Futures Price under the BOPM

- Futures prices form a martingale under the risk-neutral probability p_{f}.[a]
  - The expected futures price in the next period is

$$p_{\mathrm{f}} F u + (1 - p_{\mathrm{f}})\, F d = F \left(\frac{1-d}{u-d}\, u + \frac{u-1}{u-d}\, d \right) = F.$$

- Can be generalized to

$$F_i = E_i^{\pi}[\, F_k \,], \quad i \leq k,$$

  where F_i is the futures price at time i.

- This equation holds under stochastic interest rates, too.[b]

[a] Recall Eq. (71) on p. 526.

[b] See Exercise 13.2.11 of the textbook.

Futures Price under the BOPM (concluded)

- Futures prices do *not* form a martingale under the risk-neutral probability $p = (R - d)/(u - d)$.[a]
  - The expected futures price in the next period equals

$$
\begin{aligned}
& Fu\,\frac{R-d}{u-d} + Fd\,\frac{u-R}{u-d} \\
= \;& F\,\frac{uR-ud}{u-d} + F\,\frac{ud-Rd}{u-d} \\
= \;& FR.
\end{aligned}
$$

[a] Recall Eq. (34) on p. 257.

Martingale Pricing and Numeraire[a]

- The martingale pricing formula (77) on p. 567 uses the money market account as numeraire.
  - It expresses the price of any asset *relative to* the money market account.[b]
- The money market account is not the only choice for numeraire.
- Suppose asset S's value is *positive* at all time.

---

[a] John Law (1671–1729), "Money to be qualified for exchaning goods and for payments need not be certain in its value."

[b] Leon Walras (1834–1910).

Martingale Pricing and Numeraire (concluded)

- Choose S as numeraire.
- Martingale pricing says there exists a risk-neutral probability π under which the relative price of any asset C is a martingale:

$$\frac{C(i)}{S(i)} = E_i^{\pi}\left[\frac{C(k)}{S(k)}\right], \quad i \leq k.$$

  - $S(j)$ denotes the price of S at time j.
- So the "discount" process remains a martingale.[a]

[a]This result is related to Girsanov's theorem (1960).

Example

- Take the binomial model with two assets.
- In a period, asset one's price can go from S to S_1 or S_2.
- In a period, asset two's price can go from P to P_1 or P_2.
- Both assets must move up or down at the same time.
- Assume
$$\frac{S_1}{P_1} < \frac{S}{P} < \frac{S_2}{P_2} \tag{78}$$
to rule out arbitrage opportunities.

Example (continued)

- For any derivative security, let C_1 be its price at time one if asset one's price moves to S_1.
- Let C_2 be its price at time one if asset one's price moves to S_2.
- Replicate the derivative by solving

$$\begin{aligned} \alpha S_1 + \beta P_1 &= C_1, \\ \alpha S_2 + \beta P_2 &= C_2, \end{aligned}$$

using α units of asset one and β units of asset two.

Example (continued)

- By inequalities (78) on p. 574, α and β have unique solutions.
- In fact,

$$\alpha = \frac{P_2 C_1 - P_1 C_2}{P_2 S_1 - P_1 S_2} \quad \text{and} \quad \beta = \frac{S_2 C_1 - S_1 C_2}{S_2 P_1 - S_1 P_2}.$$

- The derivative costs

$$\begin{aligned} C &= \alpha S + \beta P \\ &= \frac{P_2 S - P S_2}{P_2 S_1 - P_1 S_2} C_1 + \frac{P S_1 - P_1 S}{P_2 S_1 - P_1 S_2} C_2. \end{aligned}$$

Example (continued)

- It is easy to verify that

$$\frac{C}{P} = p\,\frac{C_1}{P_1} + (1-p)\,\frac{C_2}{P_2}$$

with

$$p \triangleq \frac{(S/P) - (S_2/P_2)}{(S_1/P_1) - (S_2/P_2)}.$$

- By inequalities (78) on p. 574, $0 < p < 1$.
- C's price using asset two as numeraire (i.e., C/P) is a martingale under the risk-neutral probability p.
- The expected returns of the two assets are *irrelevant*.

Example (concluded)

- In the BOPM, S is the stock and P is the bond.
- Furthermore, p assumes the bond is the numeraire.
- In the binomial option pricing formula (39) on p. 277, $S \sum b(j; n, pu/R)$ uses *stock* as the numeraire.
  - Its probability measure pu/R differs from p.
- $SN(x)$ for the call and $SN(-x)$ for the put in the Black-Scholes formulas (p. 308) use stock as the numeraire as well.[a]

[a]See Exercise 13.2.12 of the textbook.

Brownian Motion[a]

- Brownian motion is a stochastic process $\{ X(t), t \geq 0 \}$ with the following properties.

  1. $X(0) = 0$, unless stated otherwise.

  2. for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables

  $$X(t_k) - X(t_{k-1})$$

  for $1 \leq k \leq n$ are independent.[b]

  3. for $0 \leq s < t$, $X(t) - X(s)$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$, where μ and $\sigma \neq 0$ are real numbers.

---

[a] Robert Brown (1773–1858).

[b] So $X(t) - X(s)$ is independent of $X(r)$ for $r \leq s < t$.

Brownian Motion (concluded)

- The existence and uniqueness of such a process is guaranteed by Wiener's theorem.[a]
- This process will be called a (μ, σ) Brownian motion with drift μ and variance σ^2.
- Although Brownian motion is a continuous function of t with probability one, it is almost nowhere differentiable.
- The $(0, 1)$ Brownian motion is called the Wiener process.
- If condition 3 is replaced by "$X(t) - X(s)$ depends only on $t - s$," we have the more general Levy process.[b]

---

[a] Norbert Wiener (1894–1964). He received his Ph.D. from Harvard in 1912.

[b] Paul Levy (1886–1971).

Example

- If $\{ X(t), t \geq 0 \}$ is the Wiener process, then

$$X(t) - X(s) \sim N(0, t - s).$$

- A (μ, σ) Brownian motion $Y = \{ Y(t), t \geq 0 \}$ can be expressed in terms of the Wiener process:

$$Y(t) = \mu t + \sigma X(t). \tag{79}$$

- Note that

$$Y(t+s) - Y(t) \sim N(\mu s, \sigma^2 s).$$

Brownian Motion as Limit of Random Walk

Claim 1 *A* (μ, σ) *Brownian motion is the limiting case of random walk.*

- A particle moves Δx to the right with probability p after Δt time.
- It moves Δx to the left with probability $1 - p$.
- Define

$$X_i \triangleq \begin{cases} +1 & \text{if the } i\text{th move is to the right,} \\ -1 & \text{if the } i\text{th move is to the left.} \end{cases}$$

  - X_i are independent with

$$\text{Prob}[\, X_i = 1\,] = p = 1 - \text{Prob}[\, X_i = -1\,].$$

Brownian Motion as Limit of Random Walk (continued)

- Recall

$$\begin{aligned} E[\,X_i\,] &= 2p-1, \\ \mathrm{Var}[\,X_i\,] &= 1-(2p-1)^2. \end{aligned}$$

- Assume $n \triangleq t/\Delta t$ is an integer.
- Its position at time t is

$$Y(t) \triangleq \Delta x\,(X_1 + X_2 + \cdots + X_n)\,.$$

Brownian Motion as Limit of Random Walk (continued)

- Therefore,

$$E[\,Y(t)\,] = n(\Delta x)(2p-1),$$

$$\mathrm{Var}[\,Y(t)\,] = n(\Delta x)^2\left[\,1-(2p-1)^2\,\right].$$

- With $\Delta x \stackrel{\Delta}{=} \sigma\sqrt{\Delta t}$ and $p \stackrel{\Delta}{=} [\,1+(\mu/\sigma)\sqrt{\Delta t}\,]/2$,[a]

$$\begin{aligned} E[\,Y(t)\,] &= n\sigma\sqrt{\Delta t}\,(\mu/\sigma)\sqrt{\Delta t} = \mu t, \\ \mathrm{Var}[\,Y(t)\,] &= n\sigma^2\Delta t\left[\,1-(\mu/\sigma)^2\Delta t\,\right] \to \sigma^2 t, \end{aligned}$$

as $\Delta t \to 0$.

---

[a]Identical to Eq. (42) on p. 300!

Brownian Motion as Limit of Random Walk (concluded)

- Thus, $\{ Y(t), t \geq 0 \}$ converges to a (μ, σ) Brownian motion by the central limit theorem.
- Brownian motion with zero drift is the limiting case of symmetric random walk by choosing $\mu = 0$.
- Similarity to the the BOPM: The p is identical to the probability in Eq. (42) on p. 300 and $\Delta x = \ln u$.
- Note that

$$\begin{aligned} & \operatorname{Var}[\, Y(t + \Delta t) - Y(t) \,] \\ = & \operatorname{Var}[\, \Delta x \, X_{n+1} \,] = (\Delta x)^2 \times \operatorname{Var}[\, X_{n+1} \,] \to \sigma^2 \Delta t. \end{aligned}$$

Geometric Brownian Motion

- Let $X \stackrel{\Delta}{=} \{ X(t), t \geq 0 \}$ be a Brownian motion process.
- The process

$$\{ Y(t) \stackrel{\Delta}{=} e^{X(t)}, t \geq 0 \},$$

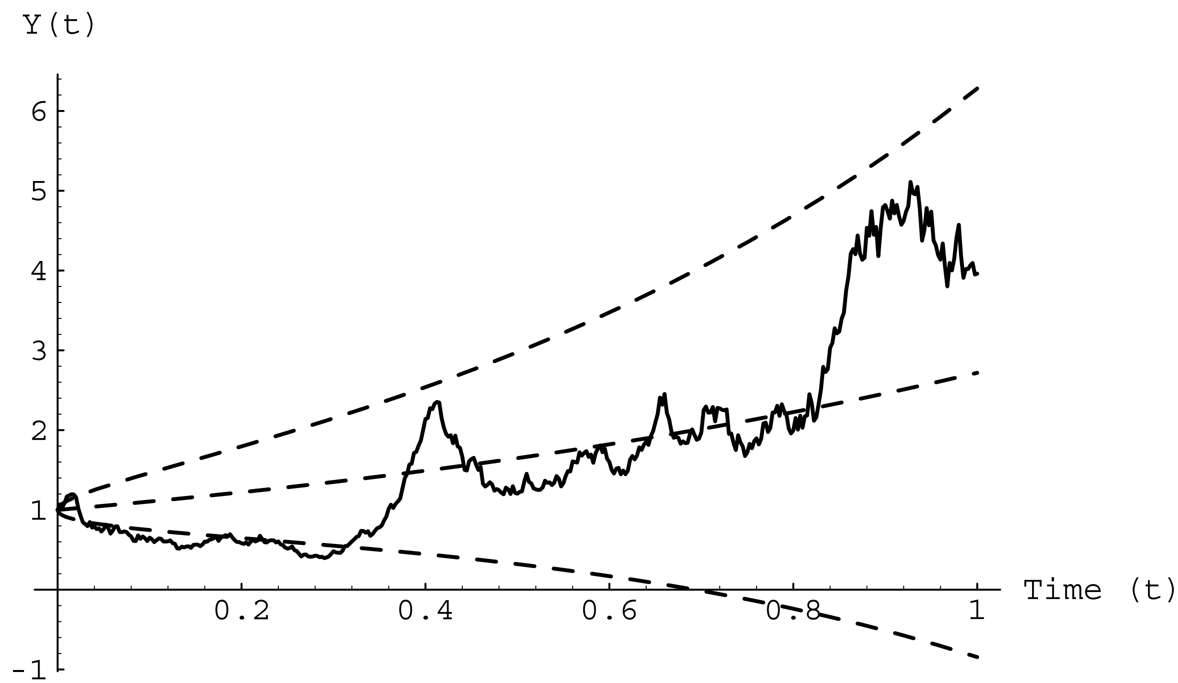
  is called geometric Brownian motion.
- Suppose further that X is a (μ, σ) Brownian motion.
- By assumption, $Y(0) = e^0 = 1$.

Geometric Brownian Motion (concluded)

- $X(t) \sim N(\mu t, \sigma^2 t)$ with moment generating function
$$E\left[e^{sX(t)}\right] = E\left[\, Y(t)^s \,\right] = e^{\mu t s + (\sigma^2 t s^2/2)}$$
from Eq. (27) on p 173.

- In particular,[a]
$$\begin{aligned} E[\, Y(t)\,] &= e^{\mu t + (\sigma^2 t/2)}, \\ \mathrm{Var}[\, Y(t)\,] &= E\left[\, Y(t)^2 \,\right] - E[\, Y(t)\,]^2 \\ &= e^{2\mu t + \sigma^2 t}\left(e^{\sigma^2 t} - 1\right). \end{aligned}$$

[a] Recall Eqs. (29) on p. 182.


Y(t)
6
5
4
3
2
1
-1
0.2
0.4
0.6
0.8
1
Time (t)

An Argument for Long-Term Investment[a]

- Suppose the stock follows the geometric Brownian motion

$$S(t) = S(0)\, e^{N(\mu t, \sigma^2 t)} = S(0)\, e^{t N(\mu, \sigma^2/t\,)}, \quad t \geq 0,$$

where $\mu > 0$.

- The annual rate of return has a normal distribution:

$$N\left(\mu, \frac{\sigma^2}{t}\right).$$

- The larger the t, the likelier the return is positive.
- The smaller the t, the likelier the return is negative.

[a] Contributed by Dr. King, Gow-Hsing on April 9, 2015. See http://www.cb.idv.tw/phpbb3/viewtopic.php?f=7&t=1025

Continuous-Time Financial Mathematics

A proof is that which convinces a reasonable man;
a rigorous proof is that which convinces an
unreasonable man.
— Mark Kac (1914–1984)

The pursuit of mathematics is a
divine madness of the human spirit.
— Alfred North Whitehead (1861–1947),
Science and the Modern World

Stochastic Integrals

- Use $W \stackrel{\Delta}{=} \{ W(t), t \geq 0 \}$ to denote the Wiener process.
- The goal is to develop integrals of X from a class of stochastic processes,[a]

$$\mathrm{I}_t(X) \stackrel{\Delta}{=} \int_0^t X \, dW, \quad t \geq 0.$$

- $\mathrm{I}_t(X)$ is a random variable called the stochastic integral of X with respect to W.
- The stochastic process $\{ \mathrm{I}_t(X), t \geq 0 \}$ will be denoted by $\int X \, dW$.

[a]Kiyoshi Ito (1915–2008).

Stochastic Integrals (concluded)

- Typical requirements for X in financial applications are:
  - $\text{Prob}[\int_0^t X^2(s)\, ds < \infty] = 1$ for all $t \geq 0$ or the stronger $\int_0^t E[X^2(s)]\, ds < \infty$.
  - The information set at time t includes the history of X and W up to that point in time.
  - But it contains nothing about the evolution of X or W after t (nonanticipating, so to speak).
  - The future cannot influence the present.

Ito Integral

- A theory of stochastic integration.
- As with calculus, it starts with step functions.
- A stochastic process $\{ X(t) \}$ is simple if there exist

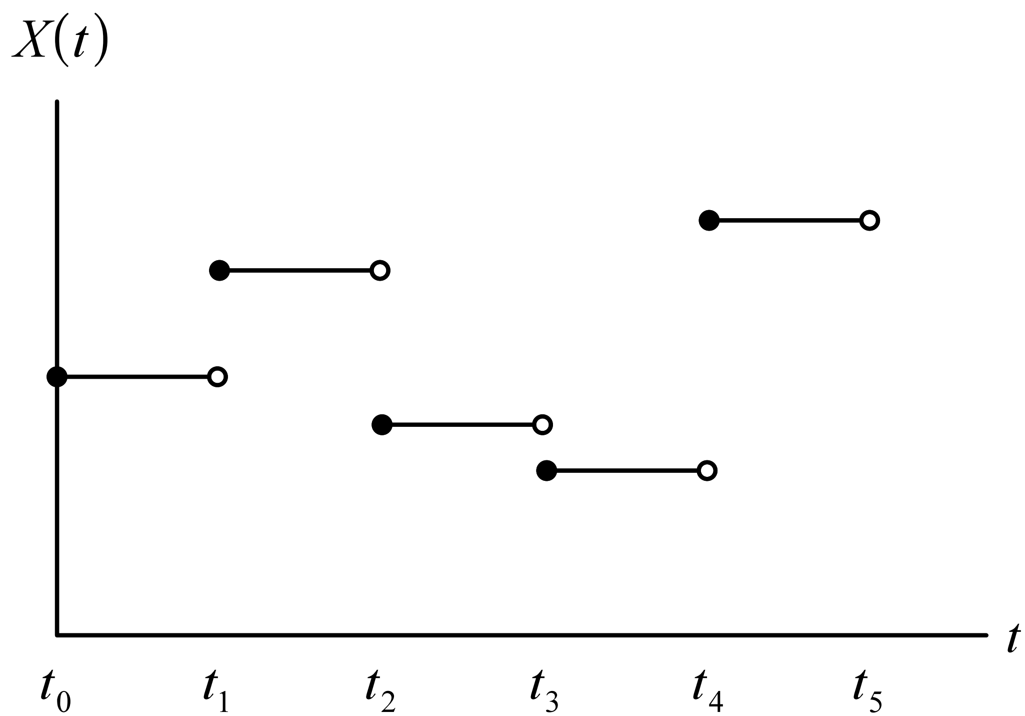
$$0 = t_0 < t_1 < \cdots$$

such that

$$X(t) = X(t_{k-1}) \text{ for } t \in [t_{k-1}, t_k),\ k = 1, 2, \ldots$$

for any realization (see figure on next page).[a]

[a]It is right-continuous.


X(t)
t
t0
t1
t2
t3
t4
t5

Ito Integral (continued)

- The Ito integral of a simple process is defined as

$$\mathrm{I}_t(X) \triangleq \sum_{k=0}^{n-1} X(t_k)[\,W(t_{k+1}) - W(t_k)\,], \tag{80}$$

  where $t_n = t$.

  – The integrand X is evaluated at t_k, not t_{k+1}.

- Define the Ito integral of more general processes as a limiting random variable of the Ito integral of simple stochastic processes.

Ito Integral (continued)

- Let $X = \{ X(t), t \geq 0 \}$ be a general stochastic process.
- There exists a random variable $\mathrm{I}_t(X)$, unique almost certainly, such that $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ for each sequence of simple stochastic processes $X_1, X_2, \ldots$ that X_n converge in probability to X.
- If X is continuous with probability one, then $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ as

$$\delta_n \stackrel{\Delta}{=} \max_{1 \leq k \leq n} (t_k - t_{k-1})$$

goes to zero.

Ito Integral (concluded)

- It is a fundamental fact that $\int X\, dW$ is continuous almost surely.
- The Ito integral is a martingale.[a]

Theorem 18 *The Ito integral* $\int X\, dW$ *is a martingale.*

- A corollary is the mean value formula
$$E\left[\int_a^b X\, dW\right] = 0.$$

[a]Exercise 14.1.1 covers simple stochastic processes.

Discrete Approximation and Nonanticipation

- Recall Eq. (80) on p. 596.
- The following *simple* stochastic process $\{\widehat{X}(t)\}$ can be used in place of X to approximate $\int_0^t X\,dW$,

$$\widehat{X}(s) \stackrel{\Delta}{=} X(t_{k-1}) \text{ for } s \in [\,t_{k-1}, t_k), \; k = 1, 2, \ldots, n.$$

- Note the *nonanticipating* feature of $\widehat{X}$.
  - The information up to time s,

$$\{\widehat{X}(t), W(t), 0 \le t \le s\},$$

    cannot determine the future evolution of X or W.

Discrete Approximation and Nonanticipation (concluded)

- Suppose, unlike Eq. (80) on p. 596, we defined the stochastic integral from

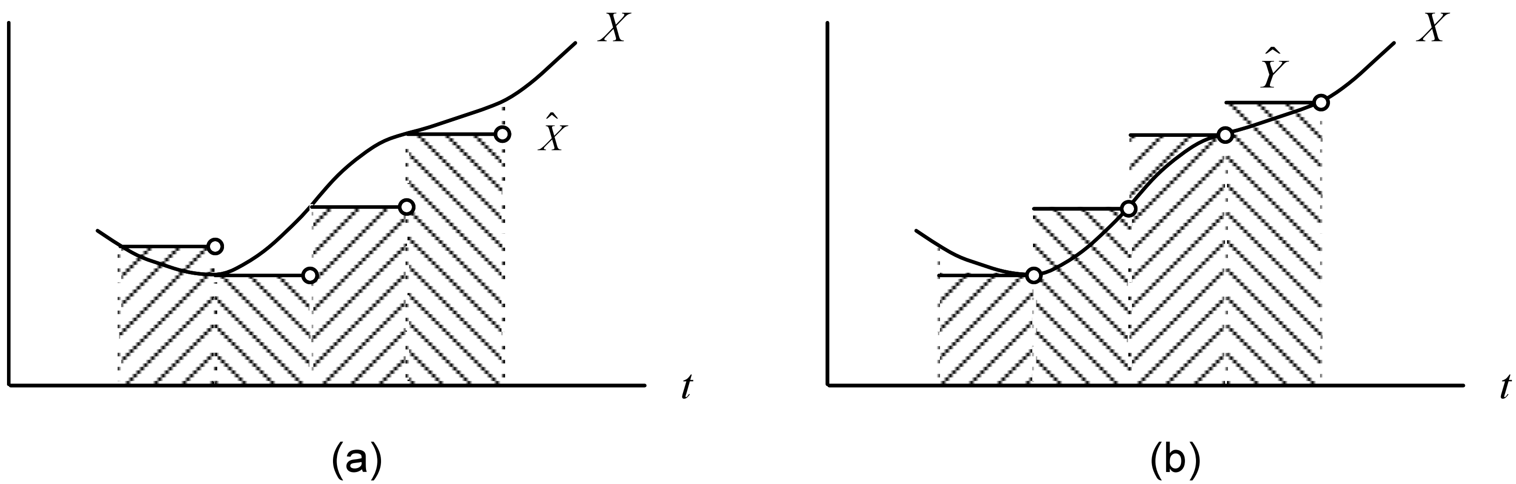
$$\sum_{k=0}^{n-1} X(t_{k+1})[\,W(t_{k+1}) - W(t_k)\,].$$

- Then we would be using the following different simple stochastic process in the approximation,

$$\widehat{Y}(s) \stackrel{\Delta}{=} X(t_k) \ \text{ for } \ s \in [\,t_{k-1}, t_k), \, k = 1, 2, \ldots, n.$$

- This clearly anticipates the future evolution of X.[a]

[a]See Exercise 14.1.2 for an example where it matters.



X
$\hat{X}$
t
(a)
X
$\hat{Y}$
t
(b)

Ito Process

- The stochastic process $X = \{ X_t, t \geq 0 \}$ that solves

$$X_t = X_0 + \int_0^t a(X_s, s)\, ds + \int_0^t b(X_s, s)\, dW_s, \quad t \geq 0$$

  is called an Ito process.

  – X_0 is a scalar starting point.

  – $\{ a(X_t, t) : t \geq 0 \}$ and $\{ b(X_t, t) : t \geq 0 \}$ are stochastic processes satisfying certain regularity conditions.

  – $a(X_t, t)$ is the drift.

  – $b(X_t, t)$ is the diffusion.

Ito Process (continued)

- Typical regularity conditions are:[a]
  - For all $T > 0$, $x \in \mathbb{R}^n$, and $0 \le t \le T$,

    $$| a(x,t) | + | b(x,t) | \le C(1 + | x |)$$

    for some constant C.[b]
  - (Lipschitz continuity) For all $T > 0$, $x \in \mathbb{R}^n$, and $0 \le t \le T$,

    $$| a(x,t) - a(y,t) | + | b(x,t) - b(y,t) | \le D | x - y |$$

    for some constant D.

---

[a] Øksendal (2007).

[b] This condition is not needed in *time-homogeneous* cases, where a and b do not depend on t.

Ito Process (continued)

- A shorthand[a] is the following stochastic differential equation[b] (SDE) for the Ito differential dX_t,

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t. \tag{81}$$

  - Or simply

$$dX_t = a_t\, dt + b_t\, dW_t.$$

  - This is Brownian motion with an *instantaneous* drift a_t and an *instantaneous* variance b_t^2.

- X is a martingale if $a_t = 0$.[c]

[a] Paul Langevin (1872–1946) in 1904.

[b] Like any equation, an SDE contains an unknown, the process X_t.

[c] Recall Theorem 18 (p. 598).

Ito Process (concluded)

- From calculus, we would expect $\int_0^t W\, dW = W(t)^2/2$.
- But $W(t)^2/2$ is not a martingale, hence wrong!
- The correct answer is $[\, W(t)^2 - t\,]/2$.
- A popular representation of Eq. (81) is

$$dX_t = a_t\, dt + b_t \sqrt{dt}\; \xi, \tag{82}$$

where $\xi \sim N(0, 1)$.

Euler Approximation

- Define $t_n \stackrel{\Delta}{=} n\Delta t$.
- The following approximation follows from Eq. (82),

$$
\begin{aligned}
&\widehat{X}(t_{n+1}) \\
=&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\,\Delta W(t_n). \qquad (83)
\end{aligned}
$$

- It is called the Euler or Euler-Maruyama method.
- Recall that $\Delta W(t_n)$ should be interpreted as

$$W(t_{n+1}) - W(t_n),$$

not $W(t_n) - W(t_{n-1})$![a]

[a] Recall Eq. (80) on p. 596.

Euler Approximation (concluded)

- With the Euler method, one can obtain a sample path

$$\widehat{X}(t_1), \widehat{X}(t_2), \widehat{X}(t_3), \ldots$$

from a sample path

$$W(t_0), W(t_1), W(t_2), \ldots.$$

- Under mild conditions, $\widehat{X}(t_n)$ converges to $X(t_n)$.

More Discrete Approximations

- Under fairly loose regularity conditions, Eq. (83) on p. 606 can be replaced by

$$\begin{aligned}&\widehat{X}(t_{n+1})\\ =&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\sqrt{\Delta t}\,Y(t_n).\end{aligned}$$

  - $Y(t_0), Y(t_1), \ldots$ are independent and identically distributed with zero mean and unit variance.

More Discrete Approximations (concluded)

- An even simpler discrete approximation scheme:

$$\begin{aligned}&\widehat{X}(t_{n+1})\\ =&\widehat{X}(t_n)+a(\widehat{X}(t_n),t_n)\,\Delta t+b(\widehat{X}(t_n),t_n)\sqrt{\Delta t}\,\xi.\end{aligned}$$

  - $\text{Prob}[\,\xi=1\,]=\text{Prob}[\,\xi=-1\,]=1/2$.
  - Note that $E[\,\xi\,]=0$ and $\text{Var}[\,\xi\,]=1$.
- This is a binomial model.
- As Δt goes to zero, $\widehat{X}$ converges to X.[a]

[a]He (1990).

Trading and the Ito Integral

- Consider an Ito process

$$d\boldsymbol{S}_t = \mu_t\, dt + \sigma_t\, dW_t.$$

  – $\boldsymbol{S}_t$ is the vector of security prices at time t.

- Let $\boldsymbol{\phi}_t$ be a trading strategy denoting the quantity of each type of security held at time t.

  – Hence the stochastic process $\boldsymbol{\phi}_t \boldsymbol{S}_t$ is the value of the portfolio $\boldsymbol{\phi}_t$ at time t.

- $\boldsymbol{\phi}_t\, d\boldsymbol{S}_t \stackrel{\Delta}{=} \boldsymbol{\phi}_t(\mu_t\, dt + \sigma_t\, dW_t)$ is the change in the portfolio value from the changes in security prices at time t.

Trading and the Ito Integral (concluded)

- The equivalent Ito integral,

$$G_T(\boldsymbol{\phi}) \triangleq \int_0^T \boldsymbol{\phi}_t \, d\boldsymbol{S}_t = \int_0^T \boldsymbol{\phi}_t \mu_t \, dt + \int_0^T \boldsymbol{\phi}_t \sigma_t \, dW_t,$$

measures the gains realized by the trading strategy over the period $[\,0, T\,]$.