Spot and Forward Rates under Continuous Compounding

- The pricing formula:

$$P = \sum_{i=1}^{n} Ce^{-iS(i)} + Fe^{-nS(n)}.$$

- The market discount function:

$$d(n) = e^{-nS(n)}.$$

- The spot rate is an arithmetic average of forward rates,[a]

$$S(n) = \frac{f(0,1) + f(1,2) + \cdots + f(n-1,n)}{n}.$$

---

[a] Compare it with formula (20) on p. 146.

Spot and Forward Rates under Continuous Compounding (continued)

- The formula for the forward rate:[a]

$$f(i,j) = \frac{jS(j) - iS(i)}{j-i}. \tag{22}$$

- The one-period forward rate:[b]

$$f(j,j+1) = -\ln \frac{d(j+1)}{d(j)}.$$

---

[a]Compare it with formula (19) on p. 140.

[b]Compare it with formula (21) on p. 146.

Spot and Forward Rates under Continuous Compounding (concluded)

- Now, the (instantaneous) forward rate curve is:

$$f(T) \triangleq \lim_{\Delta T \to 0} f(T, T + \Delta T) = S(T) + T \frac{\partial S}{\partial T}. \tag{23}$$

- So $f(T) > S(T)$ if and only if $\partial S/\partial T > 0$ (i.e., a normal spot rate curve).

- If $S(T) < -T(\partial S/\partial T)$, then $f(T) < 0$.[a]

---

[a] Consistent with the plot on p. 144. Contributed by Mr. Huang, Hsien-Chun (R03922103) on March 11, 2015.

An Example

- Let the interest rates be continuously compounded.
- Suppose the spot rate curve is[a]

$$S(T) \stackrel{\Delta}{=} 0.08 - 0.05\, e^{-0.18T}.$$

- Then by Eq. (23) on p. 153, the forward rate curve is

$$\begin{aligned} & f(T) \\ = \;& S(T) + TS'(T) \\ = \;& 0.08 - 0.05\, e^{-0.18T} + 0.009T\, e^{-0.18T}. \end{aligned}$$

[a]Hull & White (1994).

Unbiased Expectations Theory

- Forward rate equals the average future spot rate,

$$f(a, b) = E[\, S(a, b)\,]. \tag{24}$$

- It does not imply that the forward rate is an accurate predictor for the future spot rate.
- It implies the maturity strategy and the rollover strategy produce the same result at the horizon "on average."

Unbiased Expectations Theory and Spot Rate Curve

- It implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  - $f(j, j+1) > S(j+1)$ if and only if $S(j+1) > S(j)$ from formula (19) on p. 140.
  - So
    $$E[\, S(j, j+1)\,] > S(j+1) > \cdots > S(1)$$
    if and only if $S(j+1) > \cdots > S(1)$.
- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.

A "Bad" Expectations Theory

- The expected returns[a] on all possible riskless bond strategies are equal for *all* holding periods.
- So
$$(1 + S(2))^2 = (1 + S(1))\, E[\,1 + S(1,2)\,] \tag{25}$$
because of the equivalency between buying a two-period bond and rolling over one-period bonds.
- After rearrangement,
$$\frac{1}{E[\,1 + S(1,2)\,]} = \frac{1 + S(1)}{(1 + S(2))^2}.$$

[a] More precisely, the one-plus returns.

A "Bad" Expectations Theory (continued)

- Now consider two one-period strategies.
  - Strategy one buys a two-period bond for $(1 + S(2))^{-2}$ dollars and sells it after one period.
  - The expected return is

$$E[\,(1 + S(1,2))^{-1}\,]/(1 + S(2))^{-2}.$$

  - Strategy two buys a one-period bond with a return of $1 + S(1)$.

A "Bad" Expectations Theory (continued)

- The theory says the returns are equal:

$$\frac{1+S(1)}{(1+S(2))^2} = E\left[\frac{1}{1+S(1,2)}\right].$$

- Combine this with Eq. (25) on p. 157 to obtain

$$E\left[\frac{1}{1+S(1,2)}\right] = \frac{1}{E[\,1+S(1,2)\,]}.$$

A “Bad” Expectations Theory (concluded)

- But this is impossible save for a certain economy.
  - Jensen’s inequality states that $E[\,g(X)\,] > g(E[\,X\,])$ for any nondegenerate random variable X and strictly convex function g (i.e., $g''(x) > 0$).
  - Use
$$g(x) \stackrel{\Delta}{=} (1+x)^{-1}$$
    to prove our point.

Local Expectations Theory

- The expected rate of return of any bond over *a single period* equals the prevailing one-period spot rate:

$$\frac{E\left[\,(1+S(1,n))^{-(n-1)}\,\right]}{(1+S(n))^{-n}} = 1+S(1) \;\text{ for } \textit{all } n>1.$$

- This theory is the basis of many interest rate models.

Duration, in Practice

- We had assumed parallel shifts in the spot rate curve.
- To handle more general shifts, define a vector $[\,c_1, c_2, \ldots, c_n\,]$ that characterizes the shift.
  - Parallel shift: $[\,1, 1, \ldots, 1\,]$.
  - Twist: $[\,1, 1, \ldots, 1, -1, \ldots, -1\,]$, $[\,1.8, 1.6, 1.4, 1, 0, -1, -1.4, \ldots]$, etc.
  - $\ldots$.
- At least one c_i should be 1 as the reference point.

Duration in Practice (concluded)

- Let
$$P(y) \stackrel{\Delta}{=} \sum_i C_i/(1+S(i)+yc_i)^i$$
be the price associated with the cash flow $C_1, C_2, \ldots$.
- Define duration as
$$-\frac{\partial P(y)/P(0)}{\partial y}\bigg|_{y=0} \quad \text{or} \quad -\frac{P(\Delta y)-P(-\Delta y)}{2P(0)\Delta y}.$$
- Modified duration equals the above when
$$[\,c_1, c_2, \ldots, c_n\,] = [\,1, 1, \ldots, 1\,],$$
$$S(1) = S(2) = \cdots = S(n).$$

Some Loose Ends on Dates

- Holidays.
- Weekends.
- Business days ($T + 2$, etc.).
- Shall we treat a year as 1 year whether it has 365 or 366 days?

Fundamental Statistical Concepts

There are three kinds of lies:
lies, damn lies, and statistics.
— Misattributed to Benjamin Disraeli
(1804–1881)

If 50 million people believe a foolish thing,
it's still a foolish thing.
— George Bernard Shaw (1856–1950)

One death is a tragedy,
but a million deaths are a statistic.
— Josef Stalin (1879–1953)

Moments

- The variance of a random variable X is defined as

$$\mathrm{Var}[\,X\,] \stackrel{\Delta}{=} E\left[\,(X - E[\,X\,])^2\,\right].$$

- The covariance between random variables X and Y is

$$\mathrm{Cov}[\,X, Y\,] \stackrel{\Delta}{=} E\left[\,(X - \mu_X)(Y - \mu_Y)\,\right],$$

  where μ_X and μ_Y are the means of X and Y, respectively.

- Random variables X and Y are uncorrelated if

$$\mathrm{Cov}[\,X, Y\,] = 0.$$

Correlation

- The standard deviation of X is the square root of the variance,

$$\sigma_X \triangleq \sqrt{\mathrm{Var}[\,X\,]}\,.$$

- The correlation (or correlation coefficient) between X and Y is

$$\rho_{X,Y} \triangleq \frac{\mathrm{Cov}[\,X,Y\,]}{\sigma_X \sigma_Y},$$

provided both have nonzero standard deviations.[a]

[a] Wilmott (2009), "the correlations between financial quantities are notoriously unstable." It may even break down "at high-frequency time intervals" (Budish, Cramton, & Shim, 2015).

Variance of Sum

- Variance of a weighted sum of random variables equals

$$\mathrm{Var}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \,\mathrm{Cov}[\,X_i, X_j\,].$$

- It becomes

$$\sum_{i=1}^{n} a_i^2 \,\mathrm{Var}[\,X_i\,]$$

when X_i are uncorrelated.[a]

[a]Bienaymé (1853).

Conditional Expectation

- "$X \mid I$" denotes X conditional on the information set I.
- The information set can be another random variable's value or the past values of X, say.
- The conditional expectation

$$E[\,X \mid I\,]$$

is the expected value of X conditional on I.

  – It is a random variable.

- The law of iterated conditional expectations[a] says

$$E[\,X\,] = E[\,E[\,X \mid I\,]\,].$$

[a] Or the tower law.

Conditional Expectation (concluded)

- If I_2 contains at least as much information as I_1, then

$$E[\,X \,|\, I_1\,] = E[\,E[\,X \,|\, I_2\,] \,|\, I_1\,]. \tag{26}$$

  - I_1 contains price information up to time t_1, and I_2 contains price information up to a later time $t_2 > t_1$.

- In general,

$$I_1 \subseteq I_2 \subseteq \cdots$$

  means the players never forget past data so the information sets are increasing over time.[a]

[a] Hirsa & Neftci (2014). This idea is used in sigma fields and filtration in probability theory.

The Normal Distribution

- A random variable X has the normal distribution with mean μ and variance σ^2 if its probability density function is
$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}.$$
- This is expressed by $X \sim N(\mu, \sigma^2)$.
- The standard normal distribution has zero mean, unit variance, and the following distribution function
$$\text{Prob}[\, X \leq z\,] = N(z) \stackrel{\Delta}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx.$$

Moment Generating Function

- The moment generating function of random variable X is defined as
$$\theta_X(t) \stackrel{\Delta}{=} E[\, e^{tX} \,].$$
- The moment generating function of $X \sim N(\mu, \sigma^2)$ is
$$\theta_X(t) = \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]. \tag{27}$$

The Multivariate Normal Distribution

- If $X_i \sim N(\mu_i, \sigma_i^2)$ are independent, then

$$\sum_i X_i \sim N\left(\sum_i \mu_i, \sum_i \sigma_i^2\right).$$

- Let $X_i \sim N(\mu_i, \sigma_i^2)$, which may not be independent.
- Suppose

$$\sum_{i=1}^n t_i X_i \sim N\left(\sum_{i=1}^n t_i\,\mu_i, \sum_{i=1}^n \sum_{j=1}^n t_i t_j \operatorname{Cov}[\,X_i, X_j\,]\right)$$

for every linear combination $\sum_{i=1}^n t_i X_i$ with $\sum_{i=1}^n \sum_{j=1}^n t_i t_j \operatorname{Cov}[\,X_i, X_j\,] \neq 0$.

The Multivariate Normal Distribution (concluded)

- Then X_i are said to have a multivariate normal distribution.[a]
- With $M \equiv C^{-1}$ and the (i, j)th entry of the matrix M being $M_{i,j}$, the probability density function for the X_i is

$$\frac{1}{\sqrt{(2\pi)^n \det(C)}} \exp\left[-\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (X_i - \mu_i)\, M_{ij}\, (X_j - \mu_j) \right],$$

with a positive-definite covariance matrix

$$C \stackrel{\Delta}{=} [\, \mathrm{Cov}[\, X_i, X_j \,]\,]_{1 \leq i,j \leq n}.$$

[a]Corrected by Mr. Huang, Guo-Hua (R98922107) on March 10, 2010.

Generation of Univariate Normal Distributions

- Let X be uniformly distributed over $(0,1]$ so that

$$\text{Prob}[\,X \leq x\,] = x, \quad 0 < x \leq 1.$$

- Repeatedly draw two samples x_1 and x_2 from X until

$$\omega \stackrel{\Delta}{=} (2x_1 - 1)^2 + (2x_2 - 1)^2 < 1.$$

- Then $c(2x_1 - 1)$ and $c(2x_2 - 1)$ are independent standard normal variables where[a]

$$c \stackrel{\Delta}{=} \sqrt{-2(\ln \omega)/\omega}\,.$$

---

[a] As they are normally distributed, to prove independence, it suffices to prove that they are uncorrelated, which is easy. Thanks to a lively class discussion on March 5, 2014.

A Dirty Trick and a Right Attitude

- Let ξ_i are independent and uniformly distributed over $(0, 1)$.
- A simple method to generate the standard normal variable is to calculate[a]

$$\left(\sum_{i=1}^{12} \xi_i \right) - 6.$$

- But why use 12?
- Recall the mean and variance of ξ_i are $1/2$ and $1/12$, respectively.

---

[a] Jäckel (2002), "this is not a highly accurate approximation and should only be used to establish ballpark estimates."

A Dirty Trick and a Right Attitude (concluded)

- The general formula is

$$\frac{\left(\sum_{i=1}^{n} \xi_i\right) - (n/2)}{\sqrt{n/12}}.$$

- Choosing $n = 12$ yields a formula without the need of division and square-root operations.[a]
- Always blame your random number generator last.[b]
- Instead, check your programs first.

---

[a] Contributed by Mr. Chen, Shih-Hang (`R02723031`) on March 5, 2014.

[b] "The fault, dear Brutus, lies not in the stars but in ourselves that we are underlings." William Shakespeare (1564–1616), *Julius Caesar*.

Generation of Bivariate Normal Distributions

- Pairs of normally distributed variables with correlation ρ can be generated as follows.
- Let X_1 and X_2 be independent standard normal variables.
- Set

$$\begin{aligned} U &\stackrel{\Delta}{=} aX_1, \\ V &\stackrel{\Delta}{=} a\rho X_1 + a\sqrt{1-\rho^2}\, X_2. \end{aligned}$$

Generation of Bivariate Normal Distributions (continued)

- U and V are the desired random variables with

$$\begin{aligned} \text{Var}[\,U\,] &= \text{Var}[\,V\,] = a^2, \\ \text{Cov}[\,U, V\,] &= \rho a^2. \end{aligned}$$

- Note that the mapping from (X_1, X_2) to (U, V) is a one-to-one correspondence for $a \neq 0$.

Generation of Bivariate Normal Distributions (concluded)

- The mapping in matrix form is

$$\begin{bmatrix} U \\ V \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}. \tag{28}$$

The Lognormal Distribution

- A random variable Y is said to have a lognormal distribution if $\ln Y$ has a normal distribution.
- Let $X \sim N(\mu, \sigma^2)$ and $Y \stackrel{\Delta}{=} e^X$.
- The mean and variance of Y are

$$\begin{aligned} \mu_Y &= e^{\mu+\sigma^2/2}, \\ \sigma_Y^2 &= e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right), \end{aligned} \tag{29}$$

respectively.[a]

[a]They follow from $E[\,Y^n\,] = e^{n\mu+n^2\sigma^2/2}$.

The Lognormal Distribution (continued)

- Conversely, suppose Y is lognormally distributed with mean μ and variance σ^2.
- Then $\ln Y$ has a normal distribution with

$$\begin{aligned} E[\,\ln Y\,] &= \ln\left[\,\mu/\sqrt{1+(\sigma/\mu)^2}\,\right], \\ \mathrm{Var}[\,\ln Y\,] &= \ln\left[\,1+(\sigma/\mu)^2\,\right]. \end{aligned}$$

- If X and Y are joint-lognormally distributed, then

$$\begin{aligned} E[\,XY\,] &= E[\,X\,]\,E[\,Y\,]\,e^{\mathrm{Cov}[\,\ln X, \ln Y\,]}, \\ \mathrm{Cov}[\,X, Y\,] &= E[\,X\,]\,E[\,Y\,]\left(e^{\mathrm{Cov}[\,\ln X, \ln Y\,]} - 1\right). \end{aligned}$$

The Lognormal Distribution (concluded)

- Let Y be lognormally distributed such that $\ln Y \sim N(\mu, \sigma^2)$.
- Then
$$\int_a^\infty y f(y)\, dy = e^{\mu+\sigma^2/2}\, N\left(\frac{\mu - \ln a}{\sigma} + \sigma\right). \qquad (30)$$

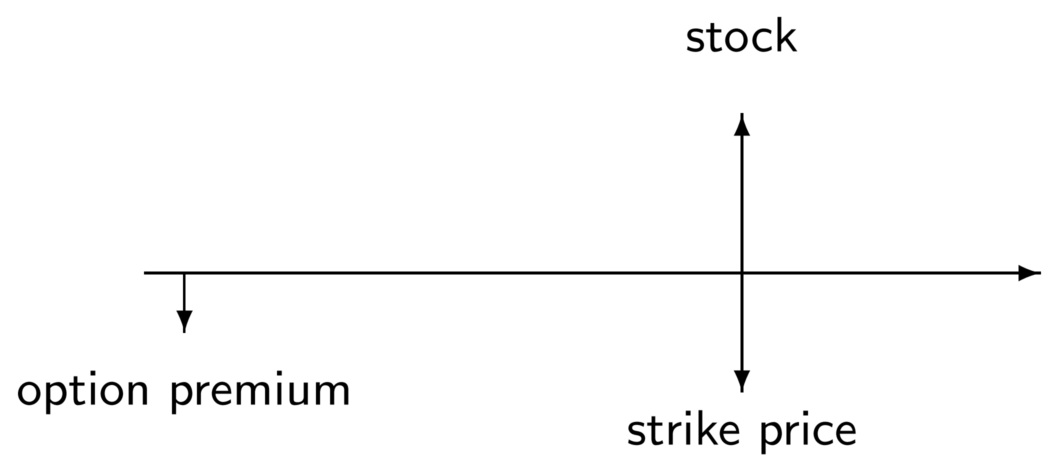
Option Basics

The shift toward options as
the center of gravity of finance [...]
— Merton H. Miller (1923–2000)

Too many potential physicists and engineers
spend their careers shifting money around
in the financial sector,
instead of applying their talents to
innovating in the real economy.
— Barack Obama (2016)

Calls and Puts

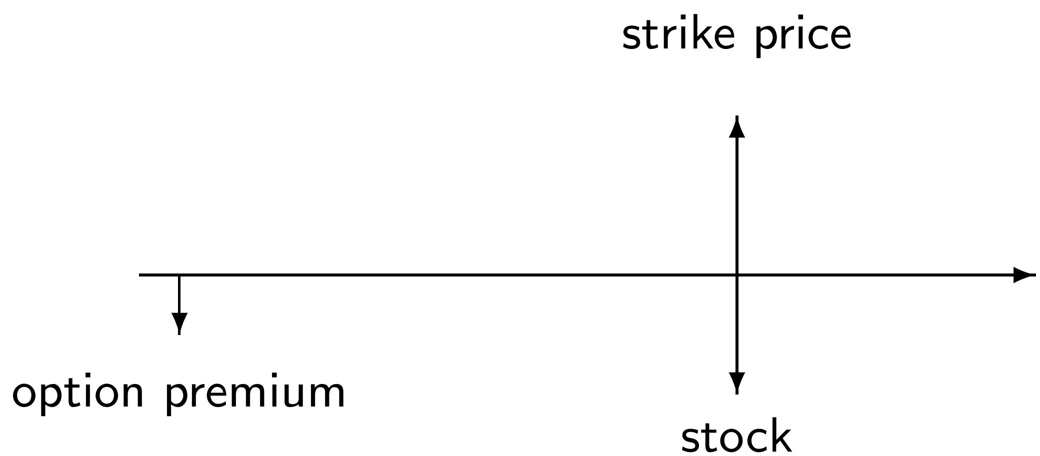
- A call gives its holder the right to *buy* a unit of the underlying asset by paying a strike price.[a]



[a]The cash flow at expiration is contingent.

Calls and Puts (continued)

- A put gives its holder the right to *sell* a unit of the underlying asset for the strike price.



Calls and Puts (concluded)

- An embedded option has to be traded along with the underlying asset.
- How to price options?
  - It can be traced to Aristotle's (384 B.C.–322 B.C.) *Politics*, if not earlier.

Exercise

- When a call is exercised, the holder pays the strike price in exchange for the stock.
- When a put is exercised, the holder receives from the writer the strike price in exchange for the stock.
- Some options can be exercised prior to the expiration date.
  - This is called early exercise.

American and European

- American options can be exercised at any time up to the expiration date.
- European options can only be exercised at expiration.
- An American option is worth at least as much as an otherwise identical European option.

Convenient Conventions

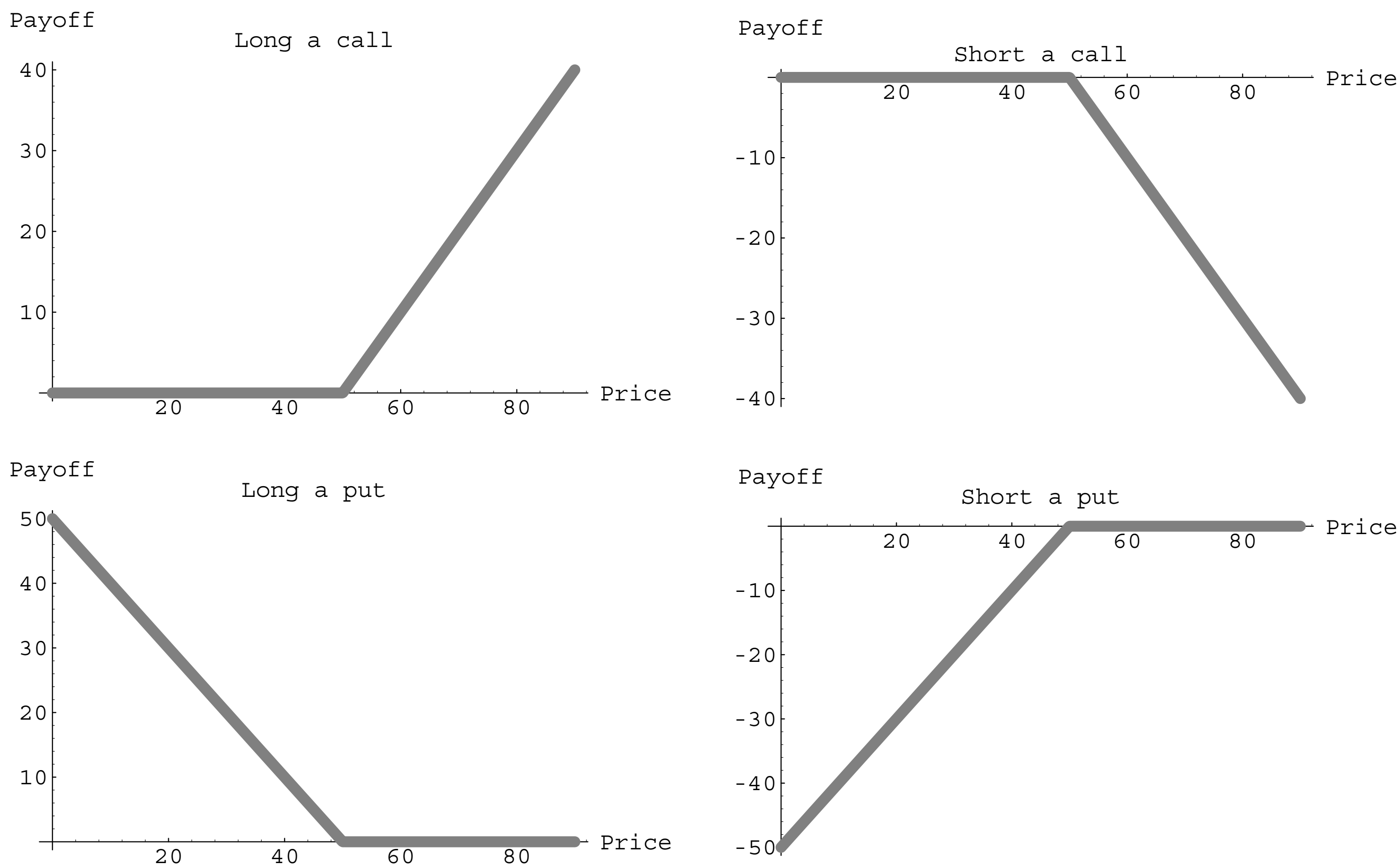
- C: call value.
- P: put value.
- X: strike price.
- S: stock price.[a]
- D: dividend.

---

[a] Assume $S \geq 0$. Contributed by Mr. Tang, Bert (`B08902102`) on March 10, 2021.

Payoff, Mathematically Speaking

- The payoff of a call at expiration is
$$C = \max(0, S - X).$$
- The payoff of a put at expiration is
$$P = \max(0, X - S).$$
- A call will be exercised only if the stock price is higher than the strike price.
- A put will be exercised only if the stock price is less than the strike price.


Payoff
Long a call
40
30
20
10
20
40
60
80
Price
Payoff
Short a call
20
40
60
80
Price
-10
-20
-30
-40
Payoff
Long a put
50
40
30
20
10
20
40
60
80
Price
Payoff
Short a put
20
40
60
80
Price
-10
-20
-30
-40
-50

Payoff, Mathematically Speaking (continued)

- At any time t before the expiration date, we call

$$\max(0, S_t - X)$$

  the intrinsic value of a call.

- At any time t before the expiration date, we call

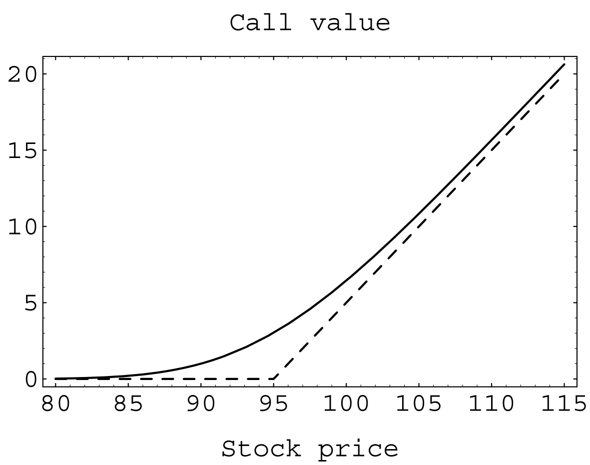
$$\max(0, X - S_t)$$

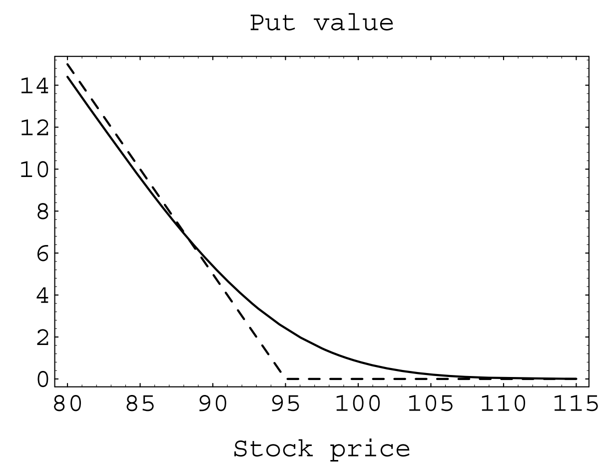
  the intrinsic value of a put.

Payoff, Mathematically Speaking (concluded)

- A call is in the money if $S > X$, at the money if $S = X$, and out of the money if $S < X$.
- A put is in the money if $S < X$, at the money if $S = X$, and out of the money if $S > X$.
- Options that are in the money at expiration should be exercised.[a]
- Finding an option's value at any time *before* expiration is a major intellectual breakthrough.

[a]About 11% of option holders let in-the-money options expire worthless.


Call value
0
5
10
15
20
80
85
90
95
100
105
110
115
Stock price


Put value
0
2
4
6
8
10
12
14
80
85
90
95
100
105
110
115
Stock price

Cash Dividends

- Exchange-traded stock options are not cash dividend-protected (or simply protected).
  - The option contract is not adjusted for *cash* dividends.
- The stock price falls by an amount roughly equal to the amount of the cash dividend as it goes ex-dividend.
- Cash dividends are detrimental for calls.
- The opposite is true for puts.

Stock Splits and Stock Dividends

- Options are adjusted for stock splits.
- After an n-for-m stock split, m shares become n shares.
- Accordingly, the strike price is only m/n times its previous value, and the number of shares covered by one option becomes n/m times its previous value.
- Exchange-traded stock options are adjusted for stock dividends.
- We assume options are unprotected.

Example

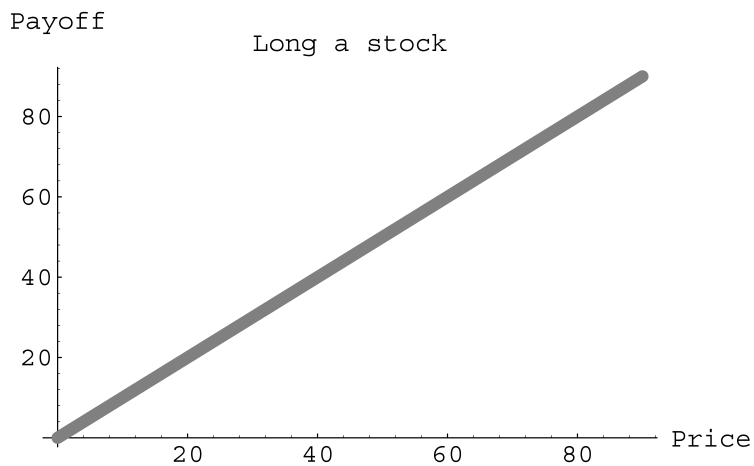
- Consider an option to buy 100 shares of a company for \$50 per share.
- A 2-for-1 split changes the term to a strike price of \$25 per share for 200 shares.

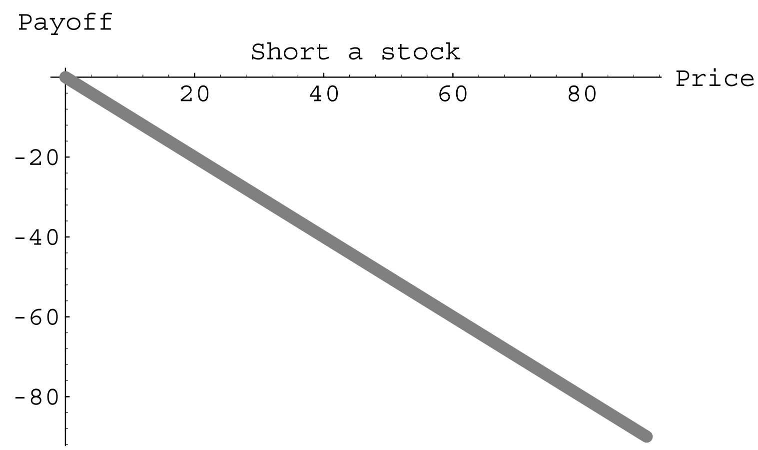
Short Selling

- Short selling[a] involves selling an asset that is *not* owned with the intention of buying it back later.
  - If you short 1,000 XYZ shares, the broker borrows them from another client to sell them in the market.
  - This action generates proceeds for the investor.
  - The investor can close out the short position by buying 1,000 XYZ shares.
- Clearly, the investor profits if the stock price falls.

[a]Or shorting. It was invented by Le Maire in 1608.

Payoff of Stock





Short Selling (concluded)

- Not all assets can be shorted.
- In reality, short selling is not simply the opposite of going long.[a]

---

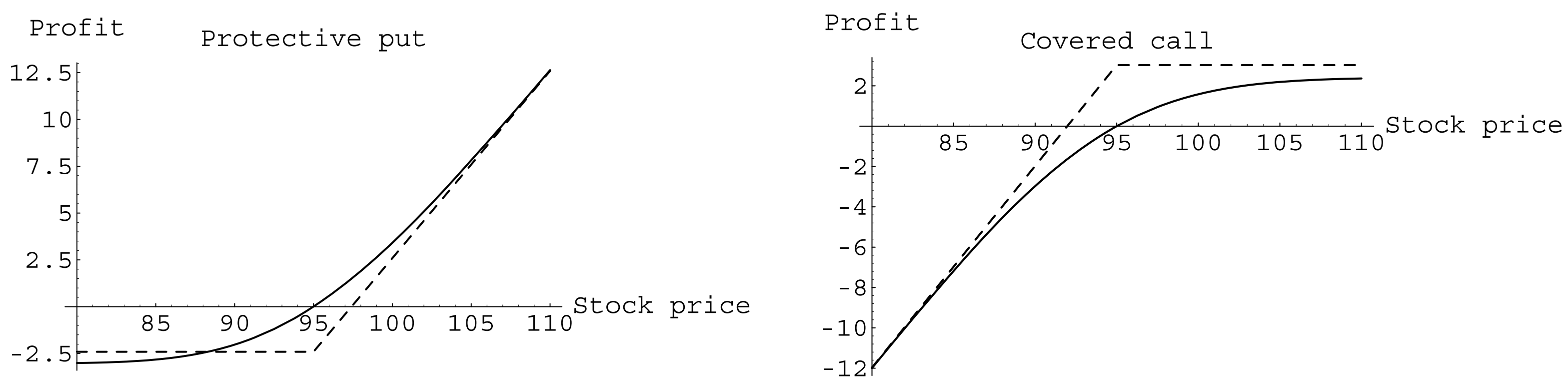
[a] Kosowski & Neftci (2015). See https://tw.news.appledaily.com/headline/daily/20180307/37950481/ for an example in Taiwan on February 6, 2018.

Covered Position: Hedge

- A hedge combines an option with its underlying stock in such a way that one protects the other against loss.
- Covered call: A long position in stock with a short call.[a]
  - It is "covered" because the stock can be delivered to the buyer of the call if the call is exercised.
- Protective put: A long position in stock with a long put.
- Both strategies break even only if the stock price rises above a certain level, so they are bullish.

---

[a] A short position has a payoff opposite in sign to that of a long position. Some ETFs offer this payoff, such as the Global X Nasdaq 100 Covered Call ETF (QYLD).



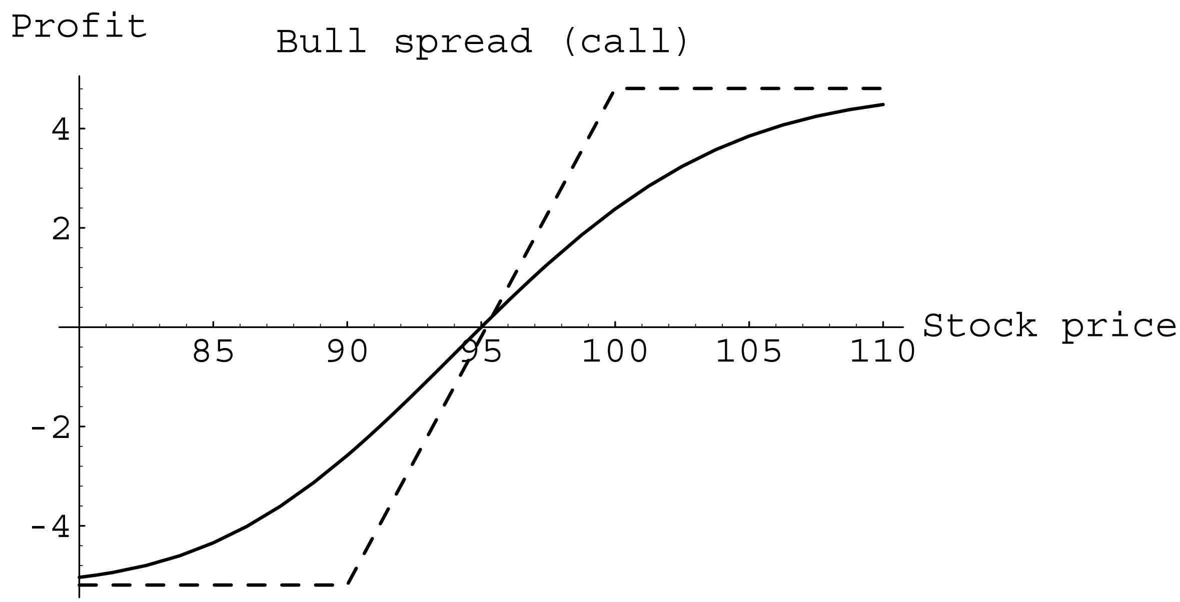
Solid lines are profits of the portfolio one month before maturity, assuming the portfolio is set up when $S = 95$ then.

Covered Position: Spread

- A spread consists of options of the same type and on the same underlying asset but with different strike prices or expiration dates.
- We use X_L, X_M, and X_H to denote the strike prices with
$$X_L < X_M < X_H.$$

Covered Position: Spread (continued)

- A bull call spread consists of a long X_L call and a short X_H call with the same expiration date.
  - The initial investment is $C_L - C_H$.
  - The payoff is nonnegative.
  - The maximum payoff is $X_H - X_L$.
    - When both are exercised at expiration.
  - The maximum profit is $(X_H - X_L) - (C_L - C_H)$.
  - The maximum loss is $C_L - C_H$.
    - When neither is exercised at expiration.


Profit
Bull spread (call)
4
2
-2
-4
85
90
95
100
105
110
Stock price

Covered Position: Spread (continued)

- If we buy $(X_H - X_L)^{-1}$ units of the bull call spread and $X_H - X_L \to 0$, a (Heaviside) step function emerges as the payoff.
- This payoff defines the binary (or digital) call.
- The binary call thus costs

$$-\frac{\partial C}{\partial X}$$

  today.
  - Recall that C is the (standard) call's price.
- This formula is model independent!

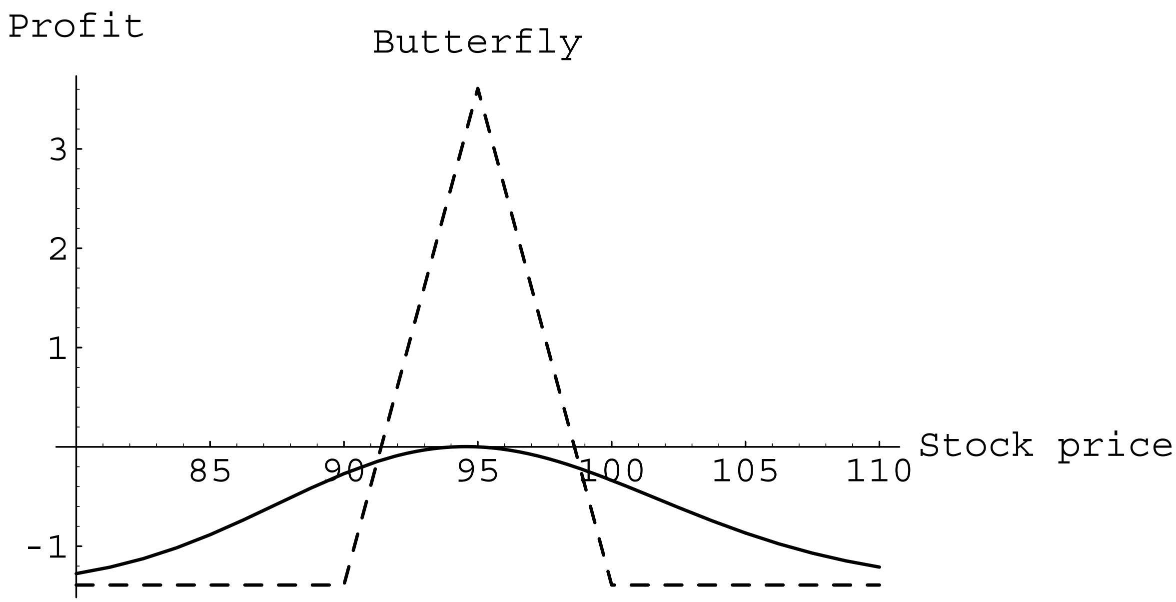
Covered Position: Spread (continued)

- Writing an X_H put and buying an X_L put with identical expiration date creates the bull put spread.[a]
- A bear spread amounts to selling a bull spread.
- It profits from declining stock prices.

[a] See `https://www.businesstoday.com.tw/article/category/80392/post/201803070` for a sad example in Taiwan on February 6, 2018.

Covered Position: Spread (continued)

- Three calls or three puts with different strike prices and the same expiration date create a butterfly spread.
  - The spread is long one X_L call, long one X_H call, and short *two* X_M calls.
- Same as long a bull call spread with strike prices X_L and X_M and short a bull call spread with strike prices X_M and X_H.
- A butterfly spread has a positive payoff at expiration only if the asset price falls between X_L and X_H.


Profit
Butterfly
3
2
1
-1
85
90
95
100
105
110
Stock price

Covered Position: Spread (continued)

- Assume $X_M = (X_H + X_L)/2$.
- Take a position in $(X_M - X_L)^{-1}$ units of the butterfly spread.
- When $X_H - X_L \to 0$, it approximates a state contingent claim,[a] which pays \$1 only in the state $S = X_M$.[b]

---

[a] Alternatively, Arrow security in honor of Kenneth Arrow (1921–2017).

[b] See Exercise 7.4.5 of the textbook.

Covered Position: Spread (concluded)

- The price of a state contingent claim is called a state price.
- The state price equals[a]

$$\frac{\partial^2 C}{\partial X^2}.$$

- In fact, the FV of $\partial^2 C/\partial X^2$ is the probability density of the stock price $S_T = X$ at option's maturity.[b]
- You can buy a butterfly spread if you believe the probability of $S_T \approx X$ is higher than this probability.

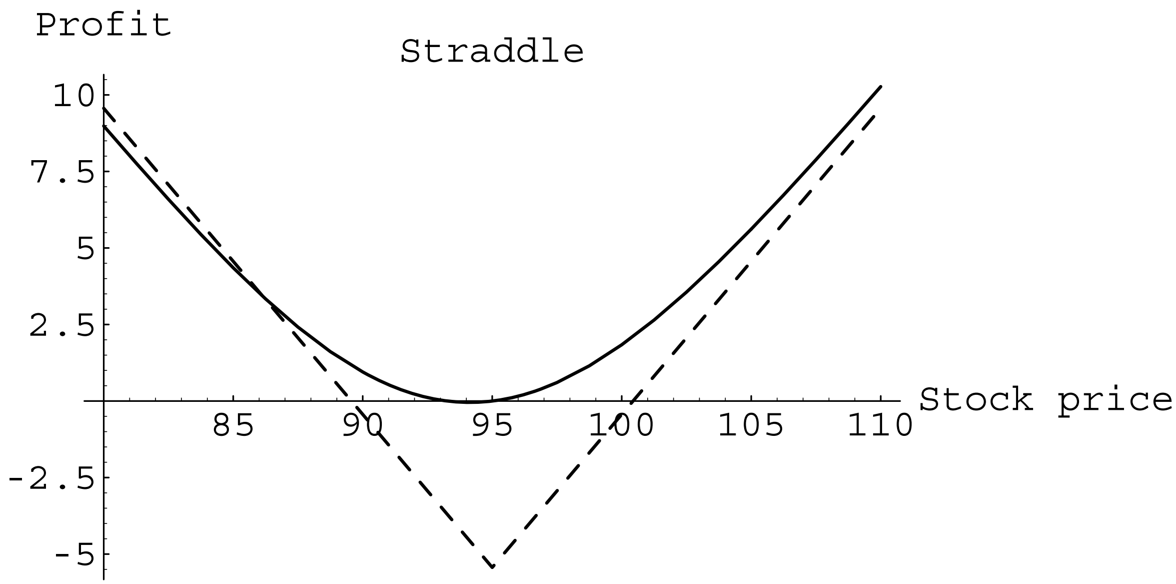
---

[a] One can also use the put (see Exercise 9.3.6 of the textbook).

[b] Breeden & Litzenberger (1978). This formula is model independent!

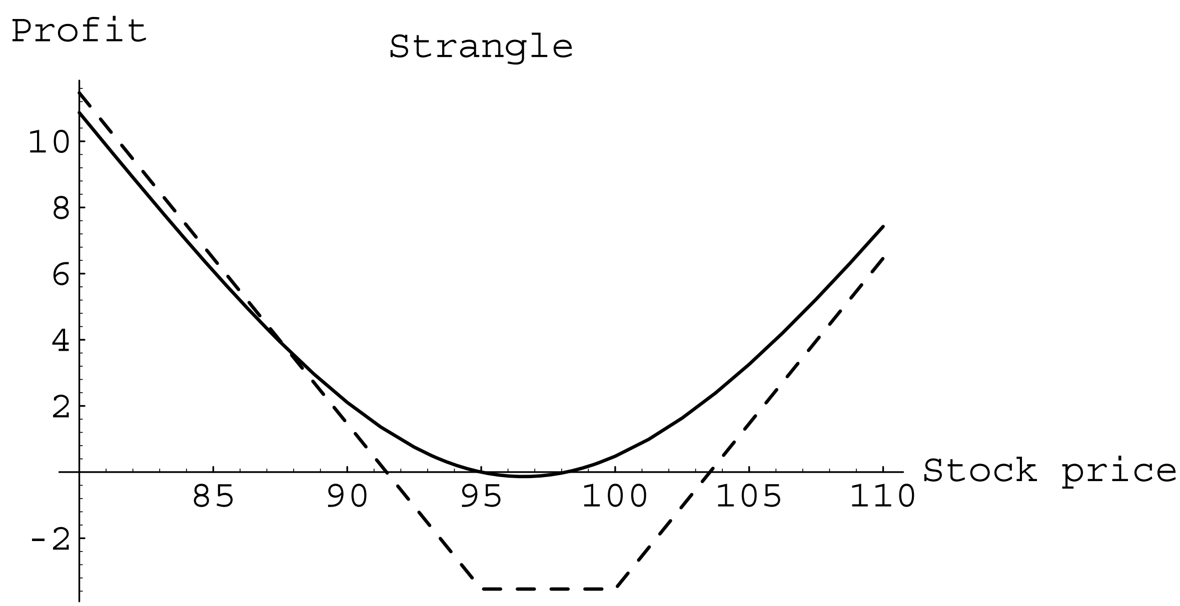
Covered Position: Combination

- A combination consists of options of different types on the same underlying asset.
  - These options must be either all bought or all written.
- Straddle: A long call and a long put with the same strike price and expiration date.
  - Since it profits from high volatility, a person who buys a straddle is "long volatility."
  - Selling a straddle benefits from low volatility.


Profit
Straddle
10
7.5
5
2.5
-2.5
-5
85
90
95
100
105
110
Stock price

Covered Position: Combination (concluded)

- Strangle: Identical to a straddle except that the call's strike price is higher than the put's.


Profit
Strangle
10
8
6
4
2
-2
85
90
95
100
105
110
Stock price

Arbitrage in Option Pricing

All general laws are
attended with inconveniences,
when applied to particular cases.
— David Hume (1711–1776)

The problem with QE is
it works in practice,
but it doesn't work in theory.
— Ben Bernanke[a] (2014)

[a] Co-winner of the 2022 Nobel Prize in Economic Sciences.

Arbitrage

- The no-arbitrage principle says there is no free lunch.
- It supplies the argument for option pricing.
- A riskless arbitrage opportunity is one that, without any initial investment, generates nonnegative returns under all circumstances and positive returns under some.
- In an efficient market, such opportunities do not exist (for long).[a]

[a]Forbes (2013), "In the real world of investments, however, there are obvious arguments against the EMH [efficient market hypothesis]. There are investors who have beaten the market—Warren Buffett."

Portfolio Dominance Principle

- Consider two portfolios A and B.
- Suppose A's payoff is at least as good as B's under *all* circumstances and better under *some*.
- Then A should be more valuable than B.

Two Simple Corollaries

- A portfolio yielding a zero return in every possible scenario must have a zero PV.[a]
  - Short the portfolio if its PV is positive.
  - Buy it if its PV is negative.
  - In both cases, a free lunch is created.
- Two portfolios that yield the same return at time T must have the same price before time T.[b]

---

[a] Lyft, Inc. (2019), "We have incurred net losses each year since our inception and we may not be able to achieve or maintain profitability in the future."

[b] Aristotle, "those who are equal should have everything alike."

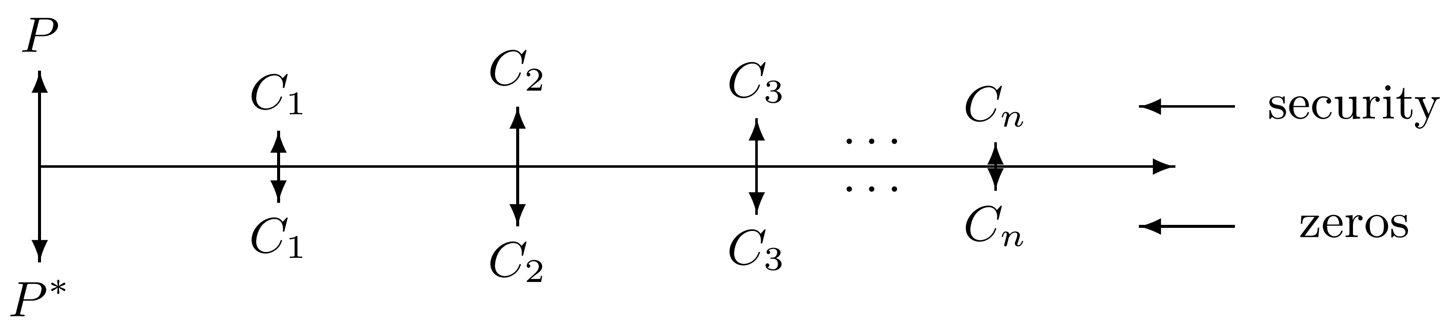
The PV Formula (p. 41) Justified

Theorem 1 *For a certain cash flow* $C_1, C_2, \ldots, C_n$,

$$P = \sum_{i=1}^{n} C_i d(i).$$

- Suppose the price $P^* < P$.
- Short[a] the n zeros that match the security's n cash flows.
- The proceeds are P dollars.

[a] A key assumption.


P
C_1
C_2
C_3
C_n
security
$\cdots$
$\cdots$
C_1
C_2
C_3
C_n
zeros
P^*

The Proof (concluded)

- Then use P^* of the proceeds to buy the security.
- The cash inflows of the security will offset exactly the obligations of the zeros.
- A riskless profit of $P - P^*$ dollars has been realized now.
- If $P^* > P$, just reverse the trades.

One More Example

Theorem 2 *A put or a call must have a nonnegative value.*

- Suppose otherwise and the option has a negative price.
- Buy the option for a positive cash flow now.
- It will end up with a nonnegative amount at expiration.
- So an arbitrage profit is realized now.

Relative Option Prices

- These relations hold regardless of the model for stock prices.
- Assume, among other things, that there are no transactions costs[a] or margin requirements, borrowing and lending are available at the riskless interest rate, interest rates are *nonnegative*, and there are no arbitrage opportunities.

[a] Schwab cut the fees of online trades of stocks and ETFs to zero on October 7, 2019.

Relative Option Prices (concluded)

- Let the current time be time zero.
- $\text{PV}(x)$ stands for the PV of x dollars at expiration.
- Hence

$$\text{PV}(x) = xd(\tau),$$

where τ is the time to expiration.

Put-Call Parity[a]

$$C = P + S - \mathrm{PV}(X). \tag{31}$$

- Consider the portfolio of:
  - One short European call;
  - One long European put;
  - One share of stock;
  - A loan of $\mathrm{PV}(X)$.
- All options are assumed to carry the same strike price X and time to expiration, τ.
- The initial cash flow is therefore

$$C - P - S + \mathrm{PV}(X).$$

[a]Castelli (1877).

The Proof (continued)

- At expiration, if the stock price $S_\tau \leq X$, the put will be worth $X - S_\tau$ and the call will expire worthless.
- The loan is now X.
- The net future cash flow is zero:

$$0 + (X - S_\tau) + S_\tau - X = 0.$$

- On the other hand, if $S_\tau > X$, the call will be worth $S_\tau - X$ and the put will expire worthless.
- The net future cash flow is again zero:

$$-(S_\tau - X) + 0 + S_\tau - X = 0.$$

The Proof (concluded)

- The net future cash flow is zero in either case.
- The no-arbitrage principle[a] implies that the initial investment to set up the portfolio must be nil as well.

---

[a] Recall p. 223.

Consequences of Put-Call Parity

- There is only one kind of European option.
  - The other can be replicated from it in combination with stock and riskless lending or borrowing.
  - Combinations such as this create synthetic securities.[a]
- $S = C - P + \text{PV}(X)$: A stock is equivalent to a portfolio containing a long call, a short put, and lending $\text{PV}(X)$.
- $C - P = S - \text{PV}(X)$: A long call and a short put amount to a long position in stock and borrowing the PV of the strike price (buying stock on margin).

---

[a]Like the synthetic bonds on p. 150.

Intrinsic Value

Lemma 3 *An American call or a European call on a non-dividend-paying stock is never worth less than its intrinsic value.*

- An American call cannot be worth less than its intrinsic value.[a]
- For European options, the put-call parity implies

$$C = (S - X) + (X - \mathrm{PV}(X)) + P \geq S - X.$$

- Recall $C \geq 0$ (p. 227).
- It follows that $C \geq \max(S - X, 0)$, the intrinsic value.

[a]See Lemma 8.3.1 of the textbook.

Intrinsic Value (concluded)

A European *put* on a non-dividend-paying stock may be worth less than its intrinsic value $X - S$.

Lemma 4 *For European puts,* $P \geq \max(\mathrm{PV}(X) - S, 0)$.

- Prove it with the put-call parity.[a]
- Can explain the right figure on p. 197 why $P < X - S$ when S is small.

---

[a]See Lemma 8.3.2 of the textbook.

Early Exercise of American Calls

European calls and American calls are identical when the underlying stock pays no dividends!

Theorem 5 (Merton, 1973) *An American call on a non-dividend-paying stock should not be exercised before expiration.*

- By Exercise 8.3.2 of the text, $C \geq \max(S - \text{PV}(X), 0)$.
- If the call is exercised, the value is $S - X$.
- But
$$\max(S - \text{PV}(X), 0) \geq S - X.$$

Remarks

- The above theorem does *not* mean American calls should be kept until maturity.
- What it does imply is that when early exercise is being considered, a *better* alternative is to sell it.
- Early exercise may become optimal for American calls on a dividend-paying stock, however.
  - Options are assumed to be unprotected.
  - Stock price declines as the stock goes ex-dividend.

Early Exercise of American Calls: Dividend Case

Surprisingly, an American call should be exercised only at a few dates.[a]

Theorem 6 (Merton, 1973) *An American call will only be exercised at expiration or just before an ex-dividend date.*

In contrast, it might be optimal to exercise an American put even if the underlying stock does not pay dividends.

[a]See Theorem 8.4.2 of the textbook.

A General Result[a]

Theorem 7 (Cox & Rubinstein, 1985) *Any piecewise linear payoff function can be replicated using a portfolio of calls and puts.*

Corollary 8 *Any sufficiently well-behaved payoff function can be approximated by a portfolio of calls and puts.*

Theorem 9 (Bakshi & Madan, 2000) *Any payoff function with bounded expectation can be replicated by a continuum of out-of-the-money European calls and puts.*

---

[a]See Exercise 8.3.6 of the textbook.