Contract Value: The Underlying Pays No Income

The value of a forward contract is

$$f = S - Xe^{-r\tau}. \tag{62}$$

- Consider a portfolio consisting of:
  - One long forward contract;
  - Cash amount $Xe^{-r\tau}$;
  - One short position in the underlying asset.

Contract Value: The Underlying Pays No Income (concluded)

- The cash will grow to X at maturity, which can be used to take delivery of the forward contract.
- The delivered asset will then close out the short position.
- Since the value of the portfolio is zero at maturity, its PV must be zero.[a]
- So a forward contract can be replicated by a long position in the underlying and a loan of $Xe^{-r\tau}$ dollars.

---

[a]Recall p. 223.

Lemma 12 (p. 494) Revisited

- Set $f = 0$ in Eq. (62) on p. 498.
- Then $X = Se^{r\tau}$, the forward price.

Forward Price: Underlying Pays Predictable Income

Lemma 13 *For a forward contract on an underlying asset providing a predictable income with a PV of I,*

$$F = (S - I)\, e^{r\tau}. \tag{63}$$

- If $F > (S - I)\, e^{r\tau}$, borrow S dollars for τ years, buy the underlying asset, and short the forward contract with delivery price F.
- Use the income to repay part of the loan.

The Proof (concluded)

- At maturity, the asset is delivered for F, and $(S - I)\, e^{r\tau}$ is used to repay the *remaining* loan.
- That leaves an arbitrage profit of

$$F - (S - I)\, e^{r\tau} > 0.$$

- If $F < (S - I)\, e^{r\tau}$, reverse the above.

Example

- Consider a 10-month forward contract on a \$50 stock.
- The stock pays a dividend of \$1 every 3 months.
- The forward price is

$$\left(50 - e^{-r_3/4} - e^{-r_6/2} - e^{-3\times r_9/4}\right) e^{r_{10}\times(10/12)}.$$

  – r_i is the annualized i-month interest rate.

Underlying Pays a Continuous Dividend Yield of q

- The value of a forward contract at any time prior to T is[a]

$$f = Se^{-q\tau} - Xe^{-r\tau}. \tag{64}$$

- One consequence of Eq. (64) is that the forward price is

$$F = Se^{(r-q)\,\tau}. \tag{65}$$

---

[a]See p. 160 of the textbook for proof.

Futures Contracts vs. Forward Contracts

- They are traded on a central exchange.
- A clearinghouse.
  - Credit risk is minimized.
- Futures contracts are standardized instruments.
- Gains and losses are marked to market daily.
  - Adjusted at the end of each trading day based on the settlement price.

Size of a Futures Contract

- The amount of the underlying asset to be delivered under the contract.
  - 5,000 bushels for the corn futures on the Chicago Board of Trade (CBOT).
  - One million U.S. dollars for the Eurodollar futures on the Chicago Mercantile Exchange (CME).[a]
- A position can be closed out (or offset) by entering into a reversing trade to the original one.
- Most futures contracts are closed out in this way rather than have the underlying asset delivered.
  - Forward contracts are meant for delivery.

[a] CME and CBOT merged on July 12, 2007.

Daily Settlements

- Price changes in the futures contract are settled daily.
- Hence the spot price rather than the initial futures price is paid on the delivery date.
- Marking to market nullifies any financial incentive for not making delivery.
  - A farmer enters into a forward contract to sell 100,000 bushels of corn at \$2.00 per bushel in November.
  - Suppose the price of corn rises to \$2.5 by November.
  - The farmer has incentive to sell his harvest in the spot market for \$2.5.

Daily Settlements (concluded)

- (continued)
  - With marking to market, the farmer has transferred \$0.5 per bushel from his futures account to that of the clearinghouse by November.[a]
  - When the farmer makes delivery, he is paid the spot price, \$2.5 per bushel.
  - The farmer has little incentive to default.
  - The *net* price remains $\$2.5 - 0.5 = 2$ per bushel, the original delivery price.

---

[a]See p. 509.

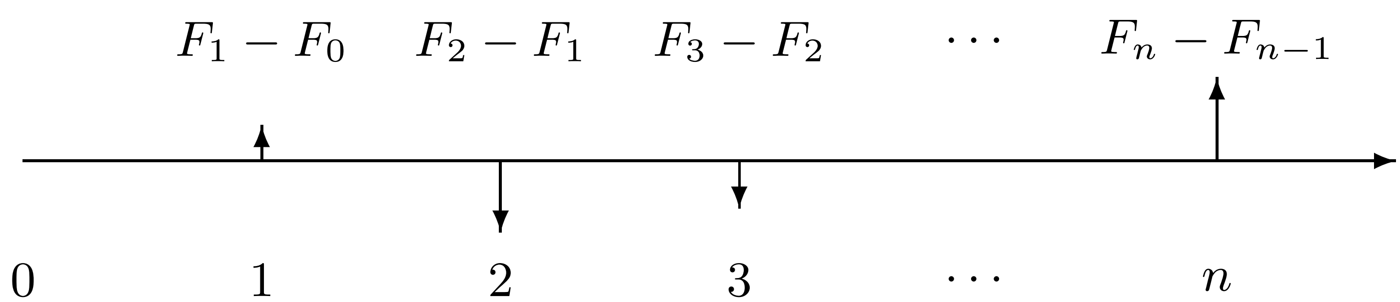
Daily Cash Flows

- Let F_i denote the futures price at the end of day i.
- The contract's cash flow on day i is $F_i - F_{i-1}$.
- The net cash flow over the life of the contract is

$$\begin{aligned}(F_1 - F_0) + (F_2 - F_1) + \cdots + (F_n - F_{n-1}) \\ = F_n - F_0 = S_T - F_0.\end{aligned} \tag{66}$$

- A futures contract has the same *accumulated* payoff $S_T - F_0$ as a forward contract.
- The actual payoff may vary because of the reinvestment of daily cash flows and how $S_T - F_0$ is distributed.

Daily Cash Flows (concluded)



Delivery and Hedging

- Futures price is the delivery price that makes the futures contract zero-valued.
- Delivery ties the futures price to the spot price.
- On the delivery date, the settlement price of the futures contract is determined by the spot price.
- Hence, when the delivery period is reached, the futures price should be very close to the spot price.[a]
- Changes in futures prices usually track those in spot price, making hedging possible.

---

[a] But since early 2006, futures for corn, wheat, and soybeans occasionally expired at a price much higher than that day's spot price (Henriques, 2008).

Forward and Futures Prices

- Surprisingly, futures price equals forward price if interest rates are nonstochastic![a]
- This result "justifies" treating a futures contract as if it were a forward contract, ignoring its marking-to-market feature.
- The West Texas Intermediate (WTI) futures price was negative on April 20, 2020![b]

---

[a]Cox, Ingersoll, & Ross (1981); see p. 164 of the textbook for proof.

[b]April 21 was the last trading day for oil delivery in May to Cushing, Oklahoma.

Remarks

- When interest rates are stochastic, forward and futures prices are no longer theoretically identical.
  - Suppose interest rates are uncertain and futures prices move in the same direction as interest rates.
  - Then futures prices will exceed forward prices.
- For short-term contracts, the differences tend to be small.
- Unless stated otherwise, assume forward and futures prices are identical.

Futures Options

- The underlying of a futures option is a futures contract.
- Upon exercise, the option holder takes a position in the futures contract with a futures price equal to the option's strike price.
  - A call holder acquires a *long* futures position.
  - A put holder acquires a *short* futures position.
- The futures contract is then marked to market.
- And the futures positions of the two parties are at the prevailing futures price (thus zero-valued).

Futures Options (concluded)

- It works as if the *call* holder received a futures contract plus cash equivalent to the prevailing futures price F_t minus the strike price X:

$$F_t - X.$$

  - This futures contract has zero value.

- It works as if the *put* holder sold a futures contract for

$$X - F_t$$

  dollars.

Forward Options

- What is delivered is now a forward contract with a delivery price equal to the option's strike price.
  - Exercising a call forward option results in a *long* position in a forward contract.
  - Exercising a put forward option results in a *short* position in a forward contract.
- Exercising a forward option incurs no immediate cash flows as there is no marking to market.

Example

- Consider an American call with strike \$100 and an expiration date in September.
- The underlying asset is a *forward* contract with a delivery date in December.
- Suppose the forward price in July is \$110.
- Upon exercise, the call holder receives a forward contract with a delivery price of \$100.[a]
- If an offsetting position is then taken in the forward market,[b] a \$10 profit *in December* will be assured.
- A call on the futures would realize the \$10 profit *in July*.

[a]Recall p. 486.

[b]The counterparty will pay you \$110 for the underlying asset.

Some Pricing Relations

- Let T be the delivery date, the current time be 0, and the option[a] have expiration date t $(t \leq T)$.
- Assume a constant, positive interest rate.
- Although forward price equals futures price, a forward option does *not* have the same value as a futures option.
- The payoffs of calls at time t are, respectively,[b]

$$\begin{aligned} \text{futures option} &= \max(F_t - X, 0), \quad (67) \\ \text{forward option} &= \max(F_t - X, 0)\, e^{-r(T-t)}. \quad (68) \end{aligned}$$

---

[a]On the futures or forward contract

[b]Recall p. 515.

Some Pricing Relations (concluded)

- A European futures option is worth the same as a European option on the underlying asset if $T = t$.
  - Futures price equals spot price at maturity.
- This conclusion is model independent.

Put-Call Parity[a]

The put-call parity is slightly different from the one in Eq. (31) on p. 230.

Theorem 14 *(1) For European options on futures contracts,*

$$C = P - (X - F)\, e^{-rt}.$$

(2) For European options on forward contracts,

$$C = P - (X - F)\, e^{-rT}.$$

[a] See Theorem 12.4.4 of the textbook for proof.

Early Exercise

The early exercise feature is not valuable for *forward* options.[a]

Theorem 15 *American forward options should not be exercised before expiration as long as the probability of their ending up out of the money is positive.*

Early exercise may be optimal for American *futures* options even if the underlying asset generates no payouts.

Theorem 16 *American futures options may be exercised optimally before expiration.*

---

[a]See Theorem 12.4.5 of the textbook for proof.

Black's Model[a]

- Formulas for European futures options:

$$C = Fe^{-rt}N(x) - Xe^{-rt}N(x - \sigma\sqrt{t}\,), \qquad (69)$$

$$P = Xe^{-rt}N(-x + \sigma\sqrt{t}\,) - Fe^{-rt}N(-x), \qquad (70)$$

where $x \stackrel{\Delta}{=} \frac{\ln(F/X)+(\sigma^2/2)\,t}{\sigma\sqrt{t}}$.

- The above formulas are related to those for options on a stock paying a continuous dividend yield.
- They are Eqs. (44) on p. 339 with q set to r and S replaced by F.

---

[a]Black (1976). It is also called the Black-76 model.

Black's Model (concluded)

- Volatility σ is that of the stock price.[a]
  - It is the same as the volatlity of the futures price.[b]
- This observation incidentally proves Theorem 16 (p. 521).
- For European forward options, just multiply the above formulas by $e^{-r(T-t)}$.
  - Forward options differ from futures options by a factor of $e^{-r(T-t)}$.[c]

---

[a] Contributed by Mr. Lu, Yu-Ming (R06723032, D08922008) on April 7, 2021.

[b] Contributed by Mr. Chen, Tung-Li (D09922014) on April 7, 2023. See also p. 634.

[c] Recall Eqs. (67)–(68) on p. 518.

Binomial Model for Forward and Futures Options

- In a risk-neutral economy, futures price behaves *like* a stock paying a continuous dividend yield of r.
  - Let the futures price at time 0 be F.
  - From Lemma 10 (p. 305), the expected value of S at time Δt in a risk-neutral economy is

$$Se^{r\Delta t}.$$

  - By Eq. (61) on p. 494, the expected futures price at time Δt is

$$Se^{r\Delta t}e^{r(T-\Delta t)} = Se^{rT} = F.$$

Binomial Model for Forward and Futures Options (continued)

- The above observation continues to hold even if S pays a dividend yield![a]
  - Let the futures price at time 0 be F.
  - From Lemma 10 (p. 305), the expected value of S at time Δt in a risk-neutral economy is

$$Se^{(r-q)\,\Delta t}.$$

  - By Eq. (65) on p. 504, the expected futures price at time Δt is

$$Se^{(r-q)\,\Delta t}e^{(r-q)(T-\Delta t)} = Se^{(r-q)\,T} = F.$$

[a] Contributed by Mr. Liu, Yi-Wei (R02723084) on April 16, 2014.

Binomial Model for Forward and Futures Options (concluded)

- Now, under the BOPM, the risk-neutral probability for the futures price is

$$p_{\mathrm{f}} \stackrel{\Delta}{=} (1-d)/(u-d) \tag{71}$$

  by Eq. (45) on p. 341.
  - The futures price moves from F to Fu with probability p_{f} and to Fd with probability $1-p_{\mathrm{f}}$.
  - The *original* u and d are used here.
- The binomial tree algorithm for *forward* options is identical except that Eq. (68) on p. 518 is the payoff.

Spot and Futures Prices under BOPM

- The futures price is related to the spot price via

$$F = Se^{rT}$$

  if the underlying asset pays no dividends.[a]

- Recall the futures price F moves to Fu with probability p_{f} per period.

- So the stock price moves from $S = Fe^{-rT}$ to

$$Fue^{-r(T-\Delta t)} = Sue^{r\Delta t}$$

  with probability p_{f} per period.

[a] Recall Lemma 12 (p. 494).

Spot and Futures Prices under BOPM (concluded)

- Similarly, the stock price moves from $S = Fe^{-rT}$ to

$$Sde^{r\Delta t}$$

  with probability $1 - p_{\mathrm{f}}$ per period.

- Note that

$$S(ue^{r\Delta t})(de^{r\Delta t}) = Se^{2r\Delta t} \neq S.$$

- This binomial model for S differs from the CRR tree.
- This model may not be suitable for pricing barrier options (why?).

Negative Probabilities Revisited

- As $0 < p_{\mathrm{f}} < 1$, we have $0 < 1 - p_{\mathrm{f}} < 1$ as well.
- The problem of negative risk-neutral probabilities is solved:
  - Build the tree for the futures price F of the futures contract expiring at the same time as the option.
  - Let the stock pay a continuous dividend yield of q.
  - By Eq. (65) on p. 504, recover S from F at each node via

$$S = Fe^{-(r-q)(T-t)}.$$

Swaps

- Swaps are agreements between two counterparties to exchange cash flows in the future according to some predetermined formula.
- There are two basic types of swaps: interest rate and currency.
- An interest rate swap occurs when two parties exchange interest payments periodically.
- Currency swaps are agreements to deliver one currency against another (our focus here).
- There are theories about why swaps exist.[a]

[a]Thanks to a lively discussion on April 16, 2014.

Currency Swaps

- A currency swap involves two parties to exchange cash flows in different currencies.
- Consider the following fixed rates available to party A and party B in U.S. dollars and Japanese yen:

	Dollars	Yen
A	$D_{\mathrm{A}}\%$	$\boxed{Y_{\mathrm{A}}\%}$
B	$\boxed{D_{\mathrm{B}}\%}$	$Y_{\mathrm{B}}\%$

- Suppose A wants to take out a fixed-rate loan in yen, and B wants to take out a fixed-rate loan in dollars.

Currency Swaps (continued)

- A straightforward scenario is for A to borrow yen at $Y_{\mathrm{A}}\%$ and B to borrow dollars at $D_{\mathrm{B}}\%$.
- But suppose A is *relatively* more competitive in the dollar market than the yen market, i.e.,

$$Y_{\mathrm{B}} - D_{\mathrm{B}} < Y_{\mathrm{A}} - D_{\mathrm{A}} \quad \text{or} \quad Y_{\mathrm{B}} - Y_{\mathrm{A}} < D_{\mathrm{B}} - D_{\mathrm{A}}.$$

- Consider this alternative arrangement:
  - A borrows dollars.
  - B borrows yen.
  - They enter into a currency swap with a bank (the swap dealer) as the intermediary.

Currency Swaps (concluded)

- The counterparties exchange principal at the beginning and the end of the life of the swap.
- This act transforms A's loan into a yen loan and B's yen loan into a dollar loan.
- The total interest rate is originally $(Y_{\mathrm{A}} + D_{\mathrm{B}})\%$.
- The new arrangement has a smaller total rate of $(D_{\mathrm{A}} + Y_{\mathrm{B}})\%$.
- So the total gain is $((D_{\mathrm{B}} - D_{\mathrm{A}}) - (Y_{\mathrm{B}} - Y_{\mathrm{A}}))\%$.
- Transactions will happen only if the gain is distributed so that the cost to each party is less than the original.

Example

- A and B face the following borrowing rates:

	Dollars	Yen
A	9%	10%
B	12%	11%

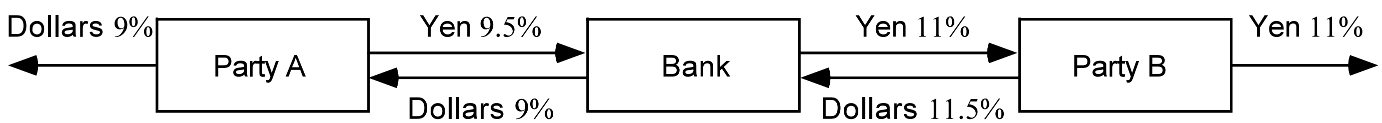
- A wants to borrow yen, and B wants to borrow dollars.
- A can borrow yen directly at 10%.
- B can borrow dollars directly at 12%.

Example (continued)

- The rate differential in dollars (3%) is different from that in yen (1%).
- So a currency swap with a total saving of $3 - 1 = 2\%$ is possible.
- A is relatively more competitive in the dollar market.
- B is relatively more competitive in the yen market.

Example (concluded)

- Next page shows an arrangement which is beneficial to all parties involved.
  - A effectively borrows yen at 9.5% (lower than 10%).
  - B borrows dollars at 11.5% (lower than 12%).
  - The gain is 0.5% for A, 0.5% for B, and, if we treat dollars and yen identically, 1% for the bank.


Dollars 9%
Party A
Yen 9.5%
Dollars 9%
Bank
Yen 11%
Dollars 11.5%
Party B
Yen 11%

As a Package of Cash Market Instruments

- Assume no default risk.
- Take B on p. 537 as an example.
- The swap is equivalent to a long position in a yen bond paying 11% annual interest and a short position in a dollar bond paying 11.5% annual interest.
- The pricing formula is $SP_{\mathrm{Y}} - P_{\mathrm{D}}$.
  - P_{D} is the dollar bond's value in dollars.
  - P_{Y} is the yen bond's value in yen.
  - S is the \$/yen spot exchange rate.

As a Package of Cash Market Instruments (concluded)

- The value of a currency swap depends on:
  - The term structures of interest rates in the currencies involved.
  - The spot exchange rate.
- It has zero value when

$$SP_{\mathrm{Y}} = P_{\mathrm{D}}.$$

Example

- Take a 3-year swap on p. 537 with principal amounts of US$1 million and 100 million yen.
- The payments are made once a year.
- The spot exchange rate is 90 yen/$ and the term structures are flat in both nations—8% in the U.S. and 9% in Japan.
- For B, the value of the swap is (in millions of USD)

$$\frac{1}{90} \times \left(11 \times e^{-0.09} + 11 \times e^{-0.09\times 2} + 111 \times e^{-0.09\times 3}\right)$$
$$- \left(0.115 \times e^{-0.08} + 0.115 \times e^{-0.08\times 2} + 1.115 \times e^{-0.08\times 3}\right) = 0.074.$$

As a Package of Forward Contracts

- From Eq. (64) on p. 504, the forward contract maturing i years from now has a *dollar* value of

$$f_i \stackrel{\Delta}{=} (SY_i)\, e^{-qi} - D_i e^{-ri}. \tag{72}$$

  - Y_i is the yen inflow at year i.
  - S is the \$/yen spot exchange rate.[a]
  - q is the yen interest rate.
  - D_i is the dollar outflow at year i.
  - r is the dollar interest rate.

---

[a]This is *not* the forward exchange rate. Contributed by Mr. Chen, Tzu-Chun (R12922037) and Mr. Wang, Wei-Li (R12922116) on April 19, 2024.

As a Package of Forward Contracts (concluded)

- For simplicity, flat term structures were assumed.
- Generalization is straightforward.

Example

- Take the swap in the example on p. 540.
- Every year, B receives 11 million yen and pays 0.115 million dollars.
- In addition, at the end of the third year, B receives 100 million yen and pays 1 million dollars.
- Each of these transactions represents a forward contract.
- $Y_1 = Y_2 = 11$, $Y_3 = 111$, $S = 1/90$, $D_1 = D_2 = 0.115$, $D_3 = 1.115$, $q = 0.09$, and $r = 0.08$.
- Plug in these numbers to get $f_1 + f_2 + f_3 = 0.074$ million dollars as before.

Stochastic Processes and Brownian Motion

Of all the intellectual hurdles which the human mind has confronted and has overcome in the last fifteen hundred years, the one which seems to me to have been the most amazing in character and the most stupendous in the scope of its consequences is the one relating to the problem of motion.

— Herbert Butterfield (1900–1979)

Stochastic Processes

- A stochastic process

$$X = \{ X(t) \}$$

is a time series of random variables.

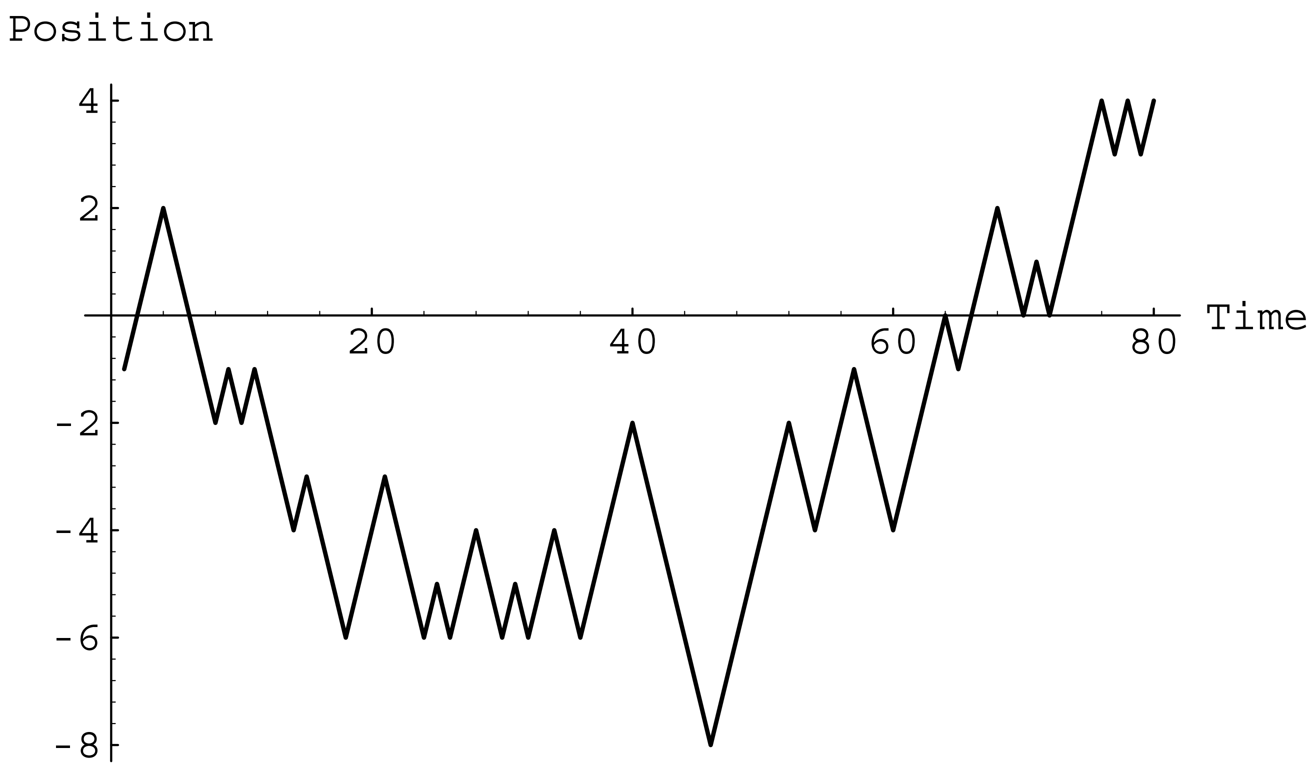
- $X(t)$ (or X_t) is a random variable for each time t and is usually called the state of the process at time t.
- A realization of X is called a sample path.

Stochastic Processes (concluded)

- If the times t form a countable set, X is called a discrete-time stochastic process or a time series.
- In this case, subscripts rather than parentheses are usually employed, as in
$$X = \{ X_n \}.$$
- If the times form a continuum, X is called a continuous-time stochastic process.

Random Walks

- The binomial model is a random walk in disguise.
- Consider a particle on the integer line, $0, \pm 1, \pm 2, \ldots$.
- In each time step, it can make one move to the right with probability p or one move to the left with probability $1 - p$.
  - This random walk is symmetric when $p = 1/2$.
- Connection with the BOPM: The particle's position denotes the number of up moves minus that of down moves up to that time.


Position
4
2
-2
-4
-6
-8
20
40
60
80
Time

Random Walk with Drift

$$X_n = \mu + X_{n-1} + \xi_n.$$

- ξ_n are independent and identically distributed with zero mean.
- Drift μ is the expected change per period.
- Note that this process is continuous in space.

Martingales[a]

- $\{ X(t), t \geq 0 \}$ is a martingale if $E[\,|\,X(t)\,|\,] < \infty$ for $t \geq 0$ and

$$E[\, X(t) \,|\, X(u), 0 \leq u \leq s \,] = X(s), \quad s \leq t. \tag{73}$$

- In the discrete-time setting, a martingale means

$$E[\, X_{n+1} \,|\, X_1, X_2, \ldots, X_n \,] = X_n. \tag{74}$$

- X_n can be interpreted as a gambler's fortune after the nth gamble.
- Identity (74) then says the expected fortune after the $(n+1)$st gamble equals the fortune after the nth gamble regardless of what may have occurred before.

[a] The origin of the name is somewhat obscure.

Martingales (concluded)

- A martingale is therefore a notion of fair games.
- Apply the law of iterated conditional expectations to both sides of Eq. (74) on p. 551 to yield
$$E[\, X_n \,] = E[\, X_1 \,] \tag{75}$$
for all n.
- Similarly,
$$E[\, X(t) \,] = E[\, X(0) \,]$$
in the continuous-time case.

Still a Martingale?

- Suppose we replace Eq. (74) on p. 551 with

$$E[\, X_{n+1} \,|\, X_n \,] = X_n.$$

- It also says past history cannot affect the future.
- But is it equivalent to the original definition (74) on p. 551?[a]

---

[a] Contributed by Mr. Hsieh, Chicheng (`M9007304`) on April 13, 2005.

Still a Martingale? (continued)

- Well, no.[a]
- Consider this random walk with drift:

$$X_i = \begin{cases} X_{i-1} + \xi_i, & \text{if } i \text{ is even,} \\ X_{i-2}, & \text{otherwise.} \end{cases}$$

- Above, ξ_n are random variables with zero mean.

[a] Contributed by Mr. Zhang, Ann-Sheng (B89201033) on April 13, 2005.

Still a Martingale? (concluded)

- It is not hard to see that

$$E[\,X_i \mid X_{i-1}\,] = \begin{cases} X_{i-1}, & \text{if } i \text{ is even,} \\ X_{i-1}, & \text{otherwise.} \end{cases}$$

  – It is a martingale by the "new" definition.

- But

$$E[\,X_i \mid \ldots, X_{i-2}, X_{i-1}\,] = \begin{cases} X_{i-1}, & \text{if } i \text{ is even,} \\ X_{i-2}, & \text{otherwise.} \end{cases}$$

  – It is not a martingale by the original definition.

Example

- Consider the stochastic process

$$\left\{ Z_n \triangleq \sum_{i=1}^{n} X_i, n \geq 1 \right\},$$

where X_i are independent random variables with zero mean.

- This process is a martingale because

$$\begin{aligned}
& E[\, Z_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n \,] \\
= \;& E[\, Z_n + X_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n \,] \\
= \;& E[\, Z_n \,|\, Z_1, Z_2, \ldots, Z_n \,] + E[\, X_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n \,] \\
= \;& Z_n + E[\, X_{n+1} \,] = Z_n.
\end{aligned}$$

Probability Measure

- A probability measure assigns probabilities to states of the world.[a]
- A martingale is defined with respect to a probability measure, under which the expectation is taken.
- Second, a martingale is defined with respect to an information set.
  - In the characterizations (73)–(74) on p. 551, the information set contains the current and past values of X by default.
  - But it need not be so.

[a] Only certain sets such as the Borel sets receive probabilities (Feller, 1972).

Probability Measure (continued)

- A stochastic process $\{ X(t), t \geq 0 \}$ is a martingale with respect to information sets $\{ I_t \}$ if, for all $t \geq 0$, $E[\,|\,X(t)\,|\,] < \infty$ and

$$E[\, X(u) \,|\, I_t \,] = X(t)$$

for all $u > t$.

- The discrete-time version: For all $n > 0$,

$$E[\, X_{n+1} \,|\, I_n \,] = X_n,$$

given the information sets $\{ I_n \}$.

Probability Measure (concluded)

- The above implies

$$E[\,X_{n+m}\,|\,I_n\,] = X_n$$

for any $m > 0$ by Eq. (26) on p. 171.

  – A typical I_n is the price information up to time n.

  – Then the above identity says the FVs of X will not deviate systematically from today's value given the price history.

Example

- Consider the stochastic process $\{ Z_n - n\mu, n \geq 1 \}$.
  - $Z_n \triangleq \sum_{i=1}^n X_i$.
  - $X_1, X_2, \ldots$ are independent random variables with mean μ.
- Now,

$$
\begin{aligned}
& E[\, Z_{n+1} - (n+1)\,\mu \,|\, X_1, X_2, \ldots, X_n \,] \\
= \; & E[\, Z_{n+1} \,|\, X_1, X_2, \ldots, X_n \,] - (n+1)\,\mu \\
= \; & E[\, Z_n + X_{n+1} \,|\, X_1, X_2, \ldots, X_n \,] - (n+1)\,\mu \\
= \; & Z_n + \mu - (n+1)\,\mu \\
= \; & Z_n - n\mu.
\end{aligned}
$$

Example (concluded)

- Define
$$I_n \stackrel{\Delta}{=} \{ X_1, X_2, \ldots, X_n \}.$$
- Then
$$\{ Z_n - n\mu, n \geq 1 \}$$
is a martingale with respect to $\{ I_n \}$.

Martingale Pricing

- Stock prices and zero-coupon bond prices are expected to rise, while call prices are expected to fall.
- They are *not* martingales.
- Why is then martingale useful?
- Recall a martingale is defined with respect to some information set *and* some probability measure.
- By modifying the probability measure, we can convert a price process into a martingale.

Martingale Pricing (continued)

- The price of a European option is the expected discounted payoff in a risk-neutral economy.[a]
- This principle can be generalized using the concept of martingale.
- Recall the recursive valuation of European option via

$$C = [\, pC_u + (1-p)\, C_d \,]/R.$$

  - p is the risk-neutral probability.
  - \$1 grows to \$$R$ in a period.

[a]Recall Eq. (37) on p. 271.

Martingale Pricing (continued)

- Let $C(i)$ denote the value of the option at time i.
- Consider the discount process

$$\left\{ \frac{C(i)}{R^i}, i = 0, 1, \ldots, n \right\}.$$

- Then,

$$\begin{aligned} E\left[\left. \frac{C(i+1)}{R^{i+1}} \,\right|\, S(i) \right] &= \frac{E[\, C(i+1) \,|\, S(i) \,]}{R^{i+1}} \\ &= \frac{pC_u + (1-p)\, C_d}{R^{i+1}} \\ &= \frac{C(i)}{R^i}. \end{aligned}$$

Martingale Pricing (continued)

- It is easy to show that

$$E\left[\left.\frac{C(k)}{R^k}\,\right|\,S(i)\right]=\frac{C(i)}{R^i},\quad i\leq k.$$

- This simplified formulation assumes:[a]
  1. The model is Markovian: The distribution of the future is determined by the present (time i) and not the past.
  2. The payoff depends only on the terminal price of the underlying asset[b] (Asian options do not qualify).

[a] Contributed by Mr. Wang, Liang-Kai (Ph.D. student, ECE, University of Wisconsin-Madison) and Mr. Hsiao, Huan-Wen (B90902081) on May 3, 2006.

[b] Recall they are called simple claims.

Martingale Pricing (continued)

- In general, the discount process is a martingale in that[a]

$$E_i^{\pi}\left[\frac{C(k)}{R^k}\right] = \frac{C(i)}{R^i}, \quad i \leq k. \tag{76}$$

  - E_i^{π} is taken under the risk-neutral probability conditional on the price information *up to time* i.

- This risk-neutral probability is also called the EMM, or the equivalent[b] martingale (probability) measure.

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[a] In this general formulation, Asian options do qualify.

[b] Two probability measures are said to be equivalent if they assign nonzero probabilities to the same set of states.

Martingale Pricing (continued)

- Equation (76) holds for all assets, not just options.
- When interest rates are stochastic, the equation becomes

$$\frac{C(i)}{M(i)} = E_i^{\pi}\left[\frac{C(k)}{M(k)}\right], \quad i \leq k. \tag{77}$$

  - $M(j)$ is the balance in the money market account at time j using the rollover strategy with an initial investment of \$1.
  - It is called the bank account process.
- It says the discount process is a martingale under π.

Martingale Pricing (continued)

- If interest rates are stochastic, then $M(j)$ is a random variable.
  - $M(0) = 1$.
  - $M(j)$ is known at time $j - 1$.[a]
- Identity (77) on p. 567 is the general formulation of risk-neutral valuation.

[a] Because the interest rate for the *next* period has been revealed then.

Martingale Pricing (concluded)

Theorem 17 *A discrete-time model is arbitrage-free if and only if there exists an equivalent probability measure*[a] *such that the discount process is a martingale.*

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[a]Called the risk-neutral probability measure.

