Toward the Black-Scholes Formula

- As n increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.
- Need to calibrate the BOPM's parameters u, d, and R to make it converge to the continuous-time model.
- We now skim through the proof.

Toward the Black-Scholes Formula (continued)

- Let τ denote the time to expiration of the option measured in years.
- Let r be the continuously compounded annual rate.
- With n periods during the option's life, each period represents a time interval of τ/n.
- Need to adjust the period-based u, d, and interest rate $\hat{r}$ to match the empirical results as $n \to \infty$.

Toward the Black-Scholes Formula (continued)

- First, $\hat{r} = r\tau/n$.
  - Each period is τ/n years long.
  - The period gross return $R = e^{\hat{r}}$.
- Let
$$\ln \frac{S_\tau}{S}$$
denote the continuously compounded rate of return of the stock.

Toward the Black-Scholes Formula (continued)

- Assume the stock's true continuously compounded rate of return has mean $\mu\tau$ and variance $\sigma^2\tau$.
- Call σ the stock's (annualized) volatility.
- We need one more condition to have a solution for u, d, q.
- Impose

$$ud = 1.$$

  - It makes nodes at the same horizontal level of the tree have identical price.[a]

[a] Other choices are possible (see text).

Toward the Black-Scholes Formula (continued)

- Pick

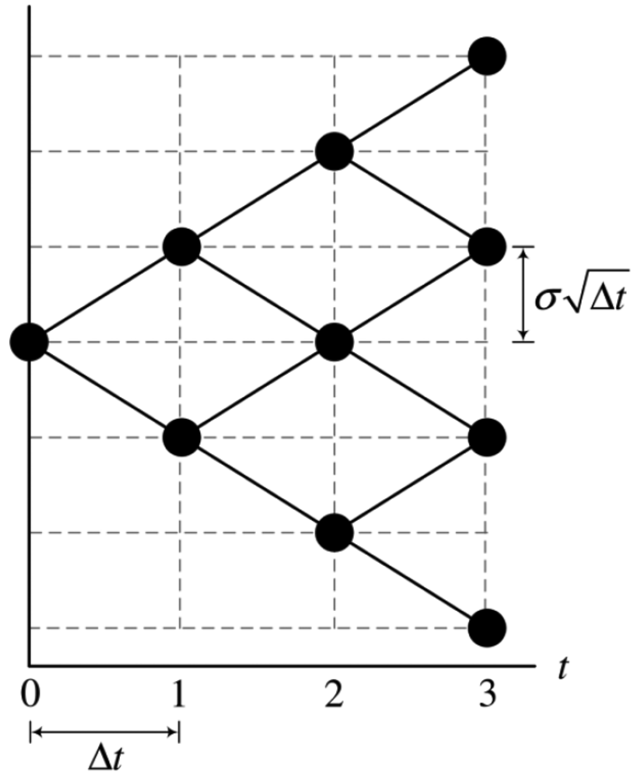
$$u = e^{\sigma\sqrt{\tau/n}}, \quad d = e^{-\sigma\sqrt{\tau/n}}, \quad q = \frac{1}{2} + \frac{1}{2}\,\frac{\mu}{\sigma}\sqrt{\frac{\tau}{n}}\,. \qquad (12)$$

- With Eqs. (12), it can be checked that the mean $\mu\tau$ is matched by the BOPM.
- Furthermore, the variance $\sigma^2\tau$ is asymptotically matched as well.

Toward the Black-Scholes Formula (continued)

- The choices (12) result in the CRR binomial model.[a]
- The no-arbitrage inequalities $d < R < u$ may not hold under Eqs. (12) on p. 80 or Eq. (8) on p. 56.
  - If this happens, the probabilities lie outside $[\,0, 1\,]$.
- The problem disappears if n is large enough.

[a]Cox, Ross, and Rubinstein (1979).



$\sigma\sqrt{\Delta t}$
t
0
1
2
3
Δt

Toward the Black-Scholes Formula (continued)

- What is the limiting probabilistic distribution of the continuously compounded rate of return $\ln(S_\tau/S)$?
- It approaches $N(\mu\tau + \ln S, \sigma^2\tau)$.
- Conclusion: S_τ has a lognormal distribution in the limit.

Toward the Black-Scholes Formula (continued)

- In the risk-neutral economy, pick

$$q = \frac{R-d}{u-d}.$$

by Eq. (8) on p. 56.

Lemma 1 *The continuously compounded rate of return* $\ln(S_\tau/S)$ *approaches the normal distribution with mean* $(r-\sigma^2/2)\,\tau$ *and variance* $\sigma^2\tau$ *in a risk-neutral economy.*[a]

[a]See Lemma 9.3.3 of the textbook.

Toward the Black-Scholes Formula (concluded)

Theorem 2 (The Black-Scholes Formula)

$$\begin{aligned} C &= SN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\ P &= Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - SN(-x), \end{aligned}$$

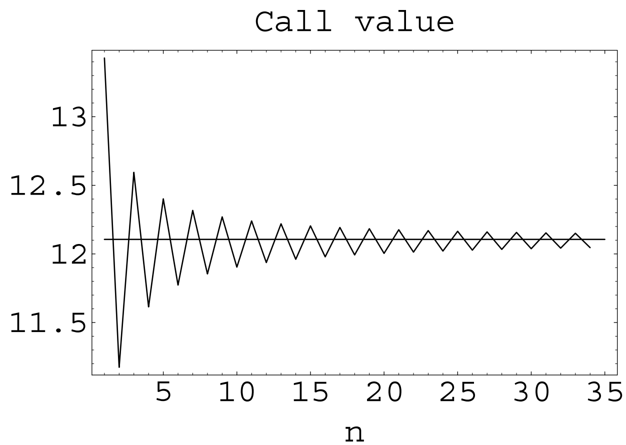
where

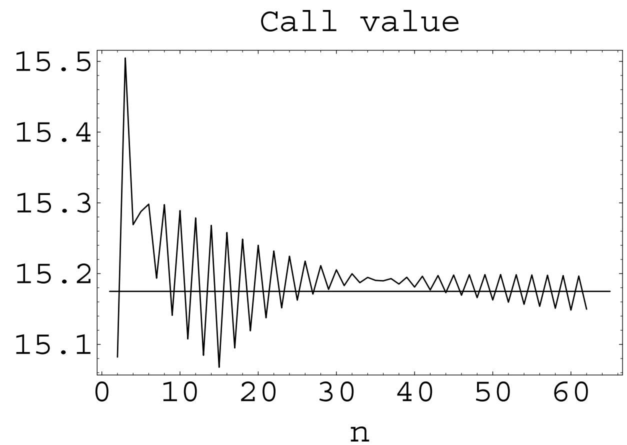
$$x \equiv \frac{\ln(S/X) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

BOPM and Black-Scholes Model

- The Black-Scholes formula needs 5 parameters: S, X, σ, τ, and r.
- Binomial tree algorithms take 6 inputs: S, X, u, d, $\hat{r}$, and n.
- The connections are

$$\begin{aligned} u &= e^{\sigma\sqrt{\tau/n}}, \\ d &= e^{-\sigma\sqrt{\tau/n}}, \\ \hat{r} &= r\tau/n. \end{aligned}$$





- $S = 100$, $X = 100$ (left), and $X = 95$ (right).

BOPM and Black-Scholes Model (concluded)

- The binomial tree algorithms converge reasonably fast.
- The error is $O(1/n)$.[a]
- Oscillations can be dealt with by the judicious choices of u and d.[b]

---

[a] Chang and Palmer (2007).

[b] See Exercise 9.3.8 of the textbook.

Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market's opinion of the volatility.
  - Solve for σ given the option price, S, X, τ, and r with numerical methods.
- Implied volatility is

  > the wrong number to put in the wrong formula to get the right price of plain-vanilla options.[a]

- It is often preferred to historical volatility in practice.

---

[a]Rebonato (2004).

Problems; the Smile

- Options written on the same underlying asset usually do not produce the same implied volatility.
- A typical pattern is a "smile" in relation to the strike price.
  - The implied volatility is lowest for at-the-money options.
  - It becomes higher the further the option is in- or out-of-the-money.
- Other patterns have also been observed.

Binomial Tree Algorithms for American Puts

- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs

  $$\max(0, X - Su^j d^{n-j})$$

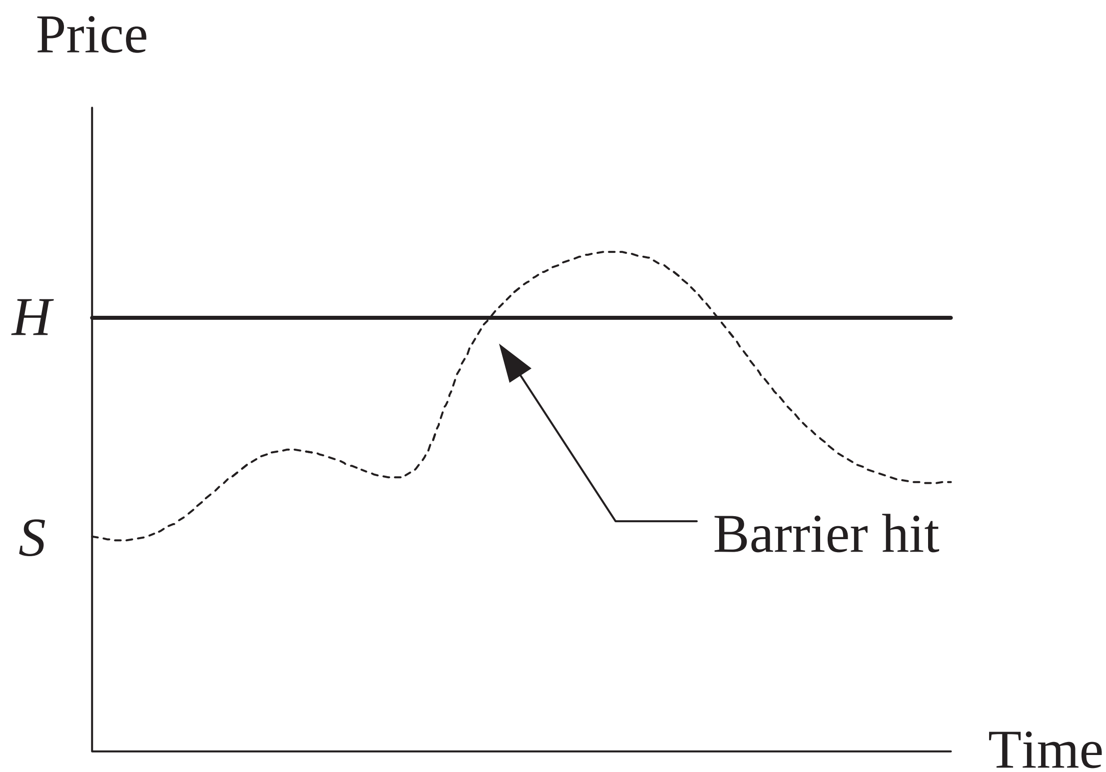
  and applies backward induction.
- At each intermediate node, it compares the payoff if exercised and the continuation value.
- It keeps the larger one.

Extensions of Options Theory

And the worst thing you can have
is models and spreadsheets.
— Warren Buffet, May 3, 2008

Barrier Options

- Their payoff depends on whether the underlying asset's price reaches a certain price level H.
- A knock-out option is like an ordinary European option.
- But it ceases to exist if the barrier H is reached by the price of its underlying asset.


Price
H
S
Barrier hit
Time

Barrier Options (concluded)

- A knock-in option comes into existence if a certain barrier is reached.
- A down-and-in option is a call knock-in option that comes into existence only when the barrier is reached and $H < S$.

A Formula for Down-and-In Calls[a]

- Assume $X \geq H$.
- The value of a European down-and-in call on a stock paying a dividend yield of q is

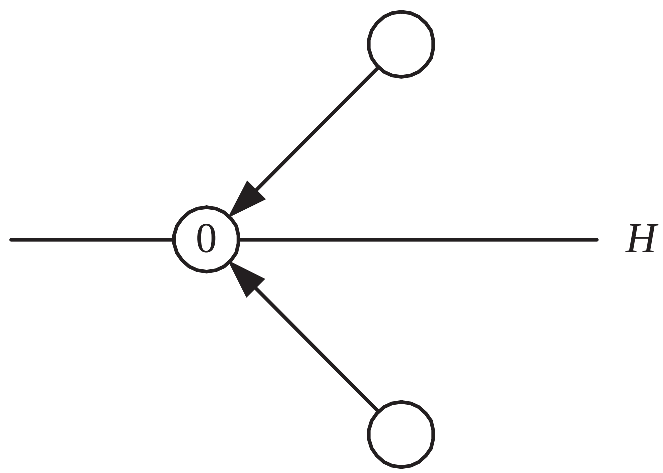
$$Se^{-q\tau} \left(\frac{H}{S}\right)^{2\lambda} N(x) - Xe^{-r\tau} \left(\frac{H}{S}\right)^{2\lambda-2} N(x - \sigma\sqrt{\tau}), \tag{13}$$

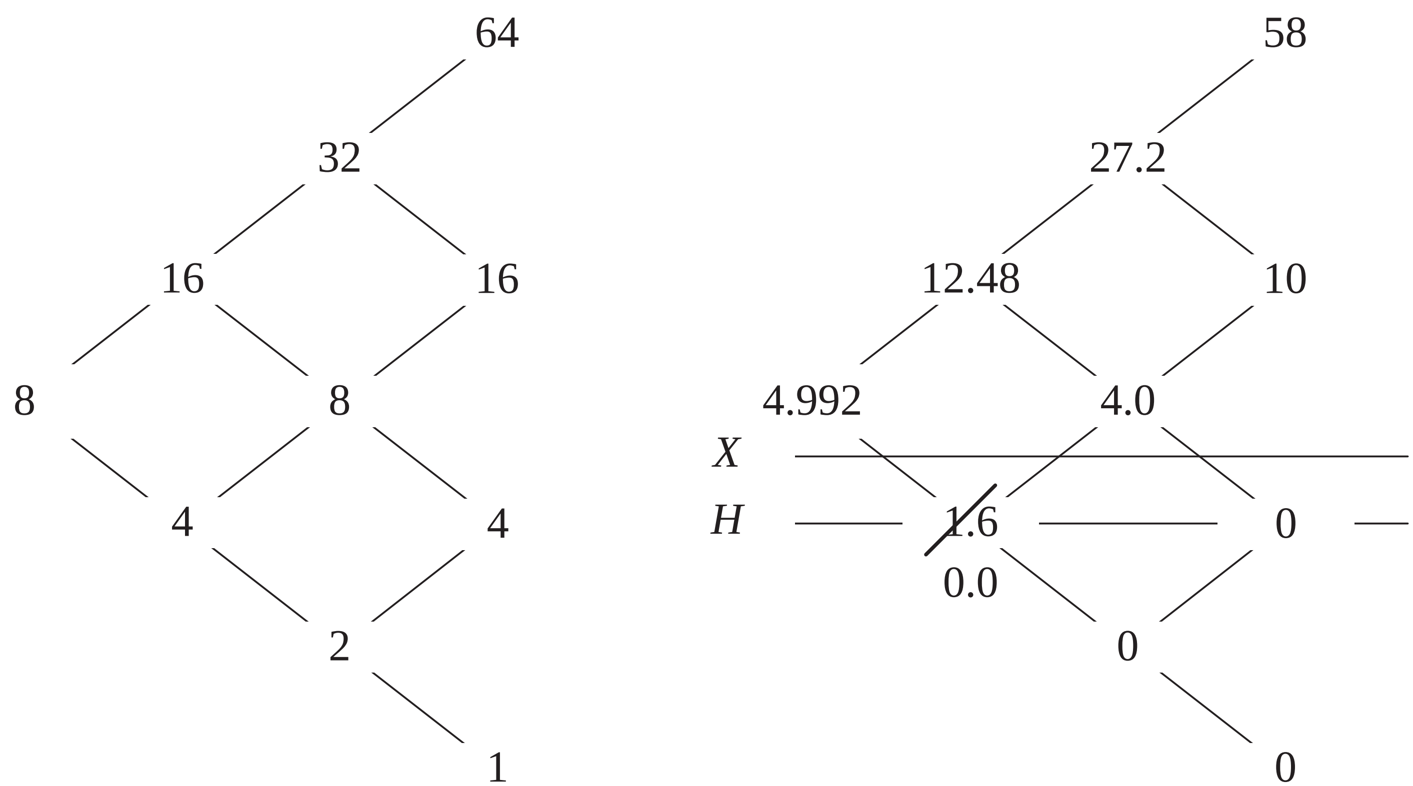
  - $x \equiv \frac{\ln(H^2/(SX)) + (r-q+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$.
  - $\lambda \equiv (r - q + \sigma^2/2)/\sigma^2$.

[a] Merton (1973).

Binomial Tree Algorithms

- Barrier options can be priced by binomial tree algorithms.
- Below is for the down-and-out option.





$S = 8$, $X = 6$, $H = 4$, $R = 1.25$, $u = 2$, and $d = 0.5$.

Backward-induction: $C = (0.5 \times C_u + 0.5 \times C_d)/1.25$.

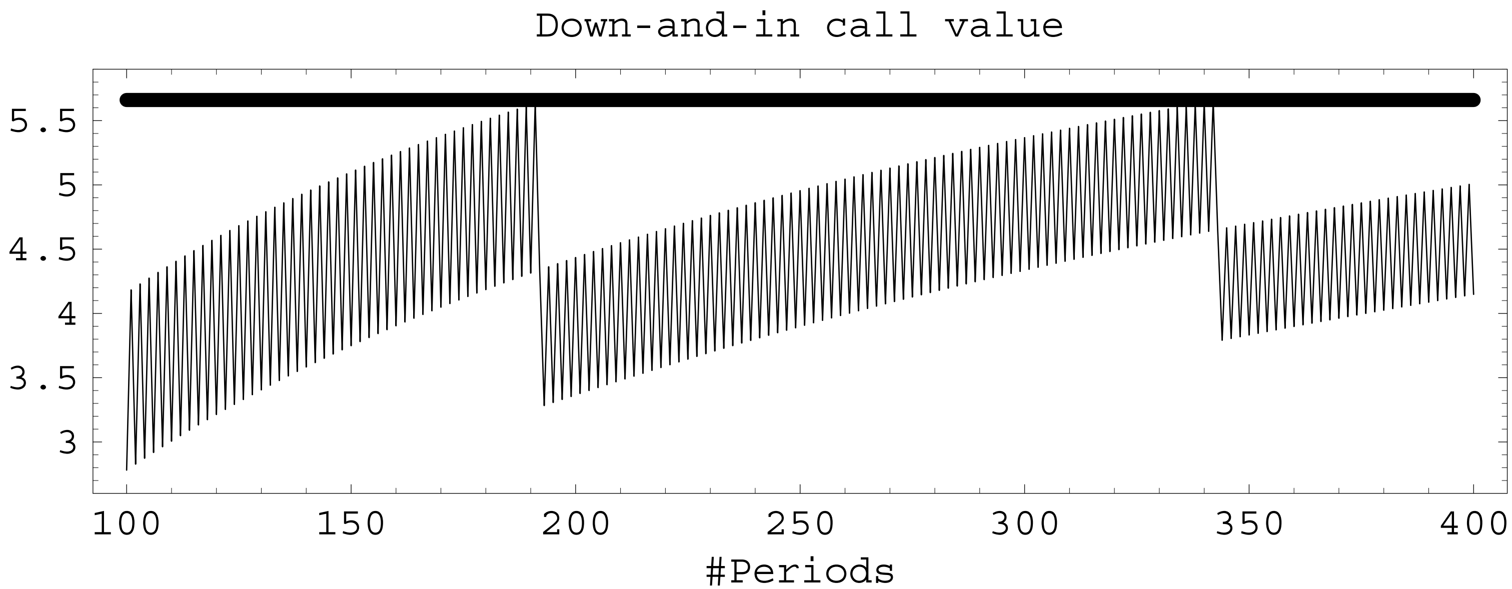
Binomial Tree Algorithms (continued)

- But convergence is erratic because H is not at a price level on the tree.[a]
  - The barrier H is moved to a node price.
  - This "effective barrier" changes as n increases.
- In fact, the binomial tree is $O(1/\sqrt{n})$ convergent.[b]

---

[a] Boyle and Lau (1994).
[b] Lin (`R95221010`) (2008).

Binomial Tree Algorithms (concluded)[a]



[a] Lyuu (1998).

Path-Dependent Derivatives

- Let $S_0, S_1, \ldots, S_n$ denote the prices of the underlying asset over the life of the option.
- S_0 is the known price at time zero.
- S_n is the price at expiration.
- The standard European call has a terminal value depending only on the last price, $\max(S_n - X, 0)$.
- Its value thus depends only on the underlying asset's terminal price regardless of how it gets there.

Path-Dependent Derivatives (continued)

- Some derivatives are path-dependent in that their terminal payoff depends *critically* on the path.
- The (arithmetic) average-rate call has this terminal value:

$$\max\left(\frac{1}{n+1}\sum_{i=0}^{n} S_i - X, 0\right).$$

- The average-rate put's terminal value is given by

$$\max\left(X - \frac{1}{n+1}\sum_{i=0}^{n} S_i, 0\right).$$

Path-Dependent Derivatives (concluded)

- Average-rate options are also called Asian options.[a]
- They are useful hedging tools for firms that will make a stream of purchases over a time period because the costs are likely to be linked to the average price.
- The averaging clause is also common in convertible bonds and structured notes.

[a]As of the late 1990s, the outstanding volume was in the range of 5–10 billion U.S. dollars (Nielsen & Sandmann, 2003).

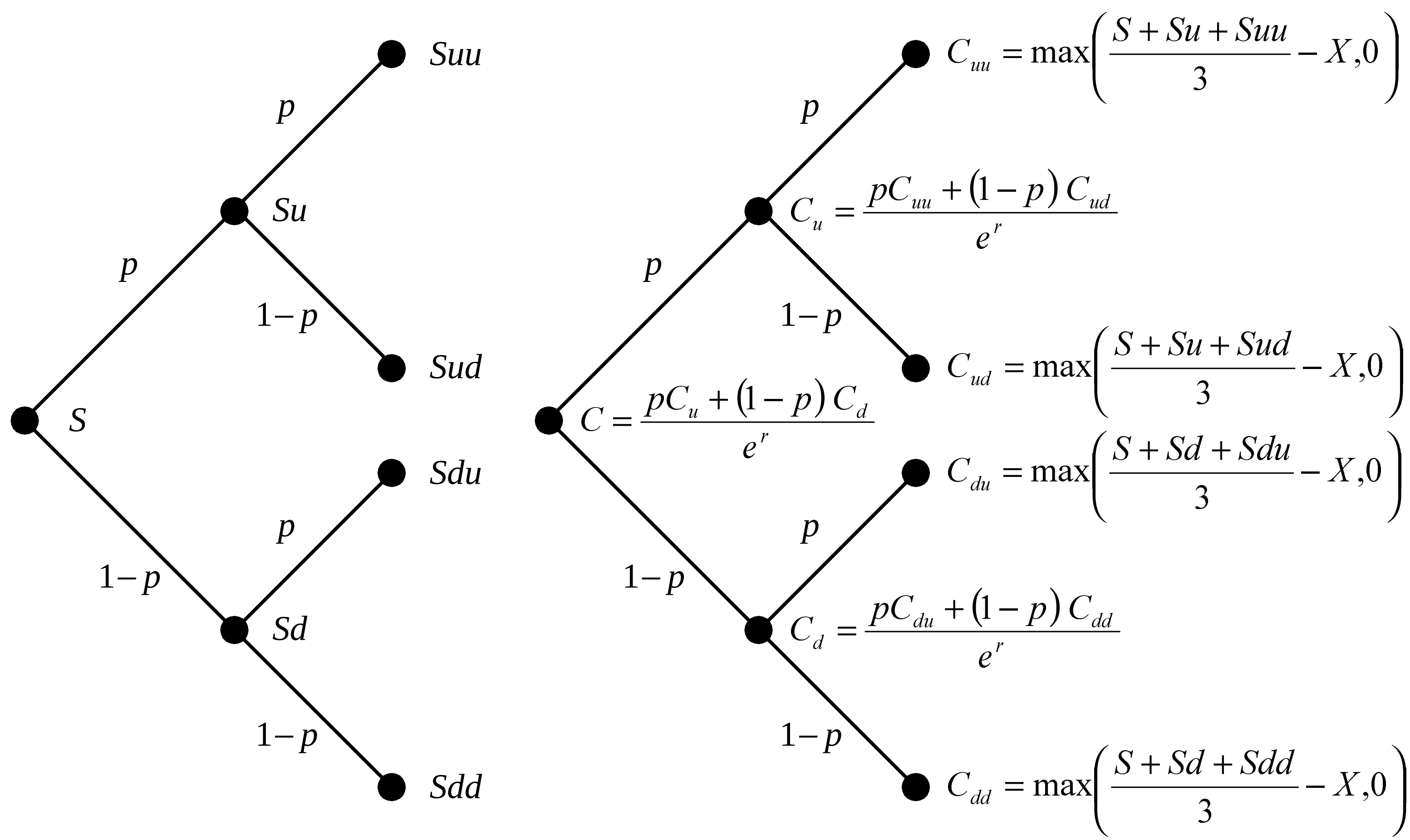
Average-Rate Options

- Average-rate options are notoriously hard to price.
- The binomial tree for the averages does not combine (see next page).
- A naive algorithm enumerates the 2^n paths for an n-period binomial tree and then averages the payoffs.[a]
- But the complexity is exponential.
- The Monte Carlo method[b] and approximation algorithms are some of the alternatives left.

---

[a]Dai (B82506025, R86526008, D8852600) and Lyuu (2007) reduce it to $2^{O(\sqrt{n})}$. Hsu (R7526001, D89922012) and Lyuu (2004) reduce it to $O(n^2)$ given some regularity assumptions.

[b]See pp. 142ff.


Suu
p
Su
p
$1-p$
Sud
S
Sdu
p
$1-p$
Sd
$1-p$
Sdd
$C_{uu} = \max\left(\frac{S+Su+Suu}{3} - X, 0\right)$
$C_u = \frac{pC_{uu} + (1-p)\,C_{ud}}{e^r}$
$C_{ud} = \max\left(\frac{S+Su+Sud}{3} - X, 0\right)$
$C = \frac{pC_u + (1-p)\,C_d}{e^r}$
$C_{du} = \max\left(\frac{S+Sd+Sdu}{3} - X, 0\right)$
$C_d = \frac{pC_{du} + (1-p)\,C_{dd}}{e^r}$
$C_{dd} = \max\left(\frac{S+Sd+Sdd}{3} - X, 0\right)$

Continuous-Time Financial Mathematics

A proof is that which convinces a reasonable man;
a rigorous proof is that which convinces an
unreasonable man.
— Mark Kac (1914–1984)

Brownian Motion[a]

- Brownian motion is a stochastic process $\{X(t), t \geq 0\}$ with the following properties.

  1. $X(0) = 0$, unless stated otherwise.

  2. for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables

  $$X(t_k) - X(t_{k-1})$$

  for $1 \leq k \leq n$ are independent.[b]

  3. for $0 \leq s < t$, $X(t) - X(s)$ is normally distributed with mean $\mu(t - s)$ and variance $\sigma^2(t - s)$, where μ and $\sigma \neq 0$ are real numbers.

---

[a] Robert Brown (1773–1858).

[b] So $X(t) - X(s)$ is independent of $X(r)$ for $r \leq s < t$.

Brownian Motion (concluded)

- The existence and uniqueness of such a process is guaranteed by Wiener's theorem.[a]
- This process will be called a (μ, σ) Brownian motion with drift μ and variance σ^2.
- The $(0, 1)$ Brownian motion is called the Wiener process.

[a]Norbert Wiener (1894–1964). He received his Ph.D. from Harvard in 1912.

Ito Process[a]

- A shorthand[b] is the following stochastic differential equation for the Ito differential dX_t,

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t. \tag{14}$$

  - Or simply

$$dX_t = a_t\, dt + b_t\, dW_t.$$

[a] Ito (1944).

[b] Paul Langevin (1872–1946) in 1904.

Ito Process (concluded)

- dW is normally distributed with mean zero and variance dt.
- An equivalent form of Eq. (14) is

$$dX_t = a_t\, dt + b_t \sqrt{dt}\; \xi, \tag{15}$$

where $\xi \sim N(0, 1)$.

Modeling Stock Prices

- The most popular stochastic model for stock prices has been the geometric Brownian motion,

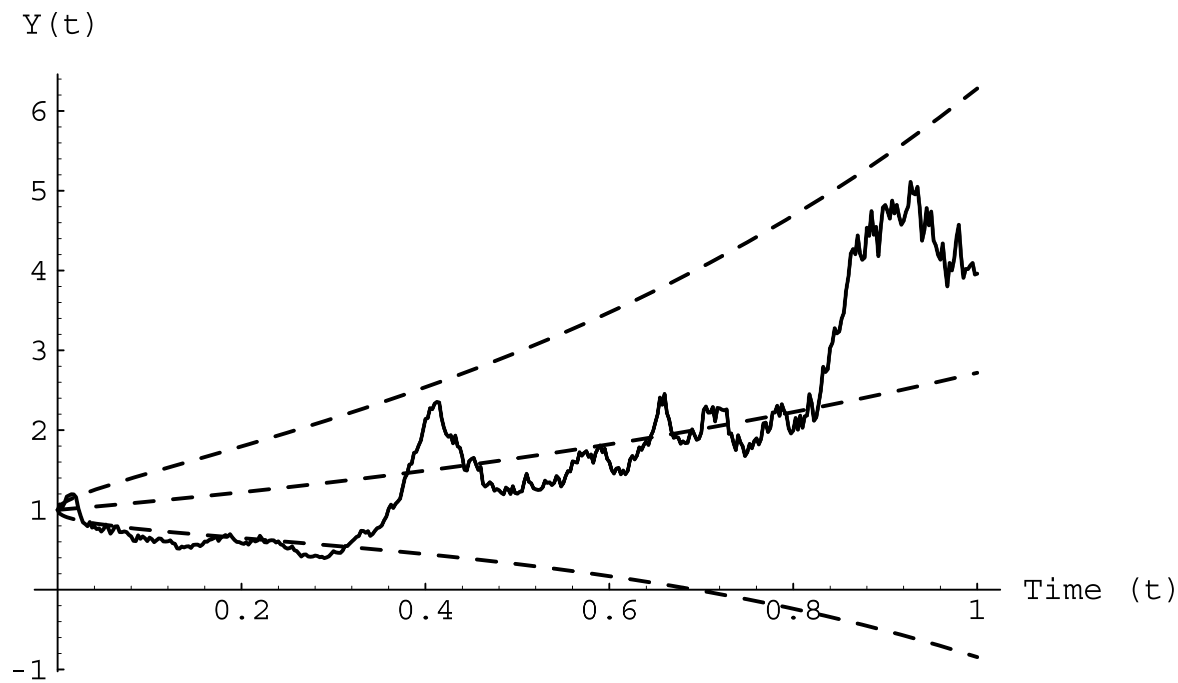
$$\frac{dS}{S} = \mu\, dt + \sigma\, dW.$$

- The continuously compounded rate of return $X \equiv \ln S$ follows

$$dX = (\mu - \sigma^2/2)\, dt + \sigma\, dW$$

by Ito's lemma.[a]

[a]Consistent with Lemma 1 (p. 84).


Y(t)
6
5
4
3
2
1
-1
0.2
0.4
0.6
0.8
1
Time (t)

Local-Volatility Models

- The more general deterministic volatility model posits

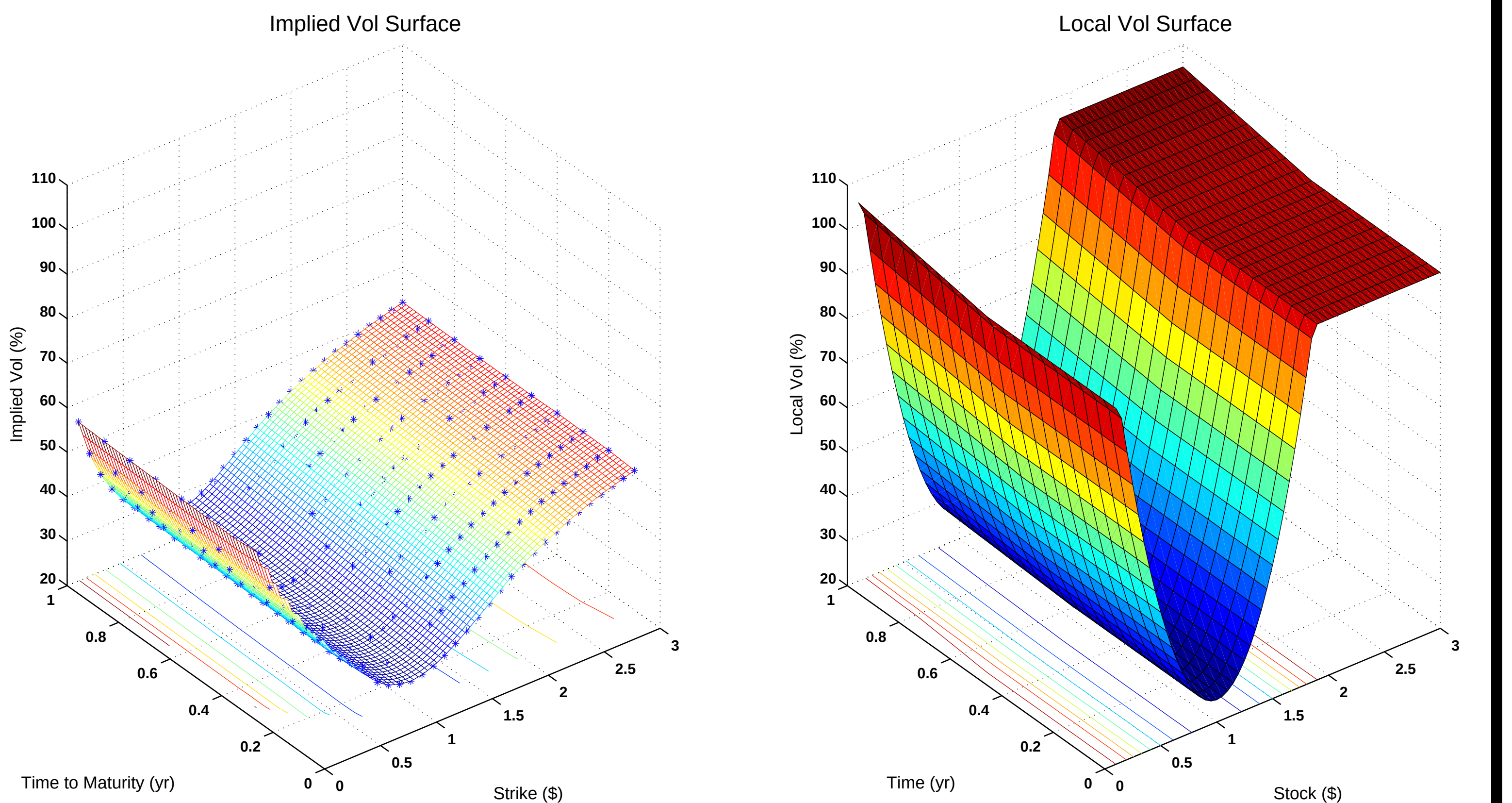
$$\frac{dS}{S} = (r_t - q_t)\,dt + \sigma(S,t)\,dW,$$

  where instantaneous volatility $\sigma(S,t)$ is called the local volatility function.[a]

- One needs to recover $\sigma(S,t)$ from the implied volatilities.

---

[a] Derman and Kani (1994); Dupire (1994).



By Mr. Lok, U Hou (D99922028) on April 5, 2014.

Implied Trees

- The trees for the local volatility model are called implied trees.[a]
- Their construction requires an implied volatility surface.
- An exponential-sized implied tree exists.[b]
- How to construct a valid implied tree with efficiency has been open for a long time.[c]

---

[a] Derman & Kani (1994); Dupire (1994); Rubinstein (1994).

[b] Charalambousa, Christofidesb, & Martzoukosa (2007).

[c] Rubinstein (1994); Derman & Kani (1994); Derman, Kani, & Chriss (1996); Jackwerth & Rubinstein (1996); Jackwerth (1997); Coleman, Kim, Li, & Verma (2000); Li (2000/2001); Moriggia, Muzzioli, & Torricelli (2009).

Implied Trees (concluded)

- It is solved for separable local volatilities σ.[a]
  - The local-volatility function $\sigma(S, V)$ is separable[b] if

$$\sigma(S, t) = \sigma_1(S)\, \sigma_2(t).$$

- A general solution is close.[c]

---

[a] Lok (`D99922028`) & Lyuu (2015, 2016).
[b] Rebonato (2004); Brace, Gątarek, & Musiela (1997).
[c] Lok (`D99922028`) & Lyuu (2016).

The Hull-White Model

- Hull and White (1987) postulate the following model,

$$\begin{aligned} \frac{dS}{S} &= r\,dt + \sqrt{V}\,dW_1, \\ dV &= \mu_{\mathrm{v}} V\,dt + bV\,dW_2. \end{aligned}$$

- Above, V is the instantaneous variance.
- They assume μ_{v} depends on V and t (but not S).

The SABR Model

- Hagan, Kumar, Lesniewski, and Woodward (2002) postulate the following model,

$$\begin{aligned}\frac{dS}{S} &= r\,dt + S^{\theta} V\,dW_1,\\ dV &= bV\,dW_2,\end{aligned}$$

for $0 \leq \theta \leq 1$.

The Hilliard-Schwartz Model

- Hilliard and Schwartz (1996) postulate the following general model,

$$\begin{aligned} \frac{dS}{S} &= r\,dt + f(S)V^a\,dW_1, \\ dV &= \mu(V)\,dt + bV\,dW_2, \end{aligned}$$

  for some well-behaved function $f(S)$ and constant a.

Heston's Stochastic-Volatility Model

- Heston (1993) assumes the stock price follows

$$\begin{aligned} \frac{dS}{S} &= (\mu - q)\, dt + \sqrt{V}\, dW_1, && (16) \\ dV &= \kappa(\theta - V)\, dt + \sigma\sqrt{V}\, dW_2. && (17) \end{aligned}$$

  - V is the instantaneous variance, which follows a square-root process.
  - dW_1 and dW_2 have correlation ρ.
  - The riskless rate r is constant.

Heston's Stochastic-Volatility Model (concluded)

- It may be the most popular continuous-time stochastic-volatility model.[a]
- For American options, we will need a tree for Heston's model.[b]
- They are all $O(n^3)$-sized.

---

[a] Christoffersen, Heston, & Jacobs (2009).

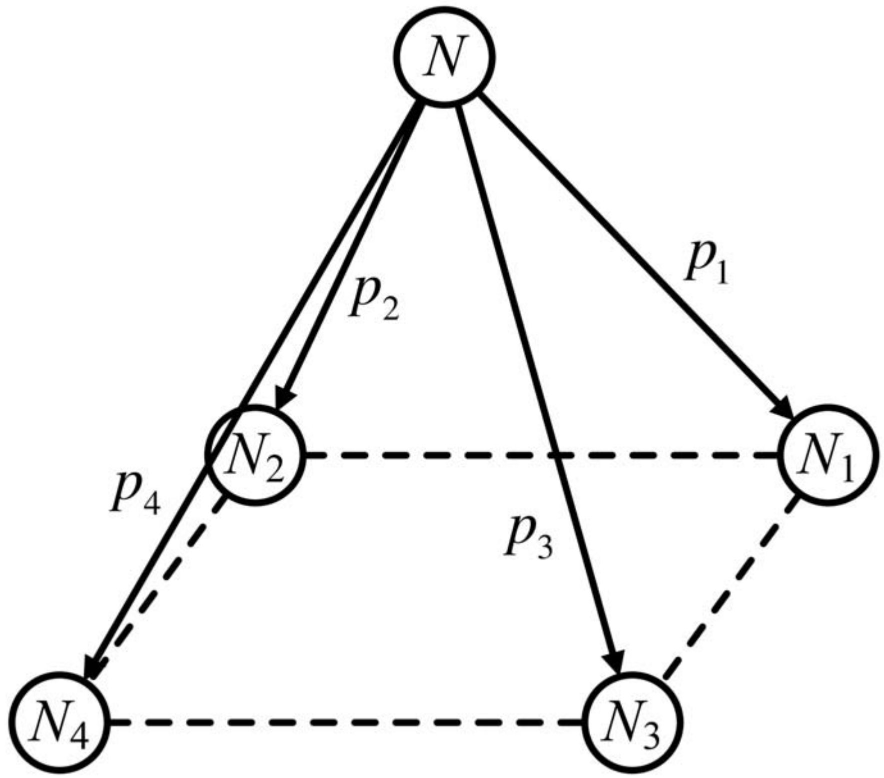
[b] Leisen (2010); Beliaeva & Nawalka (2010); Chou (`R02723073`) (2015).

Why Are Trees for Stochastic-Volatility Models Difficult?

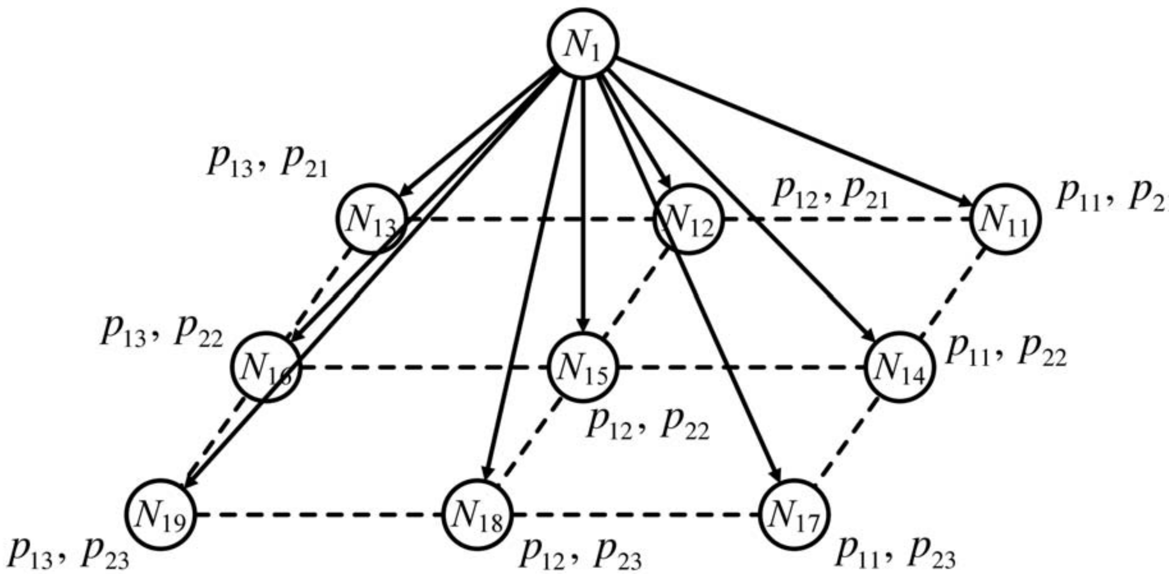
- The CRR tree is 2-dimensional.[a]
- The constant volatility makes the span from any node fixed.
- But a tree for a stochastic-volatility model must be 3-dimensional.
  - Every node is associated with a pair of stock price and a volatility.

[a]Recall p. 82.

Why Are Trees for Stochastic-Volatility Models Difficult: Binomial Case?



Why Are Trees for Stochastic-Volatility Models Difficult: Trinomial Case?



Why Are Trees for Stochastic-Volatility Models Difficult? (concluded)

- Locally, the tree looks fine for one time step.
- But the volatility regulates the spans of the nodes on the stock-price plane.
- Unfortunately, those spans differ from node to node because the volatility varies.
- So two time steps from now, the branches will not combine!

Complexities of Stochastic-Volatility Models

- A few stochastic-volatility models suffer from subexponential ($c^{\sqrt{n}}$) tree size.
- Examples include the Hull-White (1987), Hilliard-Schwartz (1996), and SABR (2002) models.[a]

[a] Chiu (R98723059) (2012).

Trees

I love a tree more than a man.
— Ludwig van Beethoven (1770–1827)

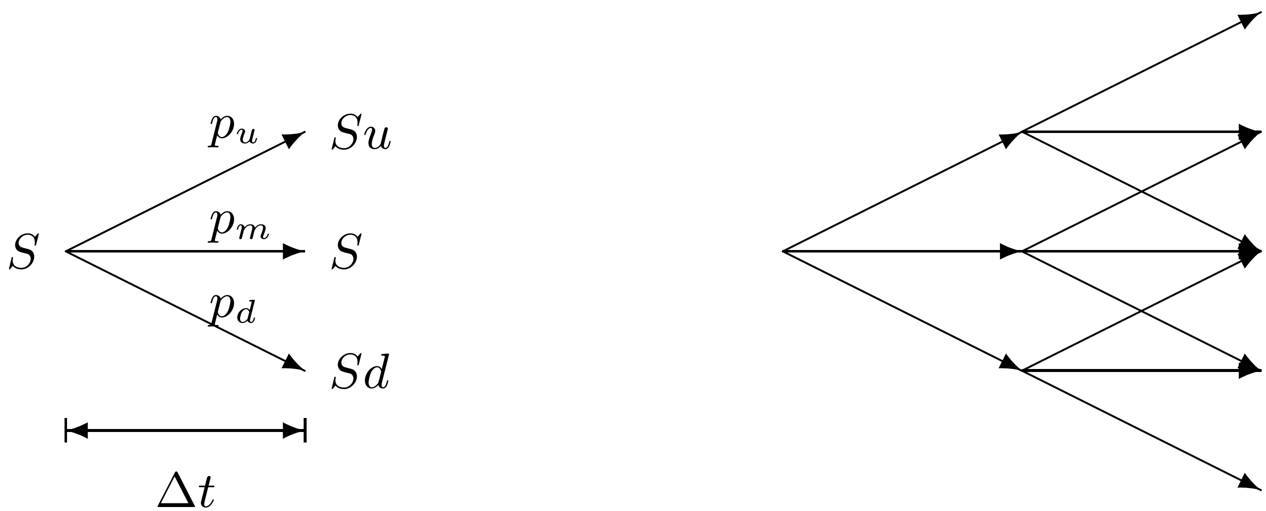
Trinomial Tree

- Set up a trinomial approximation to the geometric Brownian motion[a]

$$\frac{dS}{S} = r\,dt + \sigma\,dW.$$

- The three stock prices at time Δt are S, Su, and Sd, where $ud = 1$.
- Let the mean and variance of the stock price be SM and S^2V, respectively.

[a]Boyle (1988).


S
p_u
p_m
p_d
Su
S
Sd
Δt

Trinomial Tree (continued)

- By Eqs. (5) on p. 24,

$$
\begin{aligned}
M &\equiv e^{r\Delta t}, \\
V &\equiv M^2(e^{\sigma^2\Delta t} - 1).
\end{aligned}
$$

- Impose the matching of mean and that of variance:

$$
\begin{aligned}
1 &= p_u + p_m + p_d, \\
SM &= (p_u u + p_m + (p_d/u))\, S, \\
S^2 V &= p_u(Su - SM)^2 + p_m(S - SM)^2 + p_d(Sd - SM)^2.
\end{aligned}
$$

Trinomial Tree (concluded)

- Use linear algebra to verify that

$$
\begin{aligned}
p_u &= \frac{u\left(V + M^2 - M\right) - (M-1)}{(u-1)\left(u^2-1\right)}, \\
p_d &= \frac{u^2\left(V + M^2 - M\right) - u^3(M-1)}{(u-1)\left(u^2-1\right)}.
\end{aligned}
$$

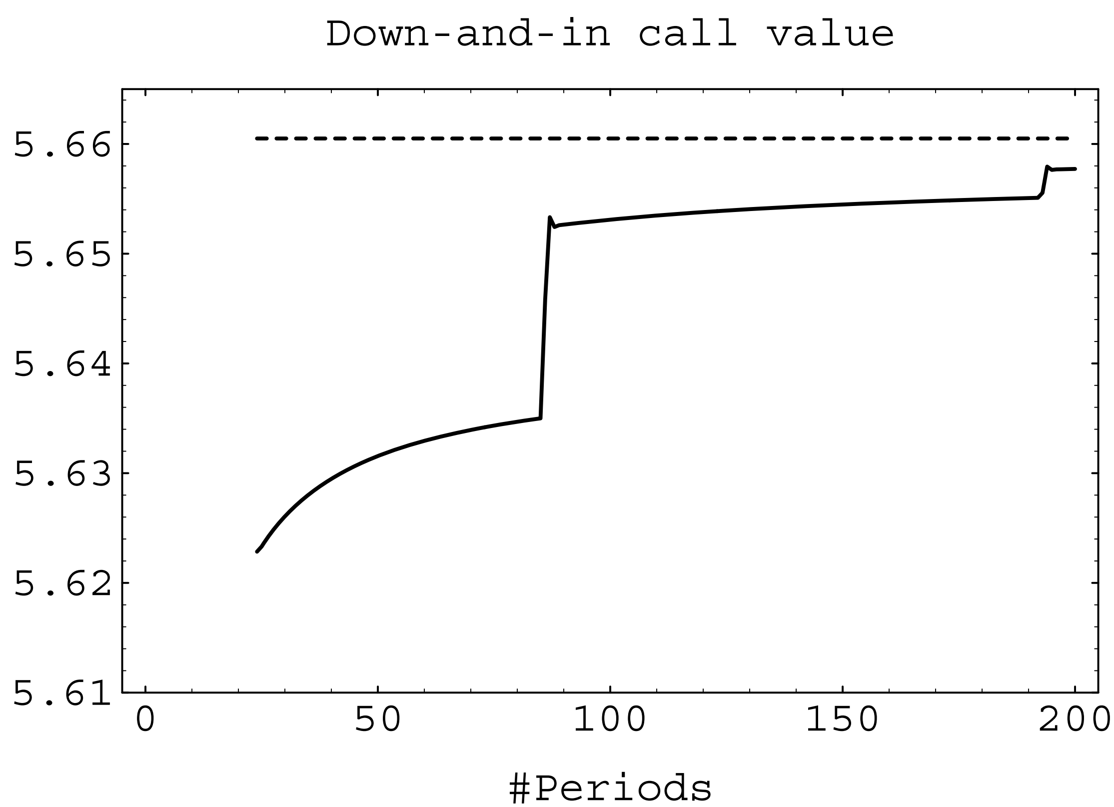
  – We must also make sure the probabilities lie between 0 and 1.

A Trinomial Tree

- Use $u = e^{\lambda\sigma\sqrt{\Delta t}}$, where $\lambda \geq 1$ is a tunable parameter.
- Then

$$\begin{aligned} p_u &\to \frac{1}{2\lambda^2} + \frac{\left(r + \sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}, \\ p_d &\to \frac{1}{2\lambda^2} - \frac{\left(r - 2\sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}. \end{aligned}$$

Barrier Options Priced by Trinomial Trees



Multivariate Contingent Claims

- They depend on two or more underlying assets.
- The basket call on m assets has the terminal payoff

$$\max\left(\sum_{i=1}^{m} \alpha_i S_i(\tau) - X, 0\right).$$

Multivariate Contingent Claims (continued)[a]

Name	Payoff
Exchange option	$\max(S_1(\tau) - S_2(\tau), 0)$
Better-off option	$\max(S_1(\tau), \ldots, S_k(\tau), 0)$
Worst-off option	$\min(S_1(\tau), \ldots, S_k(\tau), 0)$
Binary maximum option	$I\{\max(S_1(\tau), \ldots, S_k(\tau)) > X\}$
Maximum option	$\max(\max(S_1(\tau), \ldots, S_k(\tau)) - X, 0)$
Minimum option	$\max(\min(S_1(\tau), \ldots, S_k(\tau)) - X, 0)$
Spread option	$\max(S_1(\tau) - S_2(\tau) - X, 0)$
Basket average option	$\max((S_1(\tau) + \cdots + S_k(\tau))/k - X, 0)$
Multi-strike option	$\max(S_1(\tau) - X_1, \ldots, S_k(\tau) - X_k, 0)$
Pyramid rainbow option	$\max(\mid S_1(\tau) - X_1 \mid + \cdots + \mid S_k(\tau) - X_k \mid - X, 0)$
Madonna option	$\max(\sqrt{(S_1(\tau) - X_1)^2 + \cdots + (S_k(\tau) - X_k)^2} - X, 0)$

[a] Lyuu & Teng (R91723054) (2011).

Multivariate Contingent Claims (concluded)

- Trees for multivariate contingent claims typically has size exponential in the number of assets.
- This is called the curse of dimensionality.

Numerical Methods

All science is dominated
by the idea of approximation.
— Bertrand Russell

Monte Carlo Simulation[a]

- Monte Carlo simulation is a sampling scheme.
- In many important applications within finance and without, Monte Carlo is one of the few feasible tools.

[a]A top 10 algorithm according to Dongarra and Sullivan (2000).

Monte Carlo Option Pricing

- For the pricing of European options, we sample the stock prices.
- Then we average the payoffs.
- The variance of the estimator is now $1/N$ of that of the original random variable.

How about American Options?

- Standard Monte Carlo simulation is inappropriate for American options because of early exercise.
- It is difficult to determine the early-exercise point based on one single path.
- But Monte Carlo simulation can be modified to price American options with small biases.[a]
- The LSM can be easily parallelized.[b]

---

[a] Longstaff and Schwartz (2001).

[b] Huang (B96902079, R00922018) (2013); Chen (B97902046, R01922005) (2014); Chen (B97902046, R01922005), Huang (B96902079, R00922018) & Lyuu (2015).

Delta and Common Random Numbers

- In estimating delta $\partial f / \partial S$, it is natural to start with the finite-difference estimate

$$e^{-r\tau} \frac{E[\,P(S+\epsilon)\,] - E[\,P(S-\epsilon)\,]}{2\epsilon}.$$

  – $P(x)$ is the terminal payoff of the derivative security when the underlying asset's initial price equals x.

- Use simulation to estimate $E[\,P(S+\epsilon)\,]$ first.

- Use another simulation to estimate $E[\,P(S-\epsilon)\,]$.

- Finally, apply the formula to approximate the delta.

Delta and Common Random Numbers (concluded)

- This method is not recommended because of its high variance.
- A much better approach is to use common random numbers to lower the variance:

$$e^{-r\tau}\, E\left[\, \frac{P(S+\epsilon)-P(S-\epsilon)}{2\epsilon} \,\right].$$

- Here, the *same* random numbers are used for $P(S+\epsilon)$ and $P(S-\epsilon)$.

Gamma

- The finite-difference formula for gamma $\partial^2 f / \partial S^2$ is

$$e^{-r\tau}\, E\left[\, \frac{P(S+\epsilon) - 2 \times P(S) + P(S-\epsilon)}{\epsilon^2} \,\right].$$

- Choosing an ϵ of the right magnitude can be challenging.
  - If ϵ is too large, inaccurate Greeks result.
  - If ϵ is too small, unstable Greeks result.
- This phenomenon is sometimes called the curse of differentiation.[a]

[a] Aït-Sahalia and Lo (1998); Bondarenko (2003).

Gamma (continued)

- In general, suppose

$$\frac{\partial^i}{\partial \theta^i} e^{-r\tau} E[\, P(S)\,] = e^{-r\tau} E\left[\, \frac{\partial^i P(S)}{\partial \theta^i} \,\right]$$

  holds for all $i > 0$, where θ is a parameter of interest.

  – A common requirement is Lipschitz continuity.[a]

- Then formulas for the Greeks become integrals.
- As a result, we avoid ϵ, finite differences, and resimulation.

[a]Broadie and Glasserman (1996).

Gamma (concluded)

- This is indeed possible for a broad class of payoff functions.[a]
- In queueing networks, this is called infinitesimal perturbation analysis (IPA).[b]

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[a] Teng (`R91723054`) (2004) and Lyuu and Teng (`R91723054`) (2011).
[b] Cao (1985); Ho and Cao (1985).

Interest Rate Models

[Meriwether] scoring especially high marks
in mathematics — an indispensable subject
for a bond trader.
— Roger Lowenstein,
When Genius Failed (2000)

Bond market terminology was designed less
to convey meaning than to bewilder outsiders.
— Michael Lewis, *The Big Short* (2011)

The Vasicek Model[a]

- The short rate follows

$$dr = \beta(\mu - r)\, dt + \sigma\, dW.$$

- The short rate is pulled to the long-term mean level μ at rate β.
- Superimposed on this "pull" is a normally distributed stochastic term $\sigma\, dW$.

[a]Vasicek (1977).

The Cox-Ingersoll-Ross Model[a]

- It is the following square-root short rate model:

$$dr = \beta(\mu - r)\, dt + \sigma \sqrt{r}\, dW. \tag{18}$$

- The diffusion differs from the Vasicek model by a multiplicative factor $\sqrt{r}$.
- The parameter β determines the speed of adjustment.
- The short rate can reach zero only if $2\beta\mu < \sigma^2$.

[a]Cox, Ingersoll, and Ross (1985).

The Ho-Lee Model[a]

- The continuous-time limit of the Ho-Lee model is

$$dr = \theta(t)\, dt + \sigma\, dW.$$

- This is Vasicek's model with the mean-reverting drift replaced by a deterministic, time-dependent drift.
- A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying,

$$dr = \theta(t)\, dt + \sigma(t)\, dW.$$

[a]Ho and Lee (1986). Thomas Lee is a "billionaire founder" of Thomas H. Lee Partners LP, according to *Bloomberg* on May 26, 2012.

The Black-Derman-Toy Model[a]

- The continuous-time limit of the BDT model is

$$d \ln r = \left(\theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln r \right) dt + \sigma(t)\, dW.$$

- This model is extensively used by practitioners.
- The BDT short rate process is the lognormal binomial interest rate process.
- Lognormal models preclude negative short rates.

[a]Black, Derman, and Toy (BDT) (1990), but essentially finished in 1986 according to Mehrling (2005).

The Black-Karasinski Model[a]

- The BK model stipulates that the short rate follows

$$d \ln r = \kappa(t)(\theta(t) - \ln r)\, dt + \sigma(t)\, dW.$$

- This explicitly mean-reverting model depends on time through $\kappa(\cdot)$, $\theta(\cdot)$, and $\sigma(\cdot)$.
- The BK model hence has one more degree of freedom than the BDT model.
- The speed of mean reversion $\kappa(t)$ and the short rate volatility $\sigma(t)$ are independent.

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[a] Black and Karasinski (1991).

The Extended Vasicek Model[a]

- The extended Vasicek model adds time dependence to the original Vasicek model,

$$dr = (\theta(t) - a(t)\, r)\, dt + \sigma(t)\, dW.$$

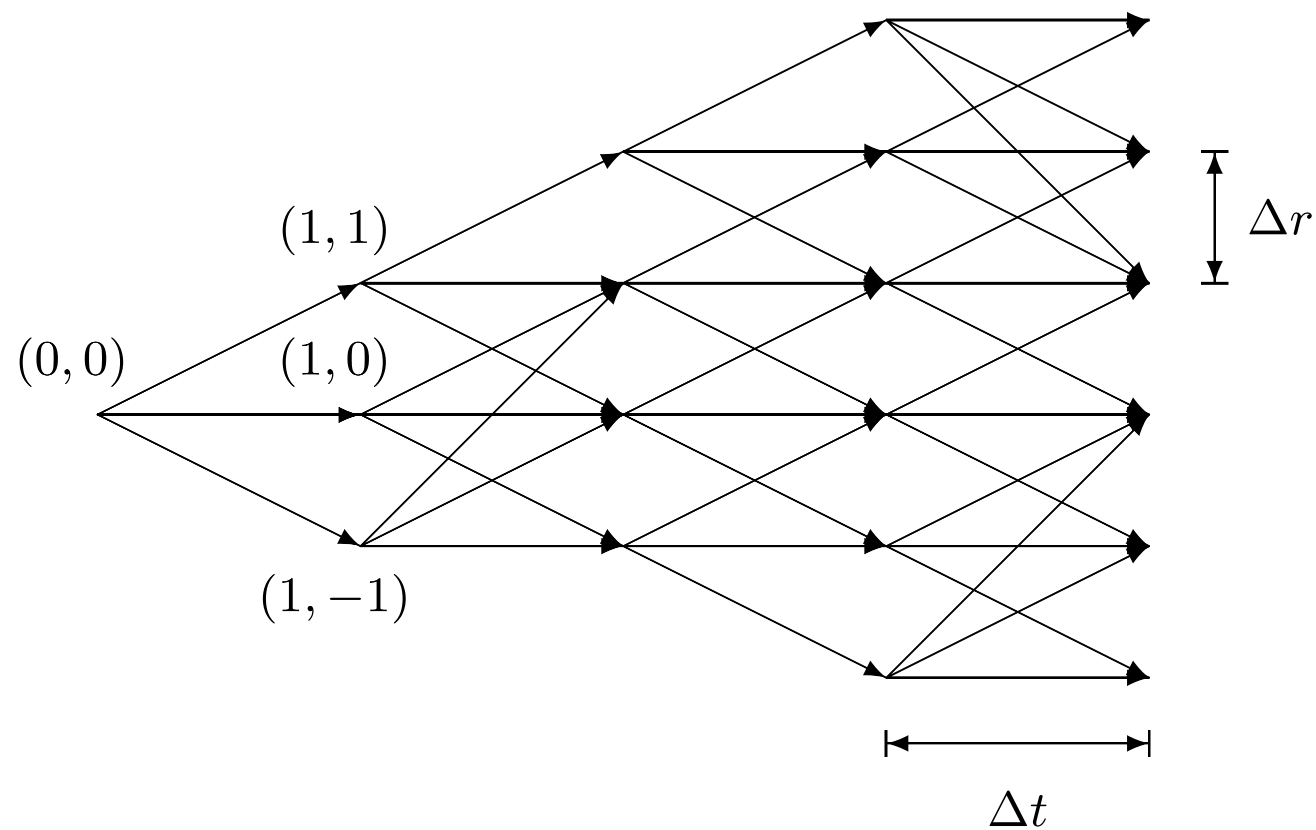
- Like the Ho-Lee model, this is a normal model.
- Many European-style securities can be evaluated analytically.
- Efficient numerical procedures can be developed for American-style securities.

[a] Hull and White (1990).

The Hull-White Model

- The Hull-White model is the following special case,

$$dr = (\theta(t) - ar)\, dt + \sigma\, dW.$$


(1, 1)
(0, 0)
(1, 0)
(1, −1)
Δr
Δt

The Extended CIR Model

- In the extended CIR model the short rate follows

$$dr = (\theta(t) - a(t)\, r)\, dt + \sigma(t) \sqrt{r}\, dW.$$

- The functions $\theta(t)$, $a(t)$, and $\sigma(t)$ are implied from market observables.
- With constant parameters, there exist analytical solutions to a small set of interest rate-sensitive securities.

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