Financial Computing

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Introduction

You must go into finance, Amory.
— F. Scott Fitzgerald (1896–1940),
This Side of Paradise (1920)

I can calculate the motions
of the heavenly bodies,
but not the madness of people.
— Isaac Newton (1642–1727)

Class Information

- Yuh-Dauh Lyuu. *Financial Engineering & Computation: Principles, Mathematics, Algorithms*. Cambridge University Press, 2002.
- Official Web page is

  `www.csie.ntu.edu.tw/~lyuu/finance3.html`

- Check

  `www.csie.ntu.edu.tw/~lyuu/capitals.html`

  for some of the software.

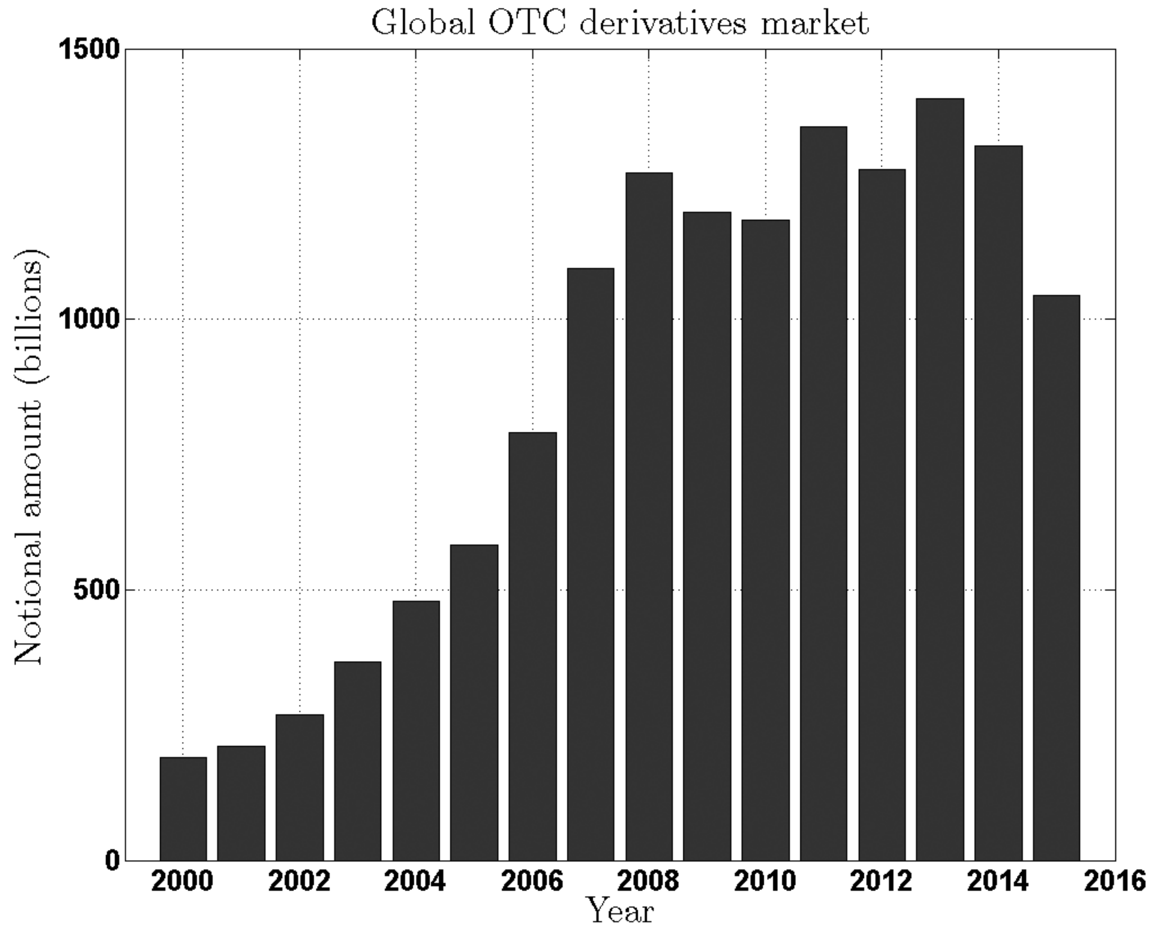
Grading

- One programming assignment.
- You are expected to write your own code and turn in your source code.
- Do not copy or collaborate with fellow students.
- Never ask your friends to write programs for you.
- Never give your code to other students or publish your code.

What This Six-Hour Course Is About

- Financial theories in pricing.
- Derivative securities.
- Pricing models.
- Efficient algorithms in pricing financial instruments.

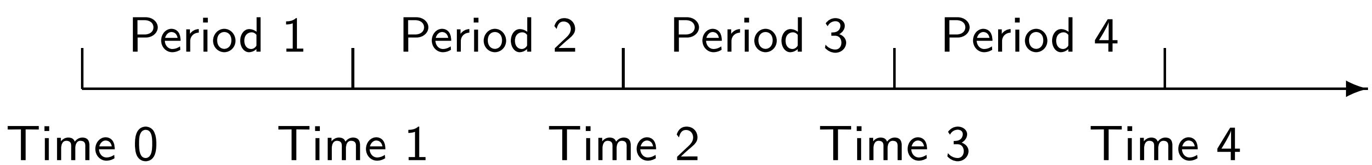
Global OTC Derivatives Market



Basic Financial Mathematics

In the fifteenth century
mathematics was mainly concerned with
questions of commercial arithmetic and
the problems of the architect.
— Joseph Alois Schumpeter (1883–1950)

The Time Line



Time Value of Money[a]

$$
\begin{aligned}
\text{FV} &= \text{PV}(1+r)^n, &(1)\\
\text{PV} &= \text{FV} \times (1+r)^{-n}.
\end{aligned}
$$

- FV (future value).
- PV (present value).
- r: interest rate.

[a] Fibonacci (1170–1240) and Irving Fisher (1867–1947).

Periodic Compounding

- Suppose the annual interest rate r is compounded m times per annum.
- Then
$$1 \to \left(1+\frac{r}{m}\right) \to \left(1+\frac{r}{m}\right)^2 \to \left(1+\frac{r}{m}\right)^3 \to \cdots$$
- Hence, after n years,
$$\text{FV} = \text{PV}\left(1+\frac{r}{m}\right)^{nm}. \tag{2}$$

Example

- Annual interest rate is 10% compounded twice per annum.
- Each dollar will grow to be

$$[\,1 + (0.1/2)\,]^2 = 1.1025$$

  one year from now.
- The rate is "equivalent to" an interest rate of 10.25% compounded once *per annum*,

$$1 + 0.1025 = 1.1025$$

Continuous Compounding[a]

- Let $m \to \infty$ so that
$$\left(1 + \frac{r}{m}\right)^m \to e^r$$
in Eq. (2) on p. 12.
- Then
$$\text{FV} = \text{PV} \times e^{rn},$$
where $e = 2.71828\ldots$.

[a] Jacob Bernoulli (1654–1705) in 1685.

The PV Formula

- The PV of the cash flow $C_1, C_2, \ldots, C_n$ at times $1, 2, \ldots, n$ is

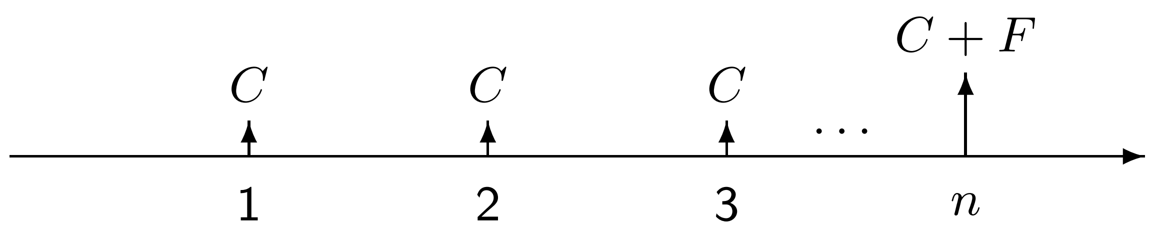
$$\text{PV} = \frac{C_1}{1+y} + \frac{C_2}{(1+y)^2} + \cdots + \frac{C_n}{(1+y)^n}. \tag{3}$$

- This formula and its variations are the engine behind most of financial calculations.[a]

[a] Cochrane (2005).

Level-Coupon Bonds

- Coupon rate.
- Par value, paid at maturity.
- F denotes the par value, and C denotes the coupon.
- Cash flow:



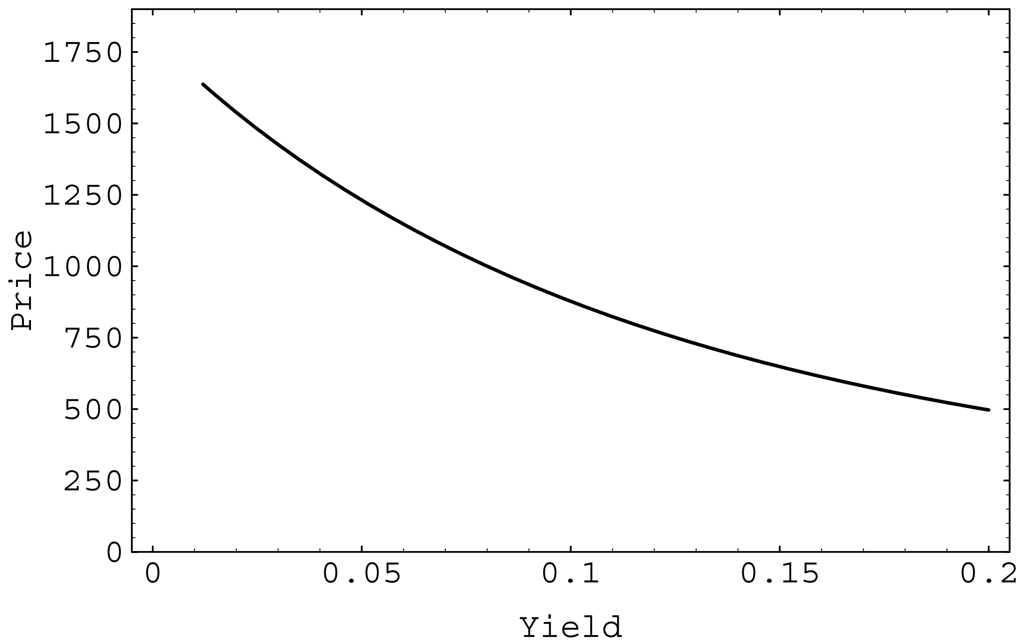
Pricing Formula

$$\begin{aligned} P &= \sum_{i=1}^{n} \frac{C}{\left(1+\frac{r}{m}\right)^{i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{n}} \\ &= C\,\frac{1-\left(1+\frac{r}{m}\right)^{-n}}{\frac{r}{m}} + \frac{F}{\left(1+\frac{r}{m}\right)^{n}}. \end{aligned} \tag{4}$$

- n: number of cash flows.
- m: number of payments per year.
- r: annual rate compounded m times per annum.
- Note $C = Fc/m$ when c is the annual coupon rate.

Price Behavior

- Bond prices fall when interest rates rise, and vice versa.
- "Only 24 percent answered the question correctly."[a]



[a]CNN, December 21, 2001.

Fundamental Statistical Concepts

There are three kinds of lies:
lies, damn lies, and statistics.
— Misattributed to Benjamin Disraeli
(1804–1881)

One death is a tragedy,
but a million deaths are a statistic.
— Josef Stalin (1879–1953)

Moments

- The variance of a random variable X is defined as

$$\mathrm{Var}[\,X\,] \equiv E\left[\,(X - E[\,X\,])^2\,\right].$$

- The covariance between random variables X and Y is

$$\mathrm{Cov}[\,X, Y\,] \equiv E\left[\,(X - \mu_X)(Y - \mu_Y)\,\right],$$

  where μ_X and μ_Y are the means of X and Y, respectively.

- Random variables X and Y are uncorrelated if

$$\mathrm{Cov}[\,X, Y\,] = 0.$$

Correlation

- The standard deviation of X is the square root of the variance,
$$\sigma_X \equiv \sqrt{\mathrm{Var}[\,X\,]}.$$
- The correlation (or correlation coefficient) between X and Y is
$$\rho_{X,Y} \equiv \frac{\mathrm{Cov}[\,X,Y\,]}{\sigma_X \sigma_Y},$$
provided both have nonzero standard deviations.

The Normal Distribution

- A random variable X has the normal distribution with mean μ and variance σ^2 if its probability density function is
$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}.$$
- This is expressed by $X \sim N(\mu, \sigma^2)$.
- The standard normal distribution has zero mean, unit variance, and the following distribution function
$$\text{Prob}[\, X \leq z\,] = N(z) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx.$$

The Lognormal Distribution

- A random variable Y is said to have a lognormal distribution if $\ln Y$ has a normal distribution.
- Let $X \sim N(\mu, \sigma^2)$ and $Y \equiv e^X$.
- The mean and variance of Y are

$$\mu_Y = e^{\mu+\sigma^2/2} \text{ and } \sigma_Y^2 = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right), \tag{5}$$

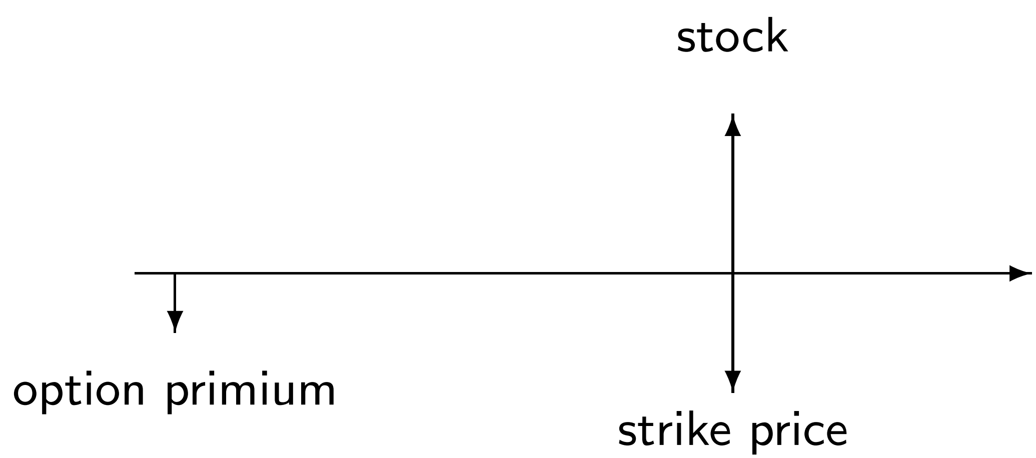
respectively.

Option Basics

The shift toward options as
the center of gravity of finance [. . .]
— Merton H. Miller (1923–2000)

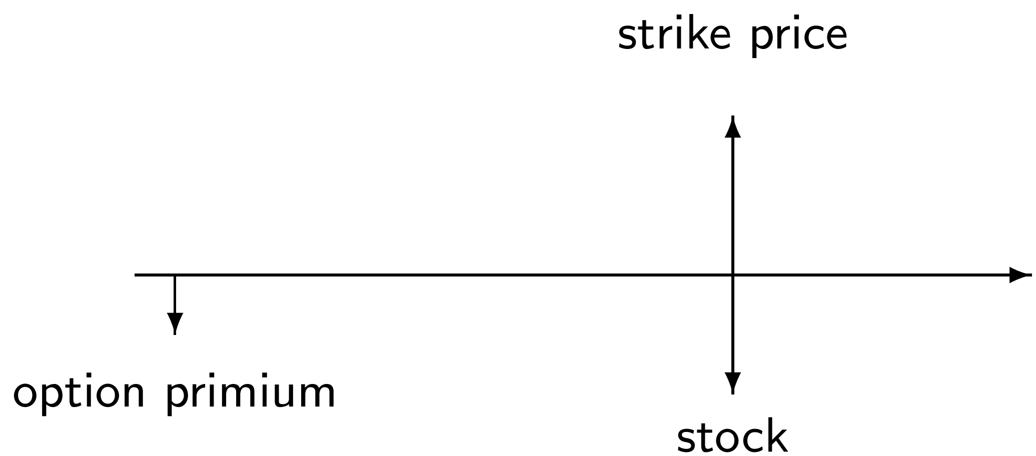
Calls and Puts

- A call gives its holder the right to *buy* a unit of the underlying asset by paying a strike price.



Calls and Puts (continued)

- A put gives its holder the right to *sell* a unit of the underlying asset for the strike price.



Calls and Puts (concluded)

- How to price options?
- Options can be traced to Aristotle's (384 B.C.–322 B.C.) *Politics*, if not earlier.

Exercise

- When a call is exercised, the holder pays the strike price in exchange for the stock.
- When a put is exercised, the holder receives from the writer the strike price in exchange for the stock.
- An option can be exercised prior to the expiration date: early exercise.

American and European

- American options can be exercised at any time up to the expiration date.
- European options can only be exercised at expiration.
- An American option is worth at least as much as an otherwise identical European option.

Convenient Conventions

- C: call value.
- P: put value.
- X: strike price.
- S: stock price.

Payoff, Mathematically Speaking

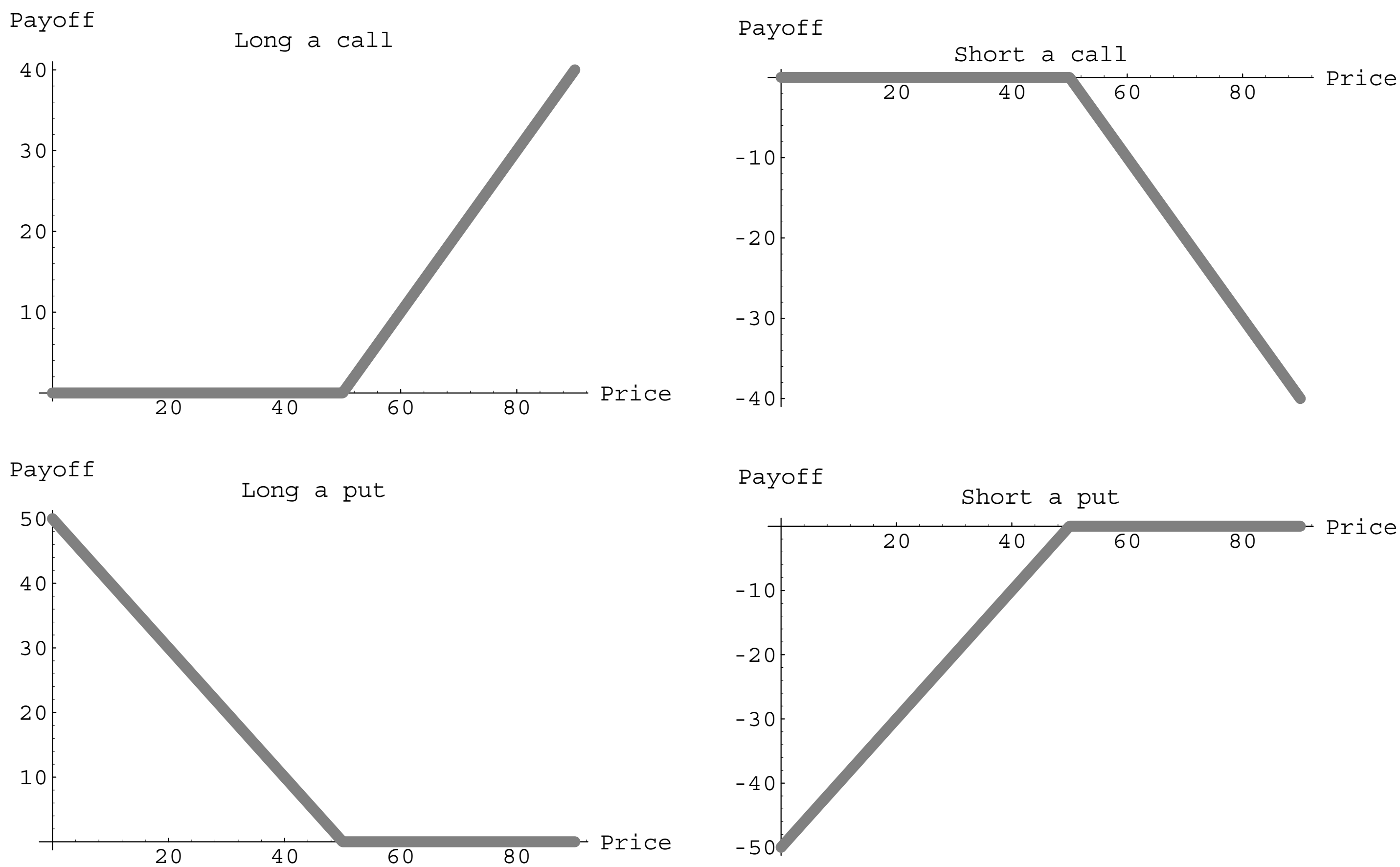
- The payoff of a call at expiration is

$$C = \max(0, S - X).$$

- The payoff of a put at expiration is

$$P = \max(0, X - S).$$

- A call will be exercised only if the stock price is higher than the strike price.
- A put will be exercised only if the stock price is less than the strike price.


Payoff
Long a call
40
30
20
10
Price
20
40
60
80
Payoff
Short a call
Price
20
40
60
80
-10
-20
-30
-40
Payoff
Long a put
50
40
30
20
10
Price
20
40
60
80
Payoff
Short a put
Price
20
40
60
80
-10
-20
-30
-40
-50

Payoff, Mathematically Speaking (concluded)

- At any time t before the expiration date, we call

$$\max(0, S_t - X)$$

  the intrinsic value of a call.

- At any time t before the expiration date, we call

$$\max(0, X - S_t)$$

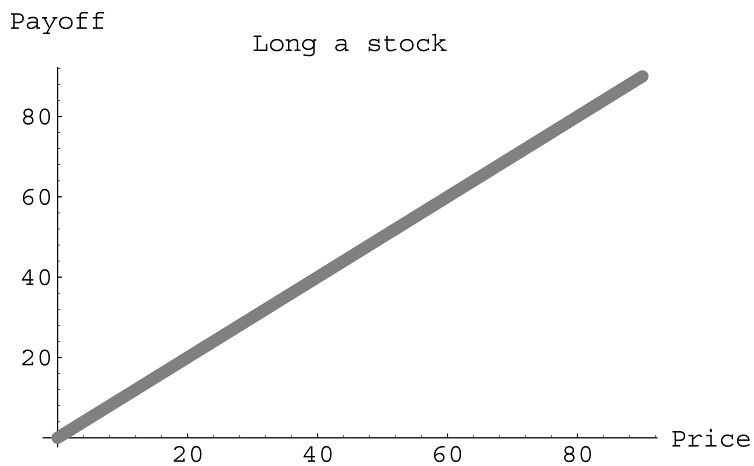
  the intrinsic value of a put.

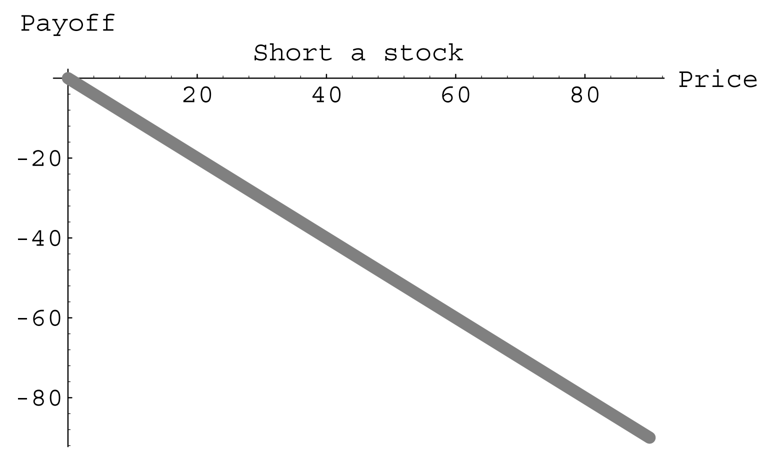
- Finding an option's value at any time *before* expiration is a major intellectual breakthrough.

Short Selling

- Short selling (or simply shorting) involves selling an asset that is *not* owned.
  - If you short 1,000 XYZ shares, the broker borrows them from another client to sell them in the market.
  - This action generates proceeds for the investor.
  - The investor can close out the short position by buying 1,000 XYZ shares.
  - Clearly, the investor profits if the stock price falls.

Payoff of Stock



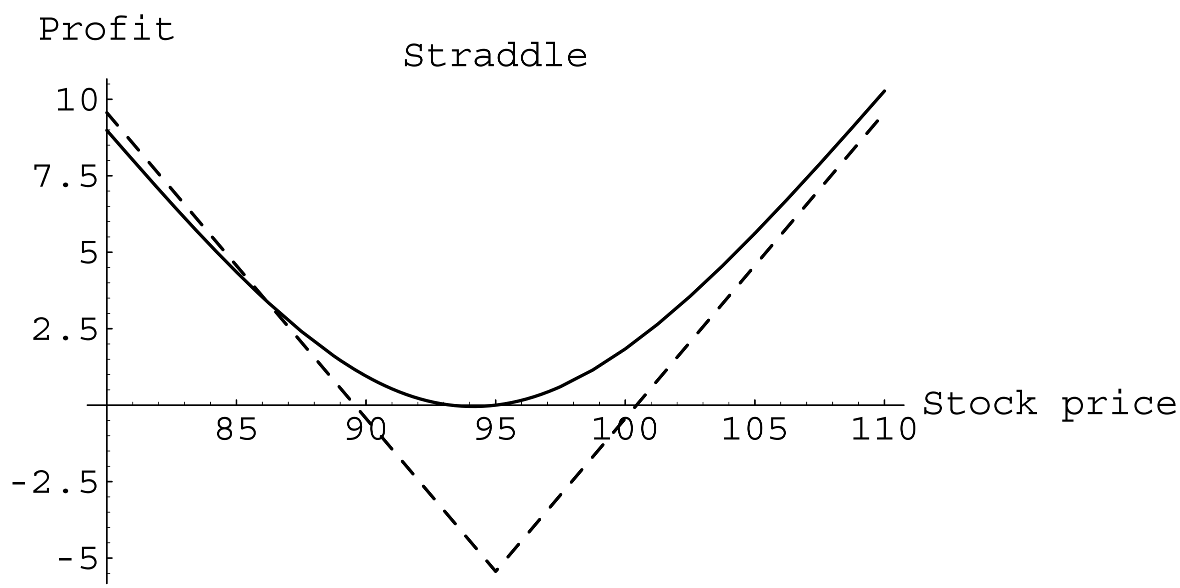


Matching Payoff Functions with Options

- We can generate any piecewise linear payoff function with options, stocks, and cash.[a]
- Straddle: A long call and a long put with the same strike price and expiration date.

[a]Cox and Rubinstein (1985). See Exercise 8.3.6 of the textbook.

Matching Payoff Functions with Options (continued)



Matching Payoff Functions with Options (concluded)

- Since it profits from high volatility, a person who buys a straddle is said to be long volatility.
- Selling a straddle benefits from low volatility.

The Black-Scholes Option Pricing Model

[Black] got the equation [in 1969] but then was unable to solve it. Had he been a better physicist he would have recognized it as a form of the familiar heat exchange equation, and applied the known solution. Had he been a better mathematician, he could have solved the equation from first principles. Certainly Merton would have known exactly what to do with the equation had he ever seen it.

— Perry Mehrling (2005)

Arbitrage

- The no-arbitrage principle says there is no free lunch.
- A riskless arbitrage opportunity is one that, without any initial investment, generates nonnegative returns under all circumstances and positive returns under some.
- In an efficient market, such opportunities do not exist (for long).

The Setting

- Need a model of probabilistic behavior of stock prices.
- Breakthrough came in 1973 when Black (1938–1995) and Scholes with help from Merton published their celebrated option pricing model.[a]
- Now known as the Black-Scholes option pricing model.

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[a]The results were obtained as early as June 1969. Merton and Scholes were winners of the 1997 Nobel Prize in Economic Sciences.

Terms and Approach

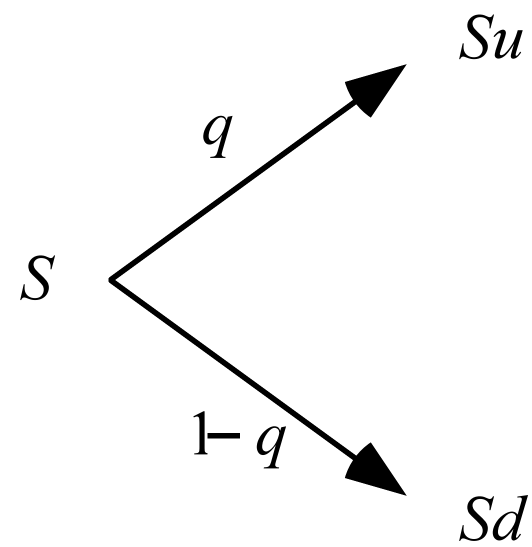
- C: call value.
- P: put value.
- X: strike price
- S: stock price
- $\hat{r} > 0$: the continuously compounded riskless rate per period.
- $R \equiv e^{\hat{r}}$: gross return.
- Start from the discrete-time binomial model.

Binomial Option Pricing Model (BOPM)

- Time is discrete and measured in periods.
- If the current stock price is S, it can go to Su with probability q and Sd with probability $1-q$, where $0 < q < 1$ and $d < u$.
  - In fact, $d < R < u$ must hold to rule out arbitrage.[a]
- Six pieces of information will suffice to determine the option value based on arbitrage considerations:

  S, u, d, X, $\hat{r}$, and the number of periods to expiration.

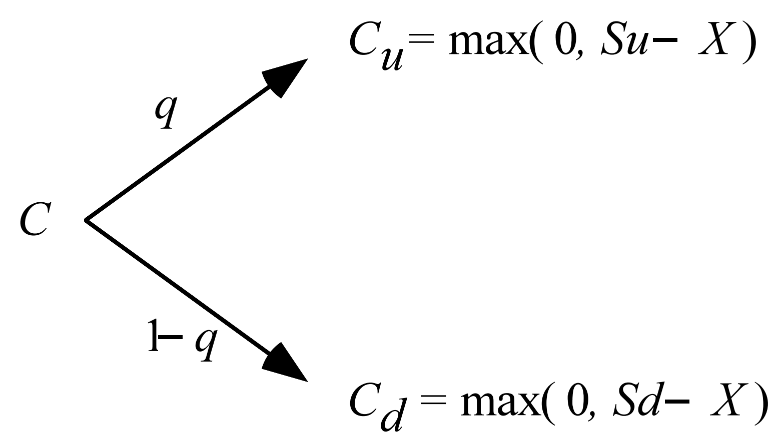
[a]See Exercise 9.2.1 of the textbook.


Su
q
S
1 − q
Sd

Call on a Non-Dividend-Paying Stock: Single Period

- The expiration date is only one period from now.
- C_u is the call price at time 1 if the stock price moves to Su.
- C_d is the call price at time 1 if the stock price moves to Sd.
- Clearly,

$$\begin{aligned} C_u &= \max(0, Su - X), \\ C_d &= \max(0, Sd - X). \end{aligned}$$


$C_u = \max(0, Su - X)$
q
C
$1 - q$
$C_d = \max(0, Sd - X)$

Call on a Non-Dividend-Paying Stock: Single Period (continued)

- Set up a portfolio of h shares of stock and B dollars in riskless bonds.
  - This costs $hS + B$.
  - We call h the hedge ratio or delta.
- The value of this portfolio at time one is

$$\begin{array}{ll} hSu + RB, & \text{up move}, \\ hSd + RB, & \text{down move}. \end{array}$$

Call on a Non-Dividend-Paying Stock: Single Period (continued)

- Choose h and B such that the portfolio *replicates* the payoff of the call,

$$\begin{aligned} hSu + RB &= C_u, \\ hSd + RB &= C_d. \end{aligned}$$

Call on a Non-Dividend-Paying Stock: Single Period (concluded)

- Solve the above equations to obtain

$$h = \frac{C_u - C_d}{Su - Sd} \geq 0, \tag{6}$$

$$B = \frac{uC_d - dC_u}{(u-d)\,R}. \tag{7}$$

- By the no-arbitrage principle, the European call should cost the same as the equivalent portfolio,

$$C = hS + B.$$

American Call Pricing in One Period

- Have to consider immediate exercise.
- $C = \max(hS + B, S - X)$.
  - When $hS + B \geq S - X$, the call should not be exercised immediately.
  - When $hS + B < S - X$, the option should be exercised immediately.

Put Pricing in One Period

- Puts can be similarly priced.
- The delta for the put is $(P_u - P_d)/(Su - Sd) \leq 0$, where

$$
\begin{aligned}
P_u &= \max(0, X - Su), \\
P_d &= \max(0, X - Sd).
\end{aligned}
$$

- Let $B = \frac{uP_d - dP_u}{(u-d)\, R}$.
- The European put is worth $hS + B$.
- The American put is worth $\max(hS + B, X - S)$.

Risk

- Surprisingly, the option value is independent of q.[a]
- The option value depends on the sizes of price changes, u and d, which the investors must agree upon.
- Then the set of possible stock prices is the same whatever q is.

[a]More precisely, not directly dependent on q.

Pseudo Probability

- After substitution and rearrangement,

$$hS + B = \frac{\left(\frac{R-d}{u-d}\right) C_u + \left(\frac{u-R}{u-d}\right) C_d}{R}.$$

- Rewrite it as

$$hS + B = \frac{pC_u + (1-p)\, C_d}{R},$$

where

$$p \equiv \frac{R-d}{u-d}. \tag{8}$$

- As $0 < p < 1$, it may be interpreted as a probability.

Risk-Neutral Probability

- The expected rate of return for the stock is equal to the riskless rate $\hat{r}$ under p as

$$pSu + (1 - p)\, Sd = RS. \tag{9}$$

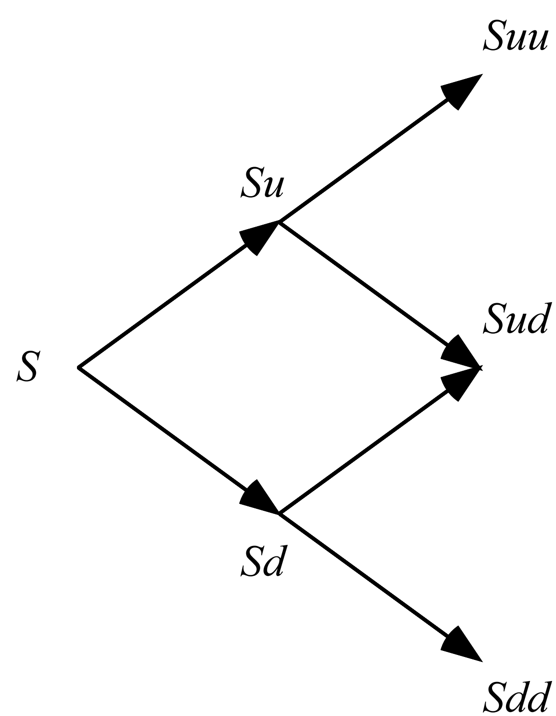
- The expected rates of return of all securities must be the riskless rate when investors are risk-neutral.
- For this reason, p is called the risk-neutral probability.
- The value of an option is the expectation of its discounted future payoff in a risk-neutral economy.

Binomial Distribution

- Denote the binomial distribution with parameters n and p by
$$b(j; n, p) \equiv \binom{n}{j} p^j (1-p)^{n-j} = \frac{n!}{j!\,(n-j)!}\, p^j (1-p)^{n-j}.$$
  - $n! = 1 \times 2 \times \cdots \times n$.
  - Convention: $0! = 1$.
- Suppose you flip a coin n times with p being the probability of getting heads.
- Then $b(j; n, p)$ is the probability of getting j heads.

Option on a Non-Dividend-Paying Stock: Multi-Period

- Consider a call with two periods remaining before expiration.
- Under the binomial model, the stock can take on three possible prices at time two: Suu, Sud, and Sdd.
  - There are 4 paths.
  - But the tree *combines* or *recombines*.


Suu
Su
Sud
S
Sd
Sdd

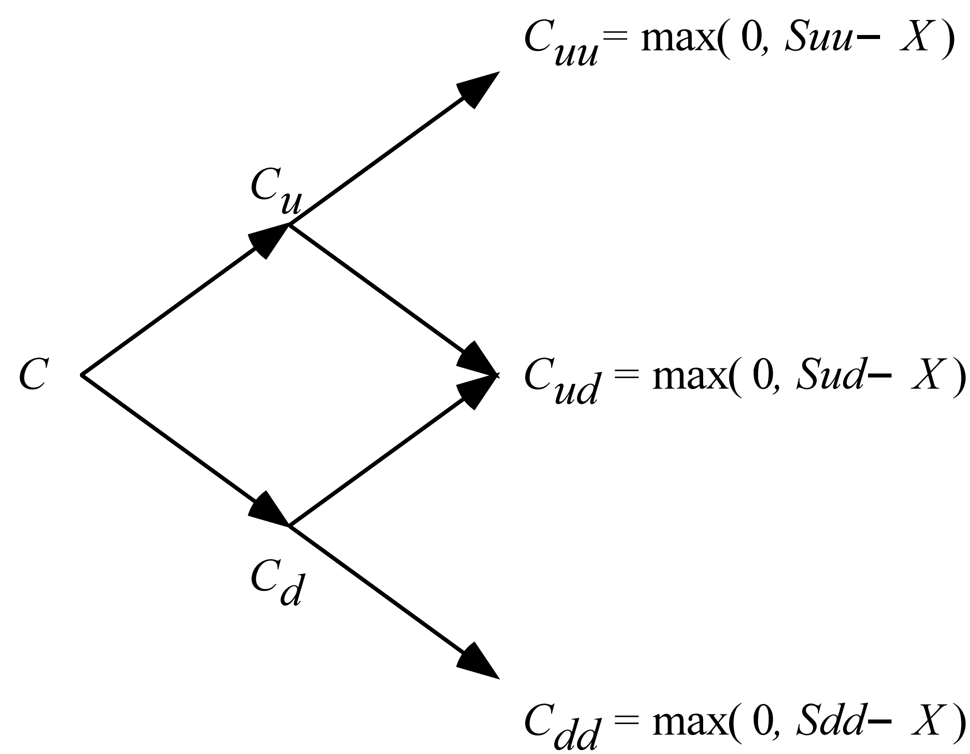
Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- Let C_{uu} be the call's value at time two if the stock price is Suu.
- Thus,

$$C_{uu} = \max(0, Suu - X).$$

- C_{ud} and C_{dd} can be calculated analogously,

$$\begin{aligned} C_{ud} &= \max(0, Sud - X), \\ C_{dd} &= \max(0, Sdd - X). \end{aligned}$$


$C_{uu} = \max(0, Suu - X)$
C_u
C
$C_{ud} = \max(0, Sud - X)$
C_d
$C_{dd} = \max(0, Sdd - X)$

Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- The call values at time 1 can be obtained by applying the same logic:

$$\begin{aligned} C_u &= \frac{pC_{uu} + (1-p)\,C_{ud}}{R}, \qquad (10) \\ C_d &= \frac{pC_{ud} + (1-p)\,C_{dd}}{R}. \end{aligned}$$

- Deltas can be derived from Eq. (6) on p. 52.
- For example, the delta at C_u is

$$\frac{C_{uu} - C_{ud}}{Suu - Sud}.$$

Option on a Non-Dividend-Paying Stock: Multi-Period (concluded)

- We now reach the current period.
- Compute
$$\frac{pC_u + (1-p)\,C_d}{R}$$
as the option price.

Early Exercise

- For European options,

$$C = hS + B = \frac{pC_u + (1-p)\, C_d}{R}.$$

- For American options,

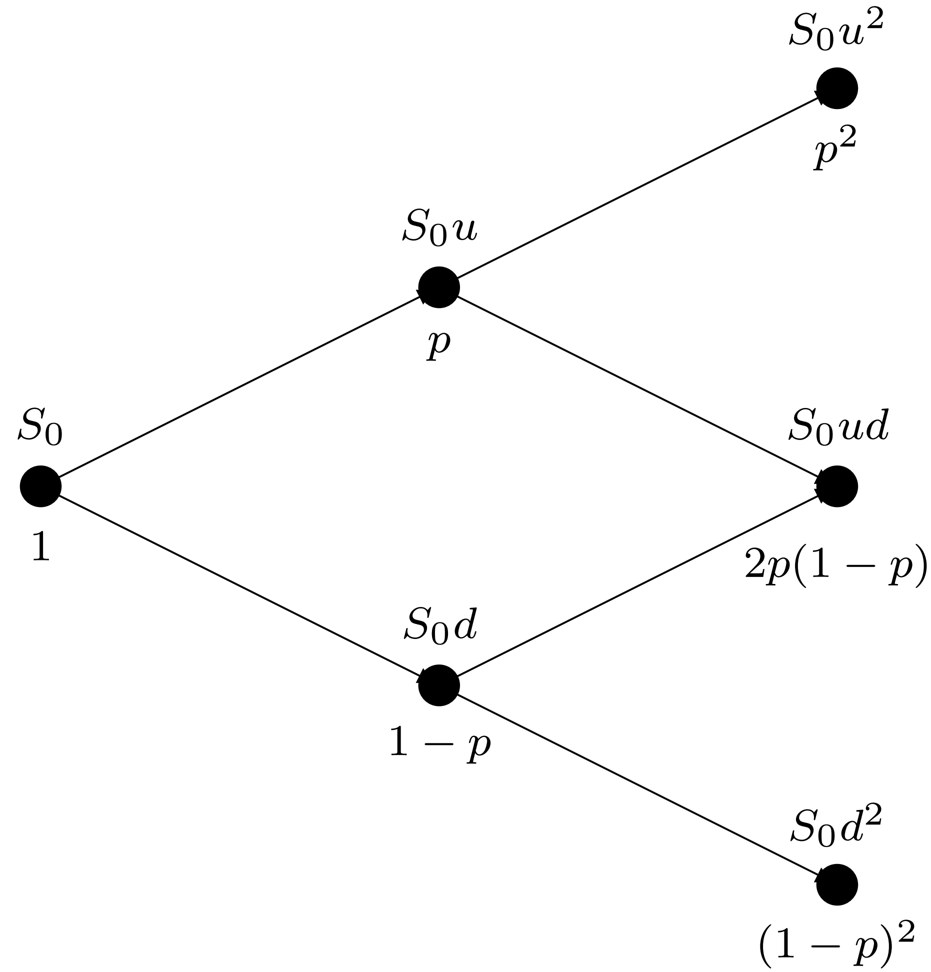
$$C = \max\left(\frac{pC_u + (1-p)\, C_d}{R}, S - X\right).$$

Backward Induction[a]

- The above expression calculates C from the two successor nodes C_u and C_d and none beyond.
- The same computation happened at C_u and C_d, too, as demonstrated in Eq. (10) on p. 63.
- This recursive procedure is called backward induction.
- If C is European, then

$$
\begin{aligned}
C &= [\,p^2 C_{uu} + 2p(1-p)\, C_{ud} + (1-p)^2 C_{dd}](1/R^2) \\
&= [\,p^2 \max\left(0, Su^2 - X\right) + 2p(1-p) \max\left(0, Sud - X\right) \\
&\quad +(1-p)^2 \max\left(0, Sd^2 - X\right)\,]/R^2.
\end{aligned}
$$

[a]Ernst Zermelo (1871–1953).


S_0u^2
p^2
S_0u
p
S_0
1
S_0ud
$2p(1-p)$
S_0d
$1-p$
S_0d^2
$(1-p)^2$

Backward Induction (concluded)

- In the n-period case,

$$C = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, Su^j d^{n-j} - X\right)}{R^n}.$$

  - The value of a European call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.

- Similarly,

$$P = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, X - Su^j d^{n-j}\right)}{R^n}.$$

- Both C and P can be calculated in $O(n)$ time.[a]

[a]See text.

The Binomial Option Pricing Formula

- The stock prices at time n are

$$Su^n, Su^{n-1}d, \ldots, Sd^n.$$

- Let a be the minimum number of upward price moves for the call to finish in the money.
- So a is the smallest nonnegative integer j such that

$$Su^j d^{n-j} \geq X,$$

or, equivalently,

$$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil.$$

The Binomial Option Pricing Formula (concluded)

- Hence,

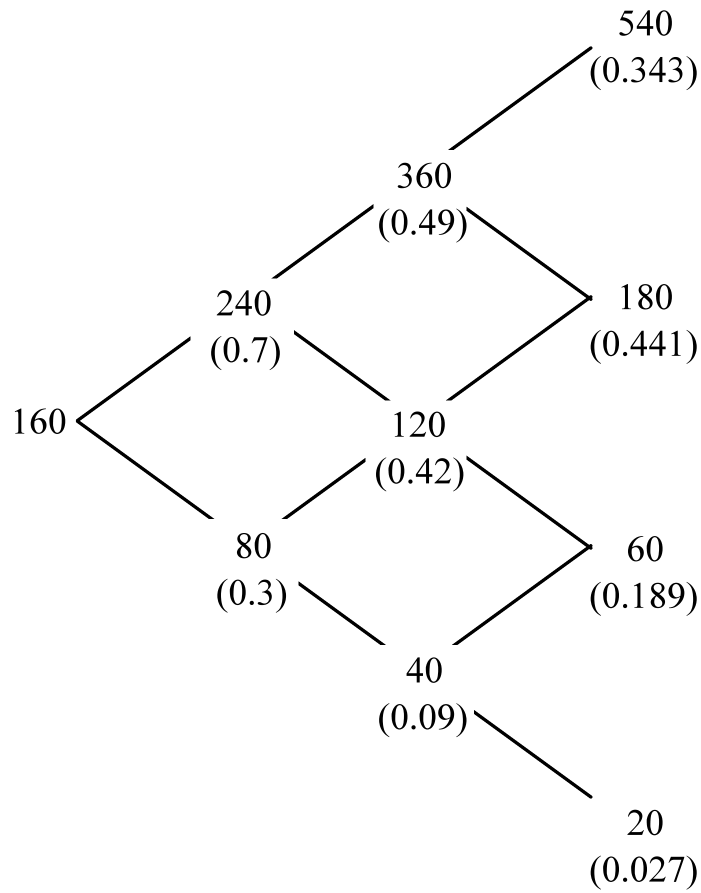
$$
\begin{aligned}
& C \\
= \quad & \frac{\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \left(S u^j d^{n-j} - X \right)}{R^n} \\
= \quad & S \sum_{j=a}^{n} \binom{n}{j} \frac{(pu)^j [\, (1-p)\, d \,]^{n-j}}{R^n} \\
& - \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \\
= \quad & S \sum_{j=a}^{n} b\,(j; n, pu/R) - X e^{-\hat{r} n} \sum_{j=a}^{n} b(j; n, p). \qquad (11)
\end{aligned}
$$

Numerical Examples

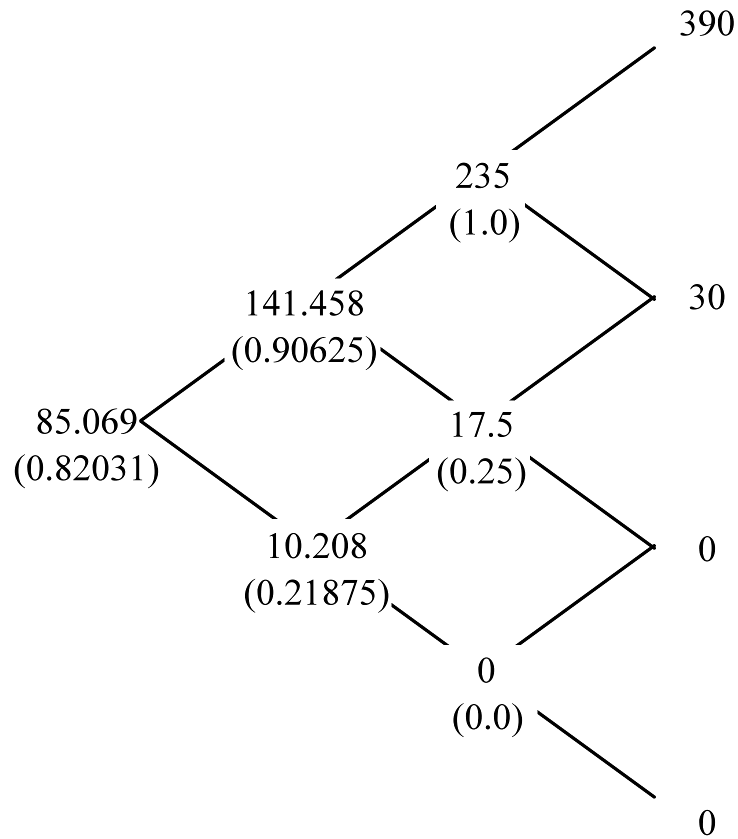
- A non-dividend-paying stock is selling for \$160.
- $u = 1.5$ and $d = 0.5$.
- $r = 18.232\%$ per period ($R = e^{0.18232} = 1.2$).
  - Hence $p = (R - d)/(u - d) = 0.7$.
- Consider a European call on this stock with $X = 150$ and $n = 3$.
- The call value is \$85.069 by backward induction.
- Or, the PV of the expected payoff at expiration:

$$\frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.$$

Binomial process for the stock price
(probabilities in parentheses)



Binomial process for the call price
(hedge ratios in parentheses)



Applications besides Exploiting Arbitrage Opportunities[a]

- Replicate an option using stocks and bonds.
  - Set up a portfolio to replicate the call with \$85.069.
- Hedge the options we issued.
  - Use \$85.069 to set up a portfolio to replicate the call to counterbalance its values exactly.[b]
- $\cdots$
- Without hedge, one may end up forking out \$390 in the worst case![c]

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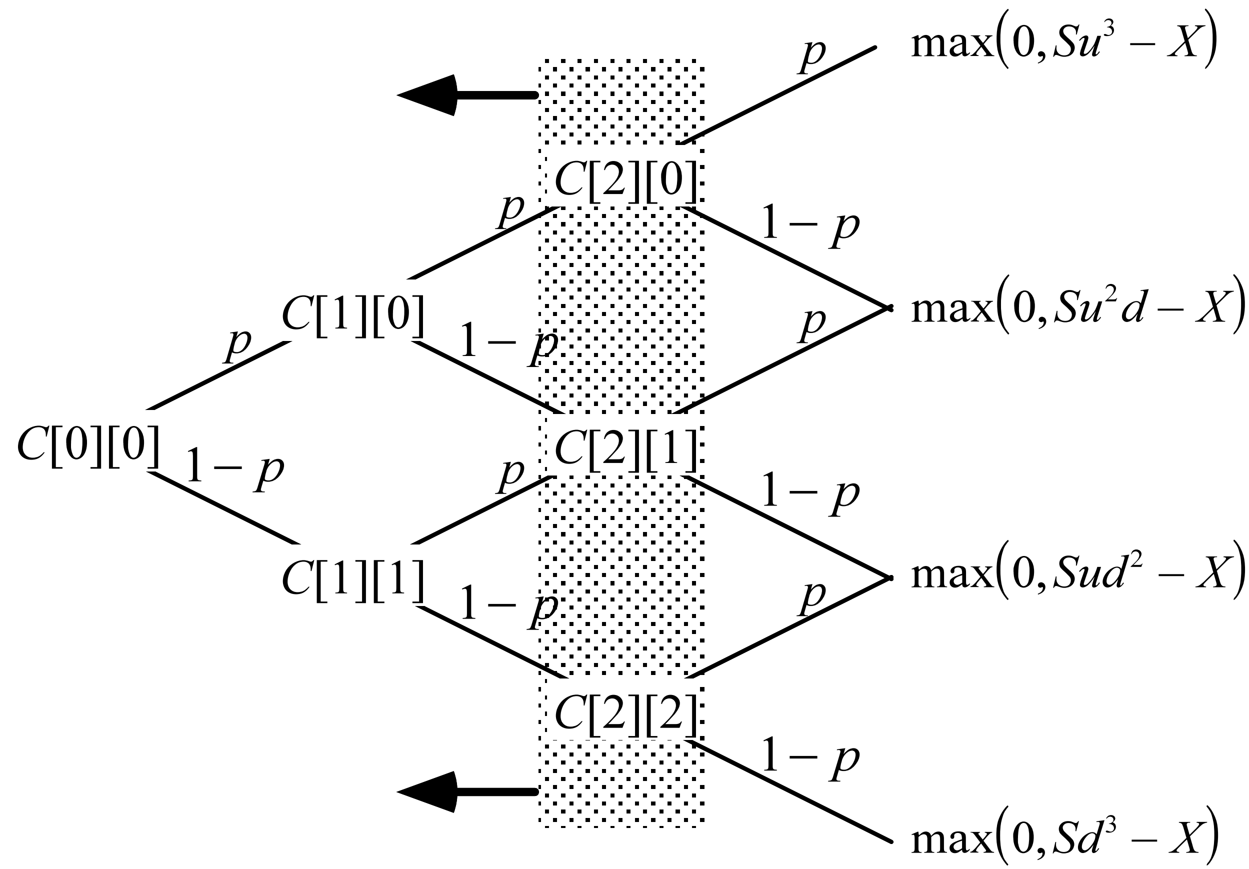
[a] Thanks to a lively class discussion on March 16, 2011.
[b] Hedging and replication are mirror images.
[c] Thanks to a lively class discussion on March 16, 2016.

Binomial Tree Algorithms for Options

- The BOPM implies the binomial tree algorithm that applies backward induction.
- The total running time is $O(n^2)$ because there are $\sim n^2/2$ nodes.
- The memory requirement is $O(n^2)$.
  - Can be easily reduced to $O(n)$ by reusing space.
- To price any other payoff function, simply replace the payoff.



$\max(0, Su^3 - X)$
$\max(0, Su^2 d - X)$
$\max(0, Sud^2 - X)$
$\max(0, Sd^3 - X)$
$C[0][0]$
$C[1][0]$
$C[1][1]$
$C[2][0]$
$C[2][1]$
$C[2][2]$
p
$1-p$