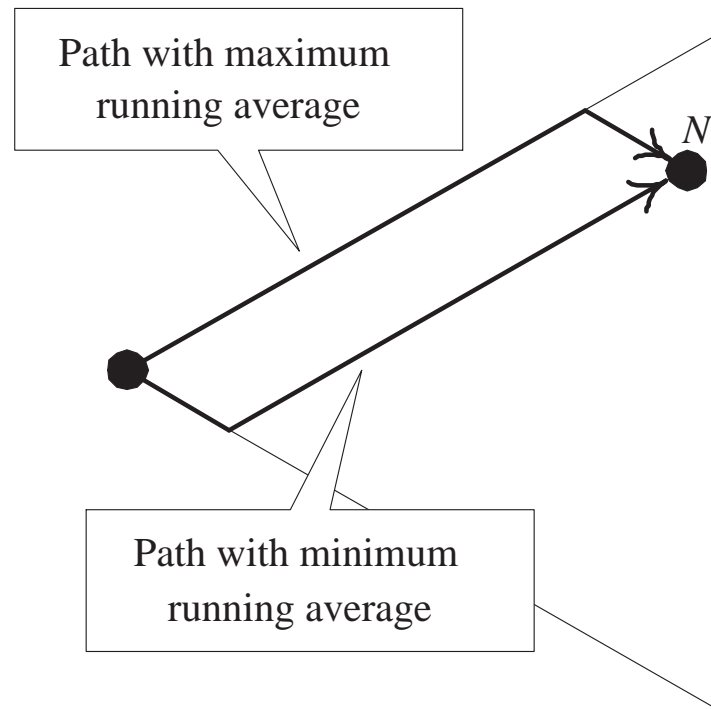# Approximation Algorithm for Asian Options

- Based on the BOPM.

- Consider a node at time j with the underlying asset price equal to $S_0 u^{j-i} d^i$.

- Name such a node $N(j, i)$.

- The running sum $\sum_{m=0}^{j} S_m$ at this node has a maximum value of

$$
S_0(\overbrace{1 + u + u^2 + \cdots + u^{j-i} + u^{j-i}d + \cdots + u^{j-i}d^i}^{j})
$$

$$
= S_0 \frac{1 - u^{j-i+1}}{1 - u} + S_0 u^{j-i} d \frac{1 - d^i}{1 - d}.
$$

Approximation Algorithm for Asian Options (continued)

- Divide this value by $j+1$ and call it $A_{\max}(j,i)$.

- Similarly, the running sum has a minimum value of

$$
\overbrace{S_0(1 + d + d^2 + \cdots + d^i + d^i u + \cdots + d^i u^{j-i}}^{j})
$$

$$
= S_0 \frac{1 - d^{i+1}}{1 - d} + S_0 d^i u \frac{1 - u^{j-i}}{1 - u}.
$$

- Divide this value by $j+1$ and call it $A_{\min}(j,i)$.

- $A_{\min}$ and $A_{\max}$ are running averages.

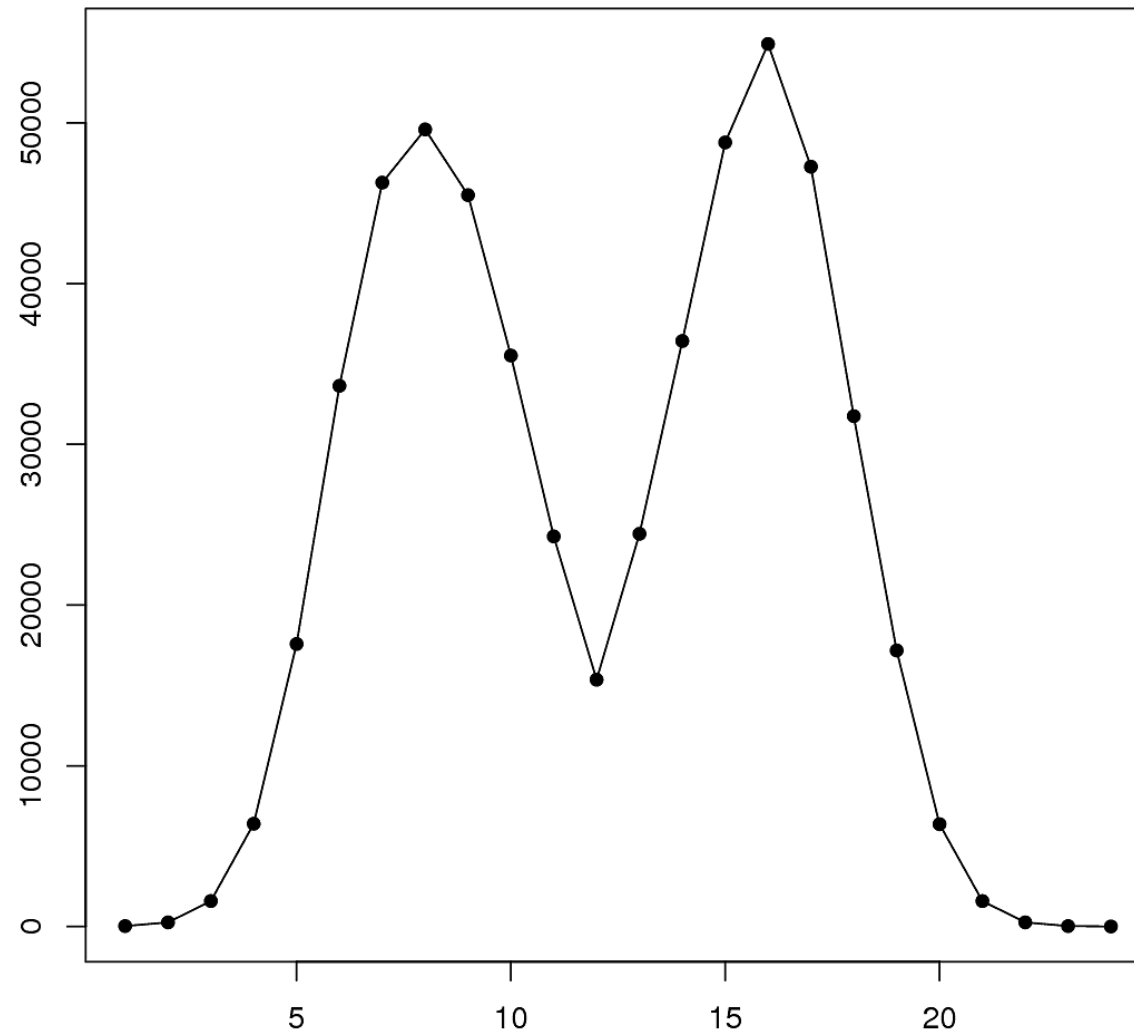# Approximation Algorithm for Asian Options (continued)

- The number of paths to $N(j, i)$ are far too many: $\binom{j}{i}$.
  - For example,
  $$\binom{j}{j/2} \sim 2^j \sqrt{2/(\pi j)}.$$

- The distribution of the running averages at any given time seems to be bimodal.[a]

  - In the plot on the next page, $n = 24$, $u = 5/4$ and $d = 4/5$.

---

[a]Contributed by Mr. Liu, Jun (`R99944027`) on April 15, 2014.

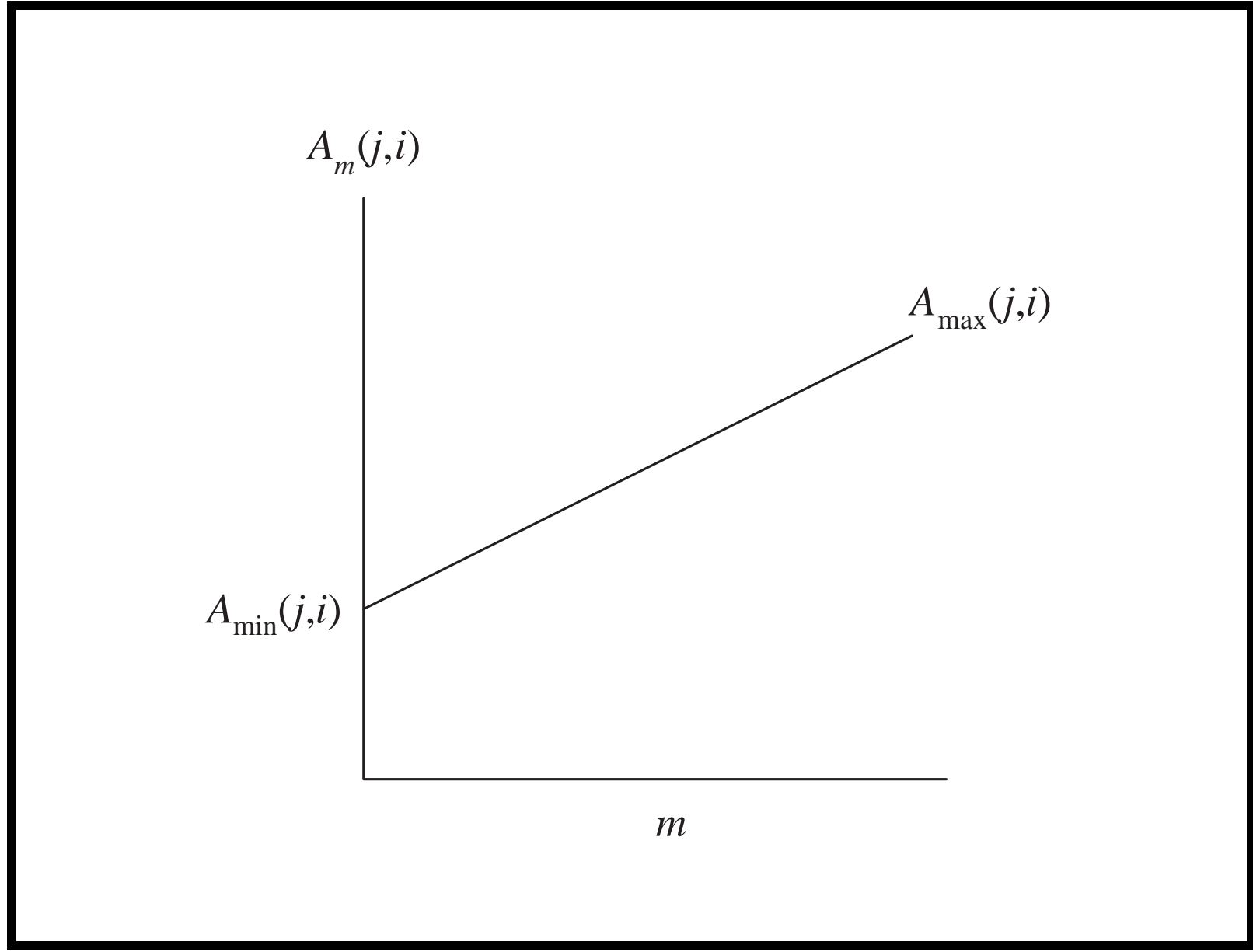Number of Averages

Approximation Algorithm for Asian Options (continued)

- But all averages must lie between $A_{\min}(j, i)$ and $A_{\max}(j, i)$.

- Pick $k + 1$ equally spaced values in this range and treat them as the true and only running averages:

$$A_m(j, i) \triangleq \left(\frac{k - m}{k} \right) A_{\min}(j, i) + \left(\frac{m}{k} \right) A_{\max}(j, i)$$

for $m = 0, 1, \ldots, k.$

$A_m(j,i)$

$A_{max}(j,i)$

$A_{min}(j,i)$

m

Approximation Algorithm for Asian Options (continued)

- Such "bucketing" or "binning " introduces errors, but it works reasonably well in practice.[a]

- A better alternative picks values whose logarithms are equally spaced.[b]

- Still other alternatives are possible, considering the distribution of averages on p. 455.

---

[a]Hull & White (1993); Ritchken, Sankarasubramanian, & Vijh (1993).
[b]Called log-linear interpolation.

Approximation Algorithm for Asian Options (continued)

- Backward induction calculates the option values at each node for the $k+1$ running averages.

- Suppose the current node is $N(j, i)$ and the running average is a.

- Assume the next node is $N(j+1, i)$, after an up move.

- As the asset price there is $S_0 u^{j+1-i} d^i$, we seek the option value corresponding to the new running average

$$A_{\mathrm{u}} \triangleq \frac{(j+1)\, a + S_0 u^{j+1-i} d^i}{j+2}.$$

Approximation Algorithm for Asian Options (continued)

- But A_u is not likely to be one of the $k+1$ running averages at $N(j+1,i)$!

- Find the 2 running averages that bracket it:

$$A_\ell(j+1,i) \le A_\mathrm{u} < A_{\ell+1}(j+1,i).$$

- In "most" cases, the fastest way to nail ℓ is via

$$\ell = \left\lfloor \frac{A_\mathrm{u} - A_\mathrm{min}(j+1,i)}{[\, A_\mathrm{max}(j+1,i) - A_\mathrm{min}(j+1,i)\,]/k} \right\rfloor.$$

$$0$$
$$\ell$$
$$\ell + 1$$
$$k$$

$$0$$
$$m$$
$$k$$

$$0$$
$$\ell'$$
$$\ell' + 1$$
$$k$$
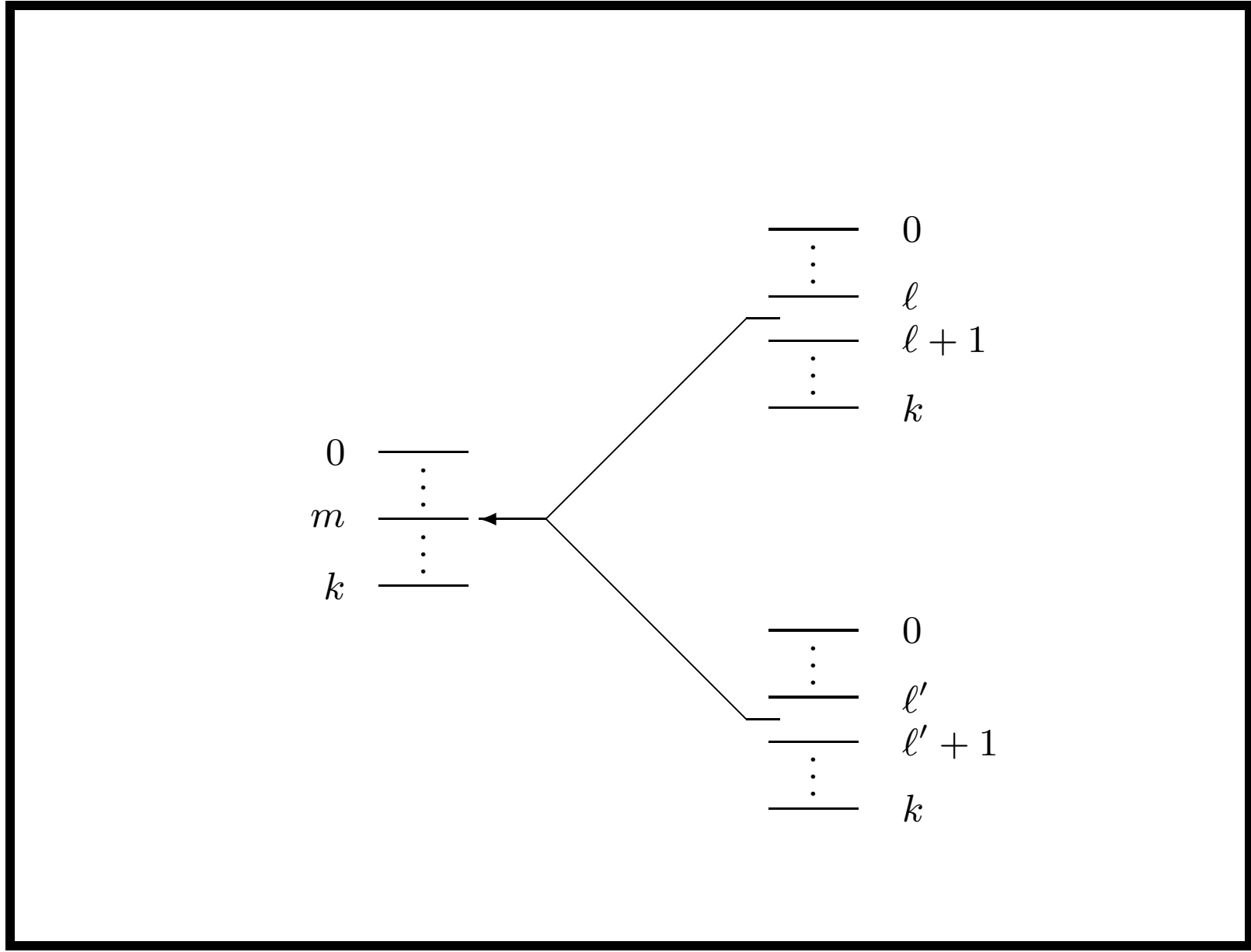
Approximation Algorithm for Asian Options (continued)

- But watch out for the rare case where

$$A_{\mathrm{u}} = A_\ell(j + 1, i)$$

  for some ℓ.

- Also watch out for the case where

$$A_{\mathrm{u}} = A_{\max}(j, i).$$

- Finally, watch out for the degenerate case where $A_0(j + 1, i) = \cdots = A_k(j + 1, i)$.

  – It happens to the two extreme paths.

Approximation Algorithm for Asian Options (continued)

- Express A_{u} as a linearly interpolated value of the two running averages,

$$A_{\mathrm{u}} = x A_{\ell}(j+1, i) + (1-x)\, A_{\ell+1}(j+1, i), \quad 0 < x \le 1.$$

- Let $C_{\ell}(t, s)$ denote the option value at node $N(t, s)$ with running average $A_{\ell}(t, s)$.

- Obtain the approximate option value given the running average A_{u} via

$$C_{\mathrm{u}} \triangleq x C_{\ell}(j+1, i) + (1-x)\, C_{\ell+1}(j+1, i).$$

Approximation Algorithm for Asian Options (continued)

- This interpolation introduces the second source of error.

    - Alternatives to linear interpolation exist.

- The same steps are repeated for the down node $N(j+1, i+1)$ to obtain another approximate option value C_{d}.

- Finally obtain the option value as

$$[\, pC_{\mathrm{u}} + (1-p)\, C_{\mathrm{d}} \,]\, e^{-r\Delta t}.$$

Approximation Algorithm for Asian Options (continued)

- The running time is $O(kn^2)$.

  - There are $O(n^2)$ nodes.

  - Each node has $O(k)$ buckets.

- No interpolation is needed for the calculations at time step $n - 1$.[a]

  - The option values are simply (for calls):

  $$
  \begin{aligned}
  C_{\mathrm{u}} &= \max(A_{\mathrm{u}} - X, 0), \\
  C_{\mathrm{d}} &= \max(A_{\mathrm{d}} - X, 0).
  \end{aligned}
  $$

  - That saves $O(nk)$ calculations.

---

[a]Contributed by Mr. Chen, Shih-Hang (`R02723031`) on April 9, 2014.

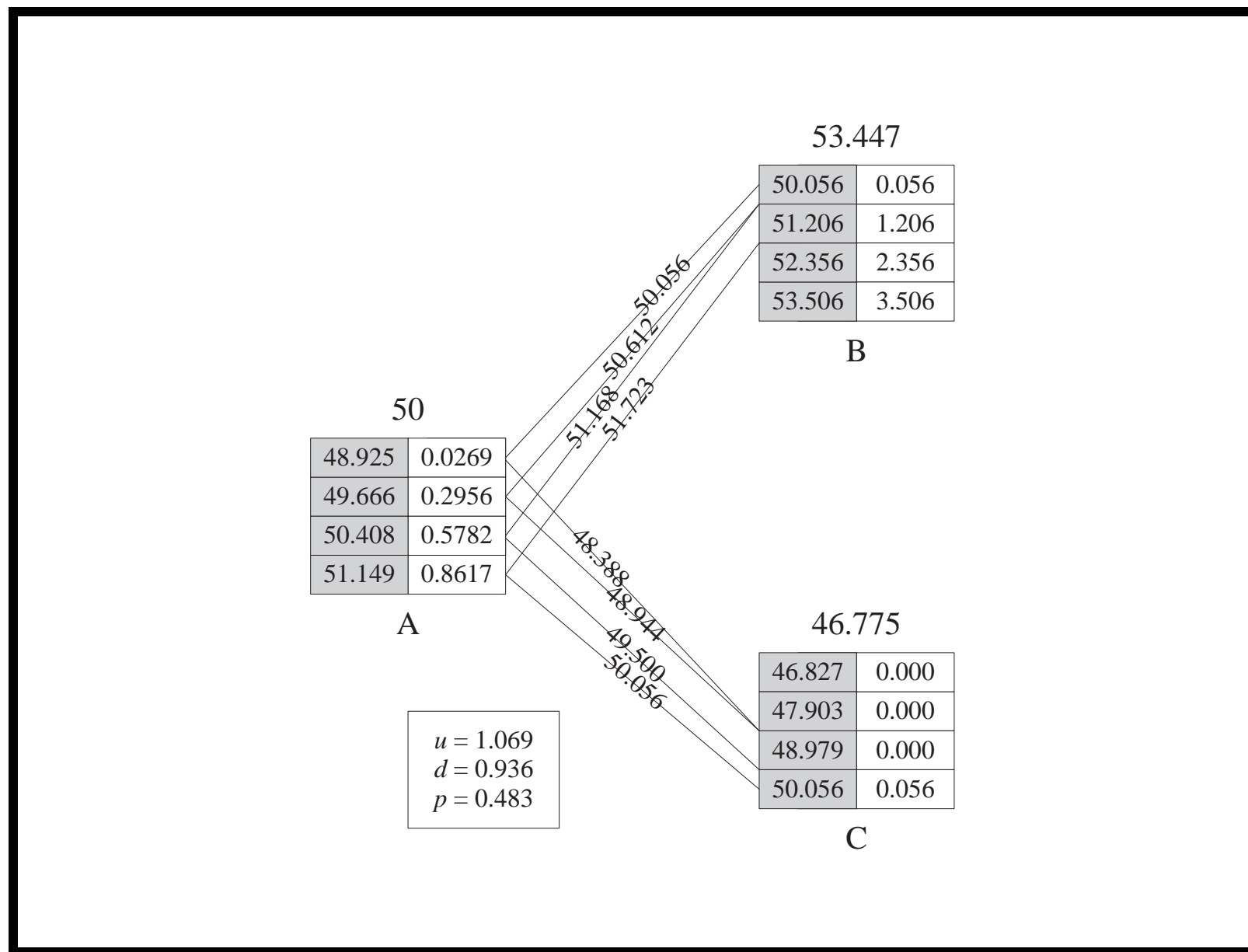## Approximation Algorithm for Asian Options (concluded)

- Arithmetic average-rate options were assumed to be newly issued: no historical average to deal with.

- This problem can be easily addressed.[a]

- How about the Greeks?[b]

---

[a]See Exercise 11.7.4 of the textbook.

[b]Thanks to lively class discussions on March 31, 2004, and April 9, 2014.

A Numerical Example

- Consider a European arithmetic average-rate call with strike price 50.

- Assume zero interest rate in order to dispense with discounting.

- The minimum running average at node A in the figure on p. 468 is 48.925.

- The maximum running average at node A in the same figure is 51.149.

53.447

50.056	0.056
51.206	1.206
52.356	2.356
53.506	3.506

B

50

48.925	0.0269
49.666	0.2956
50.408	0.5782
51.149	0.8617

A

50.056
50.612
51.168
51.723

48.388
48.944
49.500
50.056

$u = 1.069$
$d = 0.936$
$p = 0.483$

46.775

46.827	0.000
47.903	0.000
48.979	0.000
50.056	0.056

C

A Numerical Example (continued)

- Each node picks $k = 3$ for 4 equally spaced running averages.

- The same calculations are done for node A's successor nodes B and C.

- Suppose node A is 2 periods from the root node.

- Consider the up move from node A with running average 49.666.

A Numerical Example (continued)

- Because the stock price at node B is 53.447, the new running average will be

$$\frac{3 \times 49.666 + 53.447}{4} \approx 50.612.$$

- With 50.612 lying between 50.056 and 51.206 at node B, we solve

$$50.612 = x \times 50.056 + (1 - x) \times 51.206$$

to obtain $x \approx 0.517$.

A Numerical Example (continued)

- The option value corresponding to running average 50.056 at node B is 0.056.

- The option values corresponding to running average 51.206 at node B is 1.206.

- Their contribution to the option value corresponding to running average 49.666 at node A is weighted linearly as

$$x \times 0.056 + (1 - x) \times 1.206 \approx 0.611.$$

A Numerical Example (continued)

- Now consider the down move from node A with running average 49.666.

- Because the stock price at node C is 46.775, the new running average will be

$$\frac{3 \times 49.666 + 46.775}{4} \approx 48.944.$$

- With 48.944 lying between 47.903 and 48.979 at node C, we solve

$$48.944 = x \times 47.903 + (1 - x) \times 48.979$$

to obtain $x \approx 0.033$.

A Numerical Example (concluded)

- The option values corresponding to running averages 47.903 and 48.979 at node C are both 0.0.

- Their contribution to the option value corresponding to running average 49.666 at node A is 0.0.

- Finally, the option value corresponding to running average 49.666 at node A equals

$$p \times 0.611 + (1-p) \times 0.0 \approx 0.2956,$$

where $p = 0.483$.

- The remaining three option values at node A can be computed similarly.

Convergence Behavior of the Approximation Algorithm with $k = 50000$[a]
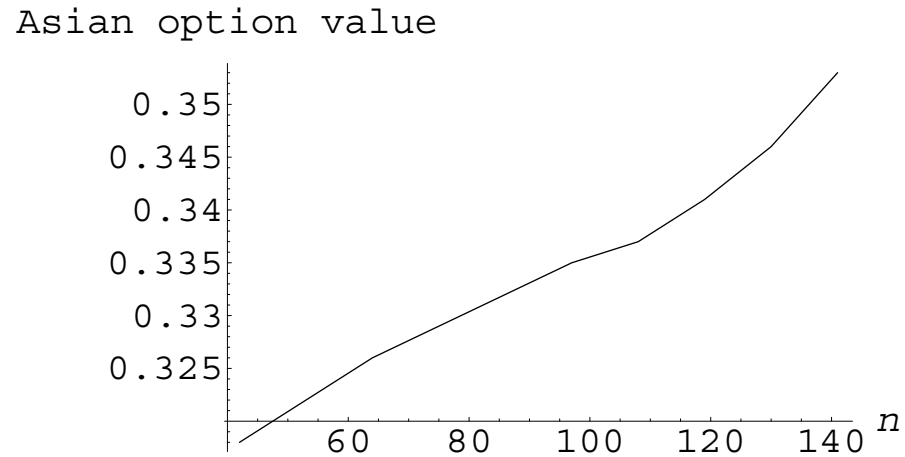
Asian option value

---
[a]T. Dai (B82506025, R86526008, D8852600) & Lyuu (2002).

Remarks on Asian Option Pricing

- Asian option pricing is an active research area.

- The above algorithm overestimates the "true" value.[a]

- To guarantee convergence, k needs to grow with n at least.[b]

- There is a convergent approximation algorithm[c] that does away with interpolation with a running time of

$$2^{O(\sqrt{n})}.$$

---

[a]T. Dai (`B82506025`, `R86526008`, `D8852600`), G. Huang (`F83506075`), & Lyuu (2002).

[b]T. Dai (`B82506025`, `R86526008`, `D8852600`), G. Huang (`F83506075`), & Lyuu (2002).

[c]T. Dai (`B82506025`, `R86526008`, `D8852600`) & Lyuu (2002, 2004).

Remarks on Asian Option Pricing (continued)

- There is an $O(kn^2)$-time algorithm with an error bound of $O(Xn/k)$ from the naive $O(2^n)$-time binomial tree algorithm in the case of European Asian options.[a]

  - k can be varied for trade-off between time and accuracy.

  - If we pick $k = O(n^2)$, then the error is $O(1/n)$, and the running time is $O(n^4)$.

---

[a]Aingworth, Motwani (1962–2009), & Oldham (2000).

# Remarks on Asian Option Pricing (continued)

- Another approximation algorithm reduces the error to $O(X\sqrt{n}/k)$.[a]

    - It varies the number of buckets per node.

    - If we pick $k = O(n)$, the error is $O(n^{-0.5})$.

    - If we pick $k = O(n^{1.5})$, then the error is $O(1/n)$, and the running time is $O(n^{3.5})$.

- Under "reasonable assumptions," an $O(n^2)$-time algorithm with an error bound of $O(1/n)$ exists.[b]

---

[a]T. Dai (B82506025, R86526008, D8852600), G. Huang (F83506075), & Lyuu (2002).

[b]Hsu (R7526001, D89922012) & Lyuu (2004).

Remarks on Asian Option Pricing (concluded)

- The basic idea is a *nonuniform* allocation of running averages instead of a uniform k.

- It strikes a tight balance between error and complexity.
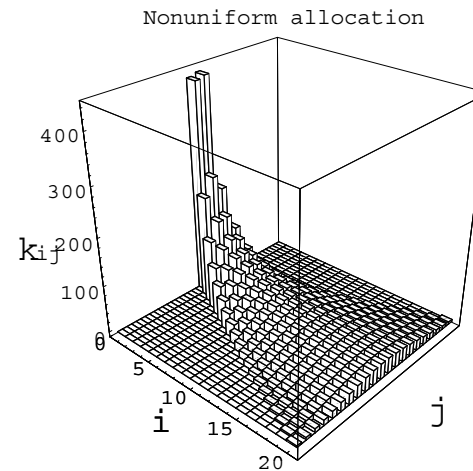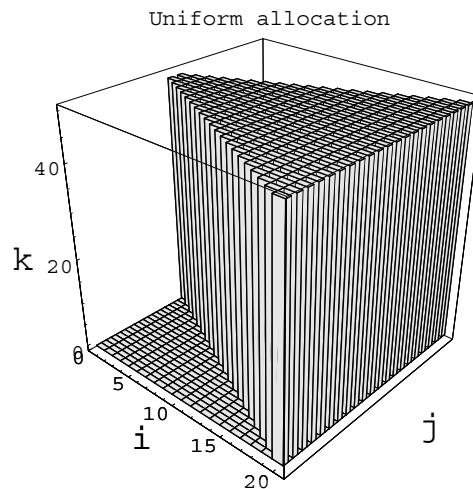
A Grand Comparison[a]

---

[a]Hsu (`R7526001`, `D89922012`) & Lyuu (2004); J. E. Zhang (2001,2003); K. Chen (`R92723061`) & Lyuu (2006).

X	σ	r	Exact	AA2	AA3	Hsu-Lyuu	Chen-Lyuu
95	0.05	0.05	7.1777275	7.1777244	7.1777279	7.178812	7.177726
100			2.7161745	2.7161755	2.7161744	2.715613	2.716168
105			0.3372614	0.3372601	0.3372614	0.338863	0.337231
95		0.09	8.8088392	8.8088441	8.8088397	8.808717	8.808839
100			4.3082350	4.3082253	4.3082331	4.309247	4.308231
105			0.9583841	0.9583838	0.9583841	0.960068	0.958331
95		0.15	11.0940944	11.0940964	11.0940943	11.093903	11.094094
100			6.7943550	6.7943510	6.7943553	6.795678	6.794354
105			2.7444531	2.7444538	2.7444531	2.743798	2.744406
90	0.10	0.05	11.9510927	11.9509331	11.9510871	11.951610	11.951076
100			3.6413864	3.6414032	3.6413875	3.642325	3.641344
110			0.3312030	0.3312563	0.3311968	0.331348	0.331074
90		0.09	13.3851974	13.3851165	13.3852048	13.385563	13.385190
100			4.9151167	4.9151388	4.9151177	4.914254	4.915075
110			0.6302713	0.6302538	0.6302717	0.629843	0.630064
90		0.15	15.3987687	15.3988062	15.3987860	15.398885	15.398767
100			7.0277081	7.0276544	7.0277022	7.027385	7.027678
110			1.4136149	1.4136013	1.4136161	1.414953	1.413286

A Grand Comparison (concluded)

X	σ	r	Exact	AA2	AA3	Hsu-Lyuu	Chen-Lyuu
90	0.20	0.05	12.5959916	12.5957894	12.5959304	12.596052	12.595602
100			5.7630881	5.7631987	5.7631187	5.763664	5.762708
110			1.9898945	1.9894855	1.9899382	1.989962	1.989242
90		0.09	13.8314996	13.8307782	13.8313482	13.831604	13.831220
100			6.7773481	6.7775756	6.7773833	6.777748	6.776999
110			2.5462209	2.5459150	2.5462598	2.546397	2.545459
90		0.15	15.6417575	15.6401370	15.6414533	15.641911	15.641598
100			8.4088330	8.4091957	8.4088744	8.408966	8.408519
110			3.5556100	3.5554997	3.5556415	3.556094	3.554687
90	0.30	0.05	13.9538233	13.9555691	13.9540973	13.953937	13.952421
100			7.9456288	7.9459286	7.9458549	7.945918	7.944357
110			4.0717942	4.0702869	4.0720881	4.071945	4.070115
90		0.09	14.9839595	14.9854235	14.9841522	14.984037	14.982782
100			8.8287588	8.8294164	8.8289978	8.829033	8.827548
110			4.6967089	4.6956764	4.6969698	4.696895	4.694902
90		0.15	16.5129113	16.5133090	16.5128376	16.512963	16.512024
100			10.2098305	10.2110681	10.2101058	10.210039	10.208724
110			5.7301225	5.7296982	5.7303567	5.730357	5.728161

Forwards, Futures, Futures Options, Swaps

Summon the nations to come to the trial.
Which of their gods can predict the future?
— Isaiah 43:9

The sure fun of the evening
outweighed the uncertain treasure[.]
— Mark Twain (1835–1910),
The Adventures of Tom Sawyer

Terms

- r will denote the riskless interest rate.

- The current time is t.

- The maturity date is T.

- The remaining time to maturity is $\tau \overset{\triangle}{=} T - t$ (years).

- The spot price is S.

- The spot price at maturity is S_T.

- The delivery price is X.

Terms (concluded)

- The forward or futures price is F for a newly written contract.

- The value of the contract is f.

- A price with a subscript t usually refers to the price at time t.

- Continuous compounding will be assumed.

Forward Contracts[a]

- *Long* forward contracts are for the *purchase* of the underlying asset for a certain delivery price on a specific time.

  - Foreign currencies, bonds, corn, etc.

- Ideal for hedging purposes.

- A farmer enters into a forward contract with a food processor to deliver 100,000 bushels of corn for $2.5 per bushel on September 27, 1995.[b]

- The farmer is assured of a buyer at an acceptable price.

- The processor knows the cost of corn in advance.

---

[a]They have a 4,000-year history (Marthinsen, 2018).
[b]The farmer assumes a *short* position.

Forward Contracts (concluded)

- A forward agreement limits both risk and rewards.

  – If the spot price of corn rises on the delivery date, the farmer will miss the opportunity of extra profits.

  – If the price declines, the processor will be paying more than it would.

- Either side has an incentive to default.

- Other problems: The food processor may go bankrupt, the farmer can go bust, the farmer might not be able to harvest 100,000 bushels of corn because of bad weather, the cost of growing corn may skyrocket, etc.

Spot and Forward Exchange Rates

- Let S denote the spot exchange rate.

- Let F denote the forward exchange rate one year from now (both in domestic/foreign terms).

- r_{f} denotes the annual interest rate of the foreign currency.

- r_{ℓ} denotes the annual interest rate of the local currency.

- Arbitrage opportunities will arise unless these four numbers satisfy an equation.

Interest Rate Parity[a]

$$\frac{F}{S} = e^{r_\ell - r_{\mathrm{f}}}. \tag{60}$$

- A holder of the local currency can do either of:

  - Lend the money in the domestic market to receive e^{r_ℓ} one year from now.

  - Convert local currency for foreign currency, lend for 1 year in foreign market, and convert foreign currency into local currency at the fixed forward exchange rate, F, by selling forward foreign currency *now*.

---

[a]Keynes (1923). John Maynard Keynes (1883–1946) was one of the greatest economists in history. The parity broke down in late 2008 (Mancini-Griffoli & Ranaldo, 2013).

Interest Rate Parity (concluded)

- No money changes hand in entering into a forward contract.

- One unit of local currency will hence become $Fe^{r_{\mathrm{f}}}/S$ one year from now in the 2nd case.

- If $Fe^{r_{\mathrm{f}}}/S > e^{r_{\ell}}$, an arbitrage profit can result from borrowing money in the domestic market and lending it in the foreign market.

- If $Fe^{r_{\mathrm{f}}}/S < e^{r_{\ell}}$, an arbitrage profit can result from borrowing money in the foreign market and lending it in the domestic market.

Forward Price

- The payoff of a forward contract at maturity is

$$S_T - X.$$

  - Contrast that with call's payoff

  $$\max(S_T - X, 0).$$

- Forward contracts do not involve any initial cash flow.

- The forward price F is the delivery price X which makes the forward contract zero valued.

  - That is,

  $$f = 0 \quad \text{when} \quad X = F.$$

Forward Price (continued)

$$S_T - X$$

0   1   2   3 $\cdots$ n

Forward Price (concluded)

- The delivery price cannot change because it is written in the contract.

- But the forward price may change after the contract comes into existence.

- So although the value of a forward contract, f, is 0 at the outset, it will fluctuate thereafter.

  - This value is enhanced when the spot price climbs.

  - It is depressed when the spot price declines.

- The forward price also varies with the maturity of the contract.

Forward Price: Underlying Pays No Income

Lemma 12 *For a forward contract on an underlying asset providing no income,*

$$F = Se^{r\tau}. \tag{61}$$

- If $F > Se^{r\tau}$:
    - Borrow S dollars for τ years.
    - Buy the underlying asset.
    - Short the forward contract with delivery price F.

Proof (concluded)

- At maturity:

  - Deliver the asset for F.

  - Use $Se^{r\tau}$ to repay the loan, leaving an arbitrage profit of

$$F - Se^{r\tau} > 0.$$

- If $F < Se^{r\tau}$, do the opposite.

Example: Zero-Coupon Bonds

- r is the annualized 3-month riskless interest rate.

- S is the spot price of the 6-month zero-coupon bond.

- A new 3-month forward contract on that 6-month zero-coupon bond should command a delivery price of $Se^{r/4}$.

- So if $r = 6\%$ and $S = 970.87$, then the delivery price is

$$970.87 \times e^{0.06/4} = 985.54.$$

Example: Options

- Suppose S is the spot price of the *European call* that expires at some time later than T.

- A τ-year forward contract on that call commands a delivery price of $Se^{r\tau}$.

- So it equals the future value of the Black-Scholes formula on p. 308.

Contract Value: The Underlying Pays No Income

The value of a forward contract is

$$f = S - Xe^{-r\tau}. \tag{62}$$

- Consider a portfolio consisting of:
  - One long forward contract;
  - Cash amount $Xe^{-r\tau}$;
  - One short position in the underlying asset.

Contract Value: The Underlying Pays No Income (concluded)

- The cash will grow to X at maturity, which can be used to take delivery of the forward contract.

- The delivered asset will then close out the short position.

- Since the value of the portfolio is zero at maturity, its PV must be zero.[a]

- So a forward contract can be replicated by a long position in the underlying and a loan of $Xe^{-r\tau}$ dollars.

---

[a]Recall p. 223.

Lemma 12 (p. 494) Revisited

- Set $f = 0$ in Eq. (62) on p. 498.

- Then $X = Se^{r\tau}$, the forward price.

Forward Price: Underlying Pays Predictable Income

Lemma 13 *For a forward contract on an underlying asset providing a predictable income with a PV of I,*

$$F = (S - I) \, e^{r\tau}. \tag{63}$$

- If $F > (S - I) \, e^{r\tau}$, borrow S dollars for τ years, buy the underlying asset, and short the forward contract with delivery price F.

- Use the income to repay part of the loan.

The Proof (concluded)

- At maturity, the asset is delivered for F, and $(S - I)\, e^{r\tau}$ is used to repay the *remaining* loan.

- That leaves an arbitrage profit of

$$F - (S - I)\, e^{r\tau} > 0.$$

- If $F < (S - I)\, e^{r\tau}$, reverse the above.

2025 Prof. Yuh-Dauh Lyuu, National Taiwan University

Example

- Consider a 10-month forward contract on a $50 stock.

- The stock pays a dividend of $1 every 3 months.

- The forward price is

$$\left(50 - e^{-r_3/4} - e^{-r_6/2} - e^{-3 \times r_9/4}\right) e^{r_{10} \times (10/12)}.$$

  – r_i is the annualized i-month interest rate.

Underlying Pays a Continuous Dividend Yield of q

- The value of a forward contract at any time prior to T is[a]

$$f = Se^{-q\tau} - Xe^{-r\tau}. \qquad (64)$$

- One consequence of Eq. (64) is that the forward price is

$$F = Se^{(r-q)\tau}. \qquad (65)$$

---

[a]See p. 160 of the textbook for proof.

Futures Contracts vs. Forward Contracts

- They are traded on a central exchange.

- A clearinghouse.

  - Credit risk is minimized.

- Futures contracts are standardized instruments.

- Gains and losses are marked to market daily.

  - Adjusted at the end of each trading day based on the settlement price.

Size of a Futures Contract

- The amount of the underlying asset to be delivered under the contract.

  - 5,000 bushels for the corn futures on the Chicago Board of Trade (CBOT).

  - One million U.S. dollars for the Eurodollar futures on the Chicago Mercantile Exchange (CME).[a]

- A position can be closed out (or offset) by entering into a reversing trade to the original one.

- Most futures contracts are closed out in this way rather than have the underlying asset delivered.

  - Forward contracts are meant for delivery.

---

[a]CME and CBOT merged on July 12, 2007.

Daily Settlements

- Price changes in the futures contract are settled daily.

- Hence the spot price rather than the initial futures price is paid on the delivery date.

- Marking to market nullifies any financial incentive for not making delivery.

  - A farmer enters into a forward contract to sell 100,000 bushels of corn at $2.00 per bushel in November.

  - Suppose the price of corn rises to $2.5 by November.

  - The farmer has incentive to sell his harvest in the spot market for $2.5.

Daily Settlements (concluded)

- (continued)

  - With marking to market, the farmer has transferred $0.5 per bushel from his futures account to that of the clearinghouse by November.[a]

  - When the farmer makes delivery, he is paid the spot price, $2.5 per bushel.

  - The farmer has little incentive to default.

  - The *net* price remains $\$2.5 - 0.5 = 2$ per bushel, the original delivery price.

---

[a]

Daily Cash Flows

- Let F_i denote the futures price at the end of day i.

- The contract's cash flow on day i is $F_i - F_{i-1}$.

- The net cash flow over the life of the contract is

$$(F_1 - F_0) + (F_2 - F_1) + \cdots + (F_n - F_{n-1}) \qquad (66)$$
$$= F_n - F_0 = S_T - F_0.$$

- A futures contract has the same *accumulated* payoff $S_T - F_0$ as a forward contract.

- The actual payoff may vary because of the reinvestment of daily cash flows and how $S_T - F_0$ is distributed.

Daily Cash Flows (concluded)

$$F_1 - F_0 \qquad F_2 - F_1 \qquad F_3 - F_2 \qquad \cdots \qquad F_n - F_{n-1}$$

$$0 \qquad\qquad 1 \qquad\qquad 2 \qquad\qquad 3 \qquad\quad \cdots \qquad\quad n$$
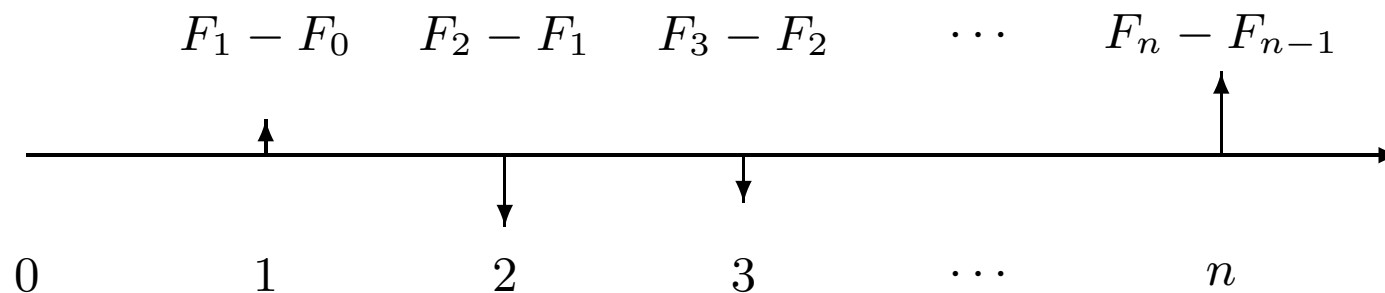
Delivery and Hedging

- Futures price is the delivery price that makes the futures contract zero-valued.

- Delivery ties the futures price to the spot price.

- On the delivery date, the settlement price of the futures contract is determined by the spot price.

- Hence, when the delivery period is reached, the futures price should be very close to the spot price.[a]

- Changes in futures prices usually track those in spot price, making hedging possible.

---

[a]But since early 2006, futures for corn, wheat, and soybeans occasionally expired at a price much higher than that day's spot price (Henriques, 2008).

Forward and Futures Prices

- Surprisingly, futures price equals forward price if interest rates are nonstochastic![a]

- This result "justifies" treating a futures contract as if it were a forward contract, ignoring its marking-to-market feature.

- The West Texas Intermediate (WTI) futures price was negative on April 20, 2020![b]

---

[a]Cox, Ingersoll, & Ross (1981); see p. 164 of the textbook for proof.
[b]April 21 was the last trading day for oil delivery in May to Cushing, Oklahoma.

Remarks

- When interest rates are stochastic, forward and futures prices are no longer theoretically identical.

  - Suppose interest rates are uncertain and futures prices move in the same direction as interest rates.

  - Then futures prices will exceed forward prices.

- For short-term contracts, the differences tend to be small.

- Unless stated otherwise, assume forward and futures prices are identical.

Futures Options

- The underlying of a futures option is a futures contract.

- Upon exercise, the option holder takes a position in the futures contract with a futures price equal to the option's strike price.

  - A call holder acquires a *long* futures position.

  - A put holder acquires a *short* futures position.

- The futures contract is then marked to market.

- And the futures positions of the two parties are at the prevailing futures price (thus zero-valued).

Futures Options (concluded)

- It works as if the *call* holder received a futures contract plus cash equivalent to the prevailing futures price F_t minus the strike price X:

$$F_t - X.$$

  - This futures contract has zero value.

- It works as if the *put* holder sold a futures contract for

$$X - F_t$$

  dollars.

Forward Options

- What is delivered is now a forward contract with a delivery price equal to the option's strike price.

  - Exercising a call forward option results in a *long* position in a forward contract.

  - Exercising a put forward option results in a *short* position in a forward contract.

- Exercising a forward option incurs no immediate cash flows as there is no marking to market.

Example

- Consider an American call with strike $100 and an expiration date in September.

- The underlying asset is a *forward* contract with a delivery date in December.

- Suppose the forward price in July is $110.

- Upon exercise, the call holder receives a forward contract with a delivery price of $100.[a]

- If an offsetting position is then taken in the forward market,[b] a $10 profit *in December* will be assured.

- A call on the futures would realize the $10 profit *in July.*

---

[a]Recall p. 486.

[b]The counterparty will pay you $110 for the underlying asset.

Some Pricing Relations

- Let T be the delivery date, the current time be 0, and the option[a] have expiration date t $(t \leq T)$.

- Assume a constant, positive interest rate.

- Although forward price equals futures price, a forward option does *not* have the same value as a futures option.

- The payoffs of calls at time t are, respectively,[b]

$$\text{futures option} = \max(F_t - X, 0), \qquad (67)$$

$$\text{forward option} = \max(F_t - X, 0) \, e^{-r(T-t)}. \quad (68)$$

---

[a]On the futures or forward contract

[b]Recall p. 515.

Some Pricing Relations (concluded)

- A European futures option is worth the same as a European option on the underlying asset if $T = t$.

    - Futures price equals spot price at maturity.

- This conclusion is model independent.

Put-Call Parity[a]

The put-call parity is slightly different from the one in Eq. (31) on p. 230.

Theorem 14 *(1) For European options on futures contracts,*

$$C = P - (X - F) \, e^{-rt}.$$

(2) For European options on forward contracts,

$$C = P - (X - F) \, e^{-rT}.$$

---

[a]See Theorem 12.4.4 of the textbook for proof.

Early Exercise

The early exercise feature is not valuable for *forward* options.[a]

Theorem 15 *American forward options should not be exercised before expiration as long as the probability of their ending up out of the money is positive.*

Early exercise may be optimal for American *futures* options even if the underlying asset generates no payouts.

Theorem 16 *American futures options may be exercised optimally before expiration.*

---

[a]See Theorem 12.4.5 of the textbook for proof.

Black's Model[a]

- Formulas for European futures options:

$$C = Fe^{-rt}N(x) - Xe^{-rt}N(x - \sigma\sqrt{t}), \qquad (69)$$

$$P = Xe^{-rt}N(-x + \sigma\sqrt{t}) - Fe^{-rt}N(-x), \quad (70)$$

where $x \triangleq \frac{\ln(F/X) + (\sigma^2/2)\, t}{\sigma\sqrt{t}}$.

- The above formulas are related to those for options on a stock paying a continuous dividend yield.

- They are Eqs. (44) on p. 339 with q set to r and S replaced by F.

---

[a]Black (1976). It is also called the Black-76 model.

Black's Model (concluded)

- Volatility σ is that of the stock price.[a]

  – It is the same as the volatlity of the futures price.[b]

- This observation incidentally proves Theorem 16 (p. 521).

- For European forward options, just multiply the above formulas by $e^{-r(T-t)}$.

  – Forward options differ from futures options by a factor of $e^{-r(T-t)}$.[c]

---

[a]Contributed by Mr. Lu, Yu-Ming (`R06723032`, `D08922008`) on April 7, 2021.

[b]Contributed by Mr. Chen, Tung-Li (`D09922014`) on April 7, 2023. See also p. 634.

[c]Recall Eqs. (67)–(68) on p. 518.

Binomial Model for Forward and Futures Options

- In a risk-neutral economy, futures price behaves *like* a stock paying a continuous dividend yield of r.

  - Let the futures price at time 0 be F.

  - From Lemma 10 (p. 305), the expected value of S at time Δt in a risk-neutral economy is

  $$Se^{r\Delta t}.$$

  - By Eq. (61) on p. 494, the expected futures price at time Δt is

  $$Se^{r\Delta t}e^{r(T-\Delta t)} = Se^{rT} = F.$$

Binomial Model for Forward and Futures Options (continued)

- The above observation continues to hold even if S pays a dividend yield![a]

    - Let the futures price at time 0 be F.

    - From Lemma 10 (p. 305), the expected value of S at time Δt in a risk-neutral economy is

    $$Se^{(r-q)\,\Delta t}.$$

    - By Eq. (65) on p. 504, the expected futures price at time Δt is

    $$Se^{(r-q)\,\Delta t}e^{(r-q)(T-\Delta t)} = Se^{(r-q)\,T} = F.$$

---

[a]Contributed by Mr. Liu, Yi-Wei (`R02723084`) on April 16, 2014.

Binomial Model for Forward and Futures Options (concluded)

- Now, under the BOPM, the risk-neutral probability for the futures price is

$$p_{\mathrm{f}} \stackrel{\Delta}{=} (1 - d)/(u - d) \qquad (71)$$

  by Eq. (45) on p. 341.

  - The futures price moves from F to Fu with probability p_{f} and to Fd with probability $1 - p_{\mathrm{f}}$.

  - The *original* u and d are used here.

- The binomial tree algorithm for *forward* options is identical except that Eq. (68) on p. 518 is the payoff.

Spot and Futures Prices under BOPM

- The futures price is related to the spot price via

$$F = Se^{rT}$$

  if the underlying asset pays no dividends.[a]

- Recall the futures price F moves to Fu with probability p_f per period.

- So the stock price moves from $S = Fe^{-rT}$ to

$$Fue^{-r(T-\Delta t)} = Sue^{r\Delta t}$$

  with probability p_f per period.

---

[a]Recall Lemma 12 (p. 494).

Spot and Futures Prices under BOPM (concluded)

- Similarly, the stock price moves from $S = Fe^{-rT}$ to

$$Sde^{r\Delta t}$$

with probability $1 - p_f$ per period.

- Note that

$$S(ue^{r\Delta t})(de^{r\Delta t}) = Se^{2r\Delta t} \neq S.$$

- This binomial model for S differs from the CRR tree.

- This model may not be suitable for pricing barrier options (why?).

Negative Probabilities Revisited

- As $0 < p_f < 1$, we have $0 < 1 - p_f < 1$ as well.

- The problem of negative risk-neutral probabilities is solved:

  - Build the tree for the futures price F of the futures contract expiring at the same time as the option.

  - Let the stock pay a continuous dividend yield of q.

  - By Eq. (65) on p. 504, recover S from F at each node via
    $$S = F e^{-(r-q)(T-t)}.$$

Swaps

- Swaps are agreements between two counterparties to exchange cash flows in the future according to some predetermined formula.

- There are two basic types of swaps: interest rate and currency.

- An interest rate swap occurs when two parties exchange interest payments periodically.

- Currency swaps are agreements to deliver one currency against another (our focus here).

- There are theories about why swaps exist.[a]

---

[a]Thanks to a lively discussion on April 16, 2014.

Currency Swaps

- A currency swap involves two parties to exchange cash flows in different currencies.

- Consider the following fixed rates available to party A and party B in U.S. dollars and Japanese yen:

	Dollars	Yen
A	$D_A\%$	$\boxed{Y_A\%}$
B	$\boxed{D_B\%}$	$Y_B\%$

- Suppose A wants to take out a fixed-rate loan in yen, and B wants to take out a fixed-rate loan in dollars.

Currency Swaps (continued)

- A straightforward scenario is for A to borrow yen at $Y_A\%$ and B to borrow dollars at $D_B\%$.

- But suppose A is *relatively* more competitive in the dollar market than the yen market, i.e.,

$$Y_B - D_B < Y_A - D_A \quad \text{or} \quad Y_B - Y_A < D_B - D_A.$$

- Consider this alternative arrangement:

  - A borrows dollars.

  - B borrows yen.

  - They enter into a currency swap with a bank (the swap dealer) as the intermediary.

Currency Swaps (concluded)

- The counterparties exchange principal at the beginning and the end of the life of the swap.

- This act transforms A's loan into a yen loan and B's yen loan into a dollar loan.

- The total interest rate is originally $(Y_{\mathrm{A}} + D_{\mathrm{B}})\%$.

- The new arrangement has a smaller total rate of $(D_{\mathrm{A}} + Y_{\mathrm{B}})\%$.

- So the total gain is $((D_{\mathrm{B}} - D_{\mathrm{A}}) - (Y_{\mathrm{B}} - Y_{\mathrm{A}}))\%$.

- Transactions will happen only if the gain is distributed so that the cost to each party is less than the original.

Example

- A and B face the following borrowing rates:

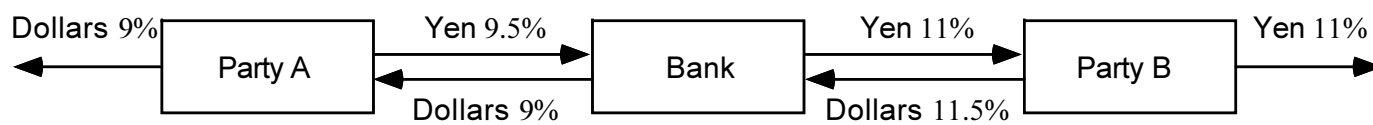|	Dollars	Yen
A	9%	10%
B	12%	11%

- A wants to borrow yen, and B wants to borrow dollars.

- A can borrow yen directly at 10%.

- B can borrow dollars directly at 12%.

Example (continued)

- The rate differential in dollars (3%) is different from that in yen (1%).

- So a currency swap with a total saving of $3 - 1 = 2\%$ is possible.

- A is relatively more competitive in the dollar market.

- B is relatively more competitive in the yen market.

Example (concluded)

- Next page shows an arrangement which is beneficial to all parties involved.

    - A effectively borrows yen at 9.5% (lower than 10%).

    - B borrows dollars at 11.5% (lower than 12%).

    - The gain is 0.5% for A, 0.5% for B, and, if we treat dollars and yen identically, 1% for the bank.

Dollars 9% ← Party A — Yen 9.5% → Bank — Yen 11% → Party B → Yen 11%
            ← Dollars 9% —   ← Dollars 11.5% —

As a Package of Cash Market Instruments

- Assume no default risk.

- Take B on p. 537 as an example.

- The swap is equivalent to a long position in a yen bond paying 11% annual interest and a short position in a dollar bond paying 11.5% annual interest.

- The pricing formula is $SP_{\mathrm{Y}} - P_{\mathrm{D}}$.

  - P_{D} is the dollar bond's value in dollars.
  - P_{Y} is the yen bond's value in yen.
  - S is the \$/yen spot exchange rate.

As a Package of Cash Market Instruments (concluded)

- The value of a currency swap depends on:

  – The term structures of interest rates in the currencies involved.

  – The spot exchange rate.

- It has zero value when

$$SP_{\mathrm{Y}} = P_{\mathrm{D}}.$$

Example

- Take a 3-year swap on p. 537 with principal amounts of US\$1 million and 100 million yen.

- The payments are made once a year.

- The spot exchange rate is 90 yen/\$ and the term structures are flat in both nations—8% in the U.S. and 9% in Japan.

- For B, the value of the swap is (in millions of USD)

$$
\frac{1}{90} \times \left(11 \times e^{-0.09} + 11 \times e^{-0.09 \times 2} + 111 \times e^{-0.09 \times 3} \right)
$$
$$
- \left(0.115 \times e^{-0.08} + 0.115 \times e^{-0.08 \times 2} + 1.115 \times e^{-0.08 \times 3} \right) \quad = \quad 0.074.
$$

As a Package of Forward Contracts

• From Eq. (64) on p. 504, the forward contract maturing i years from now has a *dollar* value of

$$f_i \triangleq (SY_i) e^{-qi} - D_i e^{-ri}. \tag{72}$$

  – Y_i is the yen inflow at year i.

  – S is the \$/yen spot exchange rate.[a]

  – q is the yen interest rate.

  – D_i is the dollar outflow at year i.

  – r is the dollar interest rate.

---

[a]This is *not* the forward exchange rate. Contributed by Mr. Chen, Tzu-Chun (R12922037) and Mr. Wang, Wei-Li (R12922116) on April 19, 2024.

As a Package of Forward Contracts (concluded)

- For simplicity, flat term structures were assumed.

- Generalization is straightforward.

Example

- Take the swap in the example on p. 540.

- Every year, B receives 11 million yen and pays 0.115 million dollars.

- In addition, at the end of the third year, B receives 100 million yen and pays 1 million dollars.

- Each of these transactions represents a forward contract.

- $Y_1 = Y_2 = 11$, $Y_3 = 111$, $S = 1/90$, $D_1 = D_2 = 0.115$, $D_3 = 1.115$, $q = 0.09$, and $r = 0.08$.

- Plug in these numbers to get $f_1 + f_2 + f_3 = 0.074$ million dollars as before.