Once α_m has been determined, the $Q_{i,j}$ for $i = m+1$ can be calculated using

$$Q_{m+1,j} = \sum_k Q_{m,k} q(k,j) exp[-(\alpha_m + k\Delta r)\Delta t]$$

where $q(k,j)$ is the probability of moving from node (m,k) to node $(m+1,j)$, and the summation is taken over all values of k for which this is non-zero.

2.3 Convergence of the Tree

As we described in the last section, the Hull-White model tree can now be constructed. But the critical questions are: Does the algorithm converge fast? What is the computing time? And is the tree stable?

In the first experiment, we implement the algorithm, and then obtain the relationship between time period n and the computing time. The result is shown in Figure 2.4 below.

In Figure 2.4 we can see that the relationship between n and computing time is essentially linear. The result is as expected. A linear-time model has the distinct advantage that the tree can be built with fine partitions. But, does the resulting tree the one we are after? We turn to this question next.

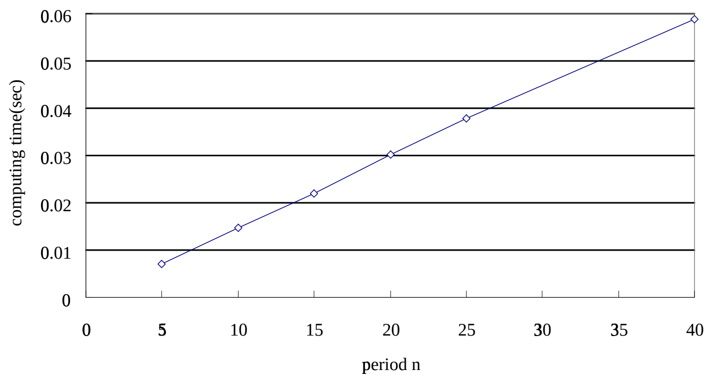


Figure 2.4: THE EXECUTION TIME OF HULL-WHITE PROCEDURE.

2.4 Stability of the Tree

As we have seen in the introduction, there is much research activity in no-arbitrage models in recent years. When a new model is proposed, everyone is concerned about the model's convergence, the computing time, and the complexity? Besides these issues, we bring up another equally important question, the stability of the model.

In the single-factor Hull-White calibration model algorithm, we first input the initial zero rate, then calculate the expected value and variance of $r(t+\Delta t)-r(t)$, and the size of the interest rate step in the tree, Δr. The next step is to calculate the probabilities of branches of each node according to the three alternative branching processes. Then introduce a new variable $Q_{i,j}$ as the present value of a security that pays off \$1 if node (i,j) is reached and zero otherwise. Using forward induction, the $Q_{i,j}$ of each node and the α_i of each period can then be calculated. After the $\alpha_i s$ have been calculated, the new interest rate tree can be constructed.

So in our implementation of the one-factor Hull-White model, the input is the initial spot rates, and the output is the adjusting amount $\alpha_i s$. By the output, α_i, we can construct the whole Hull-White model tree.

In the algorithm above, we need the initial spot rates as the input of the program, and proceed to the output, α_i, to build the whole tree. According to the tree, the present value of \$1 at maturity can then be calculated. From which we can get the yield curve. In principle, the new yield curve should fits the initial yield curve. Our question here is: Is the Hull-White model algorithm stable in the sense that the $\alpha_i s$ being reverse engineered are close to the $\alpha_i s$ that produce the yield curve in the first place?

To test the stability of the Hull-White model, we implement another program. This program does the reverse work of the Hull-White model program, named Inverse Hull-White model program. In the Hull-White model program, the yield curve is the input needed to produce the output, α_i. In the Inverse Hull-White model program, we suppose that the same tree has been constructed with the input α_i (here we don't care how to produce the $\alpha_i s$), and it produces the output, r_i (spot rates). The spot rate will feed the Hull-White model program to produce $\tilde{\alpha}_i$. We collect the $\tilde{\alpha}_i s$ produced here and compare them with the original input of the Inverse Hull-White model program, α_i, to measure the stability of the Hull-White model algorithm.

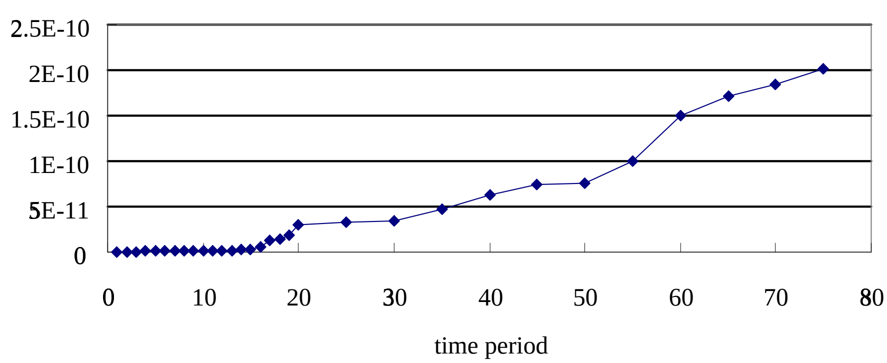


Figure 2.5: The Difference between the adjustment of short rates, with α_i=0.05.

By Figure 2.5, we can see that, when the period n increases, the value of $[\sum(\tilde{\alpha}_i - \alpha_i)^2]/n$ is only between 0 and $3.00 * 10^{-10}$. It means that when the period n increases, the sum of the difference between $\tilde{\alpha}_i s$, and $\alpha_i s$ remains small. So we are sure that when n increases, the value of $\tilde{\alpha}_i$ is almost the same as α_i.

After that, we try different $\alpha_i s$ to see if it makes the difference. Figure 2.6, and Figure 2.7 show the results from two different $\alpha_i s$.

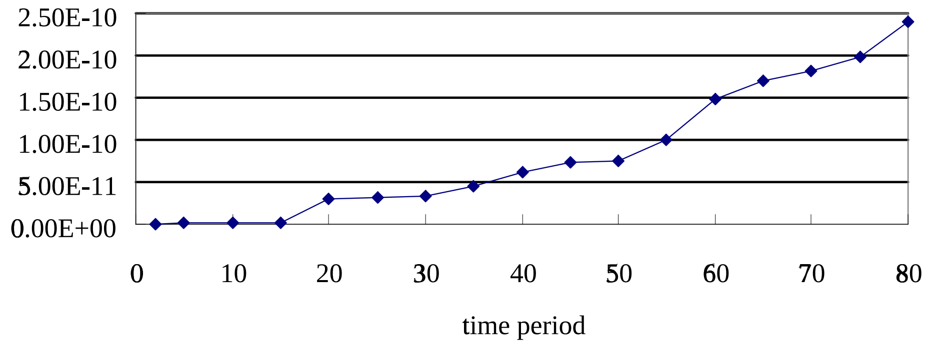


Figure 2.6: THE DIFFERENCE BETWEEN THE ADJUSTMENT OF SHORT RATES, WITH α_i=0.04.

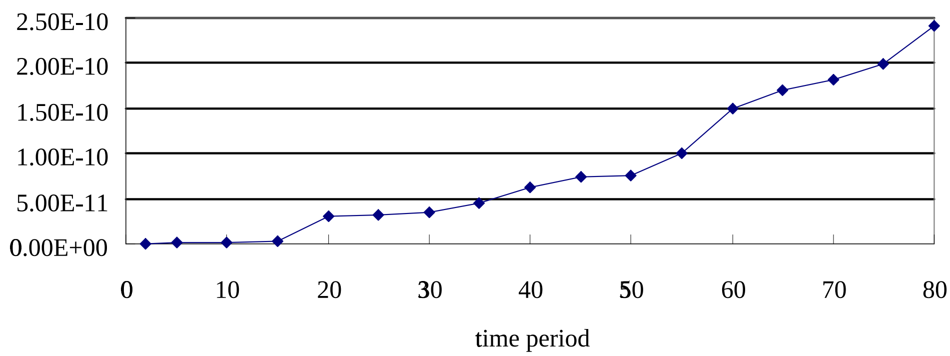


Figure 2.7: THE DIFFERENCE BETWEEN THE ADJUSTMENT OF SHORT RATES, WITH α_i=0.06.

In Figure 2.6, and Figure 2.7, different values of α_i do not change the basic relationship between $\alpha_i s$, and $\tilde{\alpha}_i s$. So, different $\alpha_i s$ do not destabilize the value of $[\sum(\tilde{\alpha}_i - \alpha_i)^2]/n$. The Hull-White model is therefore stable.

2.5 Conclusion

In this chapter, we first introduced the tree building process for the Hull-White model. In that process, the expected value of $r(t+\Delta t)-r(t)$ and the variance of $r(t+\Delta t)-r(t)$ are used to calculate the interest rate process. After that, the probabilities of each branching are calculated by the three different types of branching. In the calculation of forward rates of each nodes, we define a variable $Q_{i,j}$ as the present value of a security that pays off \$1 if node (i,j) is reached and zero otherwise. And the variable α_i is the value of adjustment of period i. The α_i and $Q_{i,j}$ are calculated using forward induction. After the calculation of α_i, the value of forward rate of node (i,j) is

$$r(i,j) = \alpha_i + j\Delta r$$

Then we have probabilities and forward rates of all the nodes of the whole tree.

After building the tree, what interests us is the convergence and stability of that the Hull-White model. To test the efficiency of the Hull-White model, we add an extra code in the original program to obtain the executing time of the calculation of $Q_{i,j}$. By the result of the test, the running time is linear. The relation is shown in Figure 2.4.

The next step is to test the stability of the Hull-White model tree, a new idea. To do this, we implement another program named Inverse Hull-White model program. In this program, the input is the α_i, and the output is the spot rates. Then, this output is used to be the input of original Hull-White model program. The Hull-White model program will then output the adjusting value, $\tilde{\alpha}_i$. We compare the input of the Inverse Hull-White model program, α_i, and the output of the Hull-White model program, $\tilde{\alpha}_i$, using mean square error $[\sum(\tilde{\alpha}_i - \alpha_i)^2]/n$ with increasing n. In Figure 2.5, when n increases to 80 years, the boundary of the curve is between 0, and $3.00 * 10^{-10}$. It means that when n increases, the output of the Hull-White model program, $\tilde{\alpha}_i$, is still almost the same as the input of the Inverse Hull-White model program, α_i. In other words, the Hull-White model is very stable.

Does the positive result above somehow depend on the α_i? To test this, we fix the input of the Inverse Hull-White model program, $\alpha_i s$, at 0.04 first, and then 0.06. Collecting the results of these tests, the curves are shown in Figure 2.6, and Figure 2.7. We find that the three curves are similar. The curve doesn't change because of different $\alpha_i s$. Concluding from the above two tests, the Hull-White model can be regarded as a stable model.

Chapter 3

Two-Factor Hull-White Model

3.1 Introduction

Certain types of interest rate derivatives require yield curves in two different countries be modeled simultaneously. Examples are diff swaps and options on diff swaps. Here we explain how the procedure in Hull and White [1994] can be extended to accommodate two correlated interest rates.

Before introducing the two-factor Hull-White model, we first describe how swaps work. After that we will show why a two-factor Hull-White model is needed.

3.2 Swap

In the last decade there has been a dramatic increase in the number of derivative securities traded on organized exchanges. In a world characterized by volatile interest rates, the limitation of existing techniques to cope with such risk has resulted in the development of new products, such as futures and options written on short-term and long-term interest rates, and swaps.

Swaps are private agreements between two companies to exchange cash flows in the future according to a prearranged formula. They can be regarded as portfolios of forward contracts. The study of swaps is therefore a natural extension of the study of forward and futures contracts.

An interest rate swap in its basic form occurs when a firm that has issued one type of debt instrument agrees to swap interest payments with a firm that has issued another type of debt instrument. For example, a firm that may have issued fixed debt agrees in a swap to make floating rate payments to a firm that issued floating rate debt. In return, this latter firm agrees to make fixed rate payments to the former firm.

Why should two companies enter into such an agreement? The most commonly reason is comparative advantages. Some companies appear to have a comparative advantage in fixed rate markets, while others have a comparative advantage in floating rate markets. A swap has the effect of transforming a fixed rate loan into a floating rate loan and vice versa.

We now give an example of how comparative advantages can lead to an interest rate swap. Suppose that two companies, A and B, both wish to borrow $10 million for 5 years and have been offered the rates shown in Figure 3.1. Assume that company B wants to borrow at a fixed rate of interest, while company A wants to borrow floating funds at a rate linked to 6-month LIBOR. Company B clearly has a lower credit rating than company A since it pays a higher rate of interest than company A in both fixed and floating markets.

	Fixed	Floating
Company A	10.00%	6-month LIBOR+0.30%
Company B	11.20%	6-month LIBOR+1.00%

Figure 3.1: BORROWING RATES.

A key feature of the rates offered to companies A and B is that the difference between the two fixed rates is greater than the difference between the two floating rates. Company B pays 1.20 percent more than company A in fixed rate markets, and only 0.70 percent more than company A in floating rate markets.

Obviously, company B has a comparative advantage in floating rate markets, while company A have a comparative advantage in fixed rate markets. A profitable swap can be negotiated. Company A borrows fixed rate funds at 10 percent per annum. Company B borrows floating rate funds at LIBOR plus 1.00 percent per annum. They then enter into a swap agreement to ensure that A ends up with floating rate funds and B ends up with fixed rate funds.

As an easy step in understanding how the swap might work, we assume that A and B get in touch with each other directly. The sort of swap they might negotiate is shown in Figure 3.2. Company A agrees to pay company B interest at 6-moth LIBOR on \$10 million. In return, company B agrees to pay company A interest at a fixed rate of 9.95 percent per annum on \$10 million.

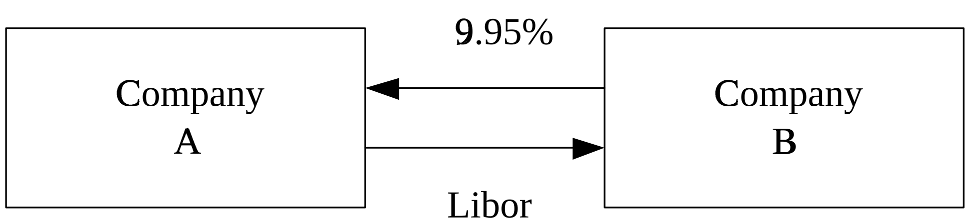


Figure 3.2: A Direct Swap Agreement between A and B.

3.3 Building and Adjusting the Tree

For ease of exposition, we suppose the two countries are the United States and Germany, and that cash flows from the derivative under consideration are to be realized in U.S. dollars (USD). We first build a tree for the USD short rate and a tree for the Deutschemark (DM) short rate using the procedure introduced in Chapter 2. As a result of the construction procedure, the USD tree describes the evolution of USD interest rates from the viewpoint of a risk-neutral USD investor, and the DM tree describes the evolution of DM rates from the viewpoint of a risk-neutral DM investor.

Since cash flows are realized in USD, the DM tree must be adjusted so that it reflects the evolution of rates from the viewpoint of a risk-neutral U.S. investor rather than a risk-neutral DM investor. Define

r_1: USD short rate from the viewpoint of a risk-neutral U.S. investor

r_2: DM short rate from the viewpoint of a risk-neutral DM investor

r_3: Pound Sterling short rate form the viewpoint of a risk-neutral Pound Sterling investor

r^*_{21}: DM short rate from the viewpoint of a risk-neutral U.S. investor

r^*_{32}: Pound Sterling short rate from the viewpoint of a risk-neutral DM investor

r^*_{31}: Pound Sterling short rate form the viewpoint of a risk-neutral U.S. investor

r^*_{321}: Pound Sterling short rate from the viewpoint of a DM investor as viewed from a USD investor

X_{12}: the exchange rate of USD and DM (USD/DM)

X_{13}: the exchange rate of USD and Pound Sterling (USD/Pound Sterling)

X_{23}: the exchange rate of DM and Pound Sterling (DM/Pound Sterling)

$\bar{\sigma}_{12}$: volatility of exchange rate X_{12}

$\bar{\sigma}_{13}$: volatility of exchange rate X_{13}

$\bar{\sigma}_{23}$: volatility of exchange rate X_{23}

$\bar{\rho}_{12}$: instantaneous coefficient of correlation between the exchange rate, X_{12}, and the DM interest rate, r_2

$\bar{\rho}_{13}$: instantaneous coefficient of correlation between the exchange rate, X_{13}, and the Pound Sterling interest rate, r_3

$\bar{\rho}_{23}$: instantaneous coefficient of correlation between the exchange rate, X_{23}, and the Pound Sterling interest rate, r_3

ρ_{12}: instantaneous coefficient of correlation between the interest rates, r_1 and r_2

ρ_{13}: instantaneous coefficient of correlation between the interest rates, r_1 and r_3

ρ_{23}: instantaneous coefficient of correlation between the interest rates, r_2 and r_3

Because USD and DM follow the Hull-White model, the processes for r_1 and r_2 are

$$dx_1 = [\theta_1(t) - a_1 x_1]dt + \sigma_1 dz_1$$

and

$$dx_2 = [\theta_2(t) - a_2 x_2]dt + \sigma_2 dz_2$$

The two processes above are interest rates from the viewpoints of USD and DM investors, respectively. Here $x_1 = f_1(r_1)$ and $x_2 = f_2(r_2)$ for some functions f_1 and f_2, and dz_1 and dz_2 are Wiener processes with correlation ρ_{12}. The reversion rate parameters, a_1 and a_2, and the standard deviations, σ_1 and σ_2, are constant. The drift parameters, θ_1 and θ_2, are functions of time. We want to calculate the process for the DM interest rate from the viewpoint of a USD investor.

3.3.1 Relating Two Economies

Define Z as the value of a variable seen from the perspective of a risk-neutral DM investor and Z^* as the value of the same variable seen from the perspective of a risk-neutral USD investor. Suppose that Z depends only on the DM risk-free rate so that

$$dZ = \mu(Z)Zdt + \sigma(Z)Zdz_2$$

where dz_2 is the Wiener process driving the DM risk-free rate, and μ and σ are functions of Z. The work of Cox, Ingersoll, and Ross [1985] and others shows that the process for Z^* has the form:

$$dZ^* = [\mu(Z^*) - \lambda\sigma(Z^*)]Z^*dt + \sigma(Z^*)dz_2$$

where the risk premium, λ, is a function of Z^*.

We first apply this result to the case where the variable under consideration is the DM price of a DM discount bond. Define P as the value of this variable from the viewpoint of a risk-neutral DM investor and P^* as that from the viewpoint of a risk-neutral USD investor. From the perspective of a risk-neutral DM investor, the variable is the price of a traded security so that:

$$dP = r_2Pdt + \sigma_pPdz_2$$

where σ_p is the volatility of P. Hence:

$$dP^* = [r_{21}^* - \lambda\sigma_p]P^*dt + \sigma_pP^*dz_2 \tag{3.1}$$

The risk-neutral process for the exchange rate, X, from the viewpoint of a risk-neutral USD investor is

$$dX = (r_1 - r_{21}^*)X\,dt + \bar{\sigma}_{12} X\,dz_x \tag{3.2}$$

where dz_x is a Wiener process.

The variable XP^* is the price in USD of the DM bond. The drift of XP^* in a risk-neutral USD world must therefore be $r_1 XP^*$. From Equations (3.1) and (3.2), this drift can also be written as

$$XP^*(r_1 - \lambda\sigma_p + \bar{\rho}_{12}\bar{\sigma}_{12}\sigma_p)$$

It follows that

$$\lambda = \bar{\rho}_{12}\bar{\sigma}_{12}$$

When moving from a risk-neutral DM investor to a risk-neutral USD investor, there is a market price of risk adjustment of $\bar{\rho}_{12}\bar{\sigma}_{12}$.

We can now apply the general result given for Z at the beginning of this solving process to the variable $f(r_2)$. We are assuming that:

$$df_2(r_2) = [\theta_2(t) - af_2(r_2)]dt + \sigma_2 dz_2$$

Hence

$$df_2(r_{21}^*) = [\theta_2(t) - \bar{\rho}_{12}\bar{\sigma}_{12}\sigma_2 - af_2(r_{21}^*)]dt + \sigma_2 dz_2$$

By relating the two economies above, we know that dx_{21}^* follows

$$dx_{21}^* = [\theta_2(t) - \bar{\rho}_{12}\sigma_2\bar{\sigma}_{12} - a_x x_{21}^*]dt + \sigma_2 dz_2$$

where $x_{21}^* = f_2(r_{21}^*)$. The effect of moving from a DM risk-neutral world to a USD risk-neutral world is to reduce the drift of x_2 by $\bar{\rho}_{12}\sigma_2\bar{\sigma}_{12}$. The expected value of x_2 at time t is reduced by

$$\int_0^t \bar{\rho}_{12}\bar{\sigma}_{12}\sigma_2 e^{-a_2(t-\tau)}d\tau = \frac{\bar{\rho}_{12}\sigma_2\bar{\sigma}_{12}}{a_2}(1 - e^{-a_2 t})$$

To adjust the DM tree so that it reflects the viewpoint of a risk-neutral U.S. investor, the value of x_2 at nodes at time $i\Delta t$ (i.e., after i time steps) should therefore be reduced by

$$\frac{\bar{\rho}_{12}\sigma_2\bar{\sigma}_{12}}{a_2}(1 - e^{-a_2 i\Delta t})$$

Note that this is true for all functions f_2, not just $f_2(r) = r$.

3.4 Arbitrage Opportunities

In the case of building the DM short rate tree from the viewpoint of a USD investor, we know that the DM short rate must be reduced by $\frac{\bar{\rho}_{12}\sigma_2\bar{\sigma}_{12}}{a_2}(1 - e^{-a_2 t})$. Now another country, England, is introduced, and that the cash flows from the derivative under consideration are to be realized in U.S. dollars (USD) as described in the last section. The Pound Sterling short rate tree will be built first using the procedure outlined in the previous chapter. By the construction, the Pound Sterling tree describes the evolution of the Pound Sterling interest rates from the viewpoint of a risk-neutral Pound Sterling investor.

By the result of the previous section, the DM interest rate tree must be adjusted so that it reflects the evolution of rates from the viewpoint of a risk-neutral U.S. investor rather than a risk-neutral DM investor. Similarly, the Pound Sterling short rate tree must be adjusted to reflect the evolution of rates from the viewpoint of a risk-neutral U.S. investor rather than a risk-neutral Pound Sterling investor.

But, there are two ways to adjust the Pound Sterling short rate tree: adjust the tree by the Pound Sterling short rate from the viewpoint of a risk-neutral U.S. investor or by the Pound Sterling short rate from the viewpoint of a DM investor as viewed from a USD investor. Could these two ways be different?

The discussion in the last section involves the processes for r_1 and r_2

$$dx_1 = [\theta_1(t) - a_1 x_1]dt + \sigma_1 dz_1$$

$$dx_2 = [\theta_2(t) - a_2 x_2]dt + \sigma_2 dz_2$$

Suppose the process for r_3 is

$$dx_3 = [\theta_3(t) - a_3 x_3]dt + \sigma_3 dz_3$$

where $x_3 = f_3(r_3)$ for some functions f_3, dz_3 is Wiener process, $\bar{\rho}_{13}$ is the correlation between dz_3 and dz_1, and $\bar{\rho}_{23}$ is the correlation between dz_3 and dz_2. The reversion rate parameter, a_3, and the standard deviation, σ_3, are constant. The drift parameter, θ_3, is a function of time. The other parameters in the processes for r_1 and r_2 are as described in the previous discussion.

As discussed in the last section, the process for r_{21}^* is

$$dx_{21}^* = [\theta_2(t) - \bar{\rho}_{12}\sigma_2\bar{\sigma}_{12} - a_2 x_{21}^*]dt + \sigma_2 dz_2$$

where $x_{21}^* = f_2(r_{21}^*)$. Reducing the drift of x_2 by $\bar{\rho}_{12}\sigma_2\bar{\sigma}_{12}$ makes the tree move from the DM risk-neutral world to the USD risk-neutral world. Similarly, we can reduce the drift of x_3 by some value to move the Pound Sterling tree from the Pound Sterling risk-neutral world to the DM risk-neutral world, or to the USD risk-neutral world.

In the previous section, we have calculated the reducing factor of the process

$$dZ = \mu(Z)Zdt + \sigma(Z)Zdz_2$$

In the same way, the reducing factor of the Pound Sterling short rate tree moving from the Pound Sterling risk-neutral investor to the USD risk-neutral investor is $\bar{\rho}_{13}\sigma_3\bar{\sigma}_{13}$, and the process for r_{31}^* is

$$dx_{31}^* = [\theta_3(t) - \bar{\rho}_{13}\sigma_3\bar{\sigma}_{13} - a_3 x_{31}^*]dt + \sigma_3 dz_3$$

where $x_{31}^* = f_3(r_{31}^*)$. The expected value of x_{31} at time t is reduced by

$$\int_0^t \bar{\rho}_{13}\sigma_3\bar{\sigma}_{13}e^{-a_3(t-\tau)}d\tau = \frac{\bar{\rho}_{13}\sigma_3\bar{\sigma}_{13}}{a_3}(1 - e^{-a_3 t})$$

and the value of x_3 at nodes at time $i\Delta t$ (i.e., after i time steps) should therefore be reduced by

$$\frac{\bar{\rho}_{13}\sigma_3\bar{\sigma}_{13}}{a_3}(1 - e^{-a_2 i\Delta t})$$

As mentioned before, this is true for all functions f_3, not just $f_3(r) = r$.

Similarly, when we want to move the Pound Sterling short rate tree from the Pound Sterling risk-neutral world to the DM risk-neutral world, we can produce the process for r_{32}^* as

$$dx_{32}^* = [\theta_3(t) - \bar{\rho}_{23}\sigma_3\bar{\sigma}_{23} - a_3 x_{32}^*]dt + \sigma_3 dz_3$$

where $x_{32}^* = f_3(r_{32}^*)$, and the expected value of x_{32} is reduced by

$$\int_0^t \bar{\rho}_{23}\sigma_3\bar{\sigma}_{23} e^{-a_2(t-\tau)} d\tau = \frac{\bar{\rho}_{23}\sigma_3\bar{\sigma}_{23}}{a_3}(1 - e^{e^{-a_3 t}})$$

r_{32}^* is now the Pound Sterling short rate from the viewpoint of a risk-neutral DM investor. It means that r_{32}^* is the short rate in the risk-neutral DM world. In the last section, we move the short rate tree from the risk-neutral DM world to the risk-neutral USD world by reducing the drift of process x_{21}^* by $\bar{\rho}_{12}\sigma_2\bar{\sigma}_{12}$. Similarly, to move the Pound Sterling interest rate tree from the risk-neutral DM world to the risk-neutral USD world requires reducing the drift of process x_{32}^* by $\bar{\rho}_{12}\sigma_2\bar{\sigma}_{12}$.

So we have the Pound Sterling short rate from the viewpoint of a DM investor as viewed from a USD investor, and the process for the short rate, r_{321}^*, is

$$dx_{321}^* = [\theta_3(t) - \bar{\rho}_{23}\sigma_3\bar{\sigma}_{23} - \bar{\rho}_{12}\sigma_2\bar{\sigma}_{12} - a_3 x_{32}^*] + \sigma_3 dz_3$$

The expected value of x_{321} is reduced by

$$\int_0^t (\bar{\rho}_{23}\sigma_3\bar{\sigma}_{23} + \bar{\rho}_{12}\sigma_2\bar{\sigma}_{12}) e^{-a_3(t-\tau)} d\tau$$
$$= \frac{\bar{\rho}_{23}\sigma_3\bar{\sigma}_{12} + \bar{\rho}_{12}\sigma_2\bar{\sigma}_{12}}{a_3}(1 - e^{-a_3 t})$$

The Pound Sterling short rate from the viewpoint of a risk-neutral USD investor is r_{31}^*, and the result of Pound Sterling short rate from the viewpoint of a DM investor as viewed from a risk-neutral USD investor is r_{321}^*. By triangular arbitrage theory, r_{31}^* must be equal to r_{321}^* which is proved below.

Case1: r_{31}^* is greater than r_{321}^*. In this case, the investors in the USD market will borrow money at the rate r^*_{321} to invest at the rate r^*_{31}. These trades will push the rate r^*_{321} high, and lower the rate r^*_{31}.

Case2: r^*_{31} is smaller than r^*_{321}. In this case, the investors in the USD market will borrow money at the lower rate r^*_{31} to invest at the higher rate r^*_{321}. These trades will push the rate r^*_{31} high, and lower the rate r^*_{321}.

Therefore, the triangular relation among r_1, r_2, and r_3 is shown in Figure 3.3 below. Once we understand the relation among the three economies, the relation among four economies is straightforward. By the process above, the relation among four economies should be similar to that among three economies, and the relation among five economies should be the same as that one, too, and so on. The relation among n economies is shown in Figure 3.4.

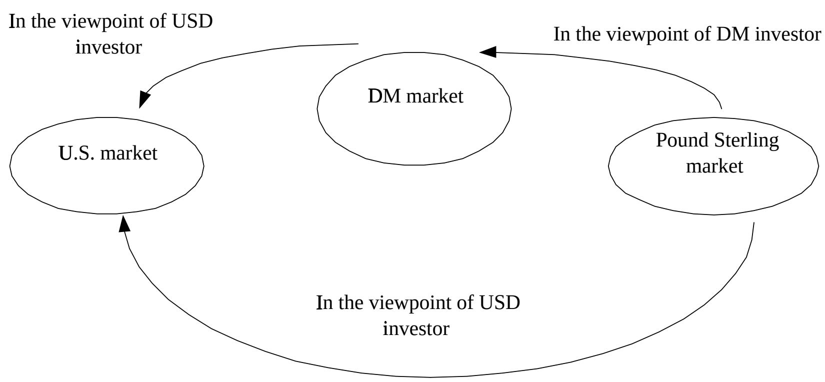


Figure 3.3: The Relationship among Three Economies.

3.5 Conclusion

At the beginning of this chapter, we introduced the derivative, swap, which requires the yield curves in two different countries to be modeled simultaneously. When the yield curves in two different countries are modeled simultaneously, the interest rate tree can not be constructed merely by the simple tree-building algorithm, because the tree of short rates involves the change of the short rates of another country.

After the introduction of swaps, we described the process of building and adjusting

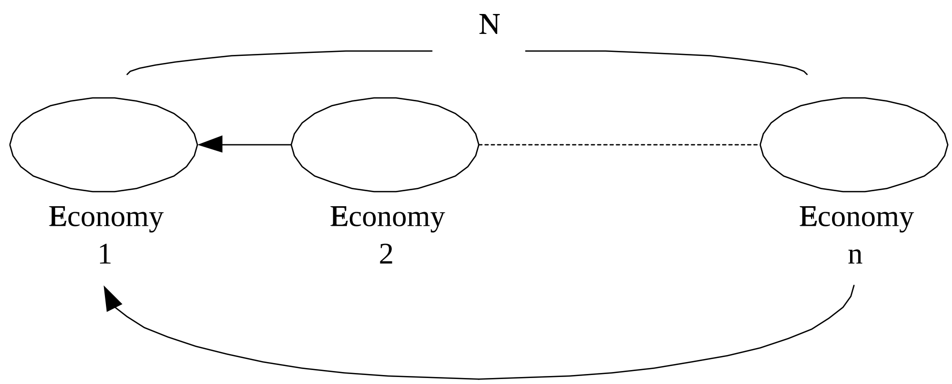


Figure 3.4: THE RELATIONSHIP AMONG N ECONOMIES.

the short rate tree. Three economies, USD, DM, and Pound Sterling, result in a number of variables, the definition of which are listed at the beginning of Section 3.1. In that section, we show how to relate two economies. Two yield curves can then be modeled at the same time. Reducing the short rate tree by $\bar{\rho}_{ij}\bar{\sigma}_{ij}\sigma_j$ can move the short rate tree from the viewpoint of the original economy i risk-neutral investor to the viewpoint of another economy j risk-neutral investor.

By the result of Section 3.1, we extend the number of economies from two to three. When there are three economies, USD, DM, Pound Sterling and considering Pound Sterling from the viewpoint of a USD risk-neutral investor, there are two ways to move the short rate tree from the Pound Sterling risk-neutral world to the USD risk-neutral world: (1) We can just consider the Pound Sterling risk-neutral world, and the USD risk-neutral world, or (2) we can adjust the short rate tree from the Pound Sterling risk-neutral world through the DM risk-neutral world to the USD risk-neutral world. In the first case, the process of Pound Sterling short rate was reduced by $\bar{\rho}_{13}\bar{\sigma}_{13}\sigma_3$. In the second case, the process was reduced by $(\bar{\rho}_{12}\bar{\sigma}_{12}\sigma_2 + \bar{\rho}_{23}\bar{\sigma}_{23}\sigma_3)$.

According to triangular arbitrage, the two adjusted process must be identical. It means that the adjusted value must be the same, in other words,

$$\bar{\rho}_{13}\bar{\sigma}_{13}\sigma_3 = (\bar{\rho}_{12}\bar{\sigma}_{12}\sigma_2 + \bar{\rho}_{23}\bar{\sigma}_{23}\sigma_3)$$

If the instantaneous coefficients of correlation between each two exchange rate do not satisfy the condition, there will be arbitrage opportunity. By the arbitrage theory, the opportunity will not hold forever, because everyone will make the same decision to take the arbitrage opportunity, and the opportunity will disappear. The instantaneous coefficients of correlation between two exchange rates therefore satisfy the above condition. The relation among three economies is shown in Figure 3.3.

After discussing the arbitrage opportunity among three economies, we extend the number of economies from 3 to n. The circumstance may be more complex, but the adjustment of each way must be the same, or there will be arbitrage opportunities.

Chapter 4

Conclusion

The Hull-White tree-building procedure is a flexible approach to constructing trees for a wide range of different one-factor models of the term structure. The tree is constructed to be exactly consistent with the initial term structure.

In the first part of this thesis we implemented the tree building procedure and benchmarked the convergence of the produce. The result is excellent, the execution time being linear in the time period n.

We have introduced a method for testing another character of a model in Chapter 2, the stability of the model. This is an important feature of the model not discussed in the literature before. We use another procedure, named Inverse Hull-White procedure doing the opposite work to the original Hull-White procedure, to test the stability of the Hull-White model. The result is reassuring. This means that the Hull-White model is stable.

In the second part of this thesis, we devoted some time to a discussion of the two-factor Hull-White model. We focused on the adjustment of interest rates when modeling two correlated interest rates following processes chosen from a family of one-factor models.

Arbitrage concern forms the last section of this thesis. We consider the situation of three correlated interest rates, USD, DM, and Pound Sterling. The three interest rates follow processes chosen from a family of one-factor models. There are two ways to adjust the Pound Sterling interest rate so that it reflects the evolution of rates from the viewpoint of a risk-neutral U.S. investor rather than a risk-neutral Pound Sterling investor. By the triangular arbitrage theory, the adjustments of both ways must be the same or arbitrage opportunity will appear. Then we extended the number of economies from three to n by using exactly the same principle.

Bibliography

[1] Black, F., E. Derman, and W. Toy. "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Options." *Financial Analysts Journal*, January-February 1990, pp. 33–39.

[2] Black, F., and P. Karasinski. "Bond and Option Pricing When Short Rates Are Lognormal." *Financial Analysts Journal*, July-August 1991, pp. 52–59.

[3] Cox, J.C., J.E. Ingersoll, Jr., and S.A. Ross. "A Theory of the Term Structure of Interest Rates." *Econometrica*, 53 (1985), pp. 385–407.

[4] Dybvig, P.H. "Bond and Bond Option Pricing Based on the Current Term Structure." *Working paper, Olin School of Business, University of Washington.*, 1988.

[5] Heathe, D., R. Jarrow, and A. Morton. "Bond Pricing and the Term Structure of Interest Rates: A New Methodology." *Econometrica*, 60, 1 (1992), pp. 77–105.

[6] Ho, T.S.Y., and S.B. Lee. "Term Structure Movements and the Pricing of Interest Rate Contingent Claims." *Journal of Finance*, 41 (December 1996), pp. 1011–1029.

[7] Hull, J.C., and A. White. "Numerical Procedures for Implementing Term Structure Models I: Single-Factor Models." *Journal of Derivatives*, 2, 1 (Fall 1993), pp. 21–31.

[8] Hull, J.C., and A. White. "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models." *Journal of Derivatives*, 2,1 (Winter 1994), pp. 37–48.

[9] Hull, J.C., and A. White. "Pricing Interest Rate Derivative Securities." *Review of Financial Studies*, 3, 4 (1990), pp. 573–592.

[10] Hull, J.C., and A. White. "One-Factor Interest Rate Models and the Valuation of Interest Rate Derivative Securities." *Journal of Financial and Quanitative Analysis*, 28 (1993), pp. 235–254.

[11] Hull, J.C., and A. White. "Using Hull-White Interest Rate Trees." *The Journal of Derivatives*, Spring 1996, pp. 26–36.

[12] Jamshidian, F. "An Exact Bond Option Formula." *Journal of Finance*, 44 (1989), pp. 205–209.

[13] Milne, F., and S. Turnbull. "A Simple Approach to Interest Rate Option Pricing." *Working Paper, Australian National University*, 1989.

[14] Vasicek, O.A. "An Equilibrium Characterization of the Term Structure." *Journal of Financial Economics*, 5 (1977), pp. 177–188.