Variance Reduction

- The *statistical* efficiency of Monte Carlo simulation can be measured by the variance of its output.
- If this variance can be lowered without changing the expected value, fewer replications are needed.
- Methods that work in this manner are called variance-reduction techniques.
- Such techniques become practical when the *added* costs are outweighed by the reduction in sampling.

Variance Reduction: Antithetic Variates

- We want to estimate $E[\, g(X_1, X_2, \ldots, X_n)\,]$.
- Let Y_1 and Y_2 be random variables with the same distribution as $g(X_1, X_2, \ldots, X_n)$.
- Then

$$\mathrm{Var}\left[\,\frac{Y_1 + Y_2}{2}\,\right] = \frac{\mathrm{Var}[\,Y_1\,]}{2} + \frac{\mathrm{Cov}[\,Y_1, Y_2\,]}{2}.$$

  - $\mathrm{Var}[\,Y_1\,]/2$ is the variance of the Monte Carlo method with *two independent* replications.
- The variance $\mathrm{Var}[\,(Y_1 + Y_2)/2\,]$ is smaller than $\mathrm{Var}[\,Y_1\,]/2$ when Y_1 and Y_2 are *negatively* correlated.

Variance Reduction: Antithetic Variates (continued)

- For each simulated sample path X, a second one is obtained by *reusing* the first path's random numbers.
- This yields a second sample path Y.
- Two estimates are then obtained: One based on X and the other on Y.
- If N independent sample paths are generated, the antithetic-variates estimator averages over $2N$ estimates.

Variance Reduction: Antithetic Variates (continued)

- Consider process $dX = a_t\, dt + b_t \sqrt{dt}\; \xi$.
- Let g be a function of n samples $X_1, X_2, \ldots, X_n$ on the sample path.
- Suppose one simulation run has realizations $\xi_1, \xi_2, \ldots, \xi_n$ for the normally distributed fluctuation term ξ.
- This generates samples $x_1, x_2, \ldots, x_n$.
- The first estimate is then $g(\boldsymbol{x})$, where $\boldsymbol{x} \stackrel{\Delta}{=} (x_1, x_2 \ldots, x_n)$.

Variance Reduction: Antithetic Variates (concluded)

- The antithetic-variates method does not sample n more numbers from ξ for the second estimate $g(\boldsymbol{x}')$.
- Instead, generate the sample path $\boldsymbol{x}' \stackrel{\Delta}{=} (x'_1, x'_2 \ldots, x'_n)$ from $-\xi_1, -\xi_2, \ldots, -\xi_n$.
- Compute $g(\boldsymbol{x}')$.
- Output $(g(\boldsymbol{x}) + g(\boldsymbol{x}'))/2$.
- Repeat the above steps.

Variance Reduction: Conditioning

- We are interested in estimating $E[\,X\,]$.
- Suppose here is a random variable Z such that $E[\,X \mid Z = z\,]$ can be efficiently and precisely computed.
- $E[\,X\,] = E[\,E[\,X \mid Z\,]\,]$ by the law of iterated conditional expectations.
- Hence the random variable $E[\,X \mid Z\,]$ is also an unbiased estimator of $E[\,X\,]$.

Variance Reduction: Conditioning (concluded)

- As
$$\mathrm{Var}[\,E[\,X \mid Z\,]\,] \leq \mathrm{Var}[\,X\,],$$
$E[\,X \mid Z\,]$ has a smaller variance than observing X directly.
- First, obtain a random observation z on Z.
- Then calculate $E[\,X \mid Z = z\,]$ as our estimate.
  - There is no need to resort to simulation in computing $E[\,X \mid Z = z\,]$.
- The procedure is repeated to reduce the variance.

Control Variates

- Use the analytic solution of a "similar" yet "simpler" problem to improve the solution.
- Suppose we want to estimate $E[X]$ and there exists a random variable Y with a known mean $\mu \triangleq E[Y]$.
- Then $W \triangleq X + \beta(Y - \mu)$ can serve as a "controlled" estimator of $E[X]$ for any constant β.
  - However β is chosen, W remains an unbiased estimator of $E[X]$ as

$$E[W] = E[X] + \beta E[Y - \mu] = E[X].$$

Control Variates (continued)

- Note that

$$\mathrm{Var}[\,W\,] = \mathrm{Var}[\,X\,] + \beta^2\,\mathrm{Var}[\,Y\,] + 2\beta\,\mathrm{Cov}[\,X, Y\,], \tag{126}$$

- Hence W is less variable than X if and only if

$$\beta^2\,\mathrm{Var}[\,Y\,] + 2\beta\,\mathrm{Cov}[\,X, Y\,] < 0. \tag{127}$$

Control Variates (concluded)

- The success of the scheme clearly depends on both β and the choice of Y.
  - For pricing American options, choose Y to be the otherwise identical European option and μ the Black-Scholes formula.[a]
  - For pricing Arithmetic Asian options, choose Y to be the otherwise identical geometric Asian option, μ the formula (59) on p. 449, and $\beta = -1$.
- This approach is often much more effective than the antithetic-variates method.[b]

---

[a]Hull & White (1988).

[b]Boyle, Broadie, & Glasserman (1997).

Choice of Y

- In general, the choice of Y is ad hoc,[a] and experiments must be performed to assess the choice.
- Try to match calls with calls and puts with puts.[b]
- On many occasions, Y is a discretized version of the derivative that gives μ.
  - Discretely monitored geometric Asian option vs. the continuously monitored version.[c]
- The discrepancy can be large (e.g., lookback options).[d]

---

[a] But see T. Dai (B82506025, R86526008, D8852600), C. Chiu (B90201037, R94922072), & Lyuu (2015, 2018).

[b] Contributed by Ms. Teng, Huei-Wen (R91723054) on May 25, 2004.

[c] Priced by formulas (59) on p. 449.

[d] Contributed by Mr. Tsai, Hwai (R92723049) on May 12, 2004.

Optimal Choice of β

- Equation (126) on p. 900 is minimized when

$$\beta = -\mathrm{Cov}[\,X, Y\,]/\mathrm{Var}[\,Y\,].$$

  - It is called beta.

- For this specific β,

$$\mathrm{Var}[\,W\,] = \mathrm{Var}[\,X\,] - \frac{\mathrm{Cov}[\,X, Y\,]^2}{\mathrm{Var}[\,Y\,]} = \left(1 - \rho_{X,Y}^2\right) \mathrm{Var}[\,X\,],$$

  where $\rho_{X,Y}$ is the correlation between X and Y.

Optimal Choice of β (continued)

- The variance can never increase with the optimal choice.
- The stronger X and Y are correlated, the greater the reduction in variance.
- For example, if this correlation is nearly perfect (± 1), we could control X almost exactly.

Optimal Choice of β (continued)

- Typically, neither $\mathrm{Var}[\,Y\,]$ nor $\mathrm{Cov}[\,X,Y\,]$ is known.
- So we cannot hope to obtain the maximum reduction in variance.
- We can guess a β and hope that the resulting W does indeed have a smaller variance than X.
- A second possibility is to use the simulated data to estimate $\mathrm{Var}[\,Y\,]$ and $\mathrm{Cov}[\,X,Y\,]$.
  - How to do it efficiently in terms of time and space?

Optimal Choice of β (concluded)

- Observe that $-\beta$ has the same sign as the correlation between X and Y.
- Hence, if X and Y are positively correlated, then $\beta < 0$ so X is adjusted downward if and only if $Y > \mu$.
- The opposite is true when X and Y are negatively correlated, in which case $\beta > 0$.
- Suppose a suboptimal $\beta + \epsilon$ is used instead.
- The variance increases by only $\epsilon^2 \text{Var}[\,Y\,]$.[a]

[a]Han & Y. Lai (2010).

A Pitfall

- A potential pitfall is to sample X and Y *independently*.
- In this case, $\text{Cov}[\,X, Y\,] = 0$.
- Equation (126) on p. 900 becomes

$$\text{Var}[\,W\,] = \text{Var}[\,X\,] + \beta^2\,\text{Var}[\,Y\,].$$

- So whatever Y is, the variance is *increased*!
- Lesson: X and Y must be correlated.

Problems with the Monte Carlo Method

- The error bound is only probabilistic.
- The probabilistic error bound of $O(1/\sqrt{N}\,)$ does not benefit from regularity of the integrand function.
- The requirement that the points be independent random samples are wasteful because of clustering.
- In reality, pseudorandom numbers generated by completely deterministic means are used.
- Monte Carlo simulation exhibits a great sensitivity on the seed of the pseudorandom-number generator.

Matrix Computation

To set up a philosophy against physics is rash;
philosophers who have done so
have always ended in disaster.
— Bertrand Russell

Definitions and Basic Results

- Let $A \stackrel{\Delta}{=} [\, a_{ij} \,]_{1 \leq i \leq m, 1 \leq j \leq n}$, or simply $A \in \boldsymbol{R}^{m \times n}$, denote an $m \times n$ matrix.
- It can also be represented as $[\, a_1, a_2, \ldots, a_n \,]$ where $a_i \in \boldsymbol{R}^m$ are vectors.
  - Vectors are column vectors unless stated otherwise.
- A is a square matrix when $m = n$.
- The rank of a matrix is the largest number of linearly independent columns.

Definitions and Basic Results (continued)

- A square matrix A is said to be symmetric if $A^{\mathrm{T}} = A$.
- A real $n \times n$ matrix

$$A \stackrel{\Delta}{=} [\, a_{ij} \,]_{i,j}$$

  is diagonally dominant if $|\, a_{ii} \,| > \sum_{j \neq i} |\, a_{ij} \,|$ for $1 \leq i \leq n$.

  - Such matrices are nonsingular.

- The identity matrix is the square matrix

$$I \stackrel{\Delta}{=} \text{diag}[\, 1, 1, \ldots, 1 \,].$$

Definitions and Basic Results (concluded)

- A matrix has full column rank if its columns are linearly independent.
- A real symmetric matrix A is positive definite if

$$x^{\mathrm{T}} A x = \sum_{i,j} a_{ij} x_i x_j > 0$$

  for any nonzero vector x.
- A matrix A is positive definite if and only if there exists a matrix W such that $A = W^{\mathrm{T}} W$ and W has full column rank.

Cholesky Decomposition

- Positive definite matrices can be factored as

$$A = LL^{\mathrm{T}},$$

called the Cholesky decomposition or Cholesky factorization.

  - Above, L is a lower triangular matrix.
  - It can be computed by Crout's algorithm in quadratic time.[a]

[a] Golub & Van Loan (1989).

Generation of Multivariate Distribution

- Let $\boldsymbol{x} \stackrel{\Delta}{=} [\, x_1, x_2, \ldots, x_n \,]^{\mathrm{T}}$ be a vector random variable with a positive definite covariance matrix C.
- As usual, assume $E[\, \boldsymbol{x} \,] = \mathbf{0}$.
- This covariance structure can be matched by $P\boldsymbol{y}$.
  - $\boldsymbol{y} \stackrel{\Delta}{=} [\, y_1, y_2, \ldots, y_n \,]^{\mathrm{T}}$ is a vector of independent random variables with a covariance matrix I.
  - $C = PP^{\mathrm{T}}$ is the Cholesky decomposition of C.[a]

[a]What if C is not positive definite? See Y. Y. Lai (`R93942114`) & Lyuu (2007).

Generation of Multivariate Distribution (concluded)

- For example, suppose

$$C = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

- Then $PP^{\mathrm{T}} = C$, where[a]

$$P = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix}.$$

---

[a] Recall Eq. (28) on p. 181.

Generation of Multivariate Normal Distribution

- Suppose we want to generate the multivariate normal distribution with a covariance matrix $C = PP^{\mathrm{T}}$.
  - First, generate independent standard normal distributions $y_1, y_2, \ldots, y_n$.
  - Then
    $$P[\, y_1, y_2, \ldots, y_n \,]^{\mathrm{T}}$$
    has the desired distribution.

Multivariate Derivatives Pricing

- Generating the multivariate normal distribution is essential for the Monte Carlo pricing of multivariate derivatives.[a]
- For example, the rainbow option on k assets has payoff

$$\max(\max(S_1, S_2, \ldots, S_k) - X, 0)$$

at maturity.

- The closed-form formula is a multi-dimensional integral.[b]

---

[a] Recall pp. 824ff.

[b] Johnson (1987); C. Y. Chen (`D95723006`) & Lyuu (2009).

Multivariate Derivatives Pricing (concluded)

- Suppose $dS_j/S_j = r\,dt + \sigma_j\,dW_j$, $1 \le j \le k$, where C is the correlation matrix for $dW_1, dW_2, \ldots, dW_k$.
- Let $C = PP^{\mathrm{T}}$.
- Let ξ consist of k independent random variables from $N(0,1)$.
- Let $\xi' = P\xi$.
- Similar to Eq. (125) on p. 863, for each asset $1 \le j \le k$,

$$S_{i+1} = S_i e^{(r-\sigma_j^2/2)\,\Delta t + \sigma_j \sqrt{\Delta t}\;\xi_j'}$$

by Eq. (125) on p. 863.

Least-Squares Problems

- The least-squares (LS) problem is concerned with

$$\min_{x \in R^n} \| Ax - b \|,$$

  where $A \in \boldsymbol{R}^{m \times n}$, $b \in \boldsymbol{R}^m$, and $m \geq n$.

- The LS problem is called regression analysis in statistics and is equivalent to minimizing the mean-square error.

- Often abbreviated as

$$Ax = b.$$

Polynomial Regression

- In polynomial regression, $x_0 + x_1 x + \cdots + x_n x^n$ is used to fit the data $\{ (a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m) \}$.
- This leads to the LS problem,

$$\begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^n \\ 1 & a_2 & a_2^2 & \cdots & a_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_m & a_m^2 & \cdots & a_m^n \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

- Consult p. 273 of the textbook for solutions.

American Option Pricing by Simulation

- The continuation value of an American option is the conditional expectation of the payoff from keeping the option alive now.
- The option holder must compare the immediate exercise value and the continuation value.
- In standard Monte Carlo simulation, each path is treated independently of other paths.
- But the exercise decision cannot be reached by looking at just one path.

The Least-Squares Monte Carlo Approach

- Estimate the continuation value from the cross-sectional information in the simulation with least squares.[a]
- The result is a function of the state for estimating it.
- Use the estimated continuation value for each path to determine its cash flow.
- This is called least-squares Monte Carlo (LSM).

[a] Longstaff & Schwartz (2001).

The Least-Squares Monte Carlo Approach (concluded)

- LSM is provably convergent.[a]
- LSM can be easily parallelized.[b]
  - Partition the paths into subproblems and perform LSM on each independently.
  - The speedup is close to linear (i.e., proportional to the number of cores).
- Surprisingly, accuracy is not affected.

---

[a] Clément, Lamberton, & Protter (2002); Stentoft (2004).

[b] K. Huang (B96902079, R00922018) (2013); Demouth (2013); C. W. Chen (B97902046, R01922005) (2014); C. W. Chen (B97902046, R01922005), K. Huang (B96902079, R00922018) & Lyuu (2015); H.-C. Chen (R03922022) (2016); L. X. Li, R.-R. Chen, and Fabozzi (2024).

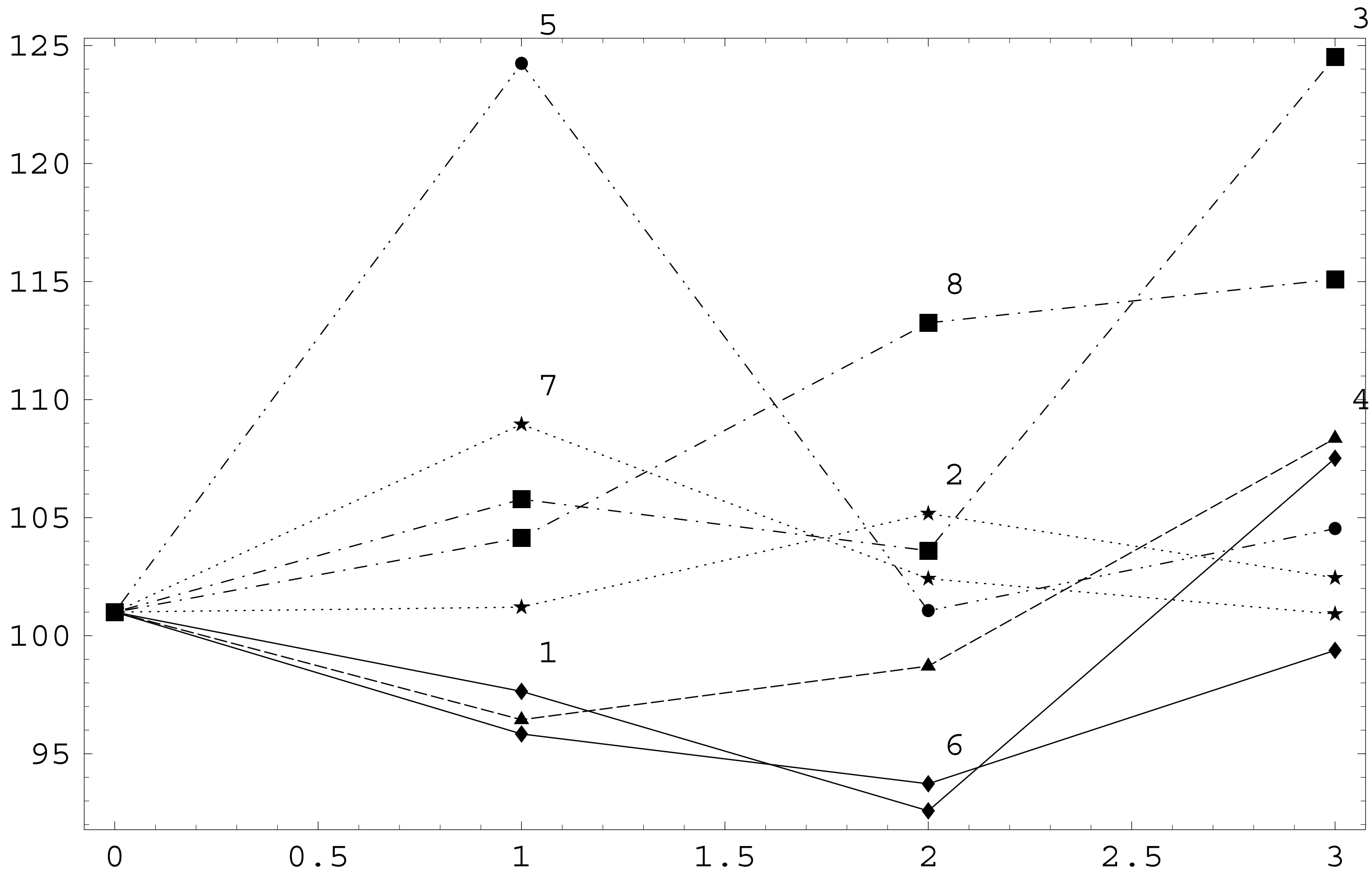
A Numerical Example

- Consider a 3-year American put on a non-dividend-paying stock.
- The put is exercisable at years 0, 1, 2, and 3.
- The strike price $X = 105$.
- The annualized riskless rate is $r = 5\%$.
  - The annual discount factor equals 0.951229.
- The current stock price is 101.
- We use 8 price paths to illustrate the algorithm.

A Numerical Example (continued)

Stock price paths

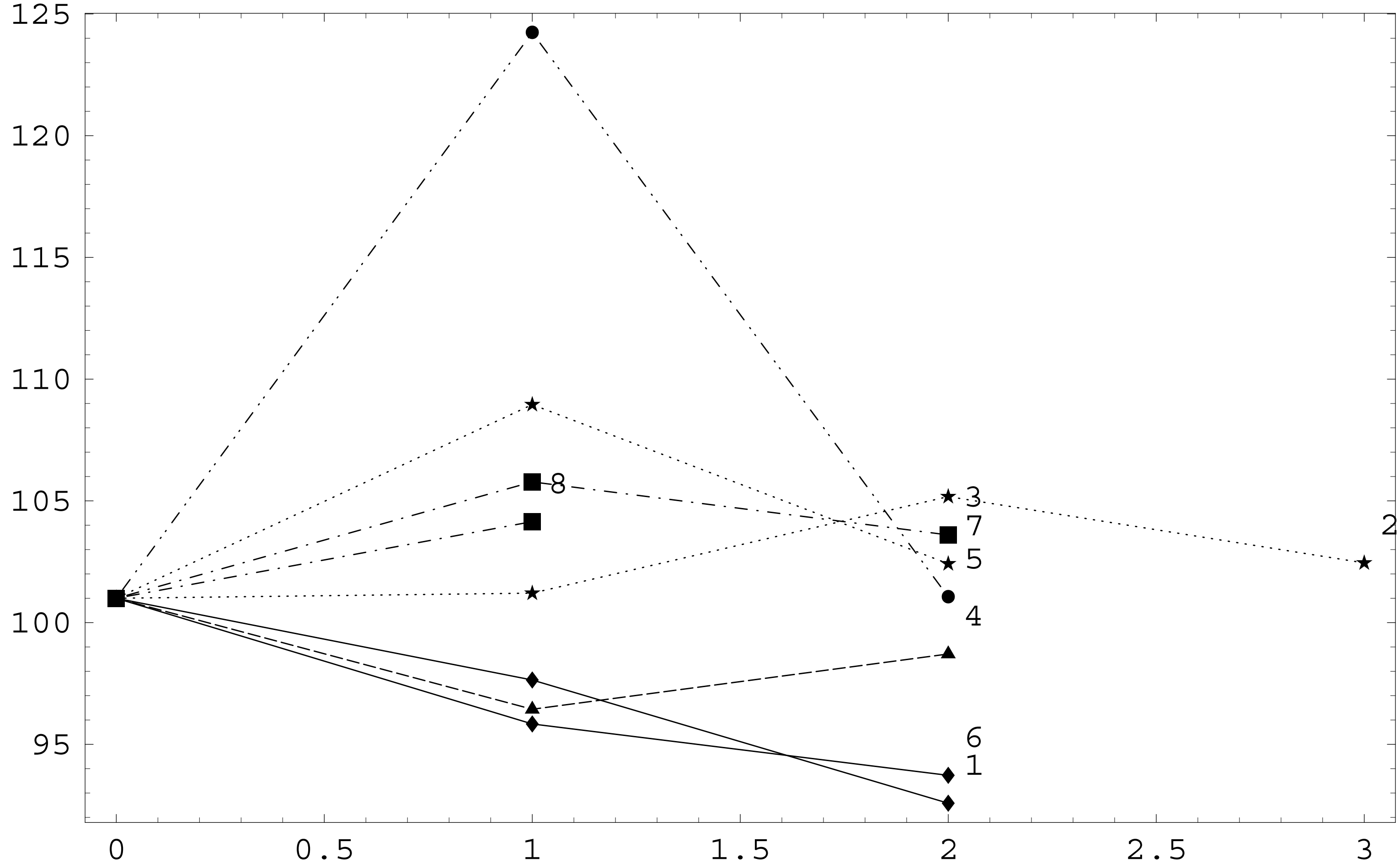
Path	Year 0	Year 1	Year 2	Year 3
1	**101**	**97.6424**	**92.5815**	107.5178
2	**101**	**101.2103**	105.1763	**102.4524**
3	**101**	105.7802	**103.6010**	124.5115
4	**101**	**96.4411**	**98.7120**	108.3600
5	**101**	124.2345	**101.0564**	**104.5315**
6	**101**	**95.8375**	**93.7270**	**99.3788**
7	**101**	108.9554	**102.4177**	**100.9225**
8	**101**	**104.1475**	113.2516	115.0994


125
120
115
110
105
100
95
0
0.5
1
1.5
2
2.5
3
5
3
8
7
4
2
1
6

A Numerical Example (continued)

- We use the basis functions $1, x, x^2$.
  - Other basis functions are possible.[a]
- The plot next page shows the final estimated optimal exercise strategy given by LSM.
- We now proceed to tackle our problem.
- The idea is to calculate the cash flow along each path, using information from *all* in-the-money paths.

[a]Laguerre polynomials, Hermite polynomials, Legendre polynomials, Chebyshev polynomials, Gedenbauer polynomials, or Jacobi polynomials.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4
6
1

A Numerical Example (continued)

Cash flows at year 3

Path	Year 0	Year 1	Year 2	Year 3
1	—	—	—	0
2	—	—	—	2.5476
3	—	—	—	0
4	—	—	—	0
5	—	—	—	0.4685
6	—	—	—	5.6212
7	—	—	—	4.0775
8	—	—	—	0

A Numerical Example (continued)

- The cash flows at year 3 are the put's payoffs.
- Only 4 paths are in the money: 2, 5, 6, 7.
- Some of the cash flows may not materialize if the put is exercised earlier, which we will find out later.
- Incidentally, the *European* counterpart has a value of

$$\begin{aligned} & 0.951229^3 \times \frac{2.5476 + 0.4685 + 5.6212 + 4.0775}{8} \\ = \; & 1.3680. \end{aligned}$$

A Numerical Example (continued)

- We move on to year 2.
- For each state that is in the money at year 2, we must decide whether to exercise it.
- There are 6 paths for which the put is in the money: 1, 3, 4, 5, 6, 7.[a]
- Only in-the-money paths will be used in the regression because they are where early exercise is possible.
  - If there were none, move on to year 1.

---

[a]Recall p. 926.

A Numerical Example (continued)

- Let x denote the stock price at year 2 for each of those 6 paths.
- Let y denote the corresponding discounted future cash flow (at year 3) if the put is *not* exercised at year 2.

A Numerical Example (continued)

Regression at year 2

Path	x	y
1	92.5815	0×0.951229
2	—	—
3	103.6010	0×0.951229
4	98.7120	0×0.951229
5	101.0564	0.4685×0.951229
6	93.7270	5.6212×0.951229
7	102.4177	4.0775×0.951229
8	—	—

A Numerical Example (continued)

- We regress y on 1, x, and x^2.
- The result is

$$f(x) = 22.08 - 0.313114 \times x + 0.00106918 \times x^2.$$

- $f(x)$ estimates the *continuation value* conditional on the stock price at year 2.
- We next compare the immediate exercise value and the estimated continuation value.[a]

[a]The $f(102.4177)$ entry on the next page was corrected by Mr. Tu, Yung-Szu (`B79503054`, `R83503086`) on May 25, 2017.

A Numerical Example (continued)

Optimal early exercise decision at year 2

Path	Exercise	Continuation
1	12.4185	$f(92.5815) = 2.2558$
2	—	—
3	1.3990	$f(103.6010) = 1.1168$
4	6.2880	$f(98.7120) = 1.5901$
5	3.9436	$f(101.0564) = 1.3568$
6	11.2730	$f(93.7270) = 2.1253$
7	2.5823	$f(102.4177) = 1.2266$
8	—	—

A Numerical Example (continued)

- The put should be exercised in all 6 paths: 1, 3, 4, 5, 6, 7.
- Now, any positive cash flow at year 3 vanishes for these paths as the put has been exercised before it.[a]
  - They are paths 5, 6, 7.
- The cash flows on p. 930 become the ones on next slide.

[a]Recall p. 926.

A Numerical Example (continued)

Cash flows at years 2 & 3

Path	Year 0	Year 1	Year 2	Year 3
1	—	—	12.4185	0
2	—	—	0	2.5476
3	—	—	1.3990	0
4	—	—	6.2880	0
5	—	—	3.9436	$\boxed{0}$
6	—	—	11.2730	$\boxed{0}$
7	—	—	2.5823	$\boxed{0}$
8	—	—	0	0

A Numerical Example (continued)

- We move on to year 1.
- For each state that is in the money at year 1, we must decide whether to exercise it.
- There are 5 paths for which the put is in the money: 1, 2, 4, 6, 8.[a]
- Only in-the-money paths will be used in the regression because they are where early exercise is possible.
  - If there were none, move on to year 0.

[a]Recall p. 926.

A Numerical Example (continued)

- Let x denote the stock price at year 1 for each of those 5 paths.
- Let y denote the corresponding discounted future cash flow if the put is not exercised at year 1.
- From p. 938, we have the following table.

A Numerical Example (continued)

Regression at year 1

Path	x	y
1	97.6424	12.4185×0.951229
2	101.2103	2.5476×0.951229^2
3	—	—
4	96.4411	6.2880×0.951229
5	—	—
6	95.8375	11.2730×0.951229
7	—	—
8	104.1475	0×0.951229

A Numerical Example (continued)

- We regress y on 1, x, and x^2.
- The result is

$$f(x) = -420.964 + 9.78113 \times x - 0.0551567 \times x^2.$$

- $f(x)$ estimates the continuation value conditional on the stock price at year 1.
- We next compare the immediate exercise value and the estimated continuation value.

A Numerical Example (continued)

Optimal early exercise decision at year 1

Path	Exercise	Continuation
1	7.3576	$f(97.6424) = 8.2230$
2	3.7897	$f(101.2103) = 3.9882$
3	—	—
4	8.5589	$f(96.4411) = 9.3329$
5	—	—
6	9.1625	$f(95.8375) = 9.83042$
7	—	—
8	0.8525	$f(104.1475) = -0.551885$

A Numerical Example (continued)

- The put should be exercised for 1 path only: 8.
  - Note that its $f(104.1475) < 0$.
- Now, any positive future cash flow vanishes for this path.
  - But there is none.
- The cash flows on p. 938 become the ones on next slide.
- They also confirm the plot on p. 929.

A Numerical Example (continued)

Cash flows at years 1, 2, & 3

Path	Year 0	Year 1	Year 2	Year 3
1	—	0	12.4185	0
2	—	0	0	2.5476
3	—	0	1.3990	0
4	—	0	6.2880	0
5	—	0	3.9436	0
6	—	0	11.2730	0
7	—	0	2.5823	0
8	—	0.8525	0	0

A Numerical Example (continued)

- We move on to year 0.
- The continuation value is, from p 945,

$$
\begin{aligned}
& (12.4185 \times 0.951229^2 + 2.5476 \times 0.951229^3 \\
& +1.3990 \times 0.951229^2 + 6.2880 \times 0.951229^2 \\
& +3.9436 \times 0.951229^2 + 11.2730 \times 0.951229^2 \\
& +2.5823 \times 0.951229^2 + 0.8525 \times 0.951229)/8 \\
= \;& 4.66263.
\end{aligned}
$$

A Numerical Example (concluded)

- As this is larger than the immediate exercise value of

$$105 - 101 = 4,$$

the put should not be exercised at year 0.

- Hence the put's value is estimated to be 4.66263.
- Compare this with the European put's value of 1.3680.[a]

---

[a]Recall p. 931.

Time Series Analysis

The historian is a prophet in reverse.
— Friedrich von Schlegel (1772–1829)

Even in my tape reading something enters
that is more than mere arithmetic.
— Edwin Lefèvre (1971–1943),
Reminiscences of a Stock Operator (1923)

GARCH Option Pricing

- Options can be priced when the underlying asset's return follows a GARCH (generalized autoregressive conditional heteroskedastic) process.[a]
- Let S_t denote the asset price at date t.
- Let h_t^2 be the *conditional* variance of the return over the period $[\,t, t+1)$ given the information at date t.
  - "One day" is merely a convenient term for any elapsed time Δt.

---

[a]Bollerslev (1986) and Taylor (1986). They are the "most popular models for time-varying volatility" (Alexander, 2001). A Bloomberg quant said to me on Feb 29, 2008, that GARCH is seldom used in trading.

GARCH Option Pricing (continued)

- Adopt the following risk-neutral process for price:[a]

$$\ln \frac{S_{t+1}}{S_t} = r - \frac{h_t^2}{2} + h_t \epsilon_{t+1}, \tag{128}$$

where

$$\begin{aligned} h_{t+1}^2 &= \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 (\epsilon_{t+1} - c)^2, &&(129)\\ \epsilon_{t+1} &\sim N(0,1) \text{ given information at date } t, \\ r &= \text{daily riskless return}, \\ c &\geq 0. \end{aligned}$$

- This is called the nonlinear asymmetric GARCH (or NGARCH) model.

[a]Duan (1995).

GARCH Option Pricing (continued)

- The five unknown parameters of the model are c, h_0, β_0, β_1, and β_2.
- It is postulated that $\beta_0, \beta_1, \beta_2 \geq 0$ to make the conditional variance positive.
- There are other inequalities to satisfy such as $\beta_1 + \beta_2 < 1$ (see text).
- It can be shown that $h_t^2 \geq \min\left[\, h_0^2, \beta_0/(1 - \beta_1)\,\right]$.[a]

[a]Lyuu & C. Wu (`R90723065`) (2005).

GARCH Option Pricing (continued)

- It captures the volatility clustering in asset returns first noted by Mandelbrot (1963).[a]
  - When $c = 0$, a large ϵ_{t+1} results in a large h_{t+1}, which in turns tends to yield a large h_{t+2}, and so on.
- It also captures the negative correlation between the asset return and changes in its (conditional) volatility.[b]
  - For $c > 0$, a positive ϵ_{t+1} (good news) tends to decrease h_{t+1}, whereas a negative ϵ_{t+1} (bad news) tends to do the opposite.

[a] "... large changes tend to be followed by large changes—of either sign—and small changes tend to be followed by small changes ..."

[b] Noted by Black (1976): Volatility tends to rise in response to "bad news" and fall in response to "good news."

GARCH Option Pricing (continued)

- This is called the leverage effect.
  - A falling stock price raises the fixed costs, relatively speaking.[a]
  - Thus c is called the leverage effect parameter.
- With $y_t \triangleq \ln S_t$ denoting the logarithmic price, the model becomes

$$y_{t+1} = y_t + r - \frac{h_t^2}{2} + h_t \epsilon_{t+1}. \tag{130}$$

- The pair (y_t, h_t^2) completely describes the current state.

[a] Black (1992).

GARCH Option Pricing (concluded)

- The conditional mean and variance of y_{t+1} are clearly

$$E[\, y_{t+1} \,|\, y_t, h_t^2 \,] = y_t + r - \frac{h_t^2}{2}, \tag{131}$$

$$\mathrm{Var}[\, y_{t+1} \,|\, y_t, h_t^2 \,] = h_t^2. \tag{132}$$

- Finally, given (y_t, h_t^2), the correlation between y_{t+1} and h_{t+1} equals

$$-\frac{2c}{\sqrt{2+4c^2}},$$

which is negative for $c > 0$.

The Ritchken-Trevor (RT) Algorithm[a]

- The GARCH model is a continuous-state model.
- To approximate it, we turn to trees with *discrete* states.
- Path dependence in GARCH makes the tree for asset prices explode exponentially.[b]
- We need to mitigate this combinatorial explosion.

---

[a] Ritchken & Trevor (1999).
[b] Why?

The RT Algorithm (continued)

- Partition a day into n periods.
- Three states follow each state (y_t, h_t^2) after a period.
- As the trinomial model combines, each state at date t is followed by $2n+1$ states at date $t+1$.[a]
- These $2n+1$ values must approximate the distribution of (y_{t+1}, h_{t+1}^2) to guarantee convergence.
- So the conditional moments (131)–(132) at date $t+1$ on p. 955 must be matched by the trinomial model.

---

[a] Recall p. 743.

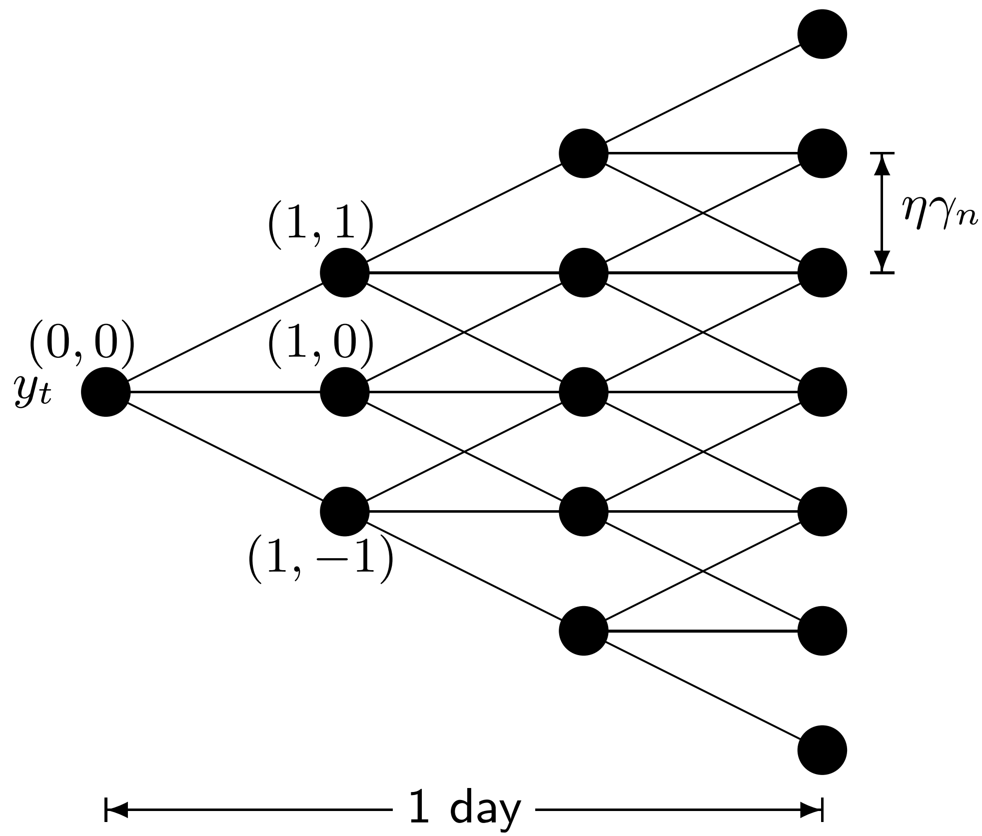
The RT Algorithm (continued)

- It remains to pick the jump size and the three branching probabilities.
- The role of σ in the Black-Scholes option pricing model is played by h_t in the GARCH model.
- As a jump size proportional to $\sigma/\sqrt{n}$ is picked in the BOPM, a comparable magnitude will be chosen here.
- Define $\gamma \stackrel{\Delta}{=} h_0$, though other multiples of h_0 are possible.
- Let

$$\gamma_n \stackrel{\Delta}{=} \frac{\gamma}{\sqrt{n}}.$$

The RT Algorithm (continued)

- The jump size will be some integer multiple η of γ_n.
- We call η the jump parameter (see next page).
- Clearly, the magnitude of η tends to grow with h_t.
- The middle branch does not change the underlying asset's price.



The seven values on the right approximate the distribution of logarithmic price y_{t+1}.

The RT Algorithm (continued)

- The probabilities for the up, middle, and down branches are

$$p_u = \frac{h_t^2}{2\eta^2\gamma^2} + \frac{r - (h_t^2/2)}{2\eta\gamma\sqrt{n}}, \tag{133}$$

$$p_m = 1 - \frac{h_t^2}{\eta^2\gamma^2}, \tag{134}$$

$$p_d = \frac{h_t^2}{2\eta^2\gamma^2} - \frac{r - (h_t^2/2)}{2\eta\gamma\sqrt{n}}. \tag{135}$$

The RT Algorithm (continued)

- It can be shown that:
  - The trinomial model takes on $2n+1$ values at date $t+1$ for y_{t+1}.
  - These values match y_{t+1}'s mean.
  - These values match y_{t+1}'s variance asymptotically.
- The central limit theorem guarantees convergence to the continuous-space model as n increases.[a]

[a] Assume the probabilities are valid.

The RT Algorithm (continued)

- We can dispense with the intermediate nodes *between* dates to create a $(2n+1)$-nomial tree.[a]
- The resulting model is multinomial with $2n+1$ branches from any state (y_t, h_t^2).
- There are two reasons behind this manipulation.
  - Interdate nodes are created merely to approximate the continuous-state model after one day.
  - Keeping the interdate nodes results in a tree that is n times larger.[b]

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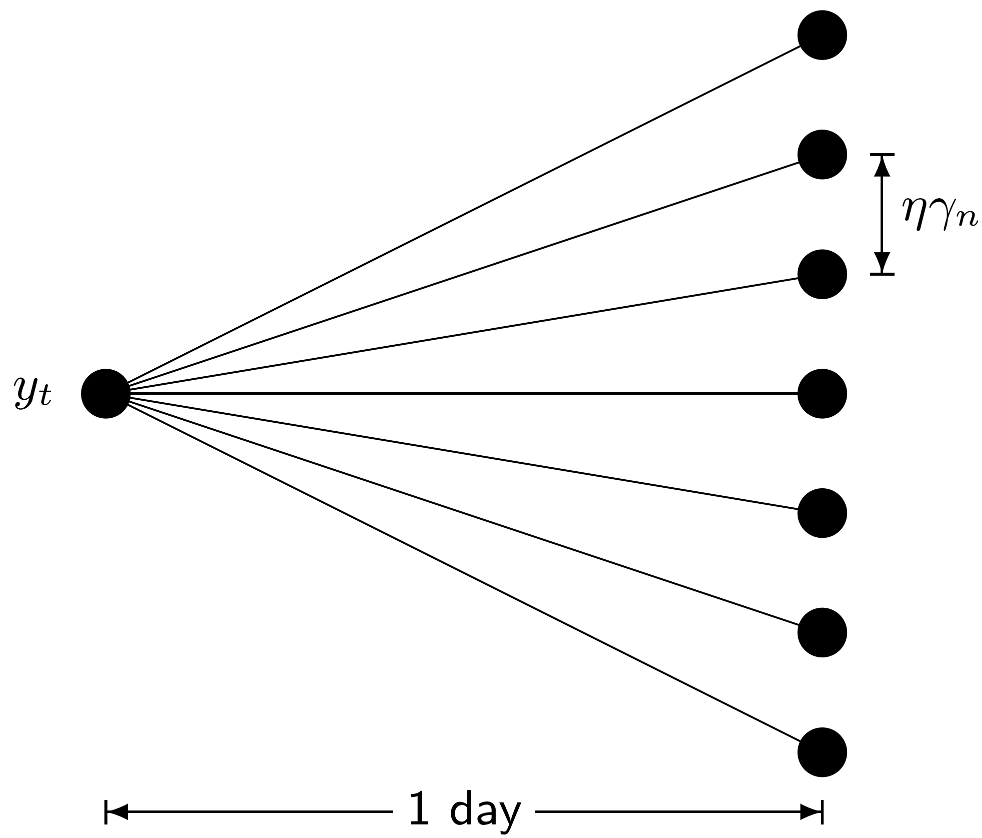
[a]See p. 964.

[b]Contrast it with the case on p. 414.



This heptanomial model is the outcome of the trinomial tree on p. 960 after the intermediate nodes are removed.

The RT Algorithm (continued)

- A node with logarithmic price $y_t + \ell\eta\gamma_n$ at date $t+1$ follows the current node at date t with price y_t, where

$$-n \leq \ell \leq n.$$

- To reach that price in n periods, the number of up moves must exceed that of down moves by exactly ℓ.
- The probability this happens is

$$P(\ell) \stackrel{\Delta}{=} \sum_{j_u, j_m, j_d} \frac{n!}{j_u!\, j_m!\, j_d!}\, p_u^{j_u} p_m^{j_m} p_d^{j_d},$$

with $j_u, j_m, j_d \geq 0$, $n = j_u + j_m + j_d$, and $\ell = j_u - j_d$.

The RT Algorithm (continued)

- A simple way to calculate the $P(\ell)$s starts by noting[a]

$$\left(p_u x + p_m + p_d x^{-1}\right)^n = \sum_{\ell=-n}^{n} P(\ell)\, x^{\ell}. \tag{136}$$

  – Convince yourself that the "accounting" is done correctly.

- So we expand $(p_u x + p_m + p_d x^{-1})^n$ and retrieve the probabilities by reading off the coefficients.

- It can be computed in $O(n^2)$ time, if not less.

[a] C. Wu (R90723065) (2003); Lyuu & C. Wu (R90723065) (2003, 2005).

The RT Algorithm (continued)

- The updating rule (129) on p. 951 must be modified to account for the adoption of the discrete-state model.
- The logarithmic price $y_t + \ell\eta\gamma_n$ at date $t+1$ following state (y_t, h_t^2) is associated with this variance:

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 (\epsilon'_{t+1} - c)^2, \qquad (137)$$

  – Above, the z-score[a]

$$\epsilon'_{t+1} = \frac{\ell\eta\gamma_n - (r - h_t^2/2)}{h_t}, \quad \ell = 0, \pm 1, \pm 2, \ldots, \pm n,$$

  is a discrete random variable with $2n+1$ values.

[a]Note that ϵ'_{t+1} has a zero mean and unit variance.

The RT Algorithm (continued)

- Different h_t^2 may require different η so that the probabilities (133)–(135) on p. 961 lie between 0 and 1.
- This implies varying jump sizes $\eta\gamma_n$.
- The necessary requirement $p_m \geq 0$ implies $\eta \geq h_t/\gamma$.
- Hence we try

$$\eta = \lceil h_t/\gamma \rceil, \lceil h_t/\gamma \rceil + 1, \lceil h_t/\gamma \rceil + 2, \ldots$$

until valid probabilities are obtained or until their nonexistence is confirmed.

The RT Algorithm (continued)

- The sufficient and necessary condition for valid probabilities to exist is[a]

$$\frac{|\, r - (h_t^2/2) \,|}{2\eta\gamma\sqrt{n}} \leq \frac{h_t^2}{2\eta^2\gamma^2} \leq \min\left(1 - \frac{|\, r - (h_t^2/2) \,|}{2\eta\gamma\sqrt{n}}, \frac{1}{2}\right).$$

- The plot on p. 970 uses $n = 1$ to illustrate our points for a 3-day model.

- For example, node $(1, 1)$ of date 1 and node $(2, 3)$ of date 2 pick $\eta = 2$.

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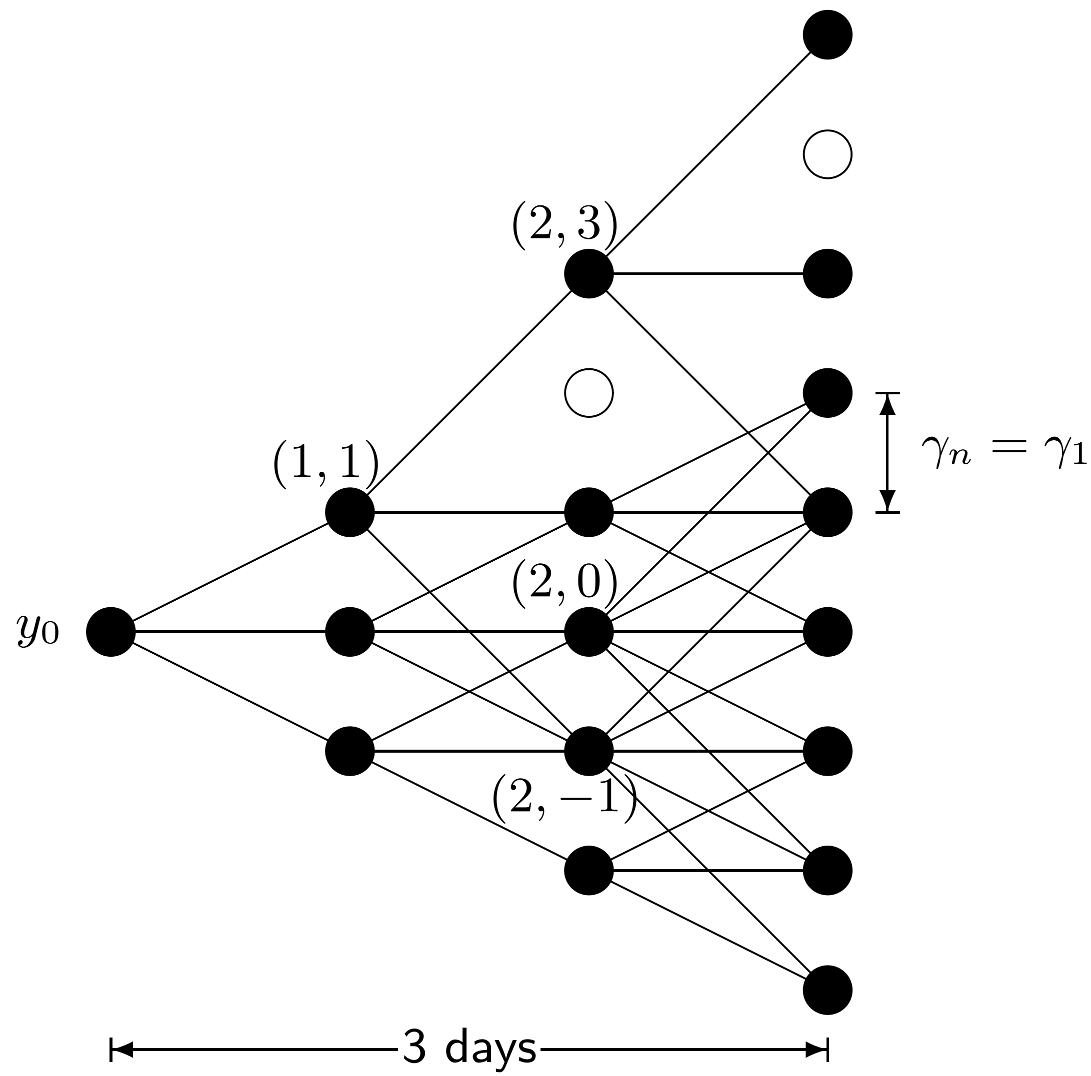
[a] C. Wu (R90723065) (2003); Lyuu & C. Wu (R90723065) (2003, 2005).



$(2,3)$
$(1,1)$
$\gamma_n = \gamma_1$
$(2,0)$
y_0
$(2,-1)$
3 days

The RT Algorithm (continued)

- The topology of the tree is not a standard combining multinomial tree.
- For example, a few nodes on p. 970 such as nodes $(2,0)$ and $(2,-1)$ have *multiple* jump sizes.
- The reason is path dependency of the model.
  - Two paths can reach node $(2,0)$ from the root node, each with a different variance h_t^2 for the node.
  - One variance results in $\eta = 1$.
  - The other results in $\eta = 2$.

The RT Algorithm (concluded)

- The number of possible values of h_t^2 at a node can be exponential.
  - Because each path may result in a different h_t^2.
- To address this problem, we record only the maximum and minimum h_t^2 at each node.[a]
- Therefore, each node on the tree contains only two states $(y_t, h_{\max}^2)$ and $(y_t, h_{\min}^2)$.
- Each of $(y_t, h_{\max}^2)$ and $(y_t, h_{\min}^2)$ carries its own η and set of $2n+1$ branching probabilities.

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[a]Cakici & Topyan (2000). But see p. 1007 for a potential problem.

Negative Aspects of the Ritchken-Trevor Algorithm[a]

- A small n may yield inaccurate option prices.
- But the tree will grow exponentially if n is large enough.
  - Specifically, $n > (1 - \beta_1)/\beta_2$ when $r = c = 0$.
- A large n has another serious problem: The tree cannot grow beyond a certain date.
- Thus the choice of n may be quite limited in practice.
- The RT algorithm can be modified to be free of invalid probabilities and, sometimes, exponential complexity.[b]

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[a]Lyuu & C. Wu (R90723065) (2003, 2005).

[b]Its size is only $O(T^2)$ if $n \leq (\sqrt{(1-\beta_1)/\beta_2} - c)^2$, where T is the number of days to maturity!