Ito's Lemma[a]

A smooth function of an Ito process is itself an Ito process.

Theorem 19 *Suppose* $f : R \to R$ *is twice continuously differentiable and* $dX = a_t\, dt + b_t\, dW$. *Then* $f(X)$ *is the Ito process,*

$$\begin{aligned} & f(X_t) \\ = \; & f(X_0) + \int_0^t f'(X_s)\, a_s\, ds + \int_0^t f'(X_s)\, b_s\, dW \\ & + \frac{1}{2} \int_0^t f''(X_s)\, b_s^2\, ds \end{aligned}$$

for $t \geq 0$.

[a] Ito (1944).

Ito's Lemma (continued)

- In differential form, Ito's lemma becomes

$$\begin{aligned} & df(X) \\ = \; & f'(X)\, a\, dt + f'(X)\, b\, dW + \frac{1}{2}\, f''(X)\, b^2\, dt \qquad (84) \\ = \; & \left[f'(X)\, a + \boxed{\tfrac{1}{2}\, f''(X)\, b^2} \right] dt + f'(X)\, b\, dW. \end{aligned}$$

- Compared with calculus, the extra term is boxed.
- A convenient formulation of Ito's lemma is

$$df(X) = f'(X)\, dX + \frac{1}{2}\, f''(X)(dX)^2. \qquad (85)$$

Ito's Lemma (continued)

- We are supposed to multiply out $(dX)^2 = (a\,dt + b\,dW)^2$ symbolically according to

$\times$	dW	dt
dW	dt	0
dt	0	0

  - The $(dW)^2 = dt$ entry is justified by a known result.
- Hence $(dX)^2 = (a\,dt + b\,dW)^2 = b^2\,dt$ in Eq. (85).
- This form is easy to remember because of its similarity to the Taylor expansion.

Ito's Lemma (continued)

Theorem 20 (Higher-Dimensional Ito's Lemma) *Let* $W_1, W_2, \ldots, W_n$ *be independent Wiener processes and* $X \stackrel{\Delta}{=} (X_1, X_2, \ldots, X_m)$ *be a vector process. Suppose* $f : R^m \to R$ *is twice continuously differentiable and* X_i *is an Ito process with* $dX_i = a_i \, dt + \sum_{j=1}^{n} b_{ij} \, dW_j$*. Then* $df(X)$ *is an Ito process with the differential,*

$$df(X) = \sum_{i=1}^{m} f_i(X) \, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X) \, dX_i \, dX_k,$$

where $f_i \stackrel{\Delta}{=} \partial f / \partial X_i$ *and* $f_{ik} \stackrel{\Delta}{=} \partial^2 f / \partial X_i \partial X_k$*.*

Ito's Lemma (continued)

- The multiplication table for Theorem 20 is

$\times$	dW_i	dt
dW_k	$\delta_{ik}\, dt$	0
dt	0	0

in which

$$\delta_{ik} = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{otherwise.} \end{cases}$$

Ito's Lemma (continued)

- In applying the higher-dimensional Ito's lemma, usually one of the variables, say X_1, is time t and $dX_1 = dt$.
- In this case, $b_{1j} = 0$ for all j and $a_1 = 1$.
- As an example, let

$$dX_t = a_t \, dt + b_t \, dW_t.$$

- Consider the process $f(X_t, t)$.

Ito's Lemma (continued)

- Then

$$
\begin{aligned}
& df \\
= \; & \frac{\partial f}{\partial X_t}\, dX_t + \frac{\partial f}{\partial t}\, dt + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2}\, (dX_t)^2 \\
= \; & \frac{\partial f}{\partial X_t}\, (a_t\, dt + b_t\, dW_t) + \frac{\partial f}{\partial t}\, dt \\
& + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2}\, (a_t\, dt + b_t\, dW_t)^2 \\
= \; & \left(\frac{\partial f}{\partial X_t}\, a_t + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2}\, b_t^2 \right) dt + \frac{\partial f}{\partial X_t}\, b_t\, dW_t. \quad (86)
\end{aligned}
$$

Ito's Lemma (continued)

Theorem 21 (Alternative Ito's Lemma) *Let* $W_1, W_2, \ldots, W_m$ *be Wiener processes and* $X \stackrel{\Delta}{=} (X_1, X_2, \ldots, X_m)$ *be a vector process. Suppose* $f : R^m \to R$ *is twice continuously differentiable and* X_i *is an Ito process with* $dX_i = a_i\, dt + b_i\, dW_i$. *Then* $df(X)$ *is the following Ito process,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X)\, dX_i\, dX_k.$$

Ito's Lemma (concluded)

- The multiplication table for Theorem 21 is

$\times$	dW_i	dt
dW_k	$\rho_{ik}\, dt$	0
dt	0	0

- Above, ρ_{ik} denotes the correlation between dW_i and dW_k.

Geometric Brownian Motion

- Consider geometric Brownian motion

$$Y(t) \stackrel{\Delta}{=} e^{X(t)}.$$

  - $X(t)$ is a (μ, σ) Brownian motion.
  - By Eq. (79) on p. 581,

$$dX = \mu\, dt + \sigma\, dW.$$

- Note that

$$\begin{aligned} \frac{\partial Y}{\partial X} &= Y, \\ \frac{\partial^2 Y}{\partial X^2} &= Y. \end{aligned}$$

Geometric Brownian Motion (continued)

- Ito's formula (84) on p. 613 implies

$$\begin{aligned} dY &= Y\,dX + (1/2)\,Y\,(dX)^2 \\ &= Y\,(\mu\,dt + \sigma\,dW) + (1/2)\,Y\,(\mu\,dt + \sigma\,dW)^2 \\ &= Y\,(\mu\,dt + \sigma\,dW) + (1/2)\,Y\sigma^2\,dt. \end{aligned}$$

- Hence[a]

$$\frac{dY}{Y} = \left(\mu + \frac{\sigma^2}{2}\right) dt + \sigma\,dW. \tag{87}$$

- The annualized *instantaneous* rate of return is $\mu + \sigma^2/2$ (not μ).[b]

---

[a]Equation (87) is an abbreviation for $dY = Y(\mu + \sigma^2/2)\,dt + \sigma\,dW$.

[b]Consistent with Lemma 10 (p. 305).

Geometric Brownian Motion (continued)

- Alternatively, from Eq. (79) on p. 581,

$$X_t = X_0 + \mu t + \sigma W_t,$$

admits an explicit (strong) solution.

- Hence

$$Y_t = Y_0 \, e^{\mu t + \sigma W_t}, \tag{88}$$

a strong solution to the SDE (87) where $Y_0 = e^{X_0}$.

Geometric Brownian Motion (concluded)

- On the other hand, suppose

$$\frac{dY}{Y} = \mu \, dt + \sigma \, dW.$$

- Then $X(t) \stackrel{\Delta}{=} \ln Y(t)$ follows

$$dX = \left(\mu - \sigma^2/2\right) dt + \sigma \, dW.$$

Exponential Martingale

- The Ito process
$$dX_t = b_t X_t \, dW_t$$
is a martingale.[a]
- It is called an exponential martingale.
- By Ito's formula (84) on p. 613,
$$X(t) = X(0) \exp\left[-\frac{1}{2} \int_0^t b_s^2 \, ds + \int_0^t b_s \, dW_s \right].$$

[a] Recall Theorem 18 (p. 598).

Product of Geometric Brownian Motion Processes

- Let

$$\frac{dY}{Y} = a\,dt + b\,dW_Y,$$
$$\frac{dZ}{Z} = f\,dt + g\,dW_Z.$$

- Assume dW_Y and dW_Z have correlation ρ.
- Consider the Ito process

$$U \triangleq YZ.$$

Product of Geometric Brownian Motion Processes (continued)

- Apply Ito's lemma (Theorem 21 on p. 619):

$$
\begin{aligned}
dU &= Z\,dY + Y\,dZ + dY\,dZ \\
&= ZY(a\,dt + b\,dW_Y) + YZ(f\,dt + g\,dW_Z) \\
&\quad + YZ(a\,dt + b\,dW_Y)(f\,dt + g\,dW_Z) \\
&= U(a + f + bg\rho)\,dt + Ub\,dW_Y + Ug\,dW_Z.
\end{aligned}
$$

- The product of correlated geometric Brownian motion processes thus remains geometric Brownian motion.

Product of Geometric Brownian Motion Processes (continued)

- Note that

$$
\begin{aligned}
Y(t) &= Y(0)\exp\left[\left(a-b^2/2\right)t+bW_Y(t)\right],\\
Z(t) &= Z(0)\exp\left[\left(f-g^2/2\right)t+gW_Z(t)\right],\\
U(t) &= U(0)\exp\left\{\left[a+f-\left(b^2+g^2\right)/2\right]t\right.\\
&\quad \left.+bW_Y(t)+gW_Z(t)\right\}.
\end{aligned}
$$

- They are the strong solutions.

Product of Geometric Brownian Motion Processes (concluded)

- $\ln U$ is Brownian motion with a mean equal to the sum of the means of $\ln Y$ and $\ln Z$.
- This holds even if Y and Z are correlated.
- Finally, $\ln Y$ and $\ln Z$ have correlation ρ.

Quotients of Geometric Brownian Motion Processes

- Suppose Y and Z are drawn from p. 626.
- Let
$$U \triangleq Y/Z.$$
- We now show that[a]
$$\frac{dU}{U} = \big(a - f + g^2 - bg\rho\big)\, dt + b\, dW_Y - g\, dW_Z. \tag{89}$$
- Keep in mind that dW_Y and dW_Z have correlation ρ.

[a]Exercise 14.3.6 of the textbook is erroneous.

Quotients of Geometric Brownian Motion Processes (concluded)

- The multidimensional Ito's lemma (Theorem 21 on p. 619) can be employed to show that[a]

$$
\begin{aligned}
& dU \\
= \quad & (1/Z)\, dY - (Y/Z^2)\, dZ - (1/Z^2)\, dY\, dZ + (Y/Z^3)\, (dZ)^2 \\
= \quad & (1/Z)(aY\, dt + bY\, dW_Y) - (Y/Z^2)(fZ\, dt + gZ\, dW_Z) \\
& -(1/Z^2)(bgYZ\rho\, dt) + (Y/Z^3)(g^2 Z^2\, dt) \\
= \quad & U(a\, dt + b\, dW_Y) - U(f\, dt + g\, dW_Z) \\
& -U(bg\rho\, dt) + U(g^2\, dt) \\
= \quad & U(a - f + g^2 - bg\rho)\, dt + Ub\, dW_Y - Ug\, dW_Z.
\end{aligned}
$$

---

[a] As $\partial^2 U/\partial Y^2 = 0$, the $(dY)^2$ term is ignored.

Forward Price

- Suppose S follows

$$\frac{dS}{S} = \mu\, dt + \sigma\, dW.$$

- Consider functional $F(S,t) \stackrel{\Delta}{=} Se^{y(T-t)}$ for constants y and T.
- As F is a function of two variables, we need the various partial derivatives of $F(S,t)$ with respect to S *and* t.[a]

---

[a]In partial differentiation with respect to one variable, other variables are held constant. Contributed by Mr. Sun, Ao (R05922147) on April 26, 2017.

Forward Prices (continued)

- Now,

$$\begin{aligned} \frac{\partial F}{\partial S} &= e^{y(T-t)}, \\ \frac{\partial^2 F}{\partial S^2} &= 0, \\ \frac{\partial F}{\partial t} &= -ySe^{y(T-t)}. \end{aligned}$$

- Then[a]

$$\begin{aligned} dF &= e^{y(T-t)}\,dS - ySe^{y(T-t)}\,dt \\ &= Se^{y(T-t)}\,(\mu\,dt + \sigma\,dW) - ySe^{y(T-t)}\,dt \\ &= F(\mu - y)\,dt + F\sigma\,dW. \end{aligned}$$

[a]One can also prove it by Eq. (86) on p. 618.

Forward Prices (concluded)

- Thus F follows

$$\frac{dF}{F} = (\mu - y)\, dt + \sigma\, dW.$$

- This result has applications in forward and futures contracts.
- In Eq. (61) on p. 494, $\mu = r = y$.
- So

$$\frac{dF}{F} = \sigma\, dW,$$

a martingale.[a]

---

[a]It is consistent with p. 570. Furthermore, it explains why Black's formulas (69)–(70) on p. 522 use the volatility σ of the stock.

Ornstein-Uhlenbeck (OU) Process

- The OU process:

$$dX = -\kappa X\, dt + \sigma\, dW,$$

where $\kappa, \sigma \geq 0$.

- For $t_0 \leq s \leq t$ and $X(t_0) = x_0$, it is known that

$$\begin{aligned}
E[\,X(t)\,] &= e^{-\kappa(t-t_0)}\, E[\,x_0\,], \\
\text{Var}[\,X(t)\,] &= \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa(t-t_0)}\right] + e^{-2\kappa(t-t_0)}\,\text{Var}[\,x_0\,], \\
\text{Cov}[\,X(s), X(t)\,] &= \frac{\sigma^2}{2\kappa}\, e^{-\kappa(t-s)}\left[1 - e^{-2\kappa(s-t_0)}\right] \\
&\quad + e^{-\kappa(t+s-2t_0)}\,\text{Var}[\,x_0\,].
\end{aligned}$$

Ornstein-Uhlenbeck Process (continued)

- $X(t)$ is normally distributed if x_0 is a constant or normally distributed.
  - $E[x_0] = x_0$ and $\text{Var}[x_0] = 0$ if x_0 is a constant.
- X is said to be a normal process.
- The OU process has the following mean-reverting property if $\kappa > 0$.
  - When $X > 0$, X is pulled toward zero.
  - When $X < 0$, it is pulled toward zero again.

Ornstein-Uhlenbeck Process (continued)

- A generalized version:

$$dX = \kappa(\mu - X)\, dt + \sigma\, dW,$$

where $\kappa, \sigma \geq 0$.

- Given $X(t_0) = x_0$, a constant, it is known that

$$\begin{aligned} E[\, X(t)\,] &= \mu + (x_0 - \mu)\, e^{-\kappa(t-t_0)}, \\ \mathrm{Var}[\, X(t)\,] &= \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa(t-t_0)}\right], \end{aligned} \tag{90}$$

for $t_0 \leq t$.

Ornstein-Uhlenbeck Process (concluded)

- The mean and standard deviation are roughly μ and $\sigma/\sqrt{2\kappa}$, respectively.
- For large t, the probability of $X < 0$ is extremely unlikely in any finite time interval when $\mu > 0$ is large relative to $\sigma/\sqrt{2\kappa}$.
- The process is mean-reverting.
  - X tends to move toward μ.
  - Useful for modeling term structure, stock price volatility, and stock price return.[a]

[a] See Knutson, Wimmer, Kuhnen, & Winkielman (2008) for the biological basis for mean reversion in financial decision making.

Square-Root Process

- Suppose X is an OU process.
- Consider
$$V \triangleq X^2.$$
- Ito's lemma says V has the differential,
$$\begin{aligned} dV &= 2X\,dX + (dX)^2 \\ &= 2\sqrt{V}\,(-\kappa\sqrt{V}\,dt + \sigma\,dW) + \sigma^2\,dt \\ &= \left(-2\kappa V + \sigma^2\right)dt + 2\sigma\sqrt{V}\,dW, \end{aligned}$$
a square-root process.

Square-Root Process (continued)

- In general, the square-root process has the SDE,

$$dX = \kappa(\mu - X)\,dt + \sigma\sqrt{X}\,dW,$$

  where $\kappa, \sigma > 0$, $\mu \geq 0$, and $X(0) \geq 0$ is a constant.

- Like the OU process, it possesses mean reversion: X tends to move toward μ, but the volatility is proportional to $\sqrt{X}$ instead of a constant.

Square-Root Process (continued)

- When X hits zero and $\mu \geq 0$, the probability is one that it will not move below zero.
  - Zero is a reflecting boundary.
- Hence, the square-root process is a good candidate for modeling interest rates.[a]
- The OU process, in contrast, allows negative interest rates.[b]
- The two processes are related.[c]

---

[a] Cox, Ingersoll, & Ross (1985).
[b] But some rates did go negative in Europe in 2015.
[c] Recall p. 639.

Square-Root Process (concluded)

- The random variable $2cX(t)$ follows the noncentral chi-square distribution,[a]

$$\chi\left(\frac{4\kappa\mu}{\sigma^2}, 2cX(0)\, e^{-\kappa t}\right),$$

where $c \stackrel{\Delta}{=} (2\kappa/\sigma^2)(1 - e^{-\kappa t})^{-1}$ and $\mu > 0$.

- Given $X(0) = x_0$, a constant,

$$\begin{aligned} E[\,X(t)\,] &= x_0 e^{-\kappa t} + \mu\left(1 - e^{-\kappa t}\right), \\ \mathrm{Var}[\,X(t)\,] &= x_0\,\frac{\sigma^2}{\kappa}\,\left(e^{-\kappa t} - e^{-2\kappa t}\right) + \mu\,\frac{\sigma^2}{2\kappa}\,\left(1 - e^{-\kappa t}\right)^2, \end{aligned}$$

for $t \geq 0$.

[a] William Feller (1906–1970) in 1951.

Modeling Stock Prices

- The most popular stochastic model for stock prices has been the geometric Brownian motion,

$$\frac{dS}{S} = \mu\, dt + \sigma\, dW.$$

- The logarithmic price $X \stackrel{\Delta}{=} \ln S$ follows

$$dX = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma\, dW$$

by Eq. (87) on p. 622.

Local-Volatility Models

- The deterministic-volatility model for "smile" posits

$$\frac{dS}{S} = (r_t - q_t)\, dt + \sigma(S, t)\, dW,$$

  where instantaneous volatility $\sigma(S, t)$ is called the local-volatility function.[a]

  – "The most popular model after Black-Scholes is a local volatility model as it is the only completely consistent volatility model."

- A (weak) solution exists if $S\sigma(S, t)$ is continuous and grows at most linearly in S and t.[b]

[a] Derman & Kani (1994); Dupire (1994).
[b] Skorokhod (1961); Achdou & Pironneau (2005).

Local-Volatility Models (continued)

- One needs to recover the local volatility surface $\sigma(S, t)$ from the implied volatility surface.[a]
- Theoretically,[b]

$$\sigma(X, T)^2 = 2 \frac{\frac{\partial C}{\partial T} + (r_T - q_T) X \frac{\partial C}{\partial X} + q_T C}{X^2 \frac{\partial^2 C}{\partial X^2}}. \tag{91}$$

  - C is the call price at time $t = 0$ (today) with strike price X and time to maturity T.
  - $\sigma(X, T)$ is the local volatility that will prevail at *future time* T and *stock price* $S_T = X$.

[a] Also called the volatility smile surface (Alexander, 2001).
[b] Dupire (1994); Andersen & Brotherton-Ratcliffe (1998).

Local-Volatility Models (continued)

- For more general models, this equation gives the expectation as seen from today, under the risk-neural probability, of the instantaneous variance at time T given that $S_T = X$.[a]

- In practice, the $\sigma(S, t)^2$ derived by Dupire's formula (91) may have spikes, vary wildly, or even be negative.

- The term $\partial^2 C/\partial X^2$ in the denominator often results in numerical instability.

---

[a] Derman & Kani (1997); R. W. Lee (2001); Derman & M. B. Miller (2016).

Local-Volatility Models (continued)

- Denote the implied volatility surface by $\Sigma(X, T)$ and the local volatility surface by $\sigma(S, t)$.
- The relation between $\Sigma(X, T)$ and $\sigma(X, T)$ is[a]

$$\sigma(X,T)^2 = \frac{\Sigma^2 + 2\Sigma\tau\left[\frac{\partial\Sigma}{\partial T} + (r_T - q_T)X\frac{\partial\Sigma}{\partial X}\right]}{\left(1 - \frac{Xy}{\Sigma}\frac{\partial\Sigma}{\partial X}\right)^2 + X\Sigma\tau\left[\frac{\partial\Sigma}{\partial X} - \frac{X\Sigma\tau}{4}\left(\frac{\partial\Sigma}{\partial X}\right)^2 + X\frac{\partial^2\Sigma}{\partial X^2}\right]},$$

$$\tau \stackrel{\Delta}{=} T - t,$$

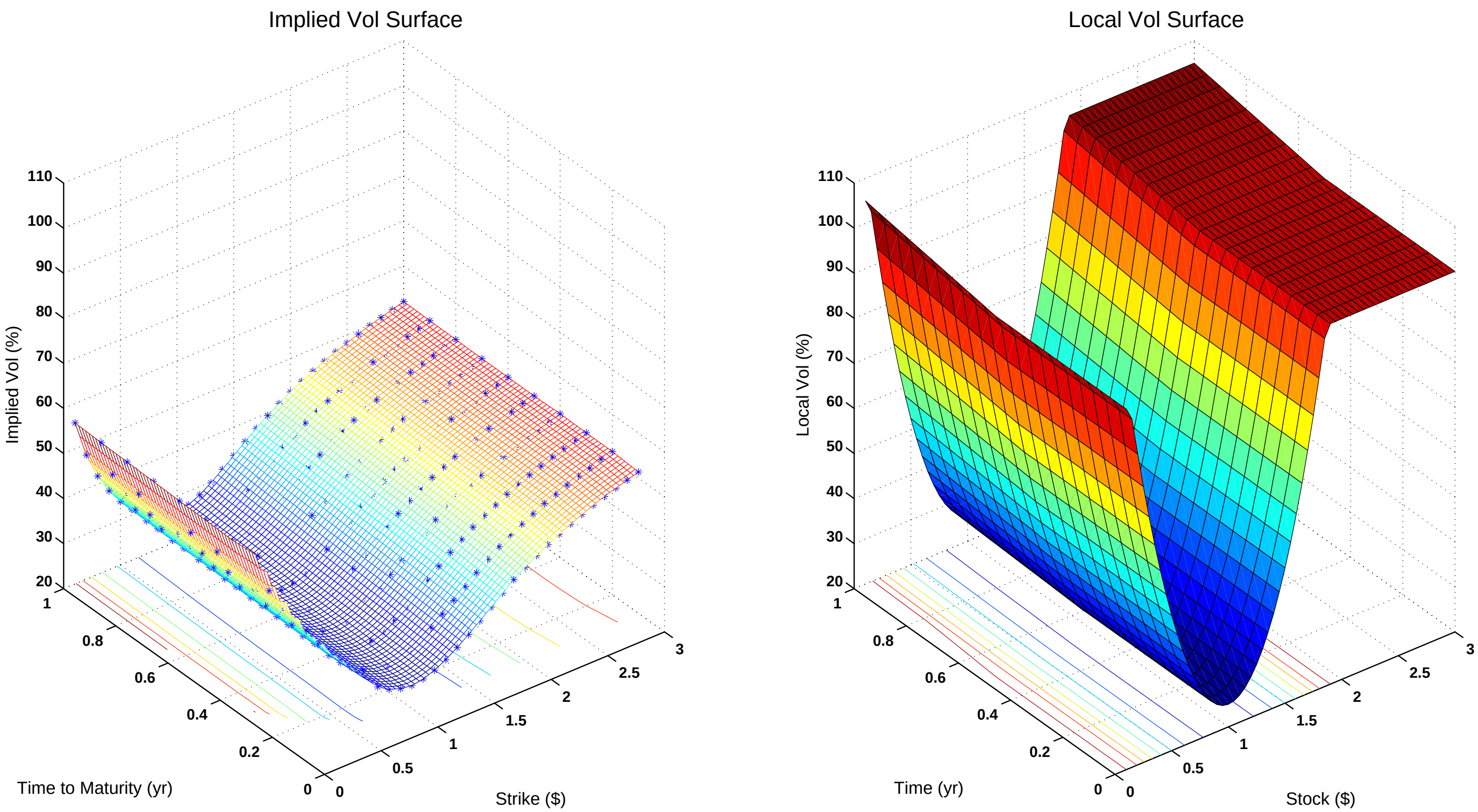
$$y \stackrel{\Delta}{=} \ln(X/S_t) + \int_t^T (q_s - r_s)\, ds.$$

[a] Andreasen (1996); Andersen & Brotherton-Ratcliffe (1998); Gatheral (2003); Wilmott (2006); Kamp (2009).

Local-Volatility Models (continued)

- Although this version may be more stable than Eq. (91) on p. 645, it is expected to suffer from the same problems.
- Small changes to the implied volatility surface may produce big changes to the local volatility surface.

Implied and Local Volatility Surfaces[a]



[a]Contributed by Mr. Lok, U Hou (D99922028) on April 5, 2014.

Local-Volatility Models (continued)

- In reality, option prices only exist for a finite set of maturities and strike prices.
- Hence interpolation and extrapolation may be needed to construct the volatility surface.[a]
- But then some implied volatility surfaces generate option prices that allow arbitrage opportunities.[b]

---

[a] Andreasen & Huge (2010). Doing it to the option prices produces worse results (Li, 2000/2001).

[b] See Rebonato (2004) for an example.

Local-Volatility Models (concluded)

- There exist conditions for a set of option prices to be arbitrage-free.[a]
- Some adopt parameterized implied volatility surfaces that guarantee freedom from certain arbitrages.[b]
- For some vanilla equity options, the Black-Scholes model seems better than the local-volatility model in predictive power.[c]
- The exact opposite is concluded for hedging in equity index markets![d]

---

[a] Kahalé (2004); Davis & Hobson (2007).
[b] Gatheral & Jacquier (2014).
[c] Dumas, Fleming, & Whaley (1998).
[d] Crépey (2004); Derman & M. B. Miller (2016).

Local-Volatility Models: Popularity

- Hirsa and Neftci (2014), "most traders and firms actively utilize this [local-volatility] model."
- Bennett (2014), "Of all the four volatility regimes, [sticky local volatility] is arguably the most realistic and fairly prices skew."
- Derman & M. B. Miller (2016), "Right or wrong, local volatility models have become popular and ubiquitous in modeling the smile."

Implied Trees

- The trees for the local-volatility model are called implied trees.[a]
- Their construction requires option prices at all strike prices and maturities.
  - That is, an implied volatility surface.
- The local volatility model does *not* imply that the implied tree must combine.
- Exponential-sized implied trees exist.[b]

[a] Derman & Kani (1994); Dupire (1994); Rubinstein (1994).

[b] Charalambousa, Christofidesb, & Martzoukosa (2007); Gong & Xu (2019).

Implied Trees (continued)

- How to construct a valid implied tree with efficiency has been open for a long time.[a]
  - Reasons may include: noise and nonsynchrony in data, arbitrage opportunities in the smoothed and interpolated/extrapolated implied volatility surface, wrong model, wrong algorithms, nonlinearity, instability, etc.
- Inversion is generally an ill-posed numerical problem.[b]

---

[a] Rubinstein (1994); Derman & Kani (1994); Derman, Kani, & Chriss (1996); Jackwerth & Rubinstein (1996); Jackwerth (1997); Coleman, Kim, Li, & Verma (2000); Li (2000/2001); Rebonato (2004); Moriggia, Muzzioli, & Torricelli (2009).

[b] Ayache, Henrotte, Nassar, & X. Wang (2004).

Implied Trees (continued)

- It is finally solved for separable local volatilities.[a]
  - The local-volatility function $\sigma(S, t)$ is separable[b] if

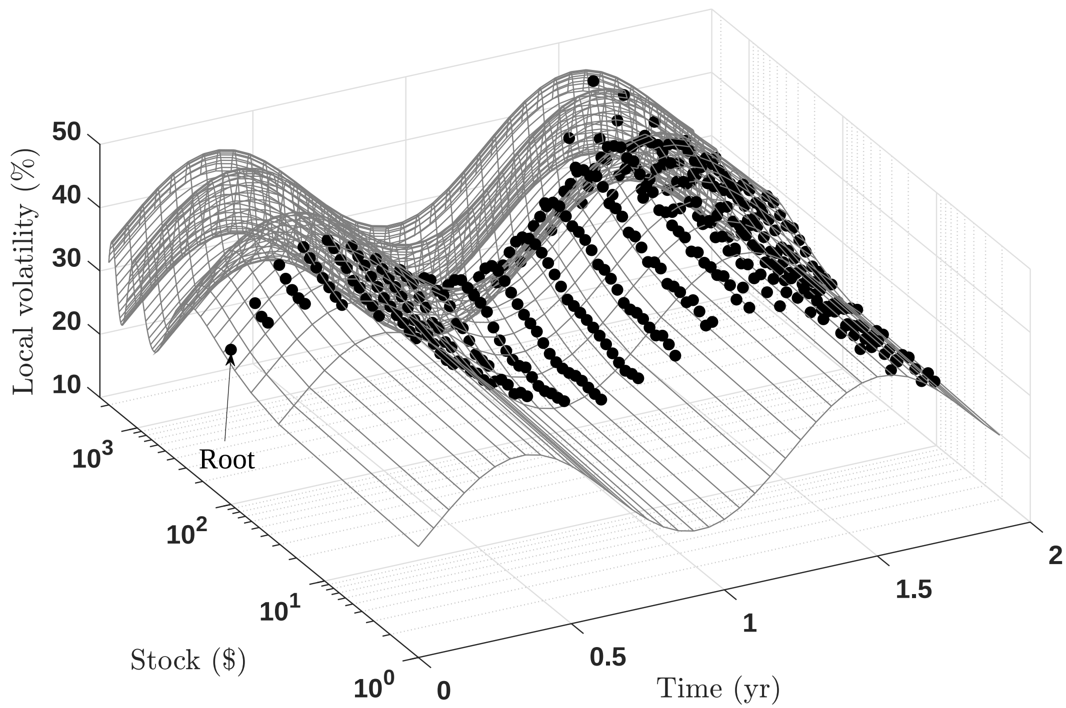
$$\sigma(S, t) = \sigma_1(S)\, \sigma_2(t).$$

- A solution is available for *any* range-bounded σ.[c]
- A combining implied trinomial tree can also be obtained from double-barrier options.[d]

---

[a] Lok (D99922028) & Lyuu (2015, 2016, 2017).
[b] Brace, Gątarek, & Musiela (1997); Rebonato (2004).
[c] Lok (D99922028) & Lyuu (2016, 2017, 2020, 2021).
[d] B. C. Chen (R09922147) (2022); Lok (D99922028) & Lyuu (2024).

Implied Trees[a] (concluded)

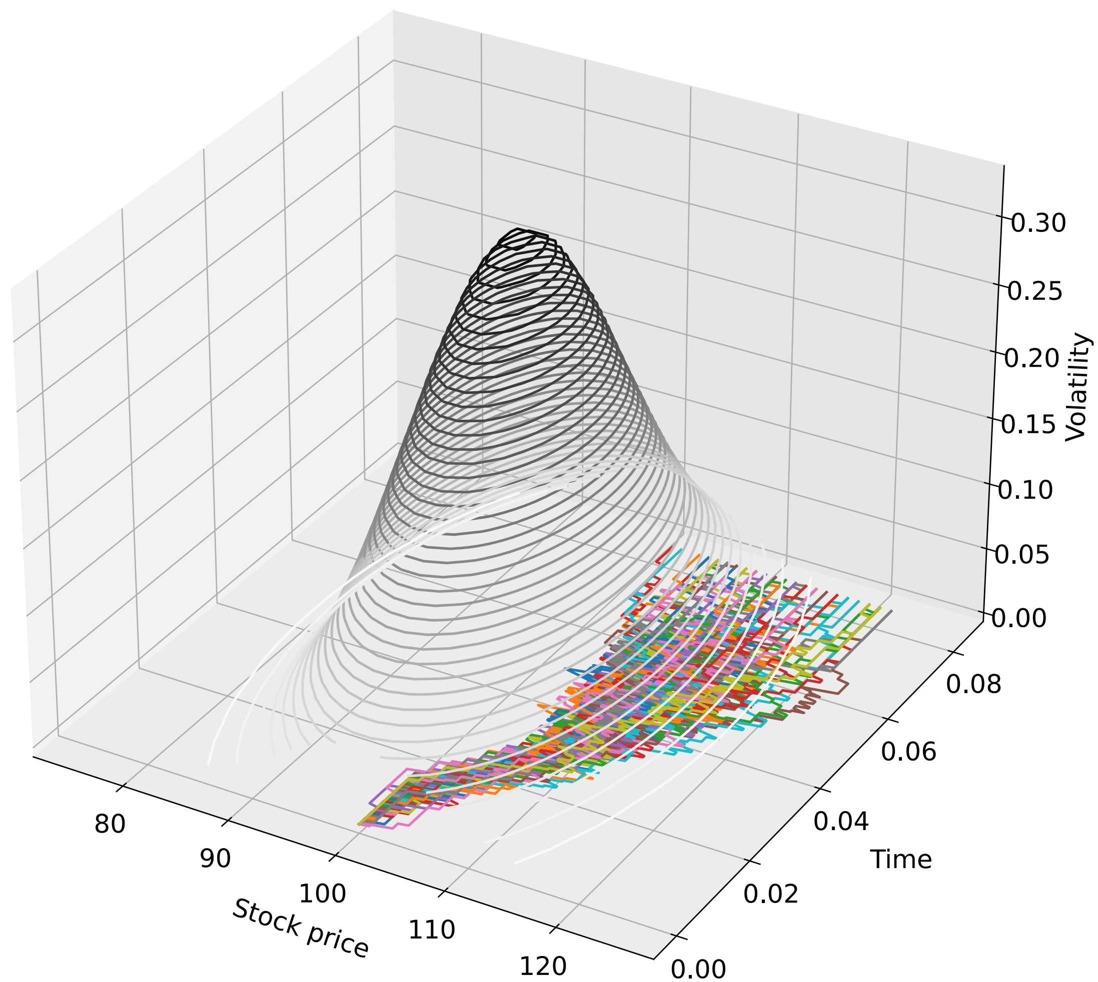


[a]Plot supplied by Prof. Lok, U Hou (D99922028) on May 4, 2019.

Delta Hedge under the Local-Volatility Model

- Delta by the implied tree differs from delta by the Black-Scholes model's implied volatility.
  - The latter is by formula (46) or (47) (p. 349) after calculating the implied volatility from the same option price by the implied tree.
- Hence the profits and losses of their delta hedges will differ.
- The next plot shows the best 100 out of 100,000 random paths where the implied tree delta outperforms the Black-Scholes delta.[a]

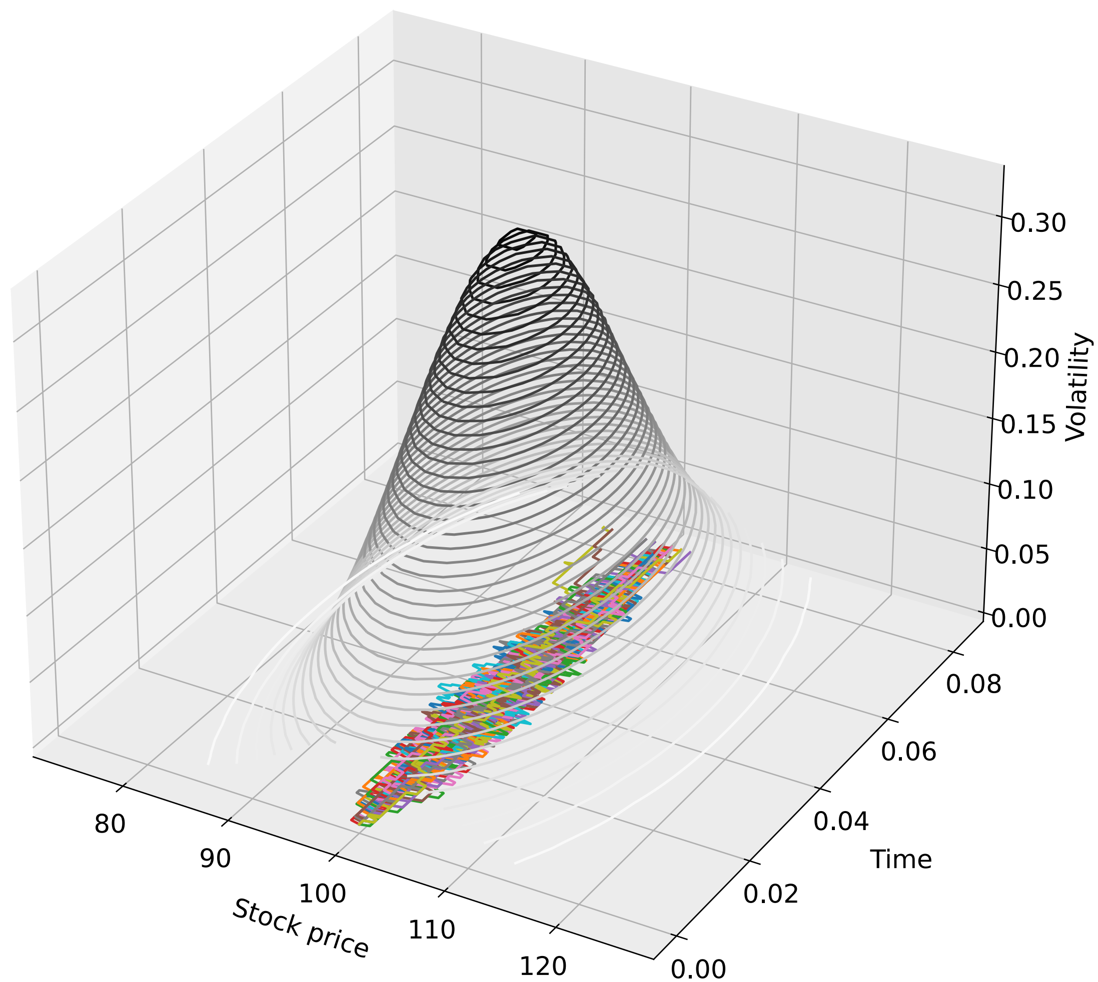
[a]In terms of profits and losses. Plot supplied by Mr. Chiu, Tzu-Hsuan (R08723061) on November 20, 2021 when hedging a long call.


0.30
0.25
0.20
0.15
0.10
0.05
0.00
Volatility
80
90
100
110
120
Stock price
0.00
0.02
0.04
0.06
0.08
Time

Delta Hedge under the Local-Volatility Model (concluded)

- The next plot shows the best 100 out of 100,000 random paths where the Black-Scholes delta outperforms the implied tree delta.[a]

[a] Plot supplied by Mr. Chiu, Tzu-Hsuan (`R08723061`) on November 20, 2021 when again hedging a long call.


0.30
0.25
0.20
0.15
0.10
0.05
0.00
Volatility
0.08
0.06
0.04
0.02
0.00
Time
80
90
100
110
120
Stock price

The Hull-White Model

- Hull and White (1987) postulate the following *stochastic-volatility* model,

$$\begin{aligned}\frac{dS}{S} &= r\,dt + \sqrt{V}\,dW_1,\\ dV &= \mu_{\mathrm{v}} V\,dt + bV\,dW_2.\end{aligned}$$

- Above, V is the instantaneous variance.
- They assume μ_{v} depends on V and t (but not S).

The Barone-Adesi–Rasmussen–Ravanelli Model

- Barone-Adesi, Rasmussen, and Ravanelli (2005) postulate the following model,

$$\begin{aligned} \frac{dS}{S} &= \mu\, dt + \sqrt{V}\, dW_1, \\ dV &= \kappa(\theta - V)\, dt + bV\, dW_2. \end{aligned}$$

- Above, W_1 and W_2 are correlated.

The Stein-Stein Model

- E. Stein and J. Stein (1991) postulate the following model,

$$\begin{aligned} \frac{dS}{S} &= r\,dt + V\,dW_1, \\ dV &= \kappa(\mu - V)\,dt + \sigma\,dW. \end{aligned}$$

- Closed-form formulas exist for European calls and puts.[a]

[a] Schöbel & J. Zhu (1999).

The SABR Model

- Hagan, Kumar, Lesniewski, and Woodward (2002) postulate the following model,

$$\frac{dS}{S} = r\,dt + S^{\theta} V\,dW_1,$$
$$dV = bV\,dW_2,$$

for $0 \leq \theta \leq 1$.

- A nice feature of this model is that the implied volatility surface has a compact, approximate closed form.

The Blacher Model

- Blacher (2001) postulates the following model,

$$\begin{aligned}\frac{dS}{S} &= r\,dt + \sigma\left[\,1 + \alpha(S - S_0) + \beta(S - S_0)^2\,\right] dW_1,\\ d\sigma &= \kappa(\theta - \sigma)\,dt + \epsilon\sigma\,dW_2.\end{aligned}$$

- The volatility σ follows a mean-reverting process to level θ.

The Hilliard-Schwartz Model

- Hilliard and Schwartz (1996) postulate the following very general model,

$$\frac{dS}{S} = r\,dt + f(S)V^a\,dW_1,$$
$$dV = \mu(V)\,dt + bV\,dW_2,$$

for some well-behaved function $f(S)$ and constant a.

- It includes all previously mentioned stochastic-volatility models as special cases.[a]

[a]H. Chiu (R98723059) (2012).

Heston's Stochastic-Volatility Model

- Heston (1993) assumes the stock price follows

$$\frac{dS}{S} = (\mu - q)\,dt + \sqrt{V}\,dW_1, \tag{92}$$

$$dV = \kappa(\theta - V)\,dt + \sigma\sqrt{V}\,dW_2. \tag{93}$$

  – V is the instantaneous variance, which follows a square-root process.

  – dW_1 and dW_2 have correlation ρ.

  – The riskless rate r is constant.

- It may be the most popular continuous-time stochastic-volatility model.[a]

[a] Christoffersen, Heston, & Jacobs (2009).

Heston's Stochastic-Volatility Model (continued)

- Heston assumes the market price of risk is $b_2\sqrt{V}$.
- So $\mu = r + b_2 V$.
- Define

$$
\begin{aligned}
dW_1^* &= dW_1 + b_2\sqrt{V}\,dt,\\
dW_2^* &= dW_2 + \rho b_2\sqrt{V}\,dt,\\
\kappa^* &= \kappa + \rho b_2\sigma,\\
\theta^* &= \frac{\theta\kappa}{\kappa + \rho b_2\sigma}.
\end{aligned}
$$

- dW_1^* and dW_2^* have correlation ρ.

Heston's Stochastic-Volatility Model (continued)

- Under the risk-neutral probability measure Q, both W_1^* and W_2^* are Wiener processes.
- Heston's model becomes, under probability measure Q,

$$\begin{aligned} \frac{dS}{S} &= (r-q)\,dt + \sqrt{V}\,dW_1^*, \\ dV &= \kappa^*(\theta^* - V)\,dt + \sigma\sqrt{V}\,dW_2^*. \end{aligned}$$

- The boundary conditions can be intricate.[a]

[a] S.-P. Zhu & C.-Y. Liu (2024).

Heston's Stochastic-Volatility Model (continued)

- Define

$$\begin{aligned}
\phi(u,\tau) &= \exp\Big\{\, \imath u(\ln S + (r-q)\,\tau) \\
&\quad + \theta^* \kappa^* \sigma^{-2} \left[(\kappa^* - \rho\sigma u\imath - d)\,\tau - 2\ln \frac{1-ge^{-d\tau}}{1-g} \right] \\
&\quad + \frac{v\sigma^{-2}(\kappa^* - \rho\sigma u\imath - d)\left(1-e^{-d\tau}\right)}{1-ge^{-d\tau}} \Big\}, \\
d &= \sqrt{(\rho\sigma u\imath - \kappa^*)^2 - \sigma^2(-\imath u - u^2)}\,, \\
g &= (\kappa^* - \rho\sigma u\imath - d)/(\kappa^* - \rho\sigma u\imath + d).
\end{aligned}$$

Heston's Stochastic-Volatility Model (continued)

The formulas for European calls and puts are[a]

$$
\begin{aligned}
C &= S\left[\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{X^{-\imath u}\phi(u-\imath,\tau)}{\imath u S e^{r\tau}}\right)du\right] \\
&\quad -Xe^{-r\tau}\left[\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{X^{-\imath u}\phi(u,\tau)}{\imath u}\right)du\right], \\
P &= Xe^{-r\tau}\left[\frac{1}{2}-\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{X^{-\imath u}\phi(u,\tau)}{\imath u}\right)du\right], \\
&\quad -S\left[\frac{1}{2}-\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{X^{-\imath u}\phi(u-\imath,\tau)}{\imath u S e^{r\tau}}\right)du\right],
\end{aligned}
$$

where $\imath=\sqrt{-1}$ and $\mathrm{Re}(x)$ denotes the real part of the complex number x.

[a]Contributed by Mr. Chen, Chun-Ying (D95723006) on August 17, 2008 and Mr. Liou, Yan-Fu (R92723060) on August 26, 2008. See Lord & Kahl (2009) and Cui, Rollin, & Germano (2017) for alternative formulas.

Heston's Stochastic-Volatility Model (concluded)

- For American options, trees are needed.
- They are all $O(n^3)$-sized and do not match all moments.[a]
- An $O(n^{2.5})$-sized 9-jump tree that matches *all* means and variances with valid probabilities is available.[b]
- The size reduces to $O(n^2)$ for knock-out double-barrier options.[c]

---

[a] Nelson & Ramaswamy (1990); Nawalkha & Beliaeva (2007); Leisen (2010); Beliaeva & Nawalkha (2010); M. Chou (`R02723073`) (2015); M. Chou (`R02723073`) & Lyuu (2016).

[b] Z. Lu (`D00922011`) & Lyuu (2018).

[c] Z. Lu (`D00922011`) & Lyuu (2018).

Stochastic-Volatility Models and Further Extensions[a]

- How to explain the October 1987 crash?
  - The Dow Jones Industrial Average fell 22.61% on October 19, 1987 (called the Black Monday).
  - The CBOE S&P 100 Volatility Index (VXO) shot up to 150%, the highest VXO ever recorded.[b]
- Stochastic-volatility models require an implausibly high-volatility level prior to *and* after the crash.
  - Because the processes are continuous.
- Discontinuous jump models *in the asset price* can alleviate the problem somewhat.[c]

---

[a] Eraker (2004).
[b] Caprio (2012).
[c] Merton (1976).

Stochastic-Volatility Models and Further Extensions (continued)

- But if the jump intensity is a constant, it cannot explain the tendency of large movements to cluster over time.
- This assumption also has no impacts on option prices.
- Jump-diffusion models combine both.
  - E.g., add a jump process to Eq. (92) on p. 667.
  - Closed-form formulas exist for GARCH-jump option pricing models.[a]

---

[a] Liou (R92723060) (2005).

Stochastic-Volatility Models and Further Extensions (concluded)

- But they still do not adequately describe the systematic variations in option prices.[a]
- Jumps *in volatility* are alternatives.[b]
  - E.g., add correlated jump processes to Eqs. (92) *and* Eq. (93) on p. 667.
- Such models allow high level of volatility caused by a jump to volatility.[c]

---

[a] Bates (2000); Pan (2002).
[b] Duffie, Pan, & Singleton (2000).
[c] Eraker, Johnnes, & Polson (2000); Y. Lin (2007); S.-P. Zhu & Lian (2012).

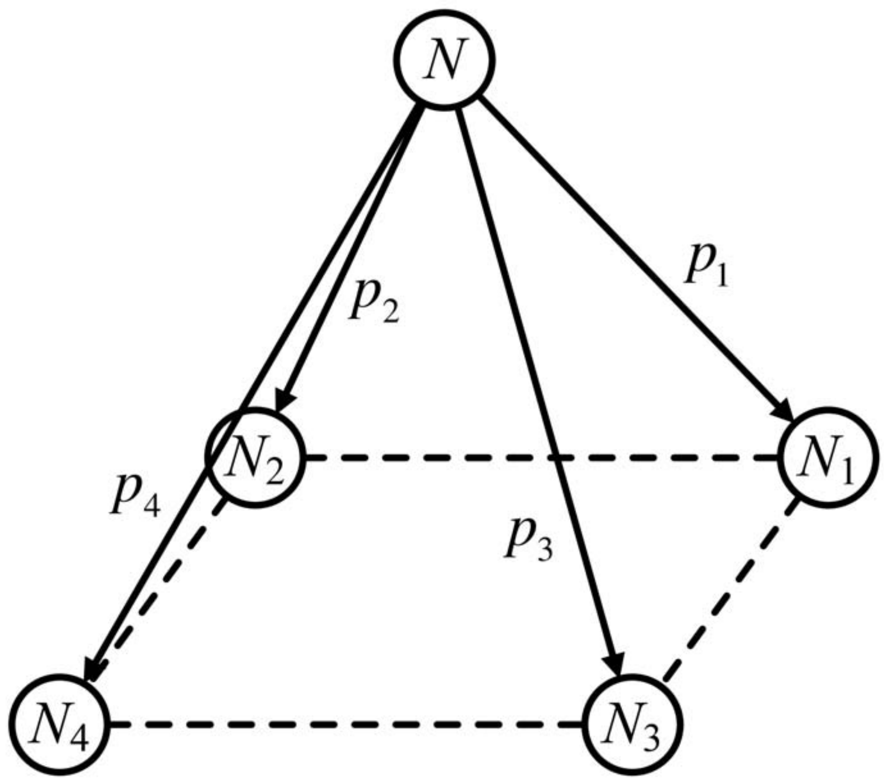
Why Are Trees for Stochastic-Volatility Models Difficult?

- The CRR tree is 2-dimensional.[a]
- The constant volatility makes the span from any node fixed.
- But a tree for a stochastic-volatility model must be 3-dimensional.
  - Every node is associated with a combination of stock price *and* volatility.

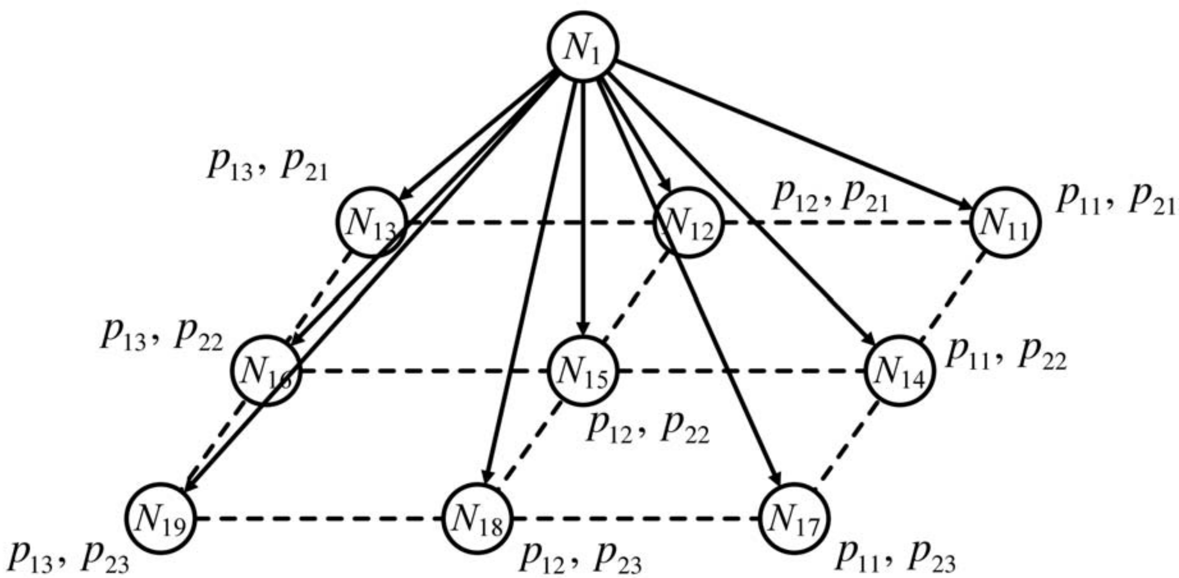
---

[a]Recall p. 302.

Why Are Trees for Stochastic-Volatility Models Difficult (Binomial Case)?



Why Are Trees for Stochastic-Volatility Models Difficult (Trinomial Case)?



Why Are Trees for Stochastic-Volatility Models Difficult? (concluded)

- *Locally*, the tree looks fine for one time step.
- But the volatility regulates the spans of the nodes on the stock-price plane.
- Unfortunately, those spans differ from node to node because the volatility varies.
- So two time steps from now, the branches will not combine!
- Smart ideas are thus needed.

Complexities of Stochastic-Volatility Models

- A few stochastic-volatility models suffer from subexponential ($c^{\sqrt{n}}$) tree size.
- Examples include the Hull-White (1987), Hilliard-Schwartz (1996), and SABR (2002) models.[a]
- Future research may extend this negative result to more stochastic-volatility models.
  - We suspect many GARCH option pricing models entertain similar problems.[b]

---

[a] H. Chiu (R98723059) (2012).

[b] Y. C. Chen (R95723051) (2008); Y. C. Chen (R95723051), Lyuu, & Wen (D94922003) (2011).

Complexities of Stochastic-Volatility Models (concluded)

- Flexible placement of nodes and removal of low-probability nodes may make the models $O(n^{2.5})$-sized![a]
- Calibration can be computationally hard.
  - Few have tried it on exotic options.[b]
- There are usually several local minima.[c]
  - They will give different prices to options not used in the calibration.
  - But which set capture the smile dynamics?

[a] Z. Lu (D00922011) & Lyuu (2018).
[b] Ayache, Henrotte, Nassar, & X. Wang (2004).
[c] Ayache (2004).

Continuous-Time Derivatives Pricing

I have hardly met a mathematician
who was capable of reasoning.
— Plato (428 B.C.–347 B.C.)

Fischer [Black] is the only real genius
I've ever met in finance. Other people,
like Robert Merton or Stephen Ross,
are just very smart and quick,
but they think like me.
Fischer came from someplace else entirely.
— John C. Cox, quoted in Mehrling (2005)

Toward the Black-Scholes Differential Equation

- The price of any derivative on a non-dividend-paying stock must satisfy a partial differential equation (PDE).
- The key step is recognizing that the same random process drives both securities.
  - Their prices are perfectly correlated.
- We then figure out the amount of stock such that the gain from it offsets exactly the loss from the derivative.
- The removal of uncertainty forces the portfolio's return to be the riskless rate.
- PDEs make many numerical methods applicable.

Assumptions[a] and Notations

- The stock price follows $dS = \mu S\, dt + \sigma S\, dW$.
- There are no dividends.
- Trading is continuous, and short selling is allowed.
- There are no transactions costs or taxes.
- All securities are infinitely divisible.
- The term structure of riskless rates is flat at r.
- There is unlimited riskless borrowing and lending.
- t is the current time, T is the expiration time, and $\tau \stackrel{\Delta}{=} T - t$.

[a]Derman & Taleb (2005) summarizes criticisms on these assumptions and the replication argument.

Black-Scholes Differential Equation

- Let C be the price of a *simple* derivative[a] on S.
- From Ito's lemma (p. 615),

$$dC = \left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S} dW.$$

  - The same W drives both C and S.
  - Unlike dS/S, the diffusion of dC/C is stochastic!
- Short one derivative and long $\partial C/\partial S$ shares of stock (call it Π).
- By construction,

$$\Pi = -C + S(\partial C/\partial S).$$

[a] Recall p. 439.

Black-Scholes Differential Equation (continued)

- The change in the value of the portfolio at time dt is[a]

$$d\Pi = -dC + \frac{\partial C}{\partial S}\, dS. \tag{94}$$

- Substitute the formulas for dC and dS into the above to yield

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} \right) dt.$$

- As this equation does not involve dW, the portfolio is riskless during dt time: $d\Pi = r\Pi\, dt$.

---

[a]Bergman (1982) and Bartels (1995) argue this is not quite right. But see Macdonald (1997). Mathematically, it is wrong (Bingham & Kiesel, 2004).

Black-Scholes Differential Equation (continued)

- So

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 C}{\partial S^2}\right) dt = r\left(C - S\,\frac{\partial C}{\partial S}\right) dt.$$

- Equate the terms to finally obtain[a]

$$\frac{\partial C}{\partial t} + rS\,\frac{\partial C}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 C}{\partial S^2} = rC.$$

- This is a backward equation, which describes the dynamics of a derivative's price *forward* in physical time.

[a]Known as the Feynman-Kac stochastic representation formula.

Black-Scholes Differential Equation (concluded)

- When there is a dividend yield q,

$$\frac{\partial C}{\partial t} + (r - q)\, S\, \frac{\partial C}{\partial S} + \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} = rC. \tag{95}$$

- Dupire's formula[a] (91) for the local-volatility model is simply its dual:[b]

$$\frac{\partial C}{\partial T} + (r_T - q_T) X \frac{\partial C}{\partial X} - \frac{1}{2}\, \sigma(X, T)^2 X^2 \frac{\partial^2 C}{\partial X^2} = -q_T C.$$

- This is a forward equation, which describes the dynamics of a derivative's price *backward* in maturity time.

---

[a]Recall p. 645.

[b]Derman & Kani (1997).

Rephrase

- The Black-Scholes differential equation can be expressed in terms of sensitivity numbers,

$$\Theta + rS\Delta + \frac{1}{2}\,\sigma^2 S^2 \Gamma = rC. \tag{96}$$

- Identity (96) leads to an alternative way of computing Θ numerically from Δ and Γ.
- When a portfolio is delta-neutral,

$$\Theta + \frac{1}{2}\,\sigma^2 S^2 \Gamma = rC.$$

  - A definite relation thus exists between Γ and Θ.

[Black] got the equation [in 1969] but then
was unable to solve it. Had he been a better
physicist he would have recognized it as a form
of the familiar heat exchange equation,
and applied the known solution. Had he been
a better mathematician, he could have
solved the equation from first principles.
Certainly Merton would have known exactly
what to do with the equation
had he ever seen it.
— Perry Mehrling (2005)

