

Stochastic Processes and Brownian Motion

Of all the intellectual hurdles which the human mind
has confronted and has overcome in the last
fifteen hundred years, the one which seems to me
to have been the most amazing in character and
the most stupendous in the scope of its
consequences is the one relating to
the problem of motion.

— Herbert Butterfield (1900–1979)

Stochastic Processes

- A stochastic process

$$X = \{ X(t) \}$$

is a time series of random variables.

- $X(t)$ (or X_t) is a random variable for each time t and is usually called the state of the process at time t .
- A realization of X is called a sample path.

Stochastic Processes (concluded)

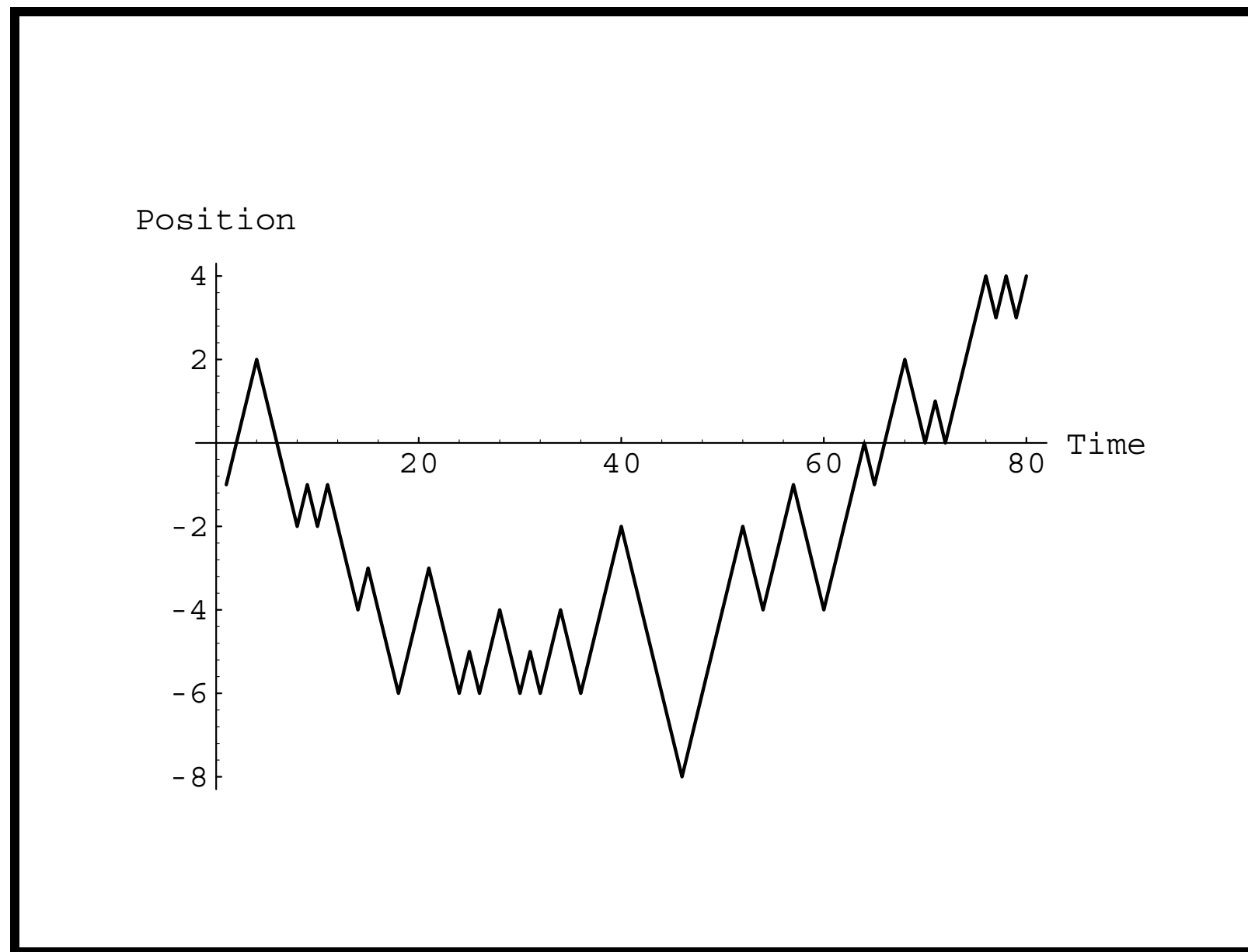
- If the times t form a countable set, X is called a discrete-time stochastic process or a time series.
- In this case, subscripts rather than parentheses are usually employed, as in

$$X = \{ X_n \}.$$

- If the times form a continuum, X is called a continuous-time stochastic process.

Random Walks

- The binomial model is a random walk in disguise.
- Consider a particle on the integer line, $0, \pm 1, \pm 2, \dots$
- In each time step, it can make one move to the right with probability p or one move to the left with probability $1 - p$.
 - This random walk is symmetric when $p = 1/2$.
- Connection with the BOPM: The particle's position denotes the number of up moves minus that of down moves up to that time.



Random Walk with Drift

$$X_n = \mu + X_{n-1} + \xi_n.$$

- ξ_n are independent and identically distributed with zero mean.
- Drift μ is the expected change per period.
- Note that this process is continuous in space.

Martingales^a

- $\{X(t), t \geq 0\}$ is a martingale if $E[|X(t)|] < \infty$ for $t \geq 0$ and

$$E[X(t) | X(u), 0 \leq u \leq s] = X(s), \quad s \leq t. \quad (73)$$

- In the discrete-time setting, a martingale means

$$E[X_{n+1} | X_1, X_2, \dots, X_n] = X_n. \quad (74)$$

- X_n can be interpreted as a gambler's fortune after the n th gamble.
- Identity (74) then says the expected fortune after the $(n+1)$ st gamble equals the fortune after the n th gamble regardless of what may have occurred before.

^aThe origin of the name is somewhat obscure.

Martingales (concluded)

- A martingale is therefore a notion of fair games.
- Apply the law of iterated conditional expectations to both sides of Eq. (74) on p. 551 to yield

$$E[X_n] = E[X_1] \quad (75)$$

for all n .

- Similarly,

$$E[X(t)] = E[X(0)]$$

in the continuous-time case.

Still a Martingale?

- Suppose we replace Eq. (74) on p. 551 with

$$E[X_{n+1} | X_n] = X_n.$$

- It also says past history cannot affect the future.
- But is it equivalent to the original definition (74) on p. 551?^a

^aContributed by Mr. Hsieh, Chicheng (M9007304) on April 13, 2005.

Still a Martingale? (continued)

- Well, no.^a
- Consider this random walk with drift:

$$X_i = \begin{cases} X_{i-1} + \xi_i, & \text{if } i \text{ is even,} \\ X_{i-2}, & \text{otherwise.} \end{cases}$$

- Above, ξ_n are random variables with zero mean.

^aContributed by Mr. Zhang, Ann-Sheng (B89201033) on April 13, 2005.

Still a Martingale? (concluded)

- It is not hard to see that

$$E[X_i | X_{i-1}] = \begin{cases} X_{i-1}, & \text{if } i \text{ is even,} \\ X_{i-1}, & \text{otherwise.} \end{cases}$$

- It is a martingale by the “new” definition.

- But

$$E[X_i | \dots, X_{i-2}, X_{i-1}] = \begin{cases} X_{i-1}, & \text{if } i \text{ is even,} \\ X_{i-2}, & \text{otherwise.} \end{cases}$$

- It is not a martingale by the original definition.

Example

- Consider the stochastic process

$$\left\{ Z_n \triangleq \sum_{i=1}^n X_i, n \geq 1 \right\},$$

where X_i are independent random variables with zero mean.

- This process is a martingale because

$$\begin{aligned} & E[Z_{n+1} \mid Z_1, Z_2, \dots, Z_n] \\ &= E[Z_n + X_{n+1} \mid Z_1, Z_2, \dots, Z_n] \\ &= E[Z_n \mid Z_1, Z_2, \dots, Z_n] + E[X_{n+1} \mid Z_1, Z_2, \dots, Z_n] \\ &= Z_n + E[X_{n+1}] = Z_n. \end{aligned}$$

Probability Measure

- A probability measure assigns probabilities to states of the world.^a
- A martingale is defined with respect to a probability measure, under which the expectation is taken.
- Second, a martingale is defined with respect to an information set.
 - In the characterizations (73)–(74) on p. 551, the information set contains the current and past values of X by default.
 - But it need not be so.

^aOnly certain sets such as the Borel sets receive probabilities (Feller, 1972).

Probability Measure (continued)

- A stochastic process $\{X(t), t \geq 0\}$ is a martingale with respect to information sets $\{I_t\}$ if, for all $t \geq 0$, $E[|X(t)|] < \infty$ and

$$E[X(u) | I_t] = X(t)$$

for all $u > t$.

- The discrete-time version: For all $n > 0$,

$$E[X_{n+1} | I_n] = X_n,$$

given the information sets $\{I_n\}$.

Probability Measure (concluded)

- The above implies

$$E[X_{n+m} | I_n] = X_n$$

for any $m > 0$ by Eq. (26) on p. 171.

- A typical I_n is the price information up to time n .
- Then the above identity says the FVs of X will not deviate systematically from today's value given the price history.

Example

- Consider the stochastic process $\{Z_n - n\mu, n \geq 1\}$.
 - $Z_n \triangleq \sum_{i=1}^n X_i$.
 - X_1, X_2, \dots are independent random variables with mean μ .
- Now,

$$\begin{aligned} & E[Z_{n+1} - (n+1)\mu \mid X_1, X_2, \dots, X_n] \\ &= E[Z_{n+1} \mid X_1, X_2, \dots, X_n] - (n+1)\mu \\ &= E[Z_n + X_{n+1} \mid X_1, X_2, \dots, X_n] - (n+1)\mu \\ &= Z_n + \mu - (n+1)\mu \\ &= Z_n - n\mu. \end{aligned}$$

Example (concluded)

- Define

$$I_n \triangleq \{ X_1, X_2, \dots, X_n \}.$$

- Then

$$\{ Z_n - n\mu, n \geq 1 \}$$

is a martingale with respect to $\{ I_n \}$.

Martingale Pricing

- Stock prices and zero-coupon bond prices are expected to rise, while call prices are expected to fall.^a
- They are *not* martingales.
- Why is then martingale useful?
- Recall a martingale is defined with respect to some information set *and* some probability measure.
- By modifying the probability measure, we can convert a price process into a martingale.

^aRecall p. 355.

Martingale Pricing (continued)

- The price of a European option is the expected discounted payoff in a risk-neutral economy.^a
- This principle can be generalized using the concept of martingale.
- Recall the recursive valuation of European option via

$$C = [pC_u + (1 - p) C_d] / R.$$

- p is the risk-neutral probability.
- \$1 grows to $\$R$ in a period.

^aRecall Eq. (37) on p. 271.

Martingale Pricing (continued)

- Let $C(i)$ denote the value of the option at time i .
- Consider the discount process

$$\left\{ \frac{C(i)}{R^i}, i = 0, 1, \dots, n \right\}.$$

- Then,

$$\begin{aligned} E \left[\frac{C(i+1)}{R^{i+1}} \mid S(i) \right] &= \frac{E[C(i+1) \mid S(i)]}{R^{i+1}} \\ &= \frac{pC_u + (1-p)C_d}{R^{i+1}} \\ &= \frac{C(i)}{R^i}. \end{aligned}$$

Martingale Pricing (continued)

- It is easy to generalize it to

$$E \left[\frac{C(k)}{R^k} \mid S(i) \right] = \frac{C(i)}{R^i}, \quad i \leq k.$$

- This simplified formulation assumes:^a
 1. The model is Markovian: The distribution of the future is determined by the present (time i) and not the past.
 2. The payoff depends only on the terminal price of the underlying asset^b (Asian options do not qualify).

^aContributed by Mr. Wang, Liang-Kai (Ph.D. student, ECE, University of Wisconsin-Madison) and Mr. Hsiao, Huan-Wen (B90902081) on May 3, 2006.

^bRecall they are called simple claims.

Martingale Pricing (continued)

- In general, the discount process is a martingale in that^a

$$E_i^\pi \left[\frac{C(k)}{R^k} \right] = \frac{C(i)}{R^i}, \quad i \leq k. \quad (76)$$

- E_i^π is taken under the risk-neutral probability conditional on the price information *up to time i*.
- This risk-neutral probability is also called the EMM, or the equivalent^b martingale (probability) measure.

^aIn this general formulation, Asian options do qualify.

^bTwo probability measures are said to be equivalent if they assign nonzero probabilities to the same set of states.

Martingale Pricing (continued)

- Equation (76) holds for all assets, not just options.
- When interest rates are stochastic, the equation becomes

$$\frac{C(i)}{M(i)} = E_i^\pi \left[\frac{C(k)}{M(k)} \right], \quad i \leq k. \quad (77)$$

- $M(j)$ is the balance in the money market account at time j using the rollover strategy with an initial investment of \$1.
- It is called the bank account process.
- It says the discount process is a martingale under π .

Martingale Pricing (continued)

- If interest rates are stochastic, then $M(j)$ is a random variable.
 - $M(0) = 1$.
 - $M(j)$ is known at time $j - 1$.^a
- Identity (77) on p. 567 is the general formulation of risk-neutral valuation.

^aBecause the interest rate for the *next* period has been revealed then.

Martingale Pricing (concluded)

Theorem 17 *A discrete-time model is arbitrage-free if and only if there exists an equivalent probability measure^a such that the discount process is a martingale.*

^aCalled the risk-neutral probability measure.

Futures Price under the BOPM

- Futures prices form a martingale under the risk-neutral probability p_f .^a
 - The expected futures price in the next period is

$$p_f F u + (1 - p_f) F d = F \left(\frac{1 - d}{u - d} u + \frac{u - 1}{u - d} d \right) = F.$$

- Can be generalized to

$$F_i = E_i^\pi [F_k], \quad i \leq k,$$

where F_i is the futures price at time i .

- This equation holds under stochastic interest rates, too.^b

^aRecall Eq. (71) on p. 526.

^bSee Exercise 13.2.11 of the textbook.

Futures Price under the BOPM (concluded)

- Futures prices do *not* form a martingale under the risk-neutral probability $p = (R - d)/(u - d)$.^a
 - The expected futures price in the next period equals

$$\begin{aligned} & Fu \frac{R - d}{u - d} + Fd \frac{u - R}{u - d} \\ = & F \frac{uR - ud}{u - d} + F \frac{ud - Rd}{u - d} \\ = & FR. \end{aligned}$$

- But they do if u and d are changed to u/R and d/R , respectively, justified by Eq. (61) on p. 494.

^aRecall Eq. (34) on p. 257.

Martingale Pricing and Numeraire^a

- The martingale pricing formula (77) on p. 567 uses the money market account as numeraire.
 - It expresses the price of any asset *relative to* the money market account.^b
- The money market account is not the only choice for numeraire.
- Suppose asset S 's value is *positive* at all time.

^aJohn Law (1671–1729), “Money to be qualified for exchanging goods and for payments need not be certain in its value.”

^bLeon Walras (1834–1910).

Martingale Pricing and Numeraire (concluded)

- Choose S as numeraire.
- Martingale pricing says there exists a risk-neutral probability π under which the relative price of any asset C is a martingale:

$$\frac{C(i)}{S(i)} = E_i^\pi \left[\frac{C(k)}{S(k)} \right], \quad i \leq k.$$

– $S(j)$ denotes the price of S at time j .

- So the “discount” process remains a martingale.^a

^aThis result is related to Girsanov’s theorem (1960).

Example

- Take the binomial model with two assets.
- In a period, asset one's price can go from S to S_1 or S_2 .
- In a period, asset two's price can go from P to P_1 or P_2 .
- Both assets must move up or down at the same time.
- Assume

$$\frac{S_1}{P_1} < \frac{S}{P} < \frac{S_2}{P_2} \quad (78)$$

to rule out arbitrage opportunities.

Example (continued)

- For any derivative security, let C_1 be its price at time one if asset one's price moves to S_1 .
- Let C_2 be its price at time one if asset one's price moves to S_2 .
- Replicate the derivative by solving

$$\alpha S_1 + \beta P_1 = C_1,$$

$$\alpha S_2 + \beta P_2 = C_2,$$

using α units of asset one and β units of asset two.

Example (continued)

- By inequalities (78) on p. 574, α and β have unique solutions.
- In fact,

$$\alpha = \frac{P_2 C_1 - P_1 C_2}{P_2 S_1 - P_1 S_2} \quad \text{and} \quad \beta = \frac{S_2 C_1 - S_1 C_2}{S_2 P_1 - S_1 P_2}.$$

- The derivative costs

$$\begin{aligned} C &= \alpha S + \beta P \\ &= \frac{P_2 S - P S_2}{P_2 S_1 - P_1 S_2} C_1 + \frac{P S_1 - P_1 S}{P_2 S_1 - P_1 S_2} C_2. \end{aligned}$$

Example (continued)

- It is easy to verify that

$$\frac{C}{P} = p \frac{C_1}{P_1} + (1 - p) \frac{C_2}{P_2}$$

with

$$p \triangleq \frac{(S/P) - (S_2/P_2)}{(S_1/P_1) - (S_2/P_2)}.$$

- By inequalities (78) on p. 574, $0 < p < 1$.
- C 's price using asset two as numeraire (i.e., C/P) is a martingale under the risk-neutral probability p .
- The expected returns of the two assets are *irrelevant*.

Example (concluded)

- In the BOPM, S is the stock and P is the bond.
- Furthermore, p assumes the bond is the numeraire.
- In the binomial option pricing formula (39) on p. 277, $S \sum b(j; n, pu/R)$ uses *stock* as the numeraire.
 - Its probability measure pu/R differs from p .
- $SN(x)$ for the call and $SN(-x)$ for the put in the Black-Scholes formulas (p. 308) use stock as the numeraire as well.^a

^aSee Exercise 13.2.12 of the textbook.

Brownian Motion^a

- Brownian motion is a stochastic process $\{X(t), t \geq 0\}$ with the following properties.

1. $X(0) = 0$, unless stated otherwise.
2. for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables

$$X(t_k) - X(t_{k-1})$$

for $1 \leq k \leq n$ are independent.^b

3. for $0 \leq s < t$, $X(t) - X(s)$ is normally distributed with mean $\mu(t - s)$ and variance $\sigma^2(t - s)$, where μ and $\sigma \neq 0$ are real numbers.

^aRobert Brown (1773–1858).

^bSo $X(t) - X(s)$ is independent of $X(r)$ for $r \leq s < t$.

Brownian Motion (concluded)

- The existence and uniqueness of such a process is guaranteed by Wiener's theorem.^a
- This process will be called a (μ, σ) Brownian motion with drift μ and variance σ^2 .
- Although Brownian motion is a continuous function of t with probability one, it is almost nowhere differentiable.
- The $(0, 1)$ Brownian motion is called the Wiener process.
- If condition 3 is replaced by “ $X(t) - X(s)$ depends only on $t - s$,” we have the more general Levy process.^b

^aNorbert Wiener (1894–1964). He received his Ph.D. from Harvard in 1912.

^bPaul Levy (1886–1971).

Example

- If $\{X(t), t \geq 0\}$ is the Wiener process, then

$$X(t) - X(s) \sim N(0, t - s).$$

- A (μ, σ) Brownian motion $Y = \{Y(t), t \geq 0\}$ can be expressed in terms of the Wiener process:

$$Y(t) = \mu t + \sigma X(t). \quad (79)$$

- Note that

$$Y(t + s) - Y(t) \sim N(\mu s, \sigma^2 s).$$

Brownian Motion as Limit of Random Walk

Claim 1 *A (μ, σ) Brownian motion is the limiting case of random walk.*

- A particle moves Δx to the right with probability p after Δt time.
- It moves Δx to the left with probability $1 - p$.
- Define

$$X_i \triangleq \begin{cases} +1 & \text{if the } i\text{th move is to the right,} \\ -1 & \text{if the } i\text{th move is to the left.} \end{cases}$$

- X_i are independent with

$$\text{Prob}[X_i = 1] = p = 1 - \text{Prob}[X_i = -1].$$

Brownian Motion as Limit of Random Walk (continued)

- Recall

$$\begin{aligned}E[X_i] &= 2p - 1, \\ \text{Var}[X_i] &= 1 - (2p - 1)^2.\end{aligned}$$

- Assume $n \triangleq t/\Delta t$ is an integer.
- Its position at time t is

$$Y(t) \triangleq \Delta x (X_1 + X_2 + \cdots + X_n).$$

Brownian Motion as Limit of Random Walk (continued)

- Therefore,

$$E[Y(t)] = n(\Delta x)(2p - 1),$$

$$\text{Var}[Y(t)] = n(\Delta x)^2 [1 - (2p - 1)^2].$$

- With $\Delta x \triangleq \sigma\sqrt{\Delta t}$ and $p \triangleq [1 + (\mu/\sigma)\sqrt{\Delta t}]/2$,^a

$$E[Y(t)] = n\sigma\sqrt{\Delta t}(\mu/\sigma)\sqrt{\Delta t} = \mu t,$$

$$\text{Var}[Y(t)] = n\sigma^2\Delta t [1 - (\mu/\sigma)^2\Delta t] \rightarrow \sigma^2 t,$$

as $\Delta t \rightarrow 0$.

^aIdentical to Eq. (42) on p. 300!

Brownian Motion as Limit of Random Walk (concluded)

- Thus, $\{Y(t), t \geq 0\}$ converges to a (μ, σ) Brownian motion by the central limit theorem.
- Brownian motion with zero drift is the limiting case of symmetric random walk by choosing $\mu = 0$.
- Similarity to the the BOPM: The p is identical to the probability in Eq. (42) on p. 300 and $\Delta x = \ln u$.
- Note that

$$\begin{aligned} & \text{Var}[Y(t + \Delta t) - Y(t)] \\ &= \text{Var}[\Delta x X_{n+1}] = (\Delta x)^2 \times \text{Var}[X_{n+1}] \rightarrow \sigma^2 \Delta t. \end{aligned}$$

Geometric Brownian Motion

- Let $X \triangleq \{X(t), t \geq 0\}$ be a Brownian motion process.
- The process

$$\{Y(t) \triangleq e^{X(t)}, t \geq 0\},$$

is called geometric Brownian motion.

- Suppose further that X is a (μ, σ) Brownian motion.
- By assumption, $Y(0) = e^0 = 1$.

Geometric Brownian Motion (concluded)

- $X(t) \sim N(\mu t, \sigma^2 t)$ with moment generating function

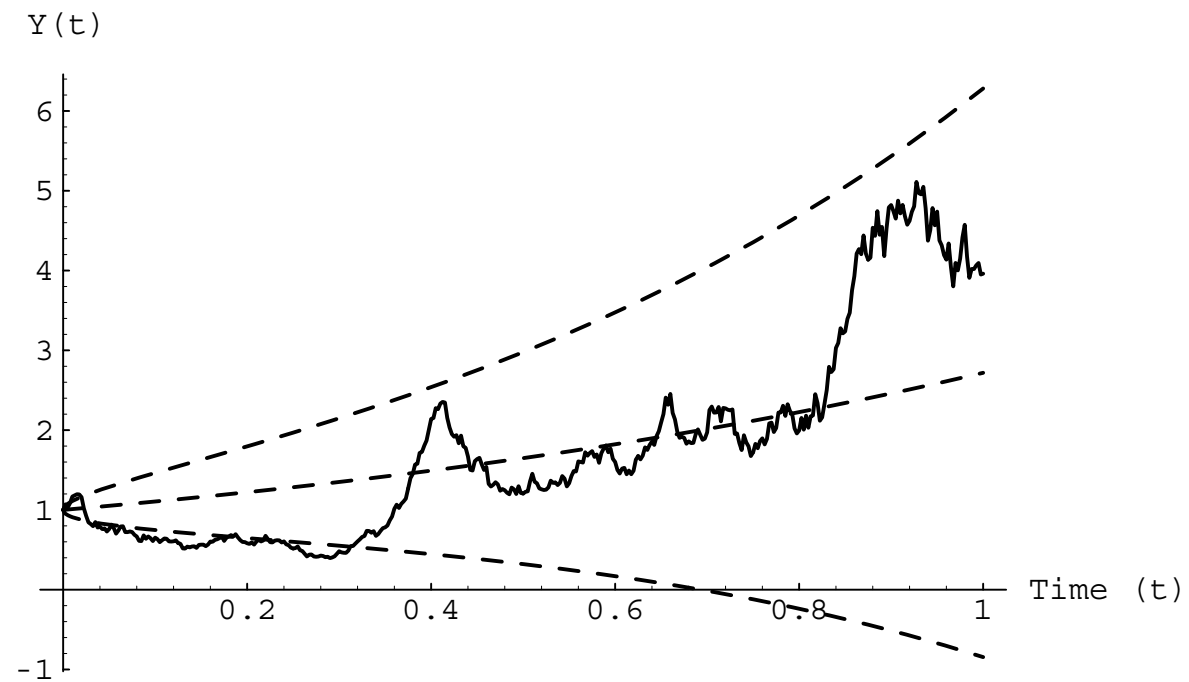
$$E \left[e^{sX(t)} \right] = E \left[Y(t)^s \right] = e^{\mu ts + (\sigma^2 ts^2/2)}$$

from Eq. (27) on p 173.

- In particular,^a

$$\begin{aligned} E[Y(t)] &= e^{\mu t + (\sigma^2 t/2)}, \\ \text{Var}[Y(t)] &= E[Y(t)^2] - E[Y(t)]^2 \\ &= e^{2\mu t + \sigma^2 t} \left(e^{\sigma^2 t} - 1 \right). \end{aligned}$$

^aRecall Eqs. (29) on p. 182.



An Argument for Long-Term Investment^a

- Suppose the stock follows the geometric Brownian motion

$$S(t) = S(0) e^{N(\mu t, \sigma^2 t)} = S(0) e^{tN(\mu, \sigma^2/t)}, \quad t \geq 0,$$

where $\mu > 0$.

- The annual rate of return has a normal distribution:

$$N\left(\mu, \frac{\sigma^2}{t}\right).$$

- The larger the t , the likelier the return is positive.
- The smaller the t , the likelier the return is negative.

^aContributed by Dr. King, Gow-Hsing on April 9, 2015. See <http://www.cb.idv.tw/phpbb3/viewtopic.php?f=7&t=1025>

Continuous-Time Financial Mathematics

A proof is that which convinces a reasonable man;
a rigorous proof is that which convinces an
unreasonable man.
— Mark Kac (1914–1984)

The pursuit of mathematics is a
divine madness of the human spirit.
— Alfred North Whitehead (1861–1947),
Science and the Modern World

Stochastic Integrals

- Use $W \triangleq \{W(t), t \geq 0\}$ to denote the Wiener process.
- The goal is to develop integrals of X from a class of stochastic processes,^a

$$I_t(X) \triangleq \int_0^t X dW, \quad t \geq 0.$$

- $I_t(X)$ is a random variable called the stochastic integral of X with respect to W .
- The stochastic process $\{I_t(X), t \geq 0\}$ will be denoted by $\int X dW$.

^aKiyoshi Ito (1915–2008).

Stochastic Integrals (concluded)

- Typical requirements for X in financial applications are:
 - $\text{Prob}[\int_0^t X^2(s) ds < \infty] = 1$ for all $t \geq 0$ or the stronger $\int_0^t E[X^2(s)] ds < \infty$.
 - The information set at time t includes the history of X and W up to that point in time.
 - But it contains nothing about the evolution of X or W after t (nonanticipating, so to speak).
 - The future cannot influence the present.

Ito Integral

- A theory of stochastic integration.
- As with calculus, it starts with step functions.
- A stochastic process $\{X(t)\}$ is simple if there exist

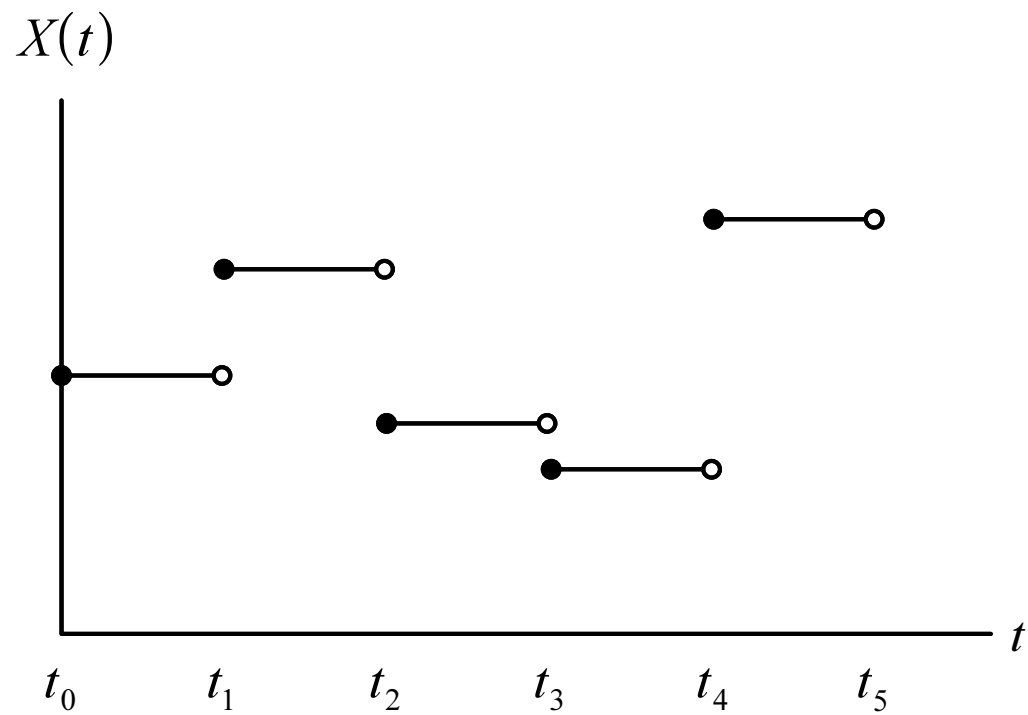
$$0 = t_0 < t_1 < \cdots$$

such that

$$X(t) = X(t_{k-1}) \quad \text{for } t \in [t_{k-1}, t_k), k = 1, 2, \dots$$

for any realization (see figure on next page).^a

^aIt is right-continuous.



Ito Integral (continued)

- The Ito integral of a simple process is defined as

$$I_t(X) \triangleq \sum_{k=0}^{n-1} X(t_k) [W(t_{k+1}) - W(t_k)], \quad (80)$$

where $t_n = t$.

- The integrand X is evaluated at t_k , not t_{k+1} .
- Define the Ito integral of more general processes as a limiting random variable of the Ito integral of simple stochastic processes.

Ito Integral (continued)

- Let $X = \{X(t), t \geq 0\}$ be a general stochastic process.
- There exists a random variable $I_t(X)$, unique almost certainly, such that $I_t(X_n)$ converges in probability to $I_t(X)$ for each sequence of simple stochastic processes X_1, X_2, \dots that X_n converge in probability to X .
- If X is continuous with probability one, then $I_t(X_n)$ converges in probability to $I_t(X)$ as

$$\delta_n \triangleq \max_{1 \leq k \leq n} (t_k - t_{k-1})$$

goes to zero.

Ito Integral (concluded)

- It is a fundamental fact that $\int X dW$ is continuous almost surely.
- The Ito integral is a martingale.^a

Theorem 18 *The Ito integral $\int X dW$ is a martingale.*

- A corollary is the mean value formula

$$E \left[\int_a^b X dW \right] = 0.$$

^aExercise 14.1.1 covers simple stochastic processes.

Discrete Approximation and Nonanticipation

- Recall Eq. (80) on p. 596.
- The following *simple* stochastic process $\{\hat{X}(t)\}$ can be used in place of X to approximate $\int_0^t X dW$,

$$\hat{X}(s) \triangleq X(t_{k-1}) \quad \text{for } s \in [t_{k-1}, t_k), k = 1, 2, \dots, n.$$

- Note the *nonanticipating* feature of \hat{X} .
 - The information up to time s ,

$$\{\hat{X}(t), W(t), 0 \leq t \leq s\},$$

cannot determine the future evolution of X or W .

Discrete Approximation and Nonanticipation (concluded)

- Suppose, unlike Eq. (80) on p. 596, we defined the stochastic integral from

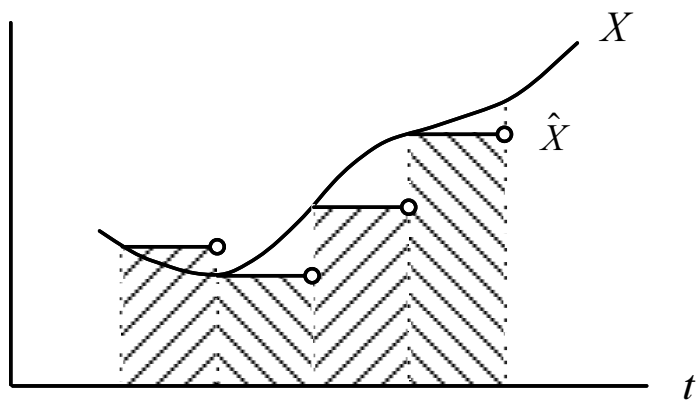
$$\sum_{k=0}^{n-1} X(t_{k+1}) [W(t_{k+1}) - W(t_k)].$$

- Then we would be using the following different simple stochastic process in the approximation,

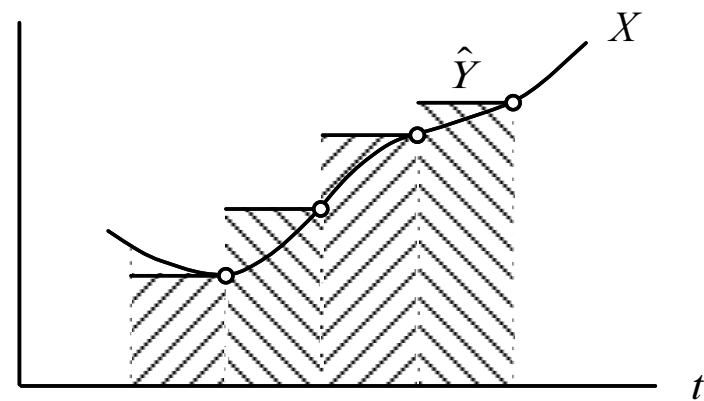
$$\hat{Y}(s) \triangleq X(t_k) \quad \text{for } s \in [t_{k-1}, t_k), k = 1, 2, \dots, n.$$

- This clearly anticipates the future evolution of X .^a

^aSee Exercise 14.1.2 for an example where it matters.



(a)



(b)

Ito Process

- The stochastic process $X = \{X_t, t \geq 0\}$ that solves

$$X_t = X_0 + \int_0^t a(X_s, s) ds + \int_0^t b(X_s, s) dW_s, \quad t \geq 0$$

is called an Ito process.

- X_0 is a scalar starting point.
- $\{a(X_t, t) : t \geq 0\}$ and $\{b(X_t, t) : t \geq 0\}$ are stochastic processes satisfying certain regularity conditions.
- $a(X_t, t)$ is the drift.
- $b(X_t, t)$ is the diffusion.

Ito Process (continued)

- Typical regularity conditions are:^a

- For all $T > 0$, $x \in \mathbb{R}^n$, and $0 \leq t \leq T$,

$$|a(x, t)| + |b(x, t)| \leq C(1 + |x|)$$

for some constant C .^b

- (Lipschitz continuity) For all $T > 0$, $x \in \mathbb{R}^n$, and $0 \leq t \leq T$,

$$|a(x, t) - a(y, t)| + |b(x, t) - b(y, t)| \leq D|x - y|$$

for some constant D .

^aØksendal (2007).

^bThis condition is not needed in *time-homogeneous* cases, where a and b do not depend on t .

Ito Process (continued)

- A shorthand^a is the following stochastic differential equation^b (SDE) for the Ito differential dX_t ,

$$dX_t = a(X_t, t) dt + b(X_t, t) dW_t. \quad (81)$$

- Or simply

$$dX_t = a_t dt + b_t dW_t.$$

- This is Brownian motion with an *instantaneous* drift a_t and an *instantaneous* variance b_t^2 .

- X is a martingale if $a_t = 0$.^c

^aPaul Langevin (1872–1946) in 1904.

^bLike any equation, an SDE contains an unknown, the process X_t .

^cRecall Theorem 18 (p. 598).

Ito Process (concluded)

- From calculus, we would expect $\int_0^t W dW = W(t)^2/2$.
- But $W(t)^2/2$ is not a martingale, hence wrong!
- The correct answer is $[W(t)^2 - t]/2$.
- A popular representation of Eq. (81) is

$$dX_t = a_t dt + b_t \sqrt{dt} \xi, \quad (82)$$

where $\xi \sim N(0, 1)$.

Euler Approximation

- Define $t_n \triangleq n\Delta t$.
- The following approximation follows from Eq. (82),

$$\begin{aligned}\hat{X}(t_{n+1}) \\ = \hat{X}(t_n) + a(\hat{X}(t_n), t_n) \Delta t + b(\hat{X}(t_n), t_n) \Delta W(t_n).\end{aligned}\tag{83}$$

- It is called the Euler or Euler-Maruyama method.
- Recall that $\Delta W(t_n)$ should be interpreted as

$$W(t_{n+1}) - W(t_n),$$

not $W(t_n) - W(t_{n-1})$!^a

^aRecall Eq. (80) on p. 596.

Euler Approximation (concluded)

- With the Euler method, one can obtain a sample path

$$\hat{X}(t_1), \hat{X}(t_2), \hat{X}(t_3), \dots$$

from a sample path

$$W(t_0), W(t_1), W(t_2), \dots$$

- Under mild conditions, $\hat{X}(t_n)$ converges to $X(t_n)$.

More Discrete Approximations

- Under fairly loose regularity conditions, Eq. (83) on p. 606 can be replaced by

$$\begin{aligned}\hat{X}(t_{n+1}) \\ = \hat{X}(t_n) + a(\hat{X}(t_n), t_n) \Delta t + b(\hat{X}(t_n), t_n) \sqrt{\Delta t} Y(t_n).\end{aligned}$$

- $Y(t_0), Y(t_1), \dots$ are independent and identically distributed with zero mean and unit variance.

More Discrete Approximations (concluded)

- An even simpler discrete approximation scheme:

$$\begin{aligned}\widehat{X}(t_{n+1}) \\ = \widehat{X}(t_n) + a(\widehat{X}(t_n), t_n) \Delta t + b(\widehat{X}(t_n), t_n) \sqrt{\Delta t} \xi.\end{aligned}$$

- $\text{Prob}[\xi = 1] = \text{Prob}[\xi = -1] = 1/2$.
- Note that $E[\xi] = 0$ and $\text{Var}[\xi] = 1$.
- This is a binomial model.
- As Δt goes to zero, \widehat{X} converges to X .^a

^aHe (1990).

Trading and the Ito Integral

- Consider an Ito process

$$d\mathbf{S}_t = \mu_t dt + \sigma_t dW_t.$$

- \mathbf{S}_t is the vector of security prices at time t .
- Let ϕ_t be a trading strategy denoting the quantity of each type of security held at time t .
 - Hence the stochastic process $\phi_t \mathbf{S}_t$ is the value of the portfolio ϕ_t at time t .
- $\phi_t d\mathbf{S}_t \triangleq \phi_t(\mu_t dt + \sigma_t dW_t)$ is the change in the portfolio value from the changes in security prices at time t .

Trading and the Ito Integral (concluded)

- The equivalent Ito integral,

$$G_T(\phi) \triangleq \int_0^T \phi_t d\mathbf{S}_t = \int_0^T \phi_t \mu_t dt + \int_0^T \phi_t \sigma_t dW_t,$$

measures the gains realized by the trading strategy over the period $[0, T]$.

Ito's Lemma^a

A smooth function of an Ito process is itself an Ito process.

Theorem 19 *Suppose $f : R \rightarrow R$ is twice continuously differentiable and $dX = a_t dt + b_t dW$. Then $f(X)$ is the Ito process,*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s) a_s ds + \int_0^t f'(X_s) b_s dW \\ &\quad + \frac{1}{2} \int_0^t f''(X_s) b_s^2 ds \end{aligned}$$

for $t \geq 0$.

^aIto (1944).

Ito's Lemma (continued)

- In differential form, Ito's lemma becomes

$$\begin{aligned} df(X) &= f'(X) a dt + f'(X) b dW + \frac{1}{2} f''(X) b^2 dt \quad (84) \\ &= \left[f'(X) a + \boxed{\frac{1}{2} f''(X) b^2} \right] dt + f'(X) b dW. \end{aligned}$$

- Compared with calculus, the extra term is boxed.
- A convenient formulation of Ito's lemma is

$$df(X) = f'(X) dX + \frac{1}{2} f''(X) (dX)^2. \quad (85)$$

Ito's Lemma (continued)

- We are supposed to multiply out $(dX)^2 = (a dt + b dW)^2$ symbolically according to

\times	dW	dt
dW	dt	0
dt	0	0

- The $(dW)^2 = dt$ entry is justified by a known result.
- Hence $(dX)^2 = (a dt + b dW)^2 = b^2 dt$ in Eq. (85).
- This form is easy to remember because of its similarity to the Taylor expansion.

Ito's Lemma (continued)

Theorem 20 (Higher-Dimensional Ito's Lemma) *Let W_1, W_2, \dots, W_n be independent Wiener processes and $X \triangleq (X_1, X_2, \dots, X_m)$ be a vector process. Suppose $f : R^m \rightarrow R$ is twice continuously differentiable and X_i is an Ito process with $dX_i = a_i dt + \sum_{j=1}^n b_{ij} dW_j$. Then $df(X)$ is an Ito process with the differential,*

$$df(X) = \sum_{i=1}^m f_i(X) dX_i + \frac{1}{2} \sum_{i=1}^m \sum_{k=1}^m f_{ik}(X) dX_i dX_k,$$

where $f_i \triangleq \partial f / \partial X_i$ and $f_{ik} \triangleq \partial^2 f / \partial X_i \partial X_k$.

Ito's Lemma (continued)

- The multiplication table for Theorem 20 is

\times	dW_i	dt
dW_k	$\delta_{ik} dt$	0
dt	0	0

in which

$$\delta_{ik} = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{otherwise.} \end{cases}$$

Ito's Lemma (continued)

- In applying the higher-dimensional Ito's lemma, usually one of the variables, say X_1 , is time t and $dX_1 = dt$.
- In this case, $b_{1j} = 0$ for all j and $a_1 = 1$.
- As an example, let

$$dX_t = a_t dt + b_t dW_t.$$

- Consider the process $f(X_t, t)$.

Ito's Lemma (continued)

- Then

$$\begin{aligned} df &= \frac{\partial f}{\partial X_t} dX_t + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} (dX_t)^2 \\ &= \frac{\partial f}{\partial X_t} (a_t dt + b_t dW_t) + \frac{\partial f}{\partial t} dt \\ &\quad + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} (a_t dt + b_t dW_t)^2 \\ &= \left(\frac{\partial f}{\partial X_t} a_t + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} b_t^2 \right) dt + \frac{\partial f}{\partial X_t} b_t dW_t. \quad (86) \end{aligned}$$

Ito's Lemma (continued)

Theorem 21 (Alternative Ito's Lemma) *Let W_1, W_2, \dots, W_m be Wiener processes and $X \triangleq (X_1, X_2, \dots, X_m)$ be a vector process. Suppose $f : R^m \rightarrow R$ is twice continuously differentiable and X_i is an Ito process with $dX_i = a_i dt + b_i dW_i$. Then $df(X)$ is the following Ito process,*

$$df(X) = \sum_{i=1}^m f_i(X) dX_i + \frac{1}{2} \sum_{i=1}^m \sum_{k=1}^m f_{ik}(X) dX_i dX_k.$$

Ito's Lemma (concluded)

- The multiplication table for Theorem 21 is

\times	dW_i	dt
dW_k	$\rho_{ik} dt$	0
dt	0	0

- Above, ρ_{ik} denotes the correlation between dW_i and dW_k .

Geometric Brownian Motion

- Consider geometric Brownian motion

$$Y(t) \triangleq e^{X(t)}.$$

- $X(t)$ is a (μ, σ) Brownian motion.
- By Eq. (79) on p. 581,

$$dX = \mu dt + \sigma dW.$$

- Note that

$$\begin{aligned}\frac{\partial Y}{\partial X} &= Y, \\ \frac{\partial^2 Y}{\partial X^2} &= Y.\end{aligned}$$

Geometric Brownian Motion (continued)

- Ito's formula (84) on p. 613 implies

$$\begin{aligned}dY &= Y dX + (1/2) Y (dX)^2 \\&= Y (\mu dt + \sigma dW) + (1/2) Y (\mu dt + \sigma dW)^2 \\&= Y (\mu dt + \sigma dW) + (1/2) Y \sigma^2 dt.\end{aligned}$$

- Hence^a

$$\frac{dY}{Y} = \left(\mu + \frac{\sigma^2}{2} \right) dt + \sigma dW. \quad (87)$$

- The annualized *instantaneous* rate of return is $\mu + \sigma^2/2$ (not μ).^b

^aEquation (87) is an abbreviation for $dY = Y(\mu + \sigma^2/2) dt + \sigma dW$.

^bConsistent with Lemma 10 (p. 305).

Geometric Brownian Motion (continued)

- Alternatively, from Eq. (79) on p. 581,

$$X_t = X_0 + \mu t + \sigma W_t,$$

admits an explicit (strong) solution.

- Hence

$$Y_t = Y_0 e^{\mu t + \sigma W_t}, \quad (88)$$

a strong solution to the SDE (87) where $Y_0 = e^{X_0}$.

Geometric Brownian Motion (concluded)

- On the other hand, suppose

$$\frac{dY}{Y} = \mu dt + \sigma dW.$$

- Then $X(t) \triangleq \ln Y(t)$ follows

$$dX = (\mu - \sigma^2/2) dt + \sigma dW.$$

Exponential Martingale

- The Ito process

$$dX_t = b_t X_t dW_t$$

is a martingale.^a

- It is called an exponential martingale.
- By Ito's formula (84) on p. 613,

$$X(t) = X(0) \exp \left[-\frac{1}{2} \int_0^t b_s^2 ds + \int_0^t b_s dW_s \right].$$

^aRecall Theorem 18 (p. 598).

Product of Geometric Brownian Motion Processes

- Let

$$\begin{aligned}\frac{dY}{Y} &= a \, dt + b \, dW_Y, \\ \frac{dZ}{Z} &= f \, dt + g \, dW_Z.\end{aligned}$$

- Assume dW_Y and dW_Z have correlation ρ .
- Consider the Ito process

$$U \triangleq YZ.$$

Product of Geometric Brownian Motion Processes (continued)

- Apply Ito's lemma (Theorem 21 on p. 619):

$$\begin{aligned}dU &= Z dY + Y dZ + dY dZ \\&= ZY(a dt + b dW_Y) + YZ(f dt + g dW_Z) \\&\quad + YZ(a dt + b dW_Y)(f dt + g dW_Z) \\&= U(a + f + bg\rho) dt + U(b dW_Y + g dW_Z).\end{aligned}$$

- The product of correlated geometric Brownian motion processes thus remains geometric Brownian motion.

Product of Geometric Brownian Motion Processes (continued)

- Note that

$$Y(t) = Y(0) \exp \left[\left(a - b^2/2 \right) t + bW_Y(t) \right],$$

$$Z(t) = Z(0) \exp \left[\left(f - g^2/2 \right) t + gW_Z(t) \right],$$

$$U(t) = U(0) \exp \left\{ \left[a + f - (b^2 + g^2)/2 \right] t + bW_Y(t) + gW_Z(t) \right\}.$$

- They are the strong solutions.

Product of Geometric Brownian Motion Processes (concluded)

- $\ln U$ is Brownian motion with a mean equal to the sum of the means of $\ln Y$ and $\ln Z$.
- This holds even if Y and Z are correlated.
- Finally, $\ln Y$ and $\ln Z$ have correlation ρ .

Quotients of Geometric Brownian Motion Processes

- Suppose Y and Z are drawn from p. 626.
- Let

$$U \triangleq Y/Z.$$

- We next show that^a

$$\frac{dU}{U} = (a - f + g^2 - bg\rho) dt + b dW_Y - g dW_Z. \quad (89)$$

- Keep in mind that dW_Y and dW_Z have correlation ρ .

^aExercise 14.3.6 of the textbook is erroneous.

Quotients of Geometric Brownian Motion Processes (concluded)

- The multidimensional Ito's lemma (Theorem 21 on p. 619) can be employed to show that^a

$$\begin{aligned}
 dU &= (1/Z) dY - (Y/Z^2) dZ - (1/Z^2) dY dZ + (Y/Z^3) (dZ)^2 \\
 &= (1/Z)(aY dt + bY dW_Y) - (Y/Z^2)(fZ dt + gZ dW_Z) \\
 &\quad - (1/Z^2)(bgYZ\rho dt) + (Y/Z^3)(g^2 Z^2 dt) \\
 &= U(a dt + b dW_Y) - U(f dt + g dW_Z) \\
 &\quad - U(bg\rho dt) + U(g^2 dt) \\
 &= U(a - f + g^2 - bg\rho) dt + Ub dW_Y - Ug dW_Z.
 \end{aligned}$$

^aAs $\partial^2 U / \partial Y^2 = 0$, the $(dY)^2$ term is ignored.