Self-Financing

- Delta changes over time.
- The maintenance of an equivalent portfolio is dynamic.
- But it does *not* depend on predicting future stock prices.
- The portfolio's value at the end of the current period is precisely the amount needed to set up the next portfolio.
- The trading strategy is *self-financing* because there is neither injection nor withdrawal of funds throughout.[a]
  - Changes in value are due entirely to capital gains.

[a]Except at the beginning, of course, when the option premium is paid before the replication starts.

Binomial Distribution

- Denote the binomial distribution with parameters n and p by

$$b(j; n, p) \triangleq \binom{n}{j} p^j (1-p)^{n-j} = \frac{n!}{j!\,(n-j)!}\, p^j (1-p)^{n-j}.$$

  - $n! = 1 \times 2 \times \cdots \times n$.
  - Convention: $0! = 1$.
- Suppose you flip a coin n times with p being the probability of getting heads.
- Then $b(j; n, p)$ is the probability of getting j heads.

The Binomial Option Pricing Formula

- The stock prices at time n are

$$Su^n, Su^{n-1}d, \ldots, Sd^n.$$

- Let a be the minimum number of upward price moves for the call to finish in the money.
- So a is the smallest nonnegative integer j such that

$$Su^j d^{n-j} \geq X,$$

or, equivalently,

$$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil.$$

The Binomial Option Pricing Formula (concluded)

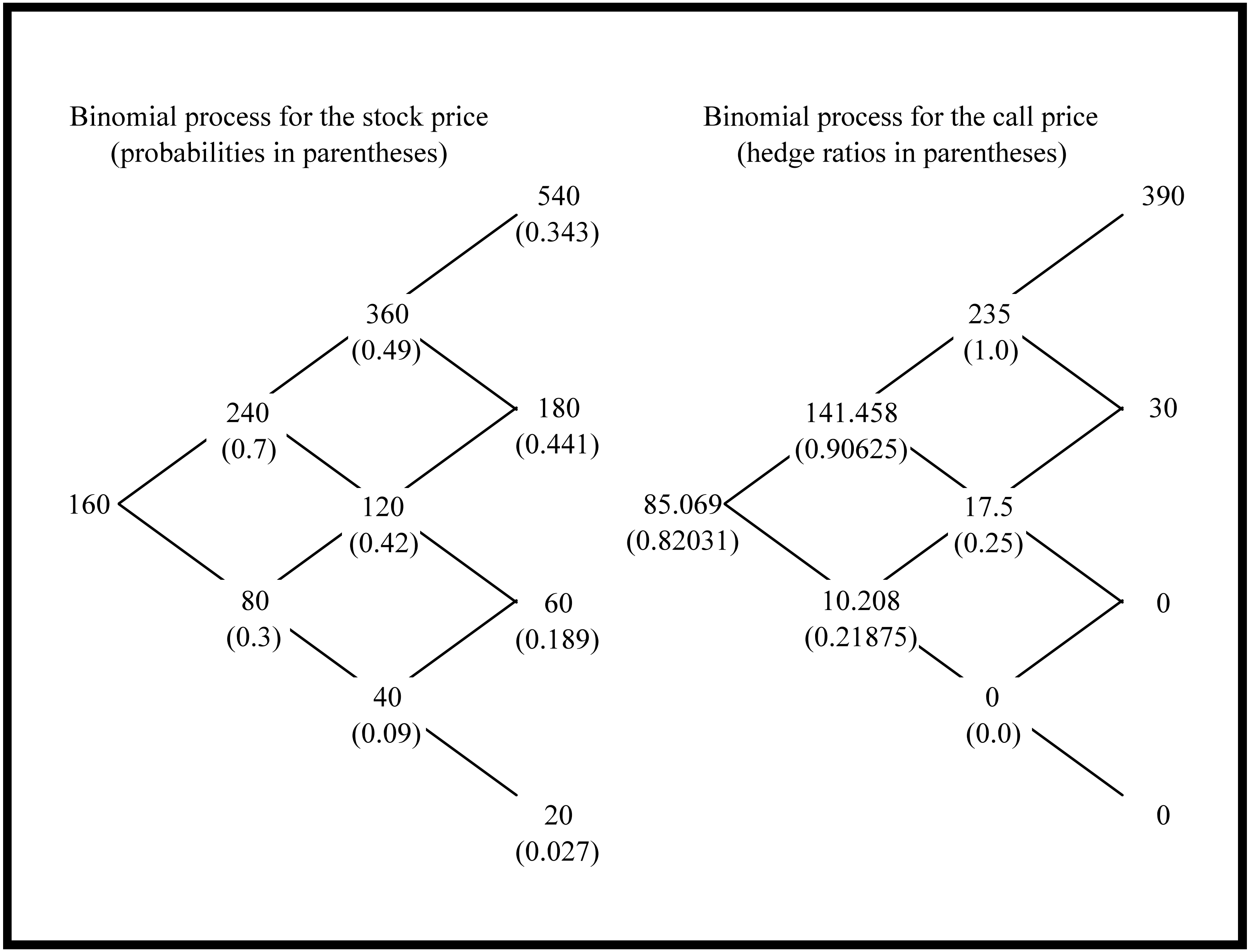
- Hence,

$$
\begin{aligned}
& C \\
= \quad & \frac{\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \left(S u^j d^{n-j} - X \right)}{R^n} && (38) \\
= \quad & S \sum_{j=a}^{n} \binom{n}{j} \frac{(pu)^j [\,(1-p)\, d\,]^{n-j}}{R^n} \\
& - \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \\
= \quad & S \sum_{j=a}^{n} b\,(j; n, pu/R) - X e^{-\hat{r} n} \sum_{j=a}^{n} b(j; n, p). && (39)
\end{aligned}
$$

Numerical Examples

- A non-dividend-paying stock is selling for \$160.
- $u = 1.5$ and $d = 0.5$.
- $r = 18.232\%$ per period ($R = e^{0.18232} = 1.2$).
  - Hence $p = (R - d)/(u - d) = 0.7$.
- Consider a European call on this stock with $X = 150$ and $n = 3$.
- The call value is \$85.069 by backward induction.
- Or, the PV of the expected payoff at expiration:

$$\frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.$$


Binomial process for the stock price
(probabilities in parentheses)
540
(0.343)
360
(0.49)
240
(0.7)
180
(0.441)
160
120
(0.42)
80
(0.3)
60
(0.189)
40
(0.09)
20
(0.027)
Binomial process for the call price
(hedge ratios in parentheses)
390
235
(1.0)
141.458
(0.90625)
30
85.069
(0.82031)
17.5
(0.25)
10.208
(0.21875)
0
0
(0.0)
0

Numerical Examples (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for \$90 instead.
- Sell the call for \$90.
- Invest \$85.069 in the *replicating* portfolio with 0.82031 shares of stock as required by the delta.
- Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.
- The fund that remains,

$$90 - 85.069 = 4.931 \text{ dollars,}$$

is the arbitrage profit, as we will see.

Numerical Examples (continued)

Time 1:

- Suppose the stock price moves to \$240.
- The new delta is 0.90625.
- Buy
$$0.90625 - 0.82031 = 0.08594$$
more shares at the cost of $0.08594 \times 240 = 20.6256$ dollars financed by borrowing.
- Debt now totals $20.6256 + 46.1806 \times 1.2 = 76.04232$ dollars.

Numerical Examples (continued)

- The trading strategy is self-financing because the portfolio has a value of

$$0.90625 \times 240 - 76.04232 = 141.45768.$$

- It matches the corresponding call value by backward induction![a]

---

[a] See p. 279.

Numerical Examples (continued)

Time 2:

- Suppose the stock price plunges to \$120.
- The new delta is 0.25.
- Sell $0.90625 - 0.25 = 0.65625$ shares.
- This generates an income of $0.65625 \times 120 = 78.75$ dollars.
- Use this income to reduce the debt to

$$76.04232 \times 1.2 - 78.75 = 12.5$$

dollars.

Numerical Examples (continued)

Time 3 (the case of rising price):

- The stock price moves to \$180.
- The call we wrote finishes in the money.
- Close out the call's short position by buying back the call or buying a share of stock for delivery.
- This results in a loss of $180 - 150 = 30$ dollars.
- Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.
- It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.

Numerical Examples (concluded)

Time 3 (the case of declining price):

- The stock price moves to \$60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of

$$0.25 \times 60 = 15$$

dollars.

- Use it to repay the debt of $12.5 \times 1.2 = 15$ dollars.

Applications besides Exploiting Arbitrage Opportunities[a]

- Replicate an option using stocks and bonds.
  - Set up a portfolio to replicate the call with \$85.069.
- Hedge the options we issued.
  - Use \$85.069 to set up a portfolio to replicate the call to counterbalance its values exactly.[b]
- $\cdots$
- Without hedge, one may end up forking out \$390 in the worst case (see p. 279)![c]

---

[a]Thanks to a lively class discussion on March 16, 2011.

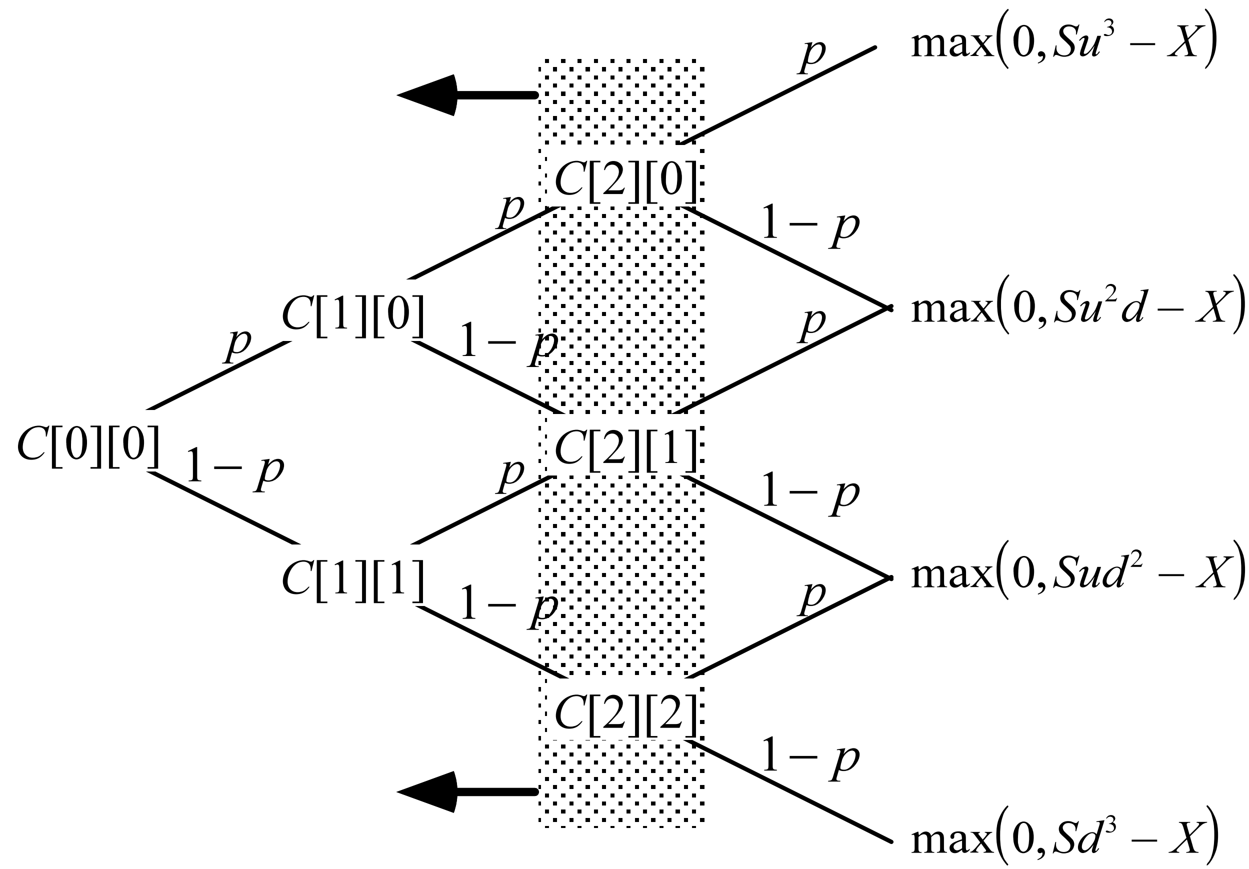
[b]Hedging and replication are mirror images.

[c]Thanks to a lively class discussion on March 16, 2016.

Binomial Tree Algorithms for European Options

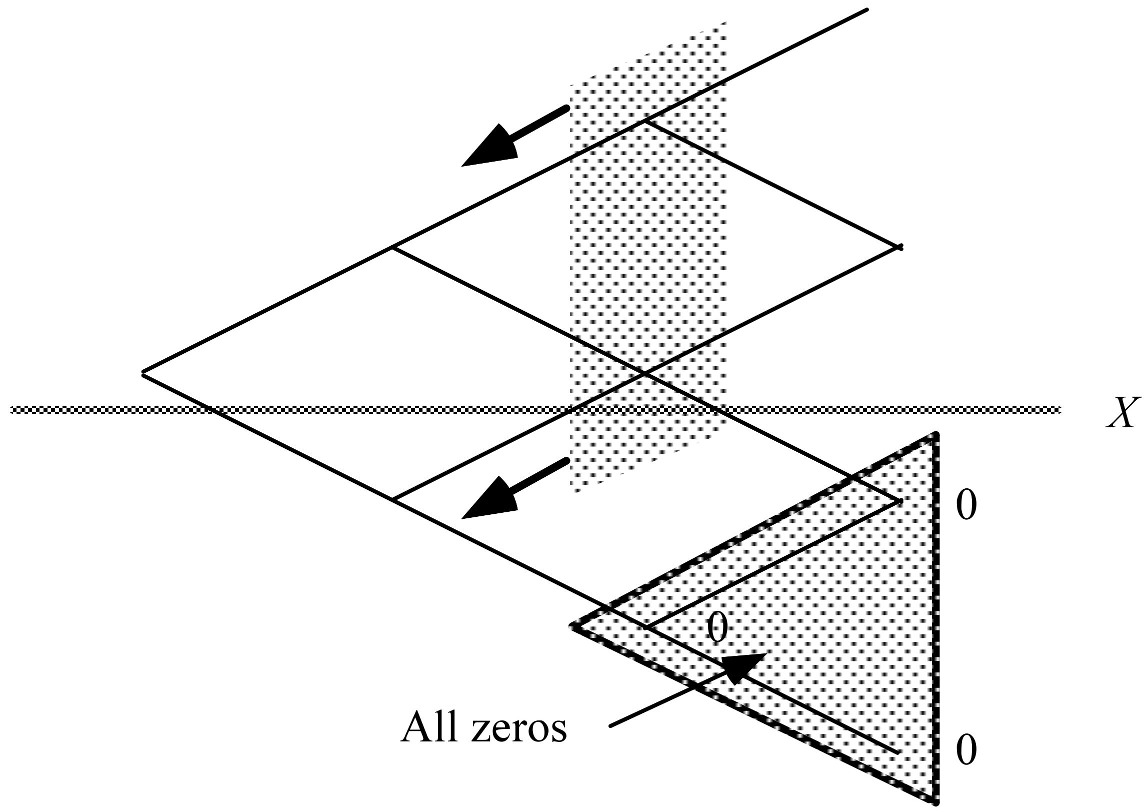
- The BOPM implies the binomial tree algorithm that applies backward induction.
- The total running time is $O(n^2)$ because there are $\sim n^2/2$ nodes.
- The memory requirement is $O(n^2)$.
  - Can be easily reduced to $O(n)$ by reusing space.[a]
- To find the hedge ratio, apply formula (32) on p. 253.
- To price European puts, simply replace the payoff.

[a]But watch out for the proper updating of array entries.



p
max(0, Su³ − X)
C[2][0]
p
1 − p
C[1][0]
1 − p
p
max(0, Su²d − X)
p
C[0][0]
1 − p
C[2][1]
p
1 − p
C[1][1]
1 − p
p
max(0, Sud² − X)
C[2][2]
1 − p
max(0, Sd³ − X)

Further Time Improvement for Calls



Optimal Algorithm

- We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.
- Note that

$$b(j; n, p) = \frac{p(n - j + 1)}{(1 - p)\, j}\, b(j - 1; n, p).$$

Optimal Algorithm (continued)

- The following program computes $b(j;n,p)$ in $b[\,j\,]$:

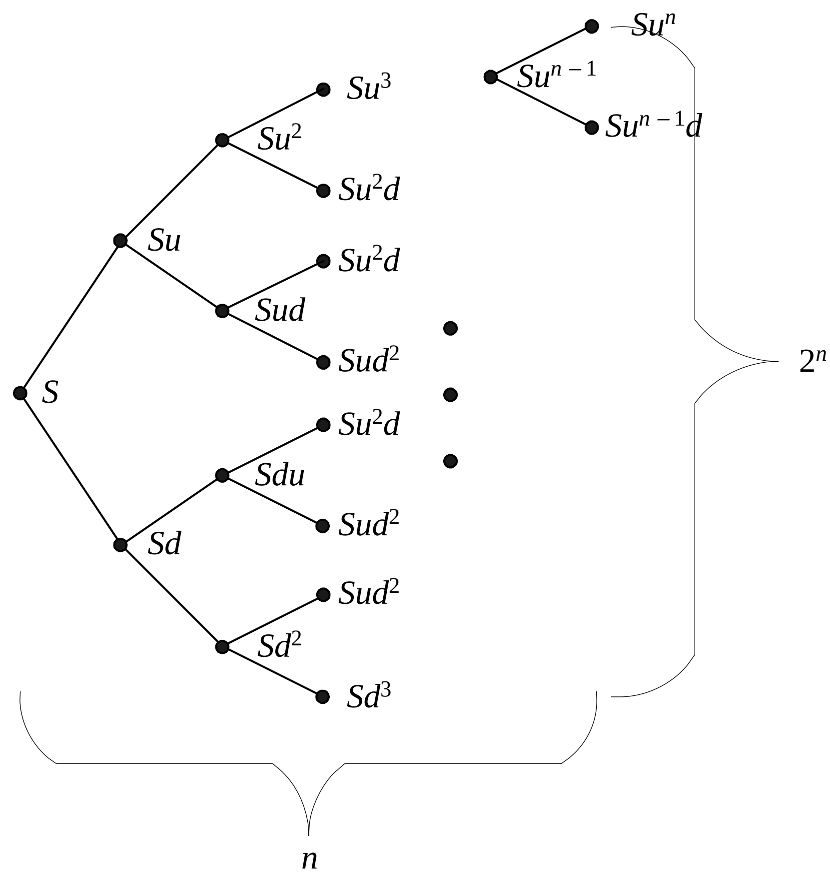
  $b[\,a\,] := \binom{n}{a}\, p^a (1-p)^{n-a}$;
  for $j = a+1, a+2, \ldots, n$ **do**
  $\quad b[\,j\,] := b[\,j-1\,] \times p \times (n-j+1)/((1-p) \times j)$;
  end for

- It runs in $O(n)$ steps.
- Alternatively, $b(j;n,p)$ can be computed by the regularized incomplete beta function.

Optimal Algorithm (concluded)

- With the $b(j; n, p)$ available, the risk-neutral valuation formula (38) on p. 277 is trivial to compute.
- But we only need a single variable to store the $b(j; n, p)$s as they are being sequentially computed.
- This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.
- This forward-induction approach *cannot* be applied to American options because of early exercise.
- So binomial tree algorithms for American options usually run in $O(n^2)$ time.

The Bushy Tree



Toward the Black-Scholes Formula

- The binomial model seems to suffer from two unrealistic assumptions.
  - The stock price takes on only two values in a period.
  - Trading occurs at discrete points in time.
- As n increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.[a]
- Need to calibrate the BOPM's parameters u, d, and R to make it converge to the continuous-time model.
- We now skim through the proof.

[a]Continuous-time trading may create arbitrage opportunities in practice (Budish, Cramton, & Shim, 2015)!

Toward the Black-Scholes Formula (continued)

- Let τ denote the time to expiration of the option measured in years.
- Let r be the continuously compounded annual rate.
- With n periods during the option's life, each period represents a time interval of τ/n.
- Need to adjust the period-based u, d, and interest rate $\hat{r}$ to match the parameters as $n \to \infty$.

Toward the Black-Scholes Formula (continued)

- First, $\hat{r} = r\tau/n$.
  - Each period is $\Delta t \triangleq \tau/n$ years long.
  - The period gross return $R = e^{\hat{r}}$.
- Let
$$\widehat{\mu} \triangleq \frac{1}{n} E\left[\ln \frac{S_\tau}{S}\right]$$
denote the expected value of the continuously compounded rate of return per period of the BOPM.
- Let
$$\widehat{\sigma}^2 \triangleq \frac{1}{n} \text{Var}\left[\ln \frac{S_\tau}{S}\right]$$
denote the variance of that return.

Toward the Black-Scholes Formula (continued)

- Under the BOPM, it is not hard to show that[a]

$$\widehat{\mu} = q \ln(u/d) + \ln d,$$
$$\widehat{\sigma}^2 = q(1-q) \ln^2(u/d).$$

- Assume the stock's *true* continuously compounded rate of return over τ years has mean $\mu\tau$ and variance $\sigma^2\tau$.
- Call σ the stock's (annualized) volatility.

[a]It follows the Bernoulli distribution.

Toward the Black-Scholes Formula (continued)

- The BOPM converges to the distribution only if

$$n\widehat{\mu} = n[\, q \ln(u/d) + \ln d\,] \to \mu\tau, \tag{40}$$

$$n\widehat{\sigma}^2 = nq(1-q)\ln^2(u/d) \to \sigma^2\tau. \tag{41}$$

- We need one more condition to have a solution for u, d, q.

Toward the Black-Scholes Formula (continued)

- Impose

$$ud = 1.$$

  - It makes nodes at the same horizontal level of the tree have identical price (review p. 289).
  - Other choices are possible (see text).

- Exact solutions for u, d, q are feasible if Eqs. (40)–(41) are replaced by equations: 3 equations for 3 variables.[a]

[a] Chance (2008).

Toward the Black-Scholes Formula (continued)

- The above requirements can be satisfied by

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad q = \frac{1}{2} + \frac{1}{2}\,\frac{\mu}{\sigma}\sqrt{\Delta t}\,. \tag{42}$$

- With Eqs. (42), it can be checked that

$$\begin{aligned} n\widehat{\mu} &= \mu\tau, \\ n\widehat{\sigma}^2 &= \left[1 - \left(\frac{\mu}{\sigma}\right)^2 \Delta t\right] \sigma^2\tau \to \sigma^2\tau. \end{aligned}$$

- With the above choice, even if σ is not calibrated correctly, the mean is still matched![a]

---

[a] Recall Eq. (35) on p. 259. So u and d are related to volatility exclusively in the CRR model. Both are independent of r and μ.

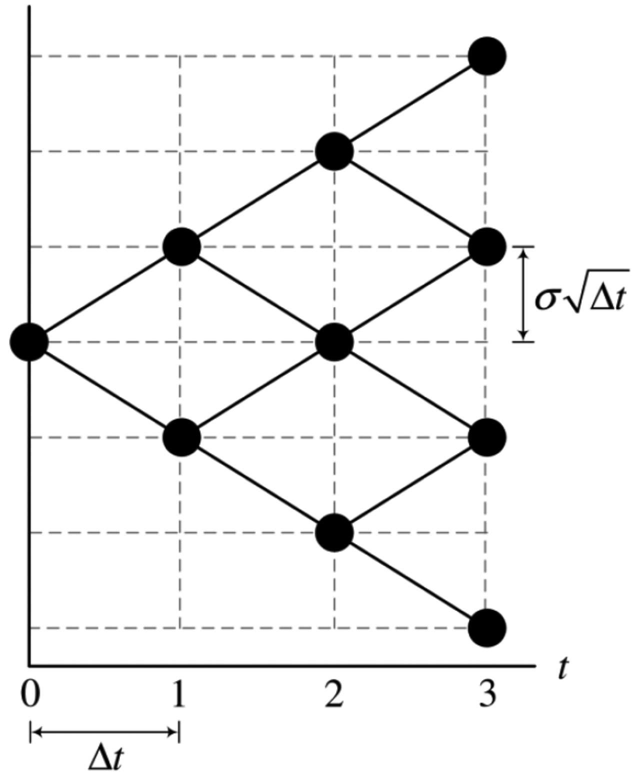
Toward the Black-Scholes Formula (continued)

- The choices (42) result in the CRR binomial model.[a]
  - Black (1992), "This method is probably used more than the original formula in practical situations."
  - OptionMetrics's (2015) IvyDB uses the CRR model.[b]
- The CRR model is best seen in logarithmic price:

$$\ln S \to \begin{cases} \ln S + \sigma\sqrt{\Delta t}, & \text{up move,} \\ \ln S - \sigma\sqrt{\Delta t}, & \text{down move.} \end{cases}$$

[a] Cox, Ross, & Rubinstein (1979).

[b] See http://www.ckgsb.com/uploads/report/file/201611/02/1478069847635278.pd



$\sigma\sqrt{\Delta t}$
t
0
1
2
3
Δt

Toward the Black-Scholes Formula (continued)

- The no-arbitrage inequalities $d < R < u$ may not hold under Eqs. (42) on p. 300 or Eq. (34) on p. 257.
  - If this happens, the probabilities lie outside $[\,0,1\,]$.[a]
- The problem disappears when n satisfies $e^{\sigma\sqrt{\Delta t}} > e^{r\Delta t}$, i.e., when

$$n > \frac{r^2}{\sigma^2}\,\tau. \tag{43}$$

  - So it goes away if n is large enough.
  - Other solutions can be found in the textbook[b] or will be presented later.

---

[a] Many papers and programs forget to check this condition!

[b] See Exercise 9.3.1 of the textbook.

Toward the Black-Scholes Formula (continued)

- The central limit theorem says $\ln(S_\tau / S)$ converges to $N(\mu\tau, \sigma^2\tau)$.[a]
- So $\ln S_\tau$ approaches $N(\mu\tau + \ln S, \sigma^2\tau)$.
- Conclusion: S_τ has a lognormal distribution in the limit.

---

[a] The normal distribution with mean $\mu\tau$ and variance $\sigma^2\tau$. As our probabilities depend on n, this argument is heuristic. But see Uspensky (1937).

Toward the Black-Scholes Formula (continued)

Lemma 10 *The continuously compounded rate of return* $\ln(S_\tau/S)$ *approaches the normal distribution with mean* $(r-\sigma^2/2)\,\tau$ *and variance* $\sigma^2\tau$ *in a risk-neutral economy.*

- Let q equal the risk-neutral probability

$$p \stackrel{\Delta}{=} (e^{r\tau/n}-d)/(u-d).$$

- Let $n\to\infty$.
- Then $\mu = r-\sigma^2/2$.[a]

[a]See Lemma 9.3.3 of the textbook. Now, $p = \frac{1}{2}+\frac{\mu}{2\sigma}\,(\Delta t)^{0.5}+\frac{\sigma^4+4\sigma^2\mu+6\mu^2}{24\sigma}\,(\Delta t)^{1.5}+O[\,(\Delta t^{2.5})\,]$, consistent with Eq. (42) on p. 300.

Toward the Black-Scholes Formula (continued)

- The expected stock price at expiration in a risk-neutral economy is[a]

$$Se^{r\tau}.$$

- The stock's expected annual rate of return is thus the riskless rate r if the rate of return means[b]

$$\frac{\ln E\left[\,\frac{S_\tau}{S}\,\right]}{\tau}.$$

---

[a] By Lemma 10 (p. 305) and Eq. (29) on p. 182.
[b] The arithmetic average rate of return.

Toward the Black-Scholes Formula (continued)

- If the rate of return means, alternatively,[a]

$$\frac{E\left[\,\ln \frac{S_\tau}{S}\,\right]}{\tau},$$

it gives $r - \sigma^2/2$ by Lemma 10.

[a] The geometric average rate of return.

Toward the Black-Scholes Formula (continued)[a]

Theorem 11 (The Black-Scholes Formula, 1973)

$$\begin{aligned} C &= SN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\ P &= Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - SN(-x), \end{aligned}$$

where

$$x \triangleq \frac{\ln(S/X) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

[a]On a United flight from San Francisco to Tokyo on March 7, 2010, a real-estate manager mentioned this formula to me!

Toward the Black-Scholes Formula (concluded)

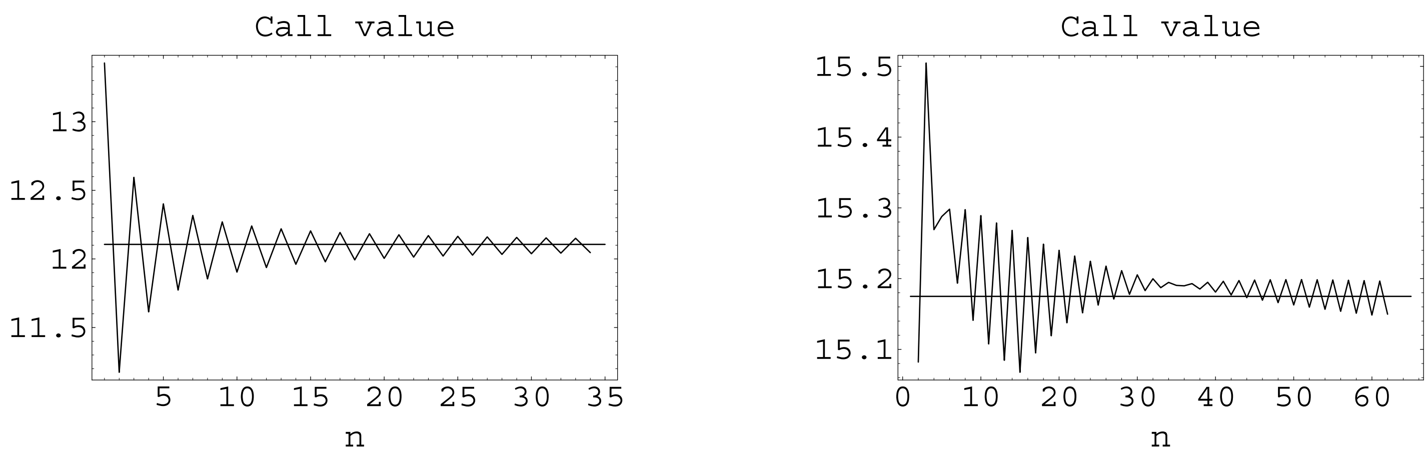
- See Eq. (39) on p. 277 for the meaning of x.
- See Exercise 13.2.12 of the textbook for an interpretation of the probability associated with $N(x)$ and $N(-x)$.

BOPM and Black-Scholes Model

- The Black-Scholes formula needs 5 parameters: S, X, σ, τ, and r.
- Binomial tree algorithms take 6 inputs: S, X, u, d, $\hat{r}$, and n.
- The connections are

$$\begin{aligned} u &= e^{\sigma\sqrt{\tau/n}}, \\ d &= e^{-\sigma\sqrt{\tau/n}}, \\ \hat{r} &= r\tau/n. \end{aligned}$$

  – This holds for the CRR model as well.



- $S = 100$, $X = 100$ (left), and $X = 95$ (right).

BOPM and Black-Scholes Model (concluded)

- The binomial tree algorithms converge reasonably fast.
- The error is $O(1/n)$.[a]
- Oscillations are inherent, however.
- Oscillations can be dealt with by judicious choices of u and d.[b]

---

[a] F. Diener & M. Diener (2004); L. Chang & Palmer (2007).

[b] See Exercise 9.3.8 of the textbook.

Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market's opinion of the volatility.[a]
  - Solve for σ given the option price, S, X, τ, and r with numerical methods.
  - How about American options?

---

[a]Implied volatility is hard to compute when τ is small (why?).

Implied Volatility (concluded)

- Implied volatility is

  the wrong number to put in the wrong formula to get the right price of plain-vanilla options.[a]

- Think of it as an alternative to quoting option prices.
- Implied volatility is often preferred to historical (statistical) volatility in practice.
  - Is using the historical volatility like driving a car with your eyes on the rearview mirror?[b]
- Volatility is meaningful only if seen through a model![c]

[a] Rebonato (2004).

[b] E.g., 1:16:23 of `https://www.youtube.com/watch?v=8TJQhQ2GZ0Y`

[c] Alexander (2001).

Problems; the Smile[a]

- Options written on the same underlying asset usually do not yield the same implied volatility.
- A typical pattern is a "smile" in relation to the strike price.
  - The implied volatility is lowest for at-the-money options.
  - It becomes higher the further the option is in- or out-of-the-money.
- This is common for foreign exchange options.

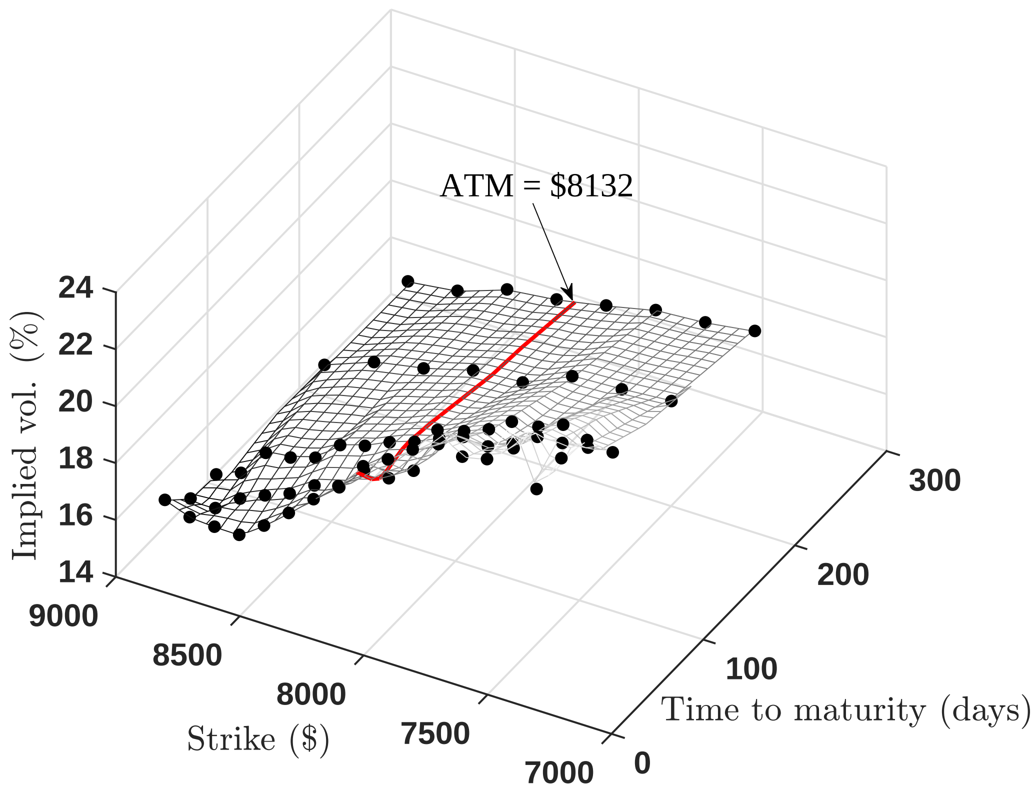
[a] Alexander (2001).

Problems; the Smile (concluded)

- Other patterns have also been observed.
- For stock options, low-strike options tend to have higher implied volatilities.
- One explanation is the high demand for insurance provided by out-of-the-money puts.
- Another reason is volatility rises when stock falls,[a] making in-the-money calls more likely to become in the money again.

[a]This is called the leverage effect (Black, 1992).

TXO Calls (September 25, 2015)[a]



[a]The underlying Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) closed at 8132. Plot supplied by Mr. Lok, U Hou (D99922028) on December 6, 2017.

Tackling the Smile

- To address this issue, volatilities are often combined to produce a composite implied volatility.
- This practice is not sound theoretically.
- The existence of different implied volatilities for options on the same underlying asset shows the Black-Scholes model is not literally true.

Binomial Tree Algorithms for American Puts

- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs

$$\max(0, X - Su^j d^{n-j})$$

  and applies backward induction.
- At each intermediate node, compare the payoff if exercised and the *continuation value*.
- Keep the larger one.

Bermudan Options

- Some American options can be exercised only at discrete time points instead of continuously.
- They are called Bermudan options.
- Their pricing algorithm is identical to that for American options.
- But early exercise is considered for only those nodes when early exercise is permitted.

Time-Dependent Volatility[a]

- Suppose the (instantaneous) volatility can change over time but otherwise predictable: $\sigma(t)$ instead of σ.
- In the limit, the variance of $\ln(S_\tau/S)$ is

$$\int_0^\tau \sigma^2(t)\, dt$$

rather than $\sigma^2\tau$.

- The annualized volatility to be used in the Black-Scholes formula should now be

$$\sqrt{\frac{\int_0^\tau \sigma^2(t)\, dt}{\tau}}\,.$$

[a]Merton (1973).

Time-Dependent Instantaneous Volatility (concluded)

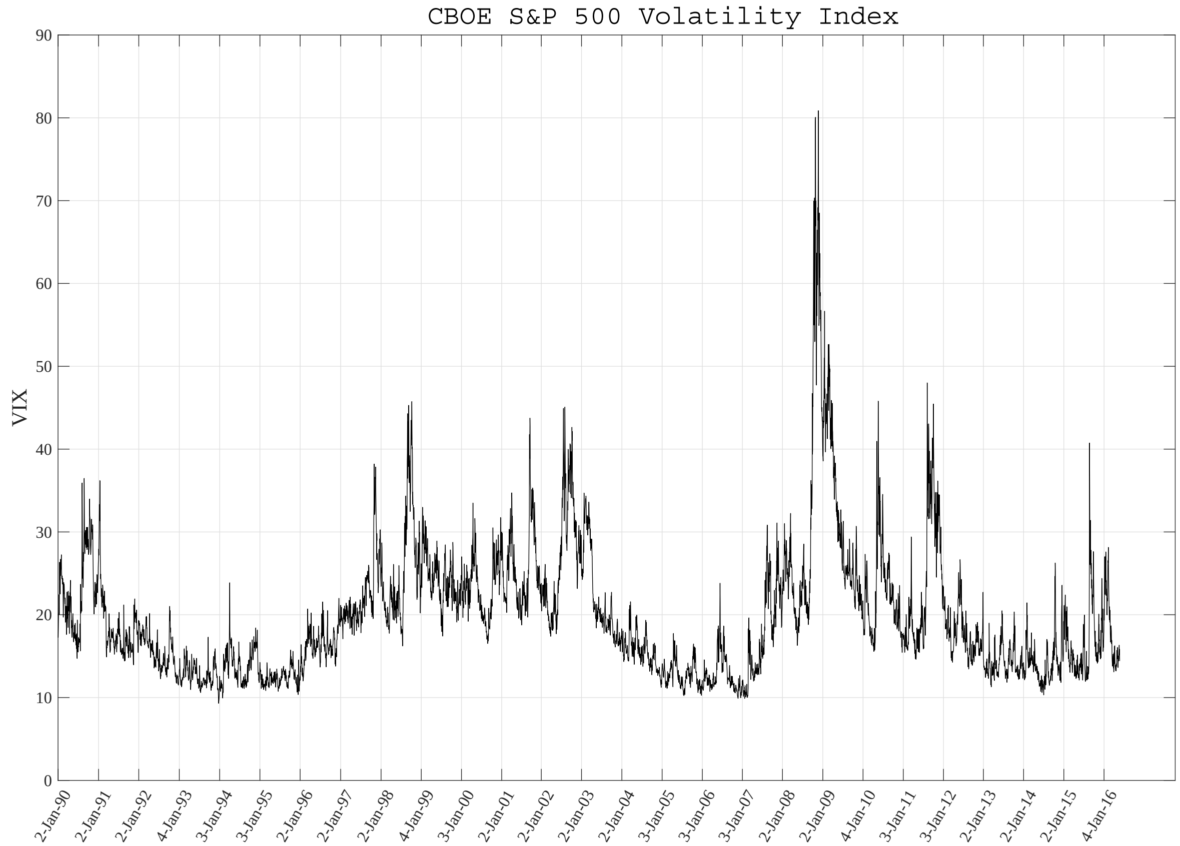
- For the binomial model, u and d depend on time:

$$u = e^{\sigma(t)\sqrt{\tau/n}},$$
$$d = e^{-\sigma(t)\sqrt{\tau/n}}.$$

- But how to make the binomial tree combine?[a]

[a] Amin (1991); C. I. Chen (R98922127) (2011).

Volatility (1990–2016)[a]



[a]Supplied by Mr. Lok, U Hou (D99922028) on July 17, 2017.

Time-Dependent Short Rates

- Suppose the short rate (i.e., the one-period spot rate or forward rate) changes over time but predictable.
- The annual riskless rate r in the Black-Scholes formula should be the spot rate with a time to maturity equal to τ.
- In other words,

$$r = \frac{\sum_{i=0}^{n-1} r_i}{\tau},$$

where r_i is the continuously compounded short rate measured in periods for period i.[a]

- Will the binomial tree fail to combine?

[a]That is, one-period forward rate.

Trading Days and Calendar Days

- Interest accrues based on the calendar day.
- But σ is usually calculated based on trading days only.
  - Stock price seems to have lower volatilities when the exchange is closed.[a]
- How to harmonize these two different times into the Black-Scholes formula and binomial tree algorithms?[b]

---

[a] Fama (1965); K. French (1980); K. French & Roll (1986).

[b] Recall p. 164 about dating issues.

Trading Days and Calendar Days (continued)

- Think of σ as measuring the *annualized* volatility of stock price *one year from now*.
- Suppose a year has m (say 253) trading days.
- We can replace σ in the Black-Scholes formula with[a]

$$\sigma \sqrt{\frac{365}{m} \times \frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}}.$$

[a]D. French (1984).

Trading Days and Calendar Days (concluded)

- This works only for European options.
- How about binomial tree algorithms?[a]

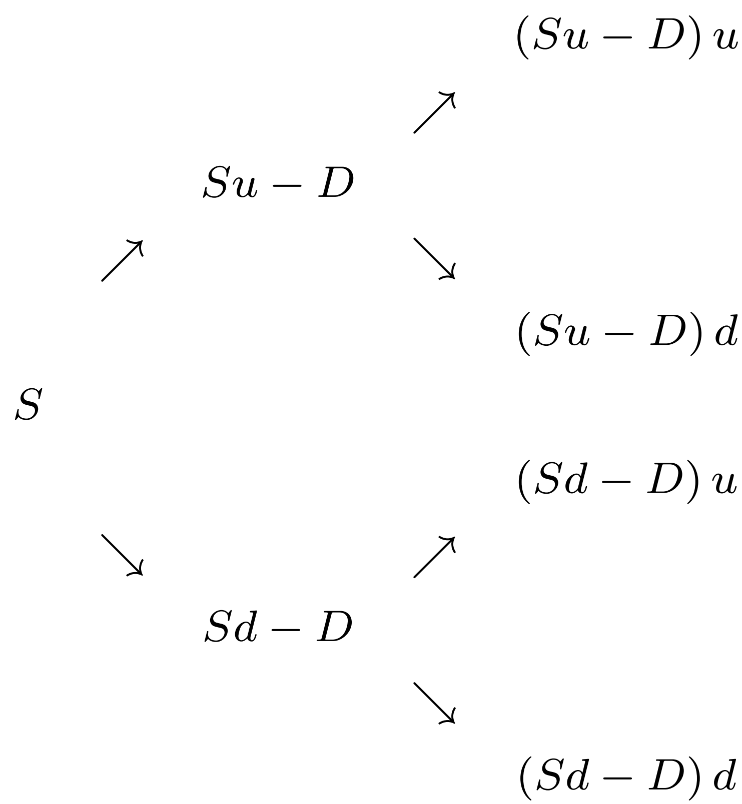
[a] Contributed by Mr. Lu, Zheng-Liang (D00922011) in 2015.

Options on a Stock That Pays Dividends

- Early exercise must be considered.
- Proportional dividend payout model is tractable (see text).
  - The dividend amount is a constant proportion of the *prevailing* stock price.
- In general, the corporate dividend policy is a complex issue.

Known Dividends

- Constant dividends introduce complications.
- Use D to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$.
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: $(Su - D)\, u$, $(Su - D)\, d$, $(Sd - D)\, u$, $(Sd - D)\, d$.
  - The binomial tree no longer combines.


$(Su - D)\, u$
$Su - D$
$(Su - D)\, d$
S
$(Sd - D)\, u$
$Sd - D$
$(Sd - D)\, d$

An Ad-Hoc Approximation

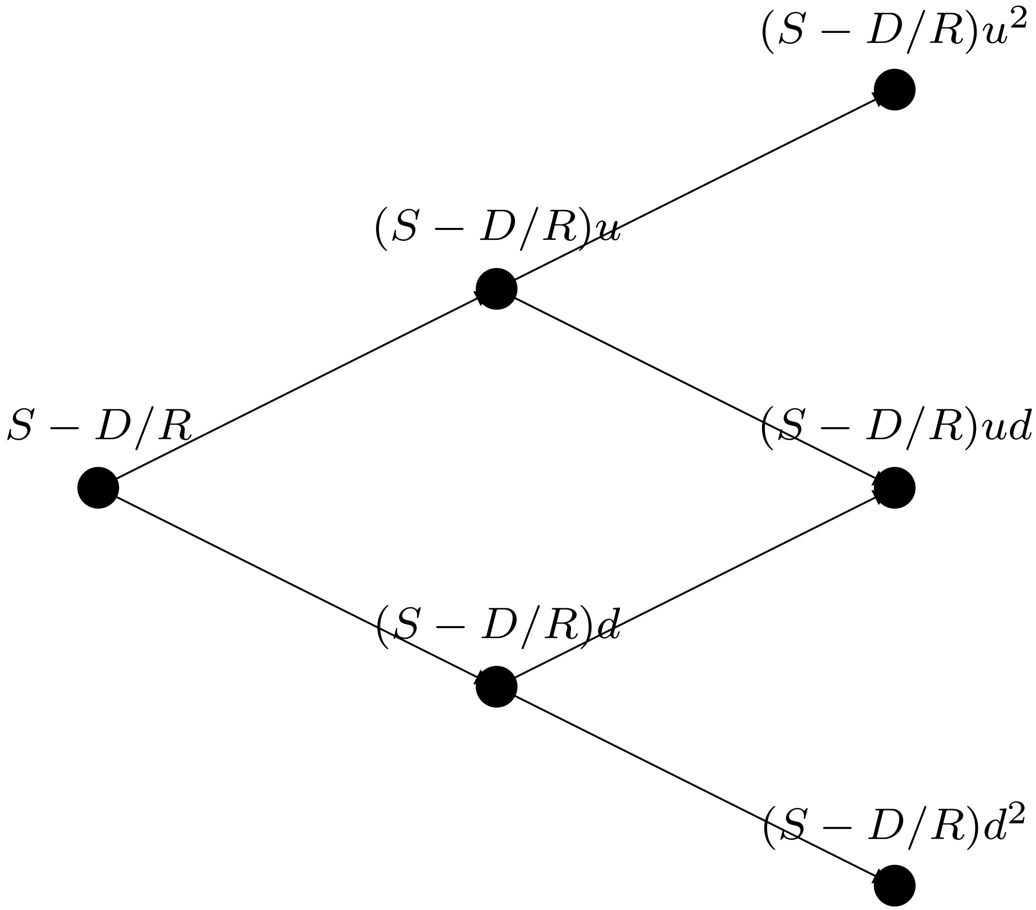
- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends.[a]
- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.
- The riskless component at any time is the PV of future dividends during the life of the option.
  - Then, σ is the volatility of the process followed by the *risky* component.
- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

[a]Roll (1977); Heath & Jarrow (1988).

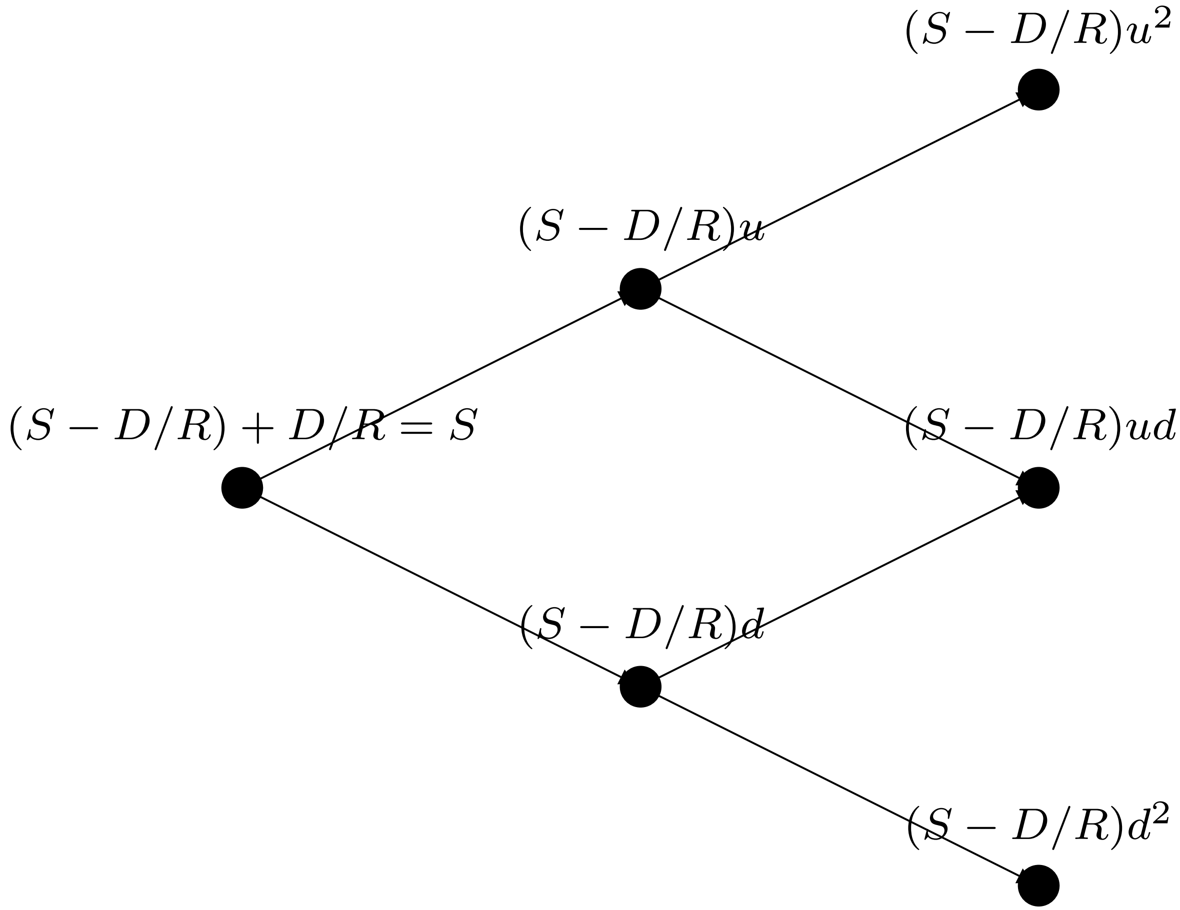
An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all *future* dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.

The Ad-Hoc Approximation vs. P. 330 (Step 1)



The Ad-Hoc Approximation vs. P. 330 (Step 2)



The Ad-Hoc Approximation vs. P. 330[a]

- The trees are different.
- The stock prices at maturity are also different.
  - $(Su - D)\,u, (Su - D)\,d, (Sd - D)\,u, (Sd - D)\,d$ (p. 330).
  - $(S - D/R)u^2, (S - D/R)ud, (S - D/R)d^2$ (ad hoc).
- Note that, as $d < R < u$,

$$\begin{aligned}(Su - D)\,u &> (S - D/R)u^2,\\ (Sd - D)\,d &< (S - D/R)d^2,\end{aligned}$$

[a]Contributed by Mr. Yang, Jui-Chung (D97723002) on March 18, 2009.

The Ad-Hoc Approximation vs. P. 330 (concluded)

- So the ad hoc approximation has a smaller dynamic range.
- This explains why in practice the volatility is usually *increased* when using the ad hoc approximation.

A General Approach[a]

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases (see pp. 792ff).
- Other approaches include adjusting σ and approximating the known dividend with a dividend yield.[b]

---

[a] T. Dai (`B82506025`, `R86526008`, `D8852600`) & Lyuu (2004). Also Arealy & Rodrigues (2013).

[b] Geske & Shastri (1985). It works well for American options but not European ones (T. Dai, 2009).

Continuous Dividend Yields

- Dividends are paid continuously.
  - Approximates a broad-based stock market portfolio.
- The payment of a continuous dividend yield at rate q reduces the growth rate of the stock price by q.
  - A stock that grows from S to S_τ with a continuous dividend yield of q would have grown from S to $S_\tau e^{q\tau}$ without the dividends.
- A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays *no* dividends.[a]

[a] In pricing European options, only the distribution of S_τ matters.

Continuous Dividend Yields (continued)

- So the Black-Scholes formulas hold with S replaced by $Se^{-q\tau}$:[a]

$$C = Se^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{44}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau} N(-x), \tag{44'}$$

where

$$x \triangleq \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

- Formulas (44) and (44′) remain valid as long as the dividend yield is predictable.

[a] Merton (1973).

Continuous Dividend Yields (continued)

- To run binomial tree algorithms, replace u with $ue^{-q\Delta t}$ and d with $de^{-q\Delta t}$, where $\Delta t \stackrel{\Delta}{=} \tau/n$.
  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.
- Other than the changes, binomial tree algorithms stay the same.
  - In particular, p should use the *original* u and d![a]

[a] Contributed by Ms. Wang, Chuan-Ju (`F95922018`) on May 2, 2007.

Continuous Dividend Yields (concluded)

- Alternatively, pick the risk-neutral probability as

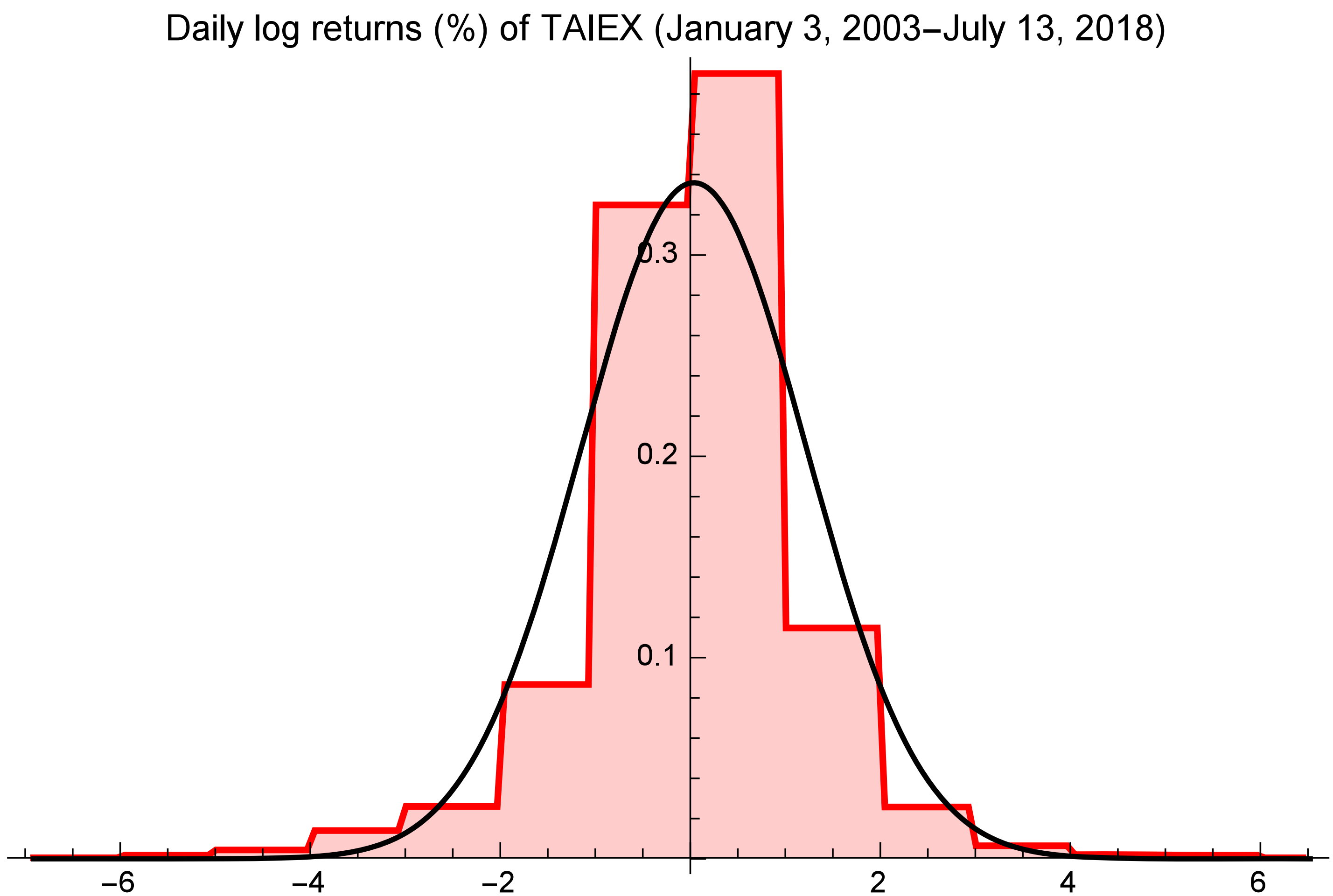
$$\frac{e^{(r-q)\,\Delta t} - d}{u - d}, \tag{45}$$

where $\Delta t \stackrel{\Delta}{=} \tau/n$.

  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.

- The u and d remain unchanged.
- Except the change in Eq. (45), binomial tree algorithms stay the same *as if there were no dividends*.

Distribution of Logarithmic Returns of TAIEX

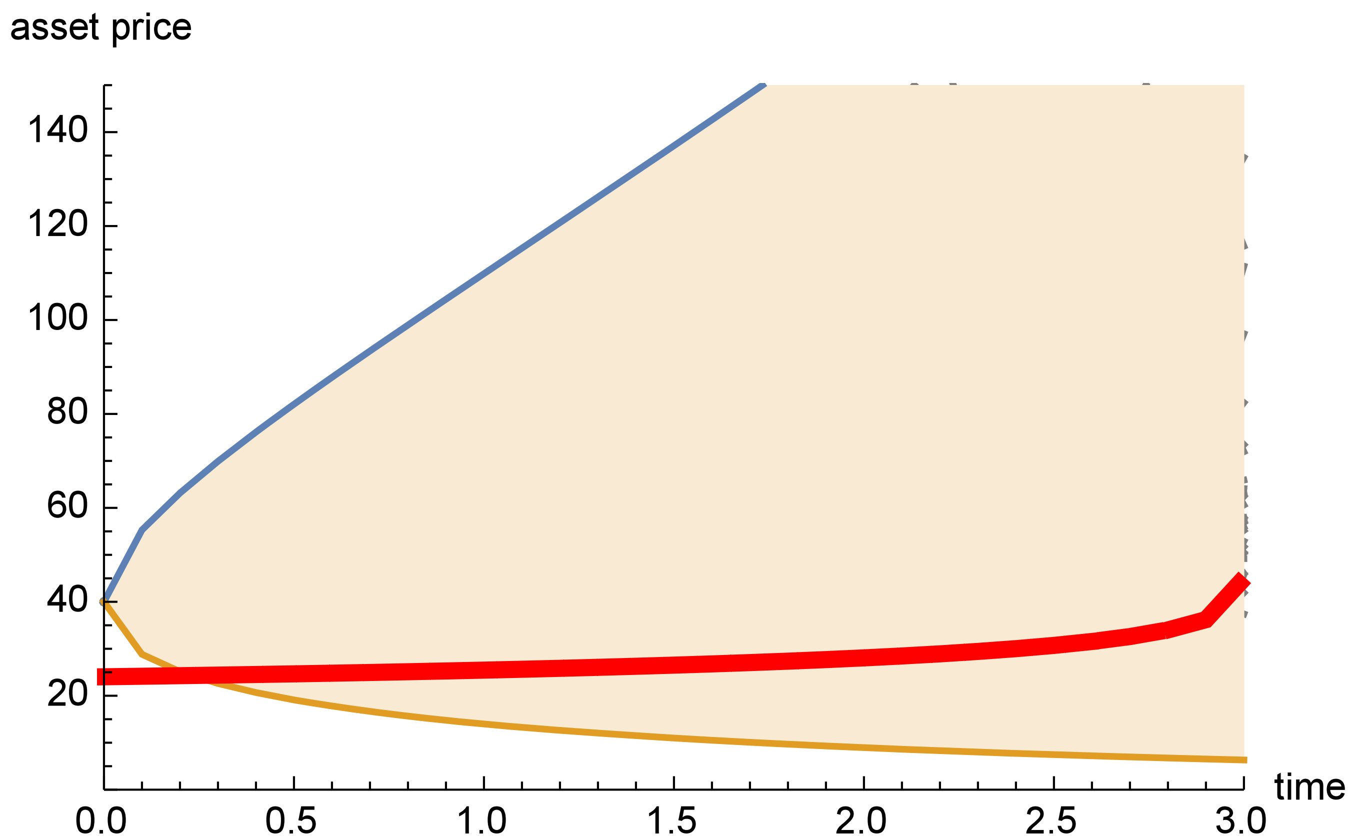


Exercise Boundaries of American Options (in the Continuous-Time Model)[a]

- The exercise boundary is a nondecreasing function of t for American *puts* (see the plot next page).
- The exercise boundary is a nonincreasing function of t for American calls.
- The exercise boundary may be approximated by multipiece exponential functions.[b]

---

[a] See Section 9.7 of the textbook for the tree analog.
[b] Ju (1998).



asset price
140
120
100
80
60
40
20
0.0
0.5
1.0
1.5
2.0
2.5
3.0
time

Sensitivity Analysis of Options

Cleopatra's nose, had it been shorter,
the whole face of the world
would have been changed.
— Blaise Pascal (1623–1662)

Sensitivity Measures ("The Greeks")

- How the value of a security changes relative to changes in a given parameter is key to hedging.
  - Duration, for instance.
- Let $x \stackrel{\Delta}{=} \frac{\ln(S/X)+(r+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$ (recall p. 308).
- Recall that

$$N'(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} > 0,$$

the density function of standard normal distribution.

Delta

- Defined as

$$\Delta \triangleq \frac{\partial f}{\partial S}.$$

  - f is the price of the derivative.
  - S is the price of the underlying asset.
- The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.[a]
- The delta used in the BOPM is the discrete analog.[b]
- The delta of a long stock is 1.

---

[a] Elementary calculus.

[b] Recall p. 251.

Delta (continued)

- The delta of a European call on a non-dividend-paying stock equals

$$\frac{\partial C}{\partial S} = N(x) > 0. \tag{46}$$

- The delta of a European put equals

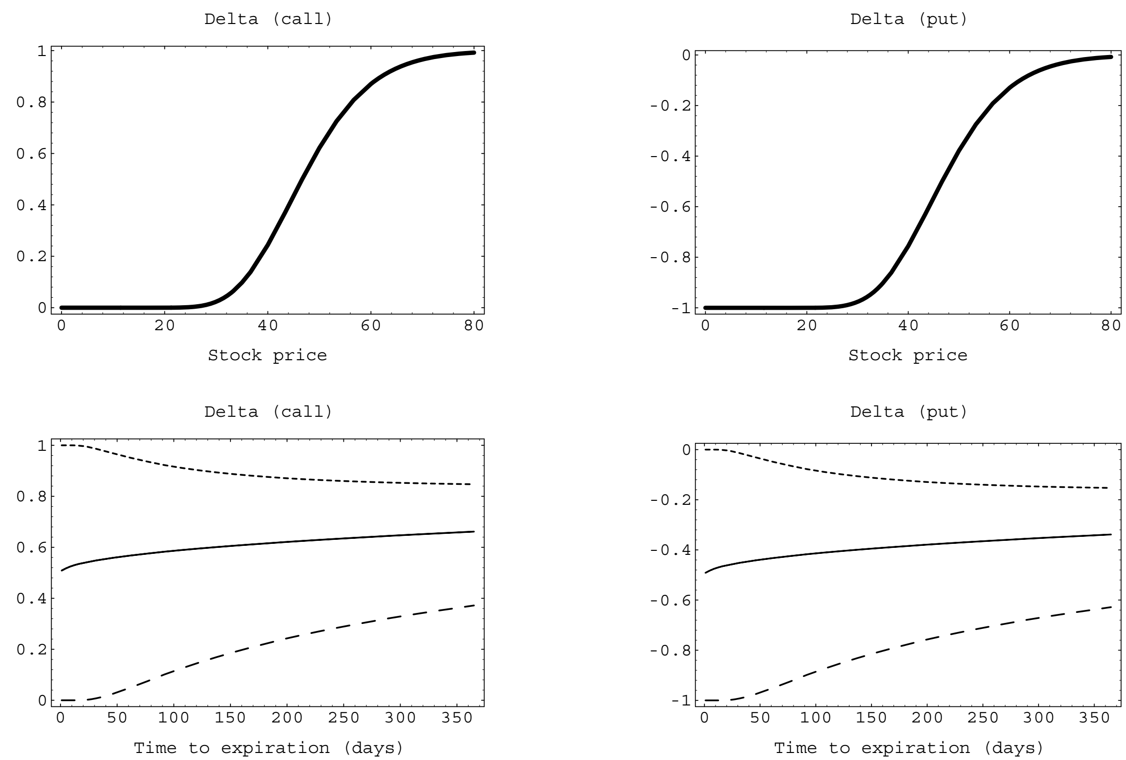
$$\frac{\partial P}{\partial S} = N(x) - 1 = -N(-x) < 0. \tag{47}$$

- So the deltas of a call and an otherwise identical put cancel each other when $N(x) = 1/2$, i.e., when[a]

$$X = Se^{(r+\sigma^2/2)\,\tau}. \tag{48}$$

---

[a]The straddle (p. 215) $C + P$ then has zero delta!



Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money options ($X = 50$).

Dashed curves: out-of-the-money calls or in-the-money puts.

Delta (continued)

- Suppose the stock pays a continuous dividend yield of q.
- Let
$$x \stackrel{\Delta}{=} \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}} \tag{49}$$
(recall p. 339).
- Then
$$\begin{aligned} \frac{\partial C}{\partial S} &= e^{-q\tau} N(x) > 0, \\ \frac{\partial P}{\partial S} &= -e^{-q\tau} N(-x) < 0. \end{aligned}$$

Delta (continued)

- Consider an X_1-strike call and an X_2-strike put, $X_1 \geq X_2$.
- They are otherwise identical.
- Let

$$x_i \triangleq \frac{\ln(S/X_i) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}. \tag{50}$$

- Then their deltas sum to zero when $x_1 = -x_2$.[a]
- That implies

$$\frac{S}{X_1} = \frac{X_2}{S}\, e^{-(2r-2q+\sigma^2)\,\tau}. \tag{51}$$

[a]The strangle (p. 217) $C + P$ then has zero delta!

Delta (concluded)

- Suppose we demand $X_1 = X_2 = X$ and have a straddle.
- Then
$$X = Se^{(r-q+\sigma^2/2)\,\tau}$$
leads to a straddle with zero delta.
  - This generalizes Eq. (48) on p. 349.
- When $C(X_1)$'s delta and $P(X_2)$'s delta sum to zero,[a] does the portfolio $C(X_1) - P(X_2)$ have zero value?
- In general, no.

---
[a] Meaning $C(X_1) + P(X_2)$ has zero delta.

Delta Neutrality

- A position with a total delta equal to 0 is delta-neutral.
  - A delta-neutral portfolio is immune to *small* price changes in the underlying asset.
- Creating one serves for hedging purposes.
  - A portfolio consisting of a call and $-\Delta$ shares of stock is delta-neutral.
  - Short Δ shares of stock to hedge a long call.
  - Long Δ shares of stock to hedge a short call.
- In general, hedge a position in a security with delta Δ_1 by shorting Δ_1/Δ_2 units of a security with delta Δ_2.

Theta (Time Decay)

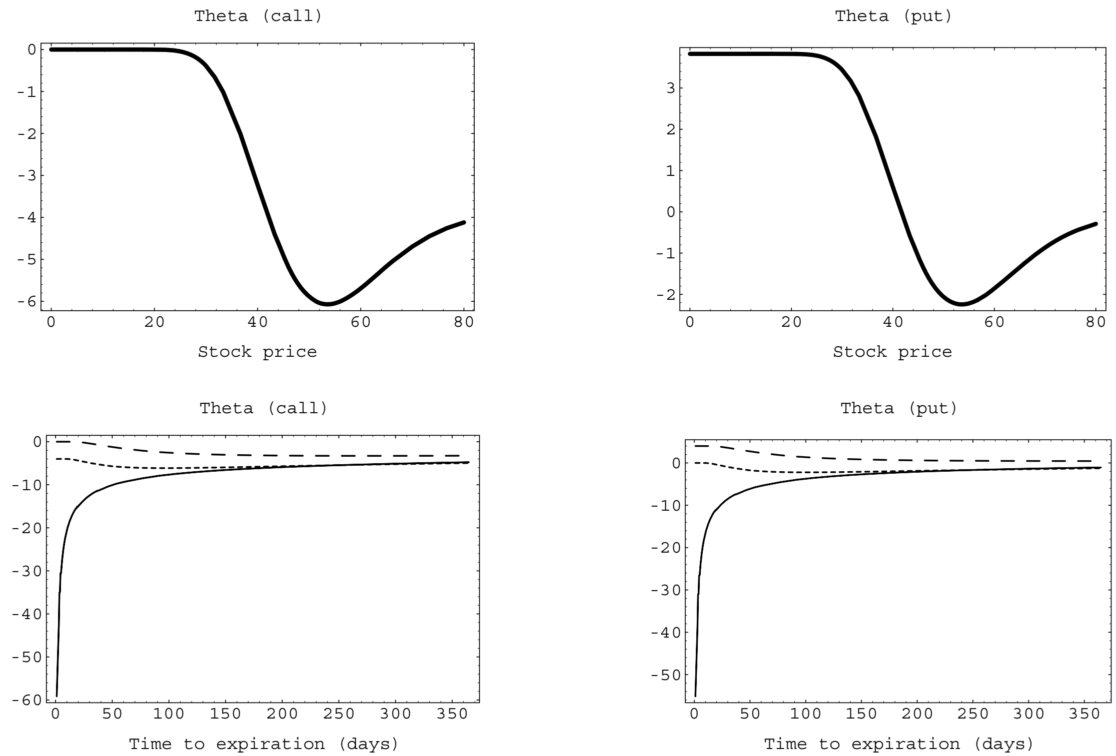
- Defined as the rate of change of a security's value with respect to time, or $\Theta \stackrel{\Delta}{=} -\partial f/\partial\tau = \partial f/\partial t$.
- For a European call on a non-dividend-paying stock,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(x - \sigma\sqrt{\tau}) < 0.$$

  - The call *loses* value with the passage of time.
- For a European put,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-x + \sigma\sqrt{\tau}).$$

  - Can be negative or positive.
- Both are consistent with the plots on p. 197.



Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money options.

Dashed curve: out-of-the-money call or in-the-money put.

Theta (concluded)

- Suppose the stock pays a continuous dividend yield of q.
- Define x as in Eq. (49) on p. 351.
- For a European call, add an extra term to the earlier formula for the theta:

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(x - \sigma\sqrt{\tau}) + qSe^{-q\tau}\,N(x).$$

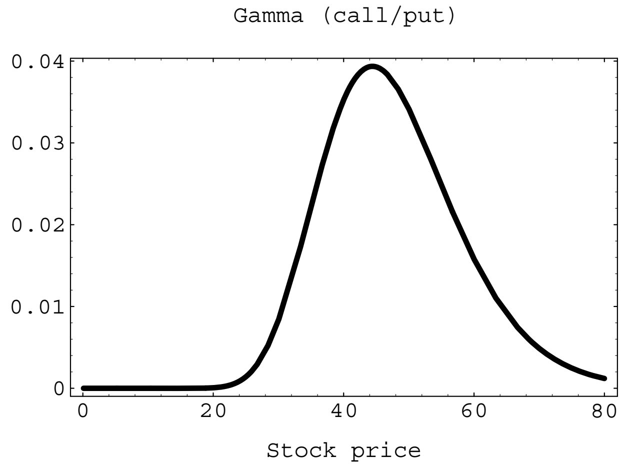
- For a European put, add an extra term to the earlier formula for the theta:

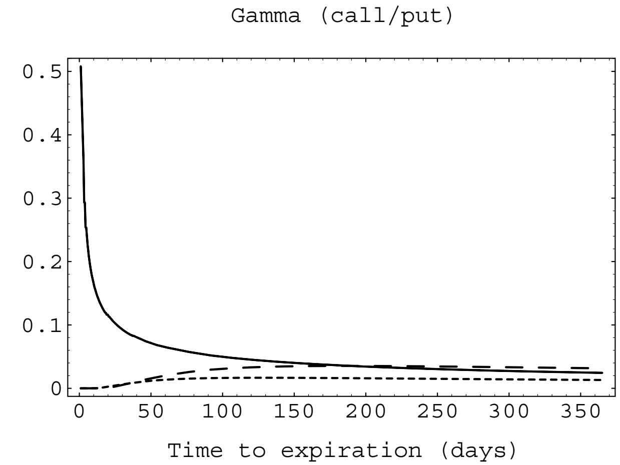
$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-x + \sigma\sqrt{\tau}) - qSe^{-q\tau}\,N(-x).$$

Gamma

- Defined as the rate of change of its delta with respect to the price of the underlying asset, or $\Gamma \stackrel{\Delta}{=} \partial^2\Pi/\partial S^2$.
- Measures how sensitive delta is to changes in the price of the underlying asset.
- In practice, a portfolio with a high gamma needs be rebalanced more often to maintain delta neutrality.
- Roughly, delta $\sim$ duration, and gamma $\sim$ convexity.
- The gamma of a European call or put on a non-dividend-paying stock is

$$N'(x)/(S\sigma\sqrt{\tau}) > 0.$$





Dotted lines: in-the-money call or out-of-the-money put.

Solid lines: at-the-money option.

Dashed lines: out-of-the-money call or in-the-money put.

Gamma (concluded)

- Gamma is maximized when the option is nearly at the money, i.e.,

$$S = X e^{-(r+3\sigma^2/2)\,\tau}.$$

- As the at-the-money option approaches expiration, its gamma tends to rise.
- The gammas of other options, however, tend to zero.

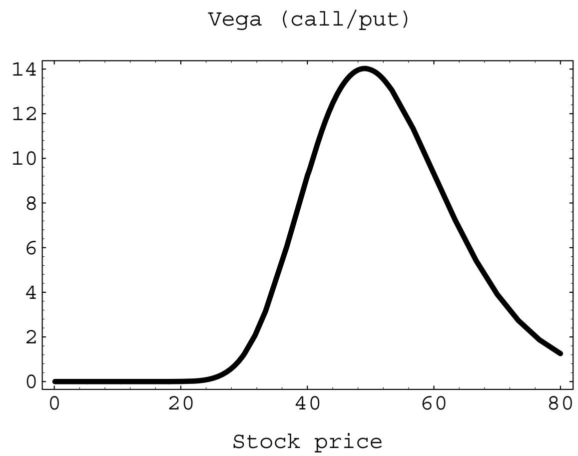
Vega[a] (Lambda, Kappa, Sigma, Zeta)

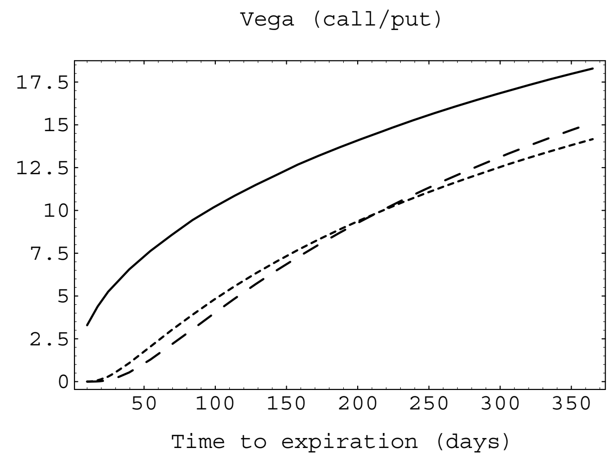
- Defined as the rate of change of a security's value with respect to the volatility of the underlying asset

$$\Lambda \triangleq \frac{\partial f}{\partial \sigma}.$$

- Volatility often changes over time.
- A security with a high vega is very sensitive to changes to or estimation error in volatility.
- The vega of a European call or put on a non-dividend-paying stock is $S\sqrt{\tau}\,N'(x) > 0$.
  - So higher volatility raises the option value.

[a]Vega is not Greek. Alexander (2001), "This is a term that was invented by Americans, and intended to sound like a Greek letter."





Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money option.

Dashed curve: out-of-the-money call or in-the-money put.

Vega (continued)

- Note that[a]

$$\Lambda = \tau\sigma S^2\Gamma. \tag{52}$$

- If the stock pays a continuous dividend yield of q, then

$$\Lambda = S e^{-q\tau} \sqrt{\tau}\, N'(x),$$

where x is defined in Eq. (49) on p. 351.

- Vega is maximized when $x = 0$, i.e., when

$$S = X e^{-(r-q+\sigma^2/2)\,\tau}.$$

- Vega declines very fast as S moves away from that peak.

[a] Reiss & Wystup (2001).

Vega (continued)

- Now consider a portfolio consisting of an X_1-strike call C and a short X_2-strike put P, $X_1 \geq X_2$.
- The options' vegas cancel out when

$$x_1 = -x_2,$$

where x_i are defined in Eq. (50) on p. 352.

- This also leads to Eq. (51) on p. 352.
  - Recall the same condition led to zero delta for the strangle $C + P$ (p. 352).

Vega (concluded)

- Note that $\tau \to 0$ implies

$$\Lambda \to 0$$

(which answers the question on p. 313).

- The Black-Scholes formula (p. 308) implies

$$C \to S,$$
$$P \to Xe^{-r\tau},$$

as $\sigma \to \infty$.[a]

- These boundary conditions are handy for some numerical methods.

[a] Recall that $C \geq \max\left(S - Xe^{-r\tau}, 0\right)$ by Exercise 8.3.2 of the text and $P \geq \max\left(Xe^{-r\tau} - S, 0\right)$ by Lemma 4 (p. 235).