

Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market's opinion of the volatility.^a
 - Solve for σ given the option price, S , X , τ , and r with numerical methods.
 - How about American options?

^aImplied volatility is hard to compute when τ is small (why?).

Implied Volatility (concluded)

- Implied volatility is
the wrong number to put in the wrong formula to
get the right price of plain-vanilla options.^a
- Think of it as an alternative to quoting option prices.
- Implied volatility is often preferred to historical
(statistical) volatility in practice.
 - Is using the historical volatility like driving a car
with your eyes on the rearview mirror?^b
- Volatility is meaningful only if seen through a model!^c

^aRebonato (2004).

^bE.g., 1:16:23 of <https://www.youtube.com/watch?v=8TJQhQ2GZOY>

^cAlexander (2001).

Problems; the Smile^a

- Options written on the same underlying asset usually do not yield the same implied volatility.
- A typical pattern is a “smile” in relation to the strike price.
 - The implied volatility is lowest for at-the-money options.
 - It becomes higher the further the option is in- or out-of-the-money.
- This is common for foreign exchange options.

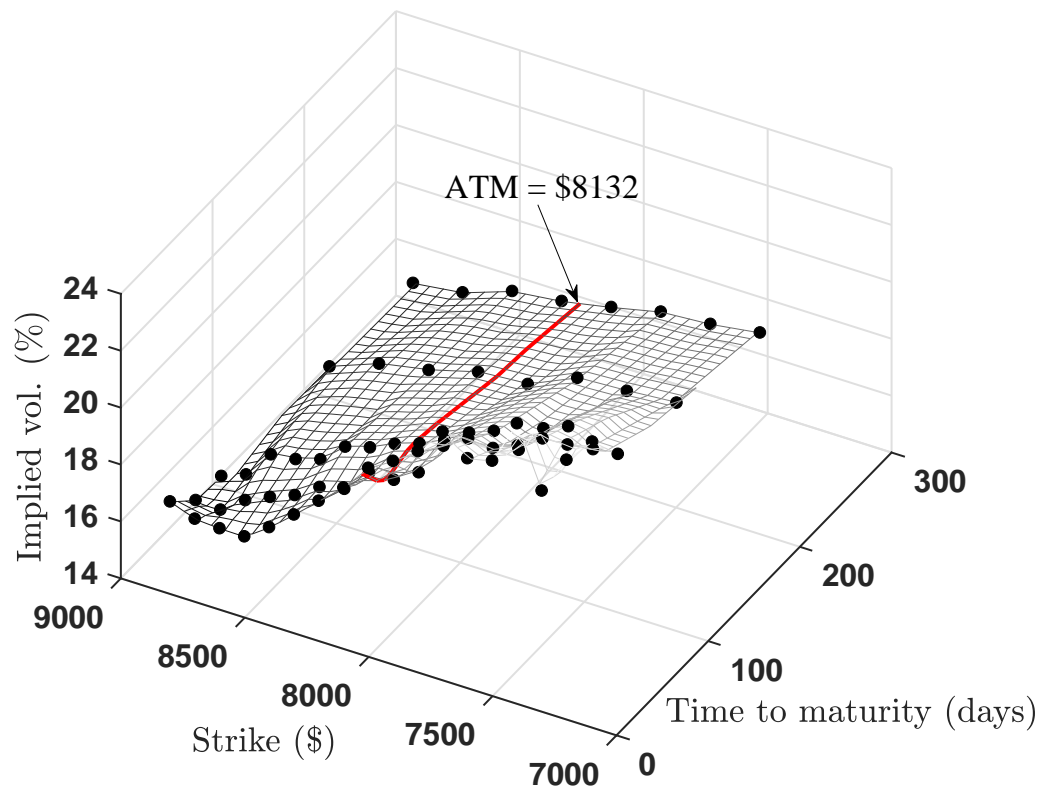
^aAlexander (2001).

Problems; the Smile (concluded)

- Other patterns have also been observed.
- For stock options, low-strike options tend to have higher implied volatilities.
- One explanation is the high demand for insurance provided by out-of-the-money puts.
- Another reason is volatility rises when stock falls,^a making in-the-money calls more likely to become in the money again.

^aThis is called the leverage effect (Black, 1992).

TXO Calls (September 25, 2015)^a



^aThe underlying Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) closed at 8132. Plot supplied by Mr. Lok, U Hou (D99922028) on December 6, 2017.

Tackling the Smile

- To address this issue, volatilities are often combined to produce a composite implied volatility.
- This practice is not sound theoretically.
- The existence of different implied volatilities for options on the same underlying asset shows the Black-Scholes model is not literally true.

Binomial Tree Algorithms for American Puts

- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs

$$\max(0, X - Su^j d^{n-j})$$

and applies backward induction.

- At each intermediate node, compare the payoff if exercised and the *continuation value*.
- Keep the larger one.

Bermudan Options

- Some American options can be exercised only at discrete time points instead of continuously.
- They are called Bermudan options.
- Their pricing algorithm is identical to that for American options.
- But early exercise is considered for only those nodes when early exercise is permitted.

Time-Dependent Volatility^a

- Suppose the (instantaneous) volatility can change over time but otherwise predictable: $\sigma(t)$ instead of σ .
- In the limit, the variance of $\ln(S_\tau/S)$ is

$$\int_0^\tau \sigma^2(t) dt$$

rather than $\sigma^2\tau$.

- The annualized volatility to be used in the Black-Scholes formula should now be

$$\sqrt{\frac{\int_0^\tau \sigma^2(t) dt}{\tau}}.$$

^aMerton (1973).

Time-Dependent Instantaneous Volatility (concluded)

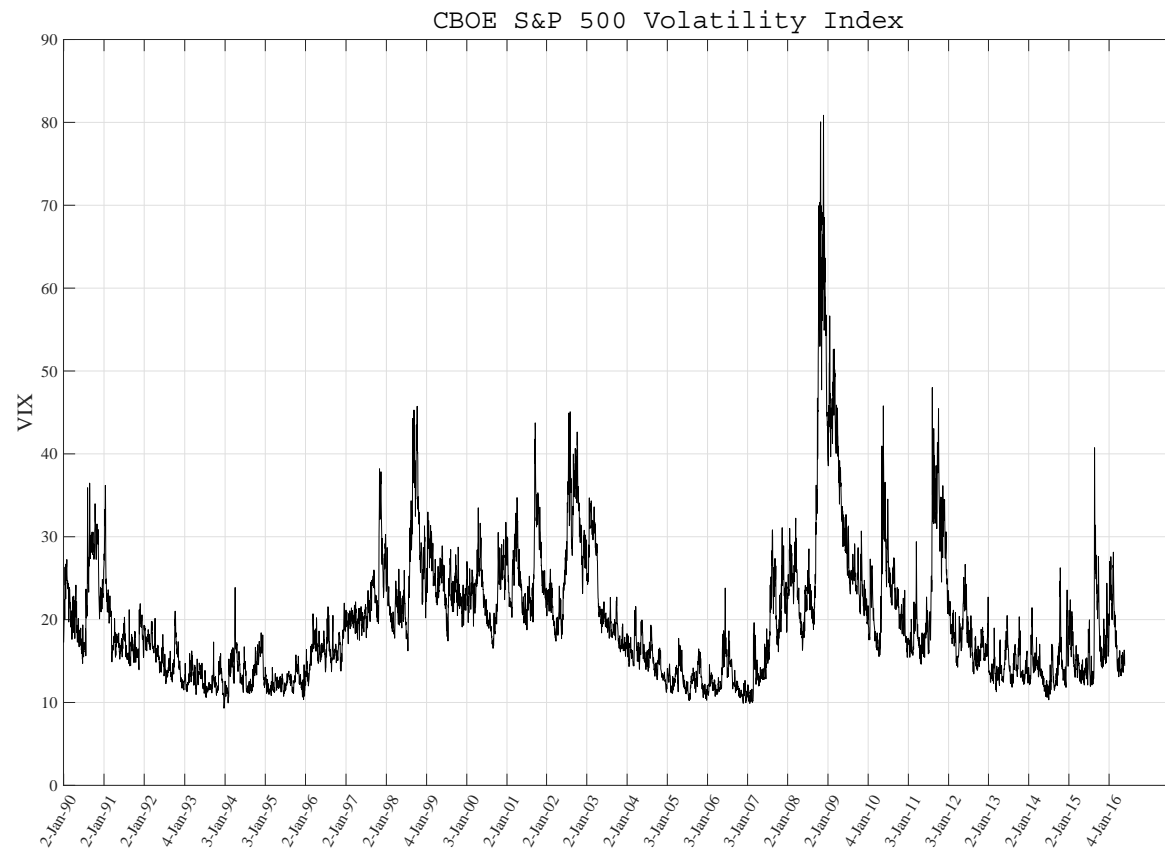
- For the binomial model, u and d depend on time:

$$\begin{aligned}u &= e^{\sigma(t)\sqrt{\tau/n}}, \\d &= e^{-\sigma(t)\sqrt{\tau/n}}.\end{aligned}$$

- But how to make the binomial tree combine?^a

^aAmin (1991); C. I. Chen (R98922127) (2011).

Volatility (1990–2016)^a



^aSupplied by Mr. Lok, U Hou (D99922028) on July 17, 2017.

Time-Dependent Short Rates

- Suppose the short rate (i.e., the one-period spot rate or forward rate) changes over time but predictable.
- The annual riskless rate r in the Black-Scholes formula should be the spot rate with a time to maturity equal to τ .
- In other words,

$$r = \frac{\sum_{i=0}^{n-1} r_i}{\tau},$$

where r_i is the continuously compounded short rate measured in periods for period i .^a

- Will the binomial tree fail to combine?

^aThat is, one-period forward rate.

Trading Days and Calendar Days

- Interest accrues based on the calendar day.
- But σ is usually calculated based on trading days only.
 - Stock price seems to have lower volatilities when the exchange is closed.^a
- How to harmonize these two different times into the Black-Scholes formula and binomial tree algorithms?^b

^aFama (1965); K. French (1980); K. French & Roll (1986).

^bRecall p. 164 about dating issues.

Trading Days and Calendar Days (continued)

- Think of σ as measuring the *annualized* volatility of stock price *one year from now*.
- Suppose a year has m (say 253) trading days.
- We can replace σ in the Black-Scholes formula with^a

$$\sigma \sqrt{\frac{365}{m} \times \frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}}.$$

^aD. French (1984).

Trading Days and Calendar Days (concluded)

- This works only for European options.
- How about binomial tree algorithms?^a

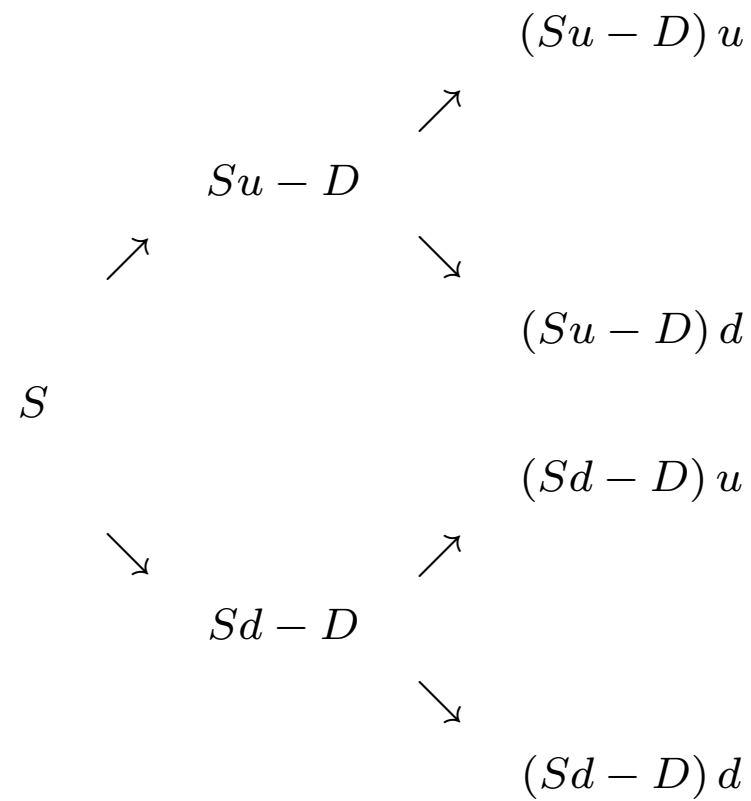
^aContributed by Mr. Lu, Zheng-Liang (D00922011) in 2015.

Options on a Stock That Pays Dividends

- Early exercise must be considered.
- Proportional dividend payout model is tractable (see text).
 - The dividend amount is a constant proportion of the *prevailing* stock price.
- In general, the corporate dividend policy is a complex issue.

Known Dividends

- Constant dividends introduce complications.
- Use D to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are $S_u - D$ and $S_d - D$.
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: $(S_u - D)u$, $(S_u - D)d$, $(S_d - D)u$, $(S_d - D)d$.
 - The binomial tree no longer combines.



An Ad-Hoc Approximation

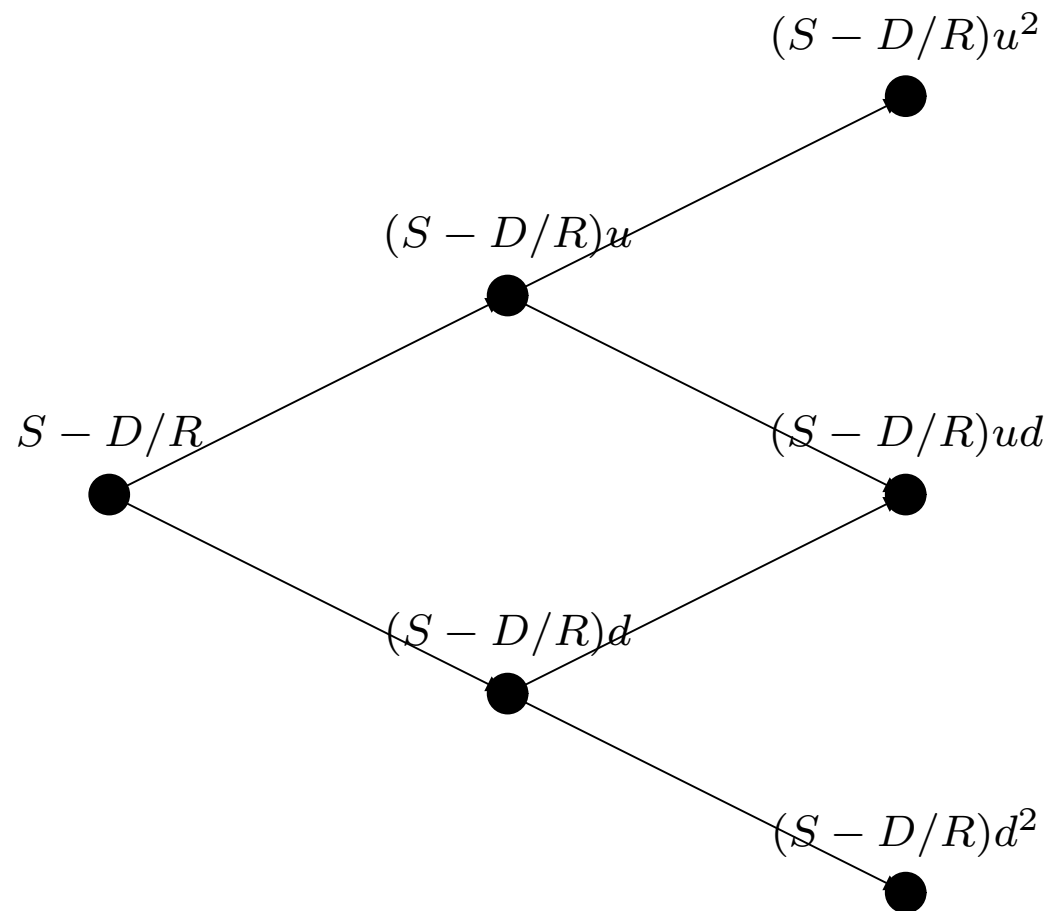
- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends.^a
- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.
- The riskless component at any time is the PV of future dividends during the life of the option.
 - Then, σ is the volatility of the process followed by the *risky* component.
- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

^aRoll (1977); Heath & Jarrow (1988).

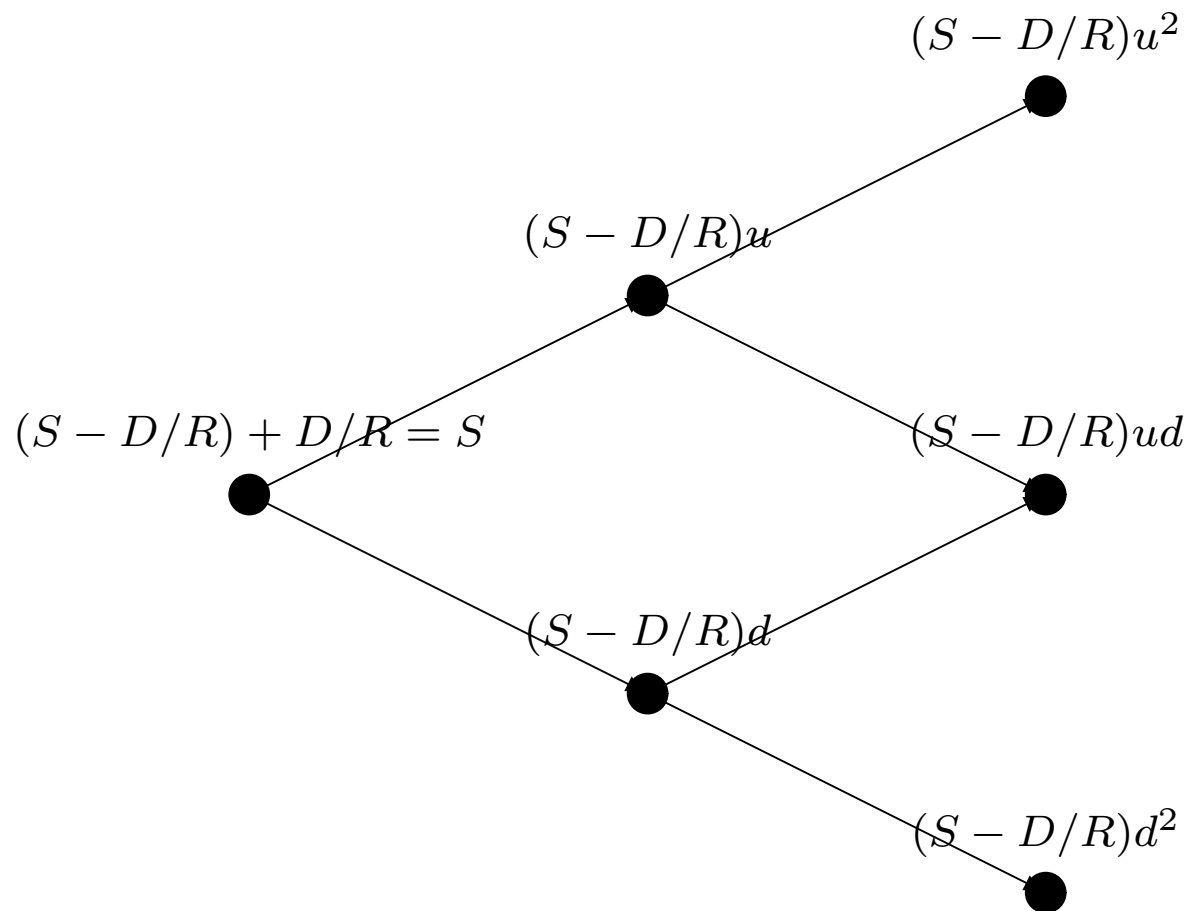
An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all *future* dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.

The Ad-Hoc Approximation vs. P. 330 (Step 1)



The Ad-Hoc Approximation vs. P. 330 (Step 2)



The Ad-Hoc Approximation vs. P. 330^a

- The trees are different.
- The stock prices at maturity are also different.
 - $(Su - D)u, (Su - D)d, (Sd - D)u, (Sd - D)d$
(p. 330).
 - $(S - D/R)u^2, (S - D/R)ud, (S - D/R)d^2$ (ad hoc).
- Note that, as $d < R < u$,

$$\begin{aligned}(Su - D)u &> (S - D/R)u^2, \\ (Sd - D)d &< (S - D/R)d^2,\end{aligned}$$

^aContributed by Mr. Yang, Jui-Chung (D97723002) on March 18, 2009.

The Ad-Hoc Approximation vs. P. 330 (concluded)

- So the ad hoc approximation has a smaller dynamic range.
- This explains why in practice the volatility is usually *increased* when using the ad hoc approximation.

A General Approach^a

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases (see pp. 792ff).
- Other approaches include adjusting σ and approximating the known dividend with a dividend yield.^b

^aT. Dai (B82506025, R86526008, D8852600) & Lyuu (2004). Also Arealy & Rodrigues (2013).

^bGeske & Shastri (1985). It works well for American options but not European ones (T. Dai, 2009).

Continuous Dividend Yields

- Dividends are paid continuously.
 - Approximates a broad-based stock market portfolio.
- The payment of a continuous dividend yield at rate q reduces the growth rate of the stock price by q .
 - A stock that grows from S to S_τ with a continuous dividend yield of q would have grown from S to $S_\tau e^{q\tau}$ without the dividends.
- A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays *no* dividends.^a

^aIn pricing European options, only the distribution of S_τ matters.

Continuous Dividend Yields (continued)

- So the Black-Scholes formulas hold with S replaced by $Se^{-q\tau}$.^a

$$C = Se^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \quad (44)$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau} N(-x), \quad (44')$$

where

$$x \triangleq \frac{\ln(S/X) + (r - q + \sigma^2/2) \tau}{\sigma\sqrt{\tau}}.$$

- Formulas (44) and (44') remain valid as long as the dividend yield is predictable.

^aMerton (1973).

Continuous Dividend Yields (continued)

- To run binomial tree algorithms, replace u with $ue^{-q\Delta t}$ and d with $de^{-q\Delta t}$, where $\Delta t \triangleq \tau/n$.
 - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.
- Other than the changes, binomial tree algorithms stay the same.
 - In particular, p should use the *original* u and d !^a

^aContributed by Ms. Wang, Chuan-Ju (F95922018) on May 2, 2007.

Continuous Dividend Yields (concluded)

- Alternatively, pick the risk-neutral probability as

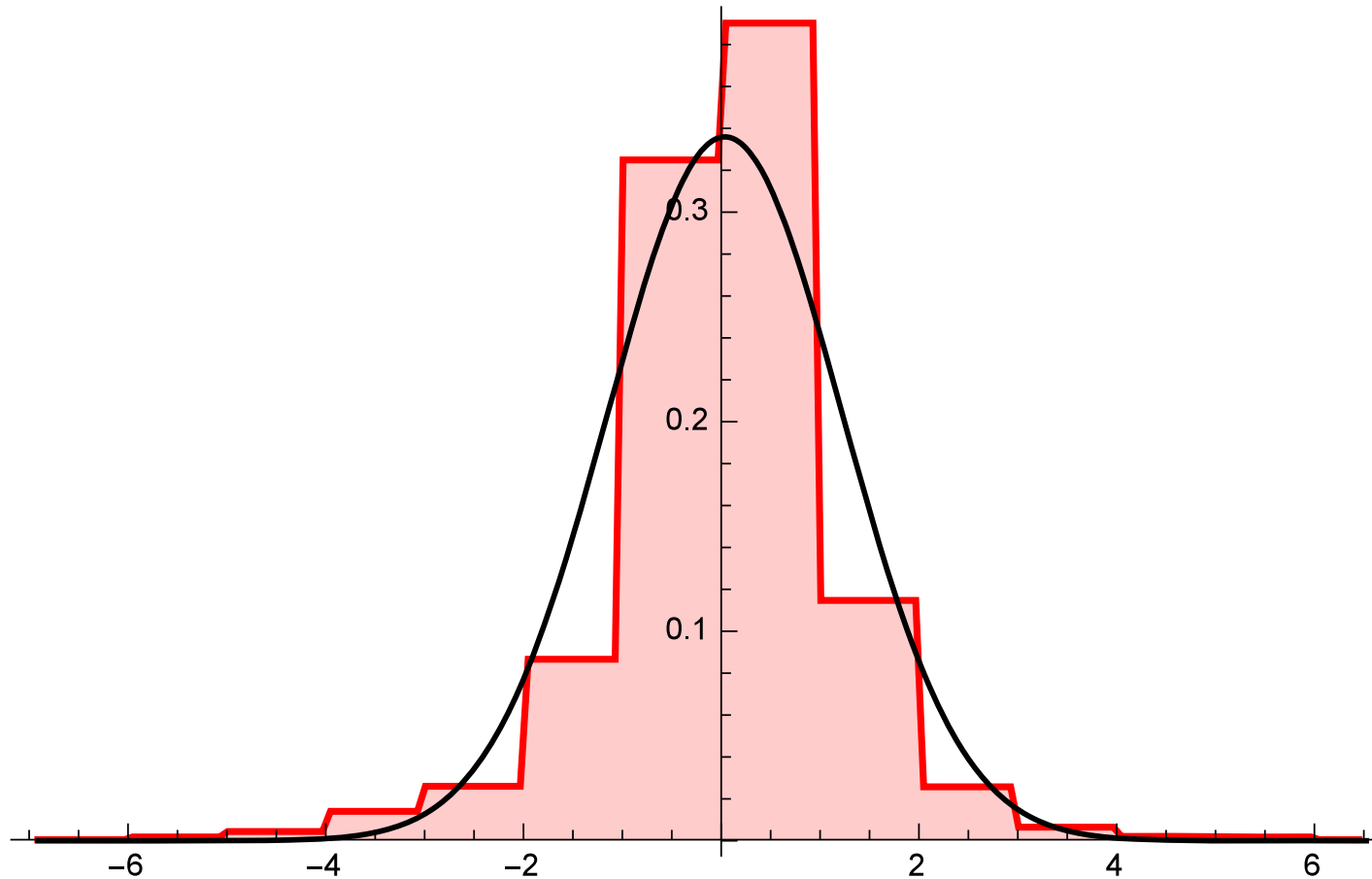
$$\frac{e^{(r-q)\Delta t} - d}{u - d}, \quad (45)$$

where $\Delta t \triangleq \tau/n$.

- The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.
- The u and d remain unchanged.
- Except the change in Eq. (45), binomial tree algorithms stay the same *as if there were no dividends*.

Distribution of Logarithmic Returns of TAIEX

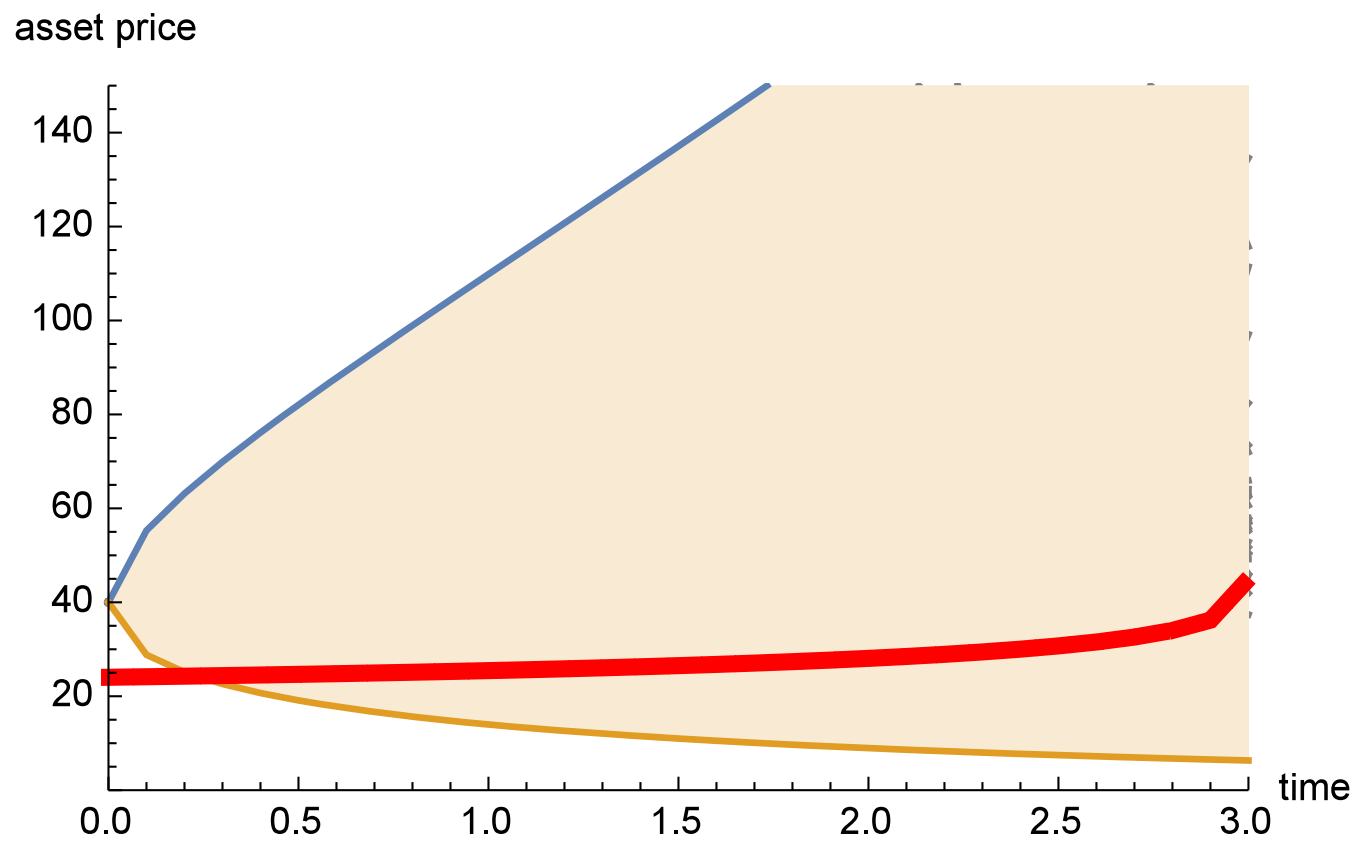
Daily log returns (%) of TAIEX (January 3, 2003–July 13, 2018)



Exercise Boundaries of American Options (in the Continuous-Time Model)^a

- The exercise boundary is a nondecreasing function of t for American *puts* (see the plot next page).
- The exercise boundary is a nonincreasing function of t for American calls.

^aSee Section 9.7 of the textbook for the tree analog.



Sensitivity Analysis of Options

Cleopatra's nose, had it been shorter,
the whole face of the world
would have been changed.
— Blaise Pascal (1623–1662)

Sensitivity Measures (“The Greeks”)

- How the value of a security changes relative to changes in a given parameter is key to hedging.
 - Duration, for instance.

- Let $x \triangleq \frac{\ln(S/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$ (recall p. 308).

- Recall that

$$N'(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} > 0,$$

the density function of standard normal distribution.

Delta

- Defined as

$$\Delta \triangleq \frac{\partial f}{\partial S}.$$

- f is the price of the derivative.
 - S is the price of the underlying asset.
- The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.^a
- The delta used in the BOPM is the discrete analog.^b
- The delta of a long stock is 1.

^aElementary calculus.

^bRecall p. 251.

Delta (continued)

- The delta of a European call on a non-dividend-paying stock equals

$$\frac{\partial C}{\partial S} = N(x) > 0. \quad (46)$$

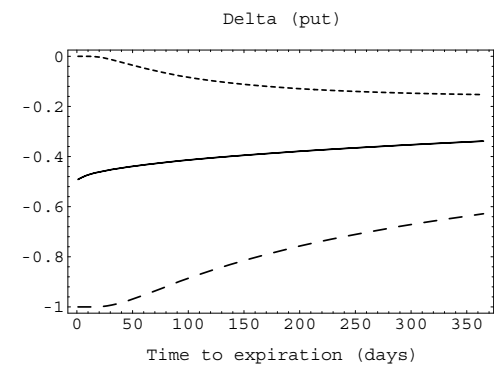
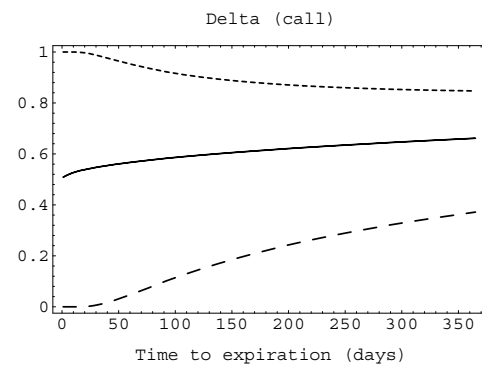
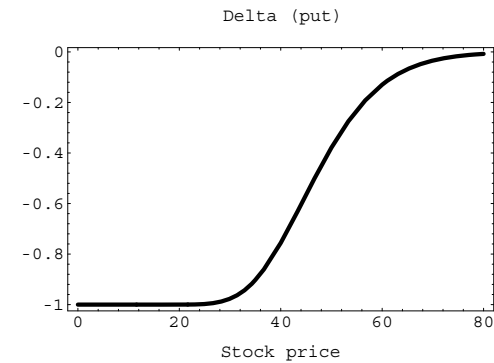
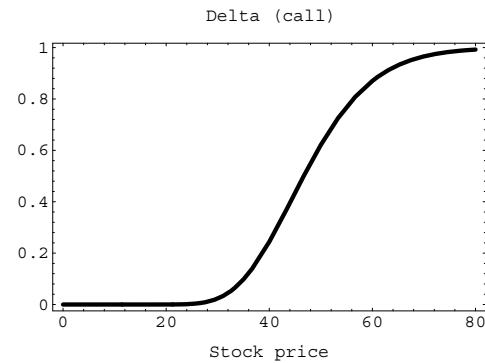
- The delta of a European put equals

$$\frac{\partial P}{\partial S} = N(x) - 1 = -N(-x) < 0. \quad (47)$$

- So the deltas of a call and an otherwise identical put cancel each other when $N(x) = 1/2$, i.e., when^a

$$X = S e^{(r+\sigma^2/2)\tau}. \quad (48)$$

^aThe straddle (p. 215) $C + P$ then has zero delta!



Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money options ($X = 50$).

Dashed curves: out-of-the-money calls or in-the-money puts.

Delta (continued)

- Suppose the stock pays a continuous dividend yield of q .
- Let

$$x \triangleq \frac{\ln(S/X) + (r - q + \sigma^2/2) \tau}{\sigma \sqrt{\tau}} \quad (49)$$

(recall p. 339).

- Then

$$\begin{aligned} \frac{\partial C}{\partial S} &= e^{-q\tau} N(x) > 0, \\ \frac{\partial P}{\partial S} &= -e^{-q\tau} N(-x) < 0. \end{aligned}$$

Delta (continued)

- Consider an X_1 -strike call and an X_2 -strike put, $X_1 \geq X_2$.
- They are otherwise identical.
- Let

$$x_i \triangleq \frac{\ln(S/X_i) + (r - q + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}. \quad (50)$$

- Then their deltas sum to zero when $x_1 = -x_2$.^a
- That implies

$$\frac{S}{X_1} = \frac{X_2}{S} e^{-(2r-2q+\sigma^2) \tau}. \quad (51)$$

^aThe strangle (p. 217) $C + P$ then has zero delta!

Delta (concluded)

- Suppose we demand $X_1 = X_2 = X$ and have a straddle.
- Then

$$X = S e^{(r-q+\sigma^2/2)\tau}$$

leads to a straddle with zero delta.

- This generalizes Eq. (48) on p. 349.
- When $C(X_1)$'s delta and $P(X_2)$'s delta sum to zero,^a does the portfolio $C(X_1) - P(X_2)$ have zero value?
- In general, no.

^aMeaning $C(X_1) + P(X_2)$ has zero delta.

Delta Neutrality

- A position with a total delta equal to 0 is delta-neutral.
 - A delta-neutral portfolio is immune to *small* price changes in the underlying asset.
- Creating one serves for hedging purposes.
 - A portfolio consisting of a call and $-\Delta$ shares of stock is delta-neutral.
 - Short Δ shares of stock to hedge a long call.
 - Long Δ shares of stock to hedge a short call.
- In general, hedge a position in a security with delta Δ_1 by shorting Δ_1/Δ_2 units of a security with delta Δ_2 .

Theta (Time Decay)

- Defined as the rate of change of a security's value with respect to time, or $\Theta \triangleq -\partial f / \partial \tau = \partial f / \partial t$.

- For a European call on a non-dividend-paying stock,

$$\Theta = -\frac{SN'(x)\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(x - \sigma\sqrt{\tau}) < 0.$$

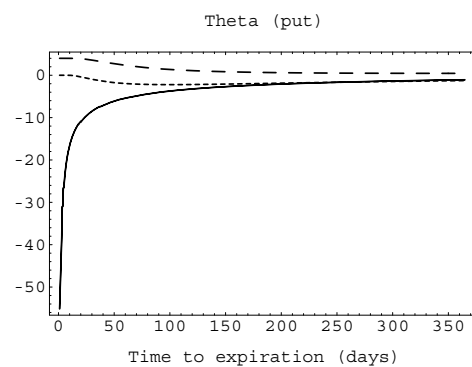
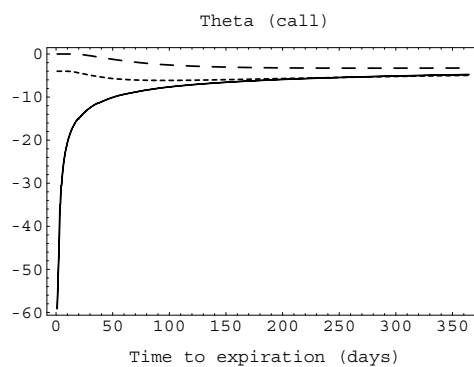
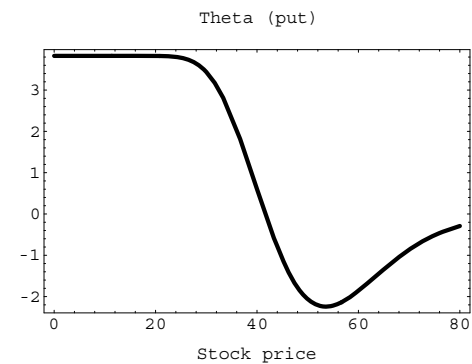
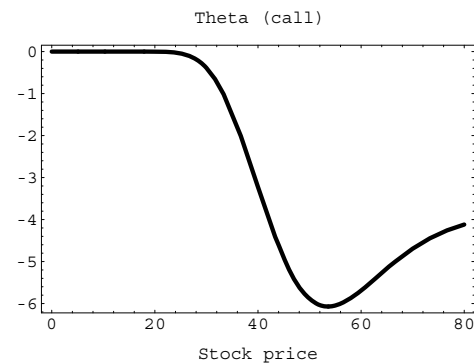
– The call *loses* value with the passage of time.

- For a European put,

$$\Theta = -\frac{SN'(x)\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-x + \sigma\sqrt{\tau}).$$

– Can be negative or positive.

- Both are consistent with the plots on p. 197.



Dotted curve: in-the-money call or out-of-the-money put.
 Solid curves: at-the-money options.
 Dashed curve: out-of-the-money call or in-the-money put.

Theta (concluded)

- Suppose the stock pays a continuous dividend yield of q .
- Define x as in Eq. (49) on p. 351.
- For a European call, add an extra term to the earlier formula for the theta:

$$\Theta = -\frac{SN'(x)\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(x - \sigma\sqrt{\tau}) + qSe^{-q\tau}N(x).$$

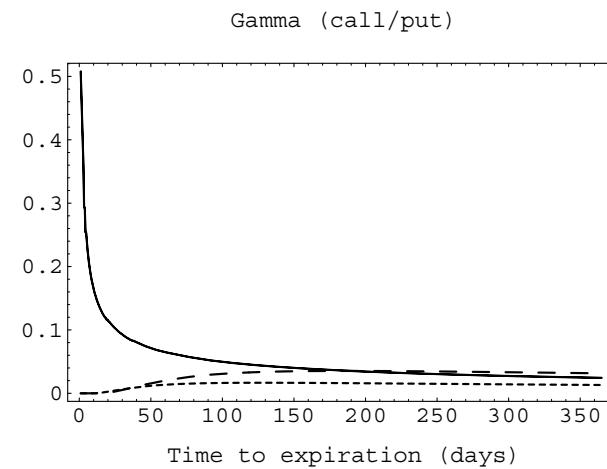
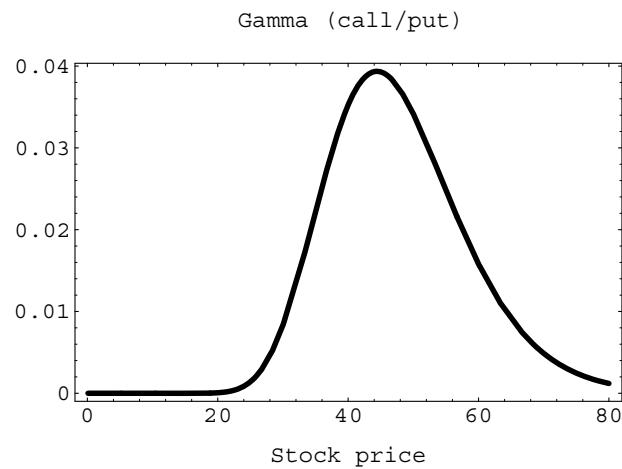
- For a European put, add an extra term to the earlier formula for the theta:

$$\Theta = -\frac{SN'(x)\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-x + \sigma\sqrt{\tau}) - qSe^{-q\tau}N(-x).$$

Gamma

- Defined as the rate of change of its delta with respect to the price of the underlying asset, or $\Gamma \triangleq \partial^2 \Pi / \partial S^2$.
- Measures how sensitive delta is to changes in the price of the underlying asset.
- In practice, a portfolio with a high gamma needs be rebalanced more often to maintain delta neutrality.
- Roughly, delta \sim duration, and gamma \sim convexity.
- The gamma of a European call or put on a non-dividend-paying stock is

$$N'(x)/(S\sigma\sqrt{\tau}) > 0.$$



Dotted lines: in-the-money call or out-of-the-money put.

Solid lines: at-the-money option.

Dashed lines: out-of-the-money call or in-the-money put.

Gamma (concluded)

- Gamma is maximized when the option is nearly at the money, i.e.,

$$S = Xe^{-(r+3\sigma^2/2)\tau}.$$

- As the at-the-money option approaches expiration, its gamma tends to rise.
- The gammas of other options, however, tend to zero.

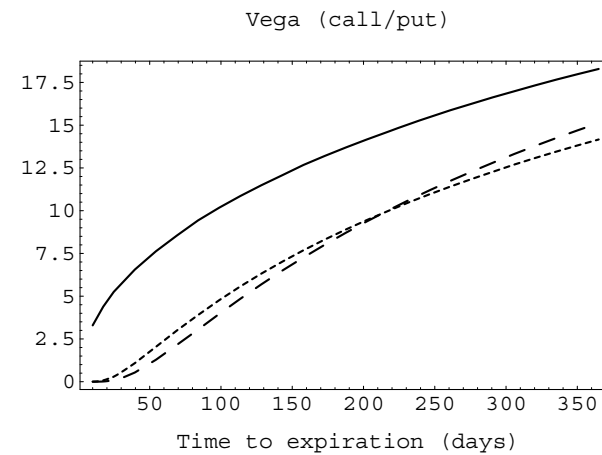
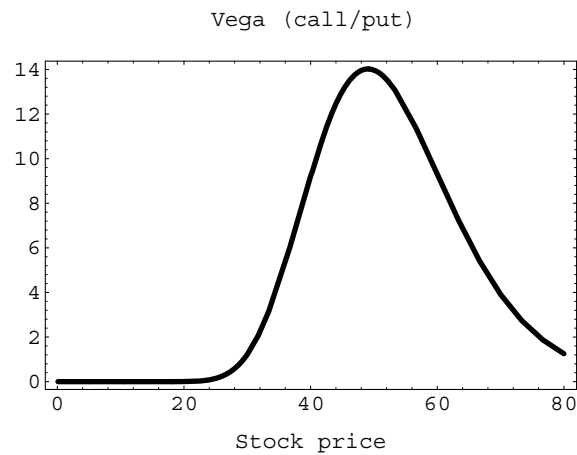
Vega^a (Lambda, Kappa, Sigma, Zeta)

- Defined as the rate of change of a security's value with respect to the volatility of the underlying asset

$$\Lambda \triangleq \frac{\partial f}{\partial \sigma}.$$

- Volatility often changes over time.
- A security with a high vega is very sensitive to changes to or estimation error in volatility.
- The vega of a European call or put on a non-dividend-paying stock is $S\sqrt{\tau} N'(x) > 0$.
 - So higher volatility raises the option value.

^aVega is not Greek. Alexander (2001), “This is a term that was invented by Americans, and intended to sound like a Greek letter.”



Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money option.

Dashed curve: out-of-the-money call or in-the-money put.

Vega (continued)

- Note that^a

$$\Lambda = \tau \sigma S^2 \Gamma. \quad (52)$$

- If the stock pays a continuous dividend yield of q , then

$$\Lambda = S e^{-q\tau} \sqrt{\tau} N'(x),$$

where x is defined in Eq. (49) on p. 351.

- Vega is maximized when $x = 0$, i.e., when

$$S = X e^{-(r-q+\sigma^2/2)\tau}.$$

- Vega declines very fast as S moves away from that peak.

^aReiss & Wystup (2001).

Vega (continued)

- Now consider a portfolio consisting of an X_1 -strike call C and a short X_2 -strike put P , $X_1 \geq X_2$.
- The options' vegas cancel out when

$$x_1 = -x_2,$$

where x_i are defined in Eq. (50) on p. 352.

- This also leads to Eq. (51) on p. 352.
 - Recall the same condition led to zero delta for the strangle $C + P$ (p. 352).

Vega (concluded)

- Note that $\tau \rightarrow 0$ implies

$$\Lambda \rightarrow 0$$

(which answers the question on p. 313).

- The Black-Scholes formula (p. 308) implies

$$\begin{aligned} C &\rightarrow S, \\ P &\rightarrow Xe^{-r\tau}, \end{aligned}$$

as $\sigma \rightarrow \infty$.^a

- These boundary conditions are handy for some numerical methods.

^aRecall that $C \geq \max(S - Xe^{-r\tau}, 0)$ by Exercise 8.3.2 of the text and $P \geq \max(Xe^{-r\tau} - S, 0)$ by Lemma 4 (p. 235).

Rho

- Defined as the rate of change in its value with respect to interest rates

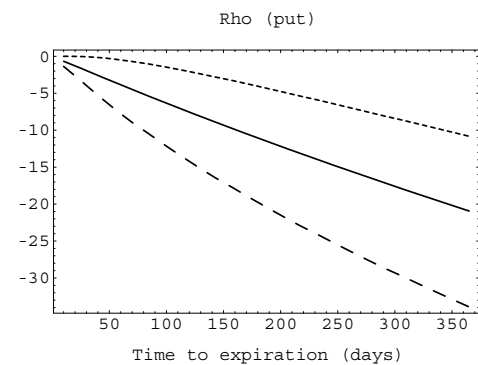
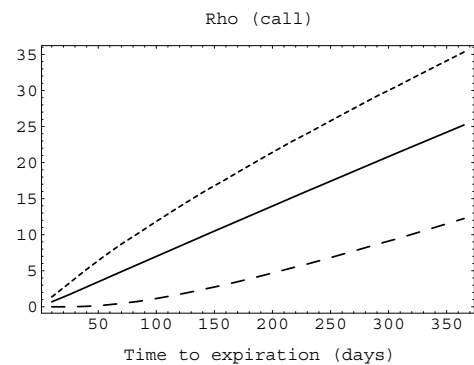
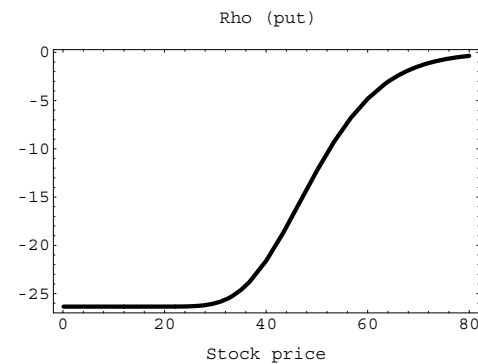
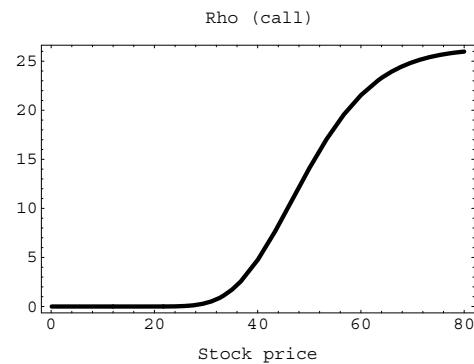
$$\rho \triangleq \frac{\partial f}{\partial r}.$$

- The rho of a European call on a non-dividend-paying stock is

$$X\tau e^{-r\tau} N(x - \sigma\sqrt{\tau}) > 0.$$

- The rho of a European put on a non-dividend-paying stock is

$$-X\tau e^{-r\tau} N(-x + \sigma\sqrt{\tau}) < 0.$$



Dotted curves: in-the-money call or out-of-the-money put.
 Solid curves: at-the-money option.
 Dashed curves: out-of-the-money call or in-the-money put.

Numerical Greeks

- Needed when closed-form formulas do not exist.
- Take delta as an example.
- A standard method computes the finite difference,

$$\frac{f(S + \Delta S) - f(S - \Delta S)}{2\Delta S}.$$

- The computation time roughly doubles that for evaluating the derivative security itself.

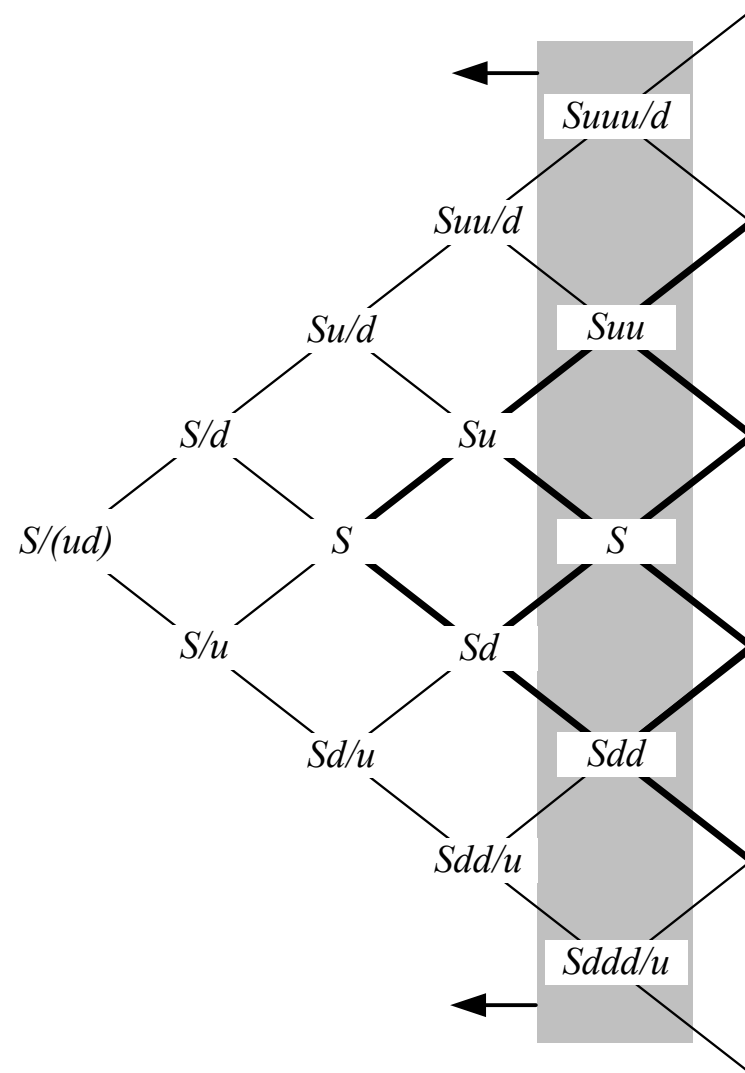
An Alternative Numerical Delta^a

- Use intermediate results of the binomial tree algorithm.
- When the algorithm reaches the end of the first period, f_u and f_d are computed.
- These values correspond to derivative values at stock prices Su and Sd , respectively.
- Delta is approximated by

$$\frac{f_u - f_d}{Su - Sd}. \quad (53)$$

- Essentially zero extra cost.

^aPelsser & Vorst (1994).



Numerical Gamma

- At the stock price $(S_{uu} + S_{ud})/2$, delta is approximately $(f_{uu} - f_{ud})/(S_{uu} - S_{ud})$.
- At the stock price $(S_{ud} + S_{dd})/2$, delta is approximately $(f_{ud} - f_{dd})/(S_{ud} - S_{dd})$.
- Gamma is the rate of change in deltas between $(S_{uu} + S_{ud})/2$ and $(S_{ud} + S_{dd})/2$, that is,

$$\frac{\frac{f_{uu} - f_{ud}}{S_{uu} - S_{ud}} - \frac{f_{ud} - f_{dd}}{S_{ud} - S_{dd}}}{(S_{uu} - S_{dd})/2}. \quad (54)$$

Alternative Numerical Delta and Gamma^a

- Let $\epsilon \equiv \ln u$.
- Think in terms of $\ln S$.
- Then

$$\left(\frac{f_u - f_d}{2\epsilon} \right) \frac{1}{S}$$

approximates the numerical delta.

- And

$$\left(\frac{f_{uu} - 2f_{ud} + f_{dd}}{\epsilon^2} - \frac{f_{uu} - f_{dd}}{2\epsilon} \right) \frac{1}{S^2}$$

approximates the numerical gamma.

^aSee p. 692.

Finite Difference Fails for Numerical Gamma

- Numerical differentiation gives

$$\frac{f(S + \Delta S) - 2f(S) + f(S - \Delta S)}{(\Delta S)^2}.$$

- It does not work (see text for the reason).
- In general, calculating gamma is a hard problem numerically.^a
- But why did the binomial tree version work?

^aRecall p. 117.

Other Numerical Greeks

- The theta can be computed as

$$\frac{f_{ud} - f}{2(\tau/n)}.$$

- In fact, the theta of a European option can be derived from delta and gamma.^a
- The vega of a European option can be derived from gamma.^b
- For rho, there seems no alternative but to run the binomial tree algorithm twice.^c

^aSee p. 690.

^bRecall identity (362) on p. 362.

^cBut see p. 872 and pp. 1066ff.

Extensions of Options Theory

As I never learnt mathematics,
so I have had to think.
— Joan Robinson (1903–1983)

Pricing Corporate Securities^a

- Interpret the underlying asset as the firm's total value.^b
- The option pricing methodology can be applied to price corporate securities.
- The result is Merton's (1974) structural model.
- Assumptions:
 - A firm can finance payouts by the sale of assets.
 - If a promised payment to an obligation other than stock is missed, the claim holders take ownership of the firm and the stockholders get nothing.

^aBlack & Scholes (1973); Merton (1974).

^bMore realistic models posit $\text{firm value} = \text{asset value} + \text{tax benefits} - \text{bankruptcy costs}$ (Leland & Toft, 1996).

Risky Zero-Coupon Bonds and Stock

- Consider XYZ.com.
- Capital structure:
 - n shares of its own common stock, S .
 - Zero-coupon bonds with an aggregate par value of X .
- What is the value of the bonds, B ?
- What is the value of the XYZ.com stock, S ?

Risky Zero-Coupon Bonds and Stock (continued)

- On the bonds' maturity date, suppose the total value of the firm V^* is less than the bondholders' claim X .
- Then the firm declares bankruptcy, and the stock becomes worthless.
- If $V^* > X$, then the bondholders obtain X and the stockholders $V^* - X$.

	$V^* \leq X$	$V^* > X$
Bonds	V^*	X
Stock	0	$V^* - X$

Risky Zero-Coupon Bonds and Stock (continued)

- The stock has the same payoff as a call!
- It is a call on the total value of the firm with a strike price of X and an expiration date equal to the bonds'.
 - This call provides the limited liability for the stockholders.
- The bonds are a covered call^a on the total value of the firm.
- Let V stand for the total value of the firm.
- Let C stand for a call on V .

^aRecall p. 204.

Risky Zero-Coupon Bonds and Stock (continued)

- Thus

$$\begin{aligned}nS &= C \text{ (market capitalization of XYZ.com),} \\ B &= V - C.\end{aligned}$$

- Knowing C amounts to knowing how the value of the firm is divided between stockholders and bondholders.
- Whatever the value of C , the total value of the stock and bonds at maturity remains V^* .
- The relative size of debt and equity is irrelevant to the firm's current value V .

Risky Zero-Coupon Bonds and Stock (continued)

- From Theorem 11 (p. 308) and the put-call parity,

$$nS = VN(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}), \quad (55)$$

$$B = VN(-x) + Xe^{-r\tau}N(x - \sigma\sqrt{\tau}). \quad (56)$$

– Above,

$$x \triangleq \frac{\ln(V/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

- The continuously compounded yield to maturity of the firm's bond is

$$\frac{\ln(X/B)}{\tau}.$$

Risky Zero-Coupon Bonds and Stock (continued)

- Define the credit spread or default premium as the yield difference between risky and riskless bonds,

$$\begin{aligned} & \frac{\ln(X/B)}{\tau} - r \\ &= -\frac{1}{\tau} \ln \left[N(-z) + \frac{1}{\omega} N(z - \sigma\sqrt{\tau}) \right]. \end{aligned}$$

$$- \omega \triangleq X e^{-r\tau} / V.$$

$$- z \triangleq \ln \omega / (\sigma\sqrt{\tau}) + (1/2) \sigma\sqrt{\tau} = -x + \sigma\sqrt{\tau}.$$

– Note that ω is the debt-to-total-value ratio.

Risky Zero-Coupon Bonds and Stock (concluded)

- In general, suppose the firm has a dividend yield at rate q and the bankruptcy costs are a constant proportion α of the remaining firm value.
- Then Eqs. (54)–(55) on p. 382 become, respectively,^a

$$nS = Ve^{-q\tau}N(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}),$$

$$B = (1 - \alpha)Ve^{-q\tau}N(-x) + Xe^{-r\tau}N(x - \sigma\sqrt{\tau}).$$

– Above,

$$x \triangleq \frac{\ln(V/X) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

^aCollin-Dufresne & Goldstein (2000).

A Numerical Example

- XYZ.com's assets consist of 1,000 shares of Merck as of March 20, 1995.
 - Merck's market value per share is \$44.5.
- It issues 1,000 shares of XYZ.com common stock and 30 zero-coupon bonds maturing on July 21, 1995.
- Each bond promises to pay \$1,000 at maturity.
- $n = 1,000$, $V = 44.5 \times n = 44,500$, and $X = 30 \times 1,000 = 30,000$.
- As the Merck calls are being traded, we do not need formulas to price them.

Option	Strike	Exp.	—Call—		—Put—	
			Vol.	Last	Vol.	Last
Merck	30	Jul	328	15 ¹ / ₄
44 ¹ / ₂	35	Jul	150	9 ¹ / ₂	10	1/16
44 ¹ / ₂	40	Apr	887	43/4	136	1/16
44 ¹ / ₂	40	Jul	220	5 ¹ / ₂	297	1/4
44 ¹ / ₂	40	Oct	58	6	10	1/2
44 ¹ / ₂	45	Apr	3050	7/8	100	11/8
44 ¹ / ₂	45	May	462	13/8	50	13/8
44 ¹ / ₂	45	Jul	883	115/16	147	13/4
44 ¹ / ₂	45	Oct	367	23/4	188	21/16

A Numerical Example (continued)

- The Merck option relevant for pricing is the July call with a strike price of $X/n = 30$ dollars.
- Such a call is selling for \$15.25.
- So XYZ.com's stock is worth $15.25 \times n = 15,250$ dollars.
- The entire bond issue is worth

$$B = 44,500 - 15,250 = 29,250$$

dollars.

– Or \$975 per bond.

A Numerical Example (continued)

- The XYZ.com bonds are equivalent to a default-free zero-coupon bond with $\$X$ par value plus n written European puts on Merck at a strike price of \$30.
 - By the put-call parity.^a
- The difference between B and the price of the default-free bond is the value of these puts.
- The next table shows the total market values of the XYZ.com stock and bonds under various debt amounts X .

^aRecall p. 230.

Promised payment to bondholders	Current market value of bonds	Current market value of stock	Current total value of firm
X	B	nS	V
30,000	29,250.0	15,250.0	44,500
35,000	35,000.0	9,500.0	44,500
40,000	39,000.0	5,500.0	44,500
45,000	42,562.5	1,937.5	44,500

A Numerical Example (continued)

- Suppose the promised payment to bondholders is \$45,000.
- Then the relevant option is the July call with a strike price of $45,000/n = 45$ dollars.
- Since that option is selling for $\$1_{15/16}$, the market value of the XYZ.com stock is $(1 + 15/16) \times n = 1,937.5$ dollars.
- The market value of the stock decreases as the debt-to-equity ratio increases.

A Numerical Example (continued)

- There are conflicts between stockholders and bondholders.
- An option's terms cannot be changed after issuance.
- But a firm can change its capital structure.
- There lies one key difference between options and corporate securities.
 - Parameters such volatility,^a dividend, and strike price are under partial control of the stockholders or boards.

^aThis is called the asset substitution problem (Myers, 1977).

A Numerical Example (continued)

- Suppose XYZ.com issues 15 more bonds with the same terms to buy back stock.
- The total debt is now $X = 45,000$ dollars.
- The table on p. 389 says the total market value of the bonds should be \$42,562.5.
- The *new* bondholders pay

$$42,562.5 \times (15/45) = 14,187.5$$

dollars.

- The remaining stock is worth \$1,937.5.

A Numerical Example (continued)

- The stockholders therefore gain

$$14,187.5 + 1,937.5 - 15,250 = 875$$

dollars.

- The *original* bondholders lose an equal amount,

$$29,250 - \frac{30}{45} \times 42,562.5 = 875.$$

– This is called claim dilution.^a

^aFama & M. H. Miller (1972).

A Numerical Example (continued)

- Suppose the stockholders sell $(1/3) \times n$ Merck shares to fund a \$14,833.3 cash dividend.
- The stockholders now have \$14,833.3 in cash plus a call on $(2/3) \times n$ Merck shares.
- The strike price remains $X = 30,000$.
- This is equivalent to owning $2/3$ of a call on n Merck shares with a strike price of \$45,000.
- n such calls are worth \$1,937.5 (p. 389).
- So the total market value of the XYZ.com stock is $(2/3) \times 1,937.5 = 1,291.67$ dollars.

A Numerical Example (concluded)

- The market value of the XYZ.com bonds is hence

$$(2/3) \times n \times 44.5 - 1,291.67 = 28,375$$

dollars.

- Hence the stockholders gain

$$14,833.3 + 1,291.67 - 15,250 \approx 875$$

dollars.

- The bondholders watch their value drop from \$29,250 to \$28,375, a loss of \$875.

Further Topics

- Other examples:^a
 - Stock as compound call when company issues coupon bonds.
 - Subordinated debts as bull call spreads.
 - Warrants as calls.
 - Callable bonds as American calls with 2 strike prices.
 - Convertible bonds.
 - Bonds with safety covenants as barrier options.

^aCox & Rubinstein (1985); Geske (1977).

Further Topics (concluded)

- Securities issued by firms with a complex capital structure must be solved by numerical methods such as trees.^a

^aT. Dai (B82506025, R86526008, D8852600), Lyuu, & C. Wang (F95922018) (2010).