Interest Rates and Bond Prices: Which Determines Which?[a]

- If you have one, you have the other.
- So they are just two names given to the same thing: cost of fund.
- Traders most likely work with prices.
- Banks most likely work with interest rates.

[a]Contributed by Mr. Wang, Cheng (R01741064) on March 5, 2014.

Term Structure of Interest Rates

Why is it that the interest of money is lower,
when money is plentiful?
— Samuel Johnson (1709–1784)

If you have money, don't lend it at interest.
Rather, give [it] to someone
from whom you won't get it back.
— Thomas Gospel 95

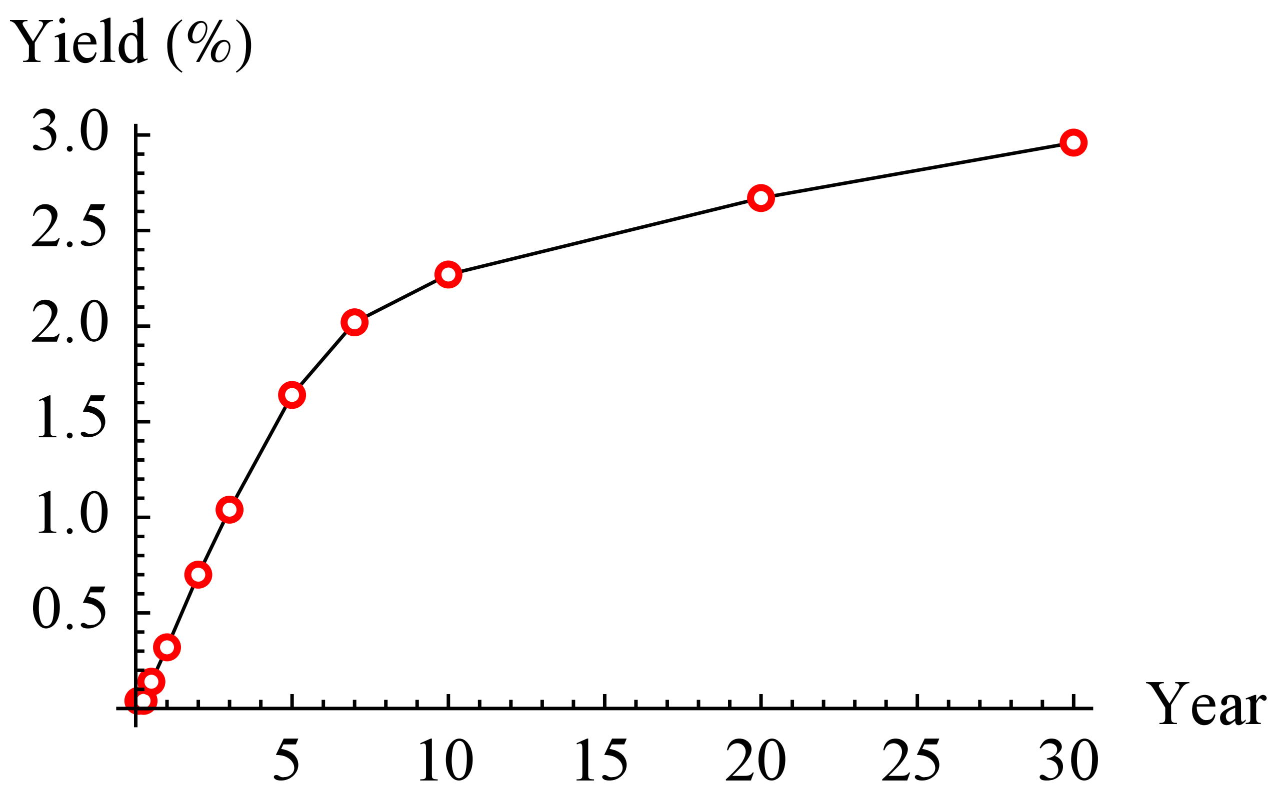
Term Structure of Interest Rates

- Concerned with how interest rates change with maturity.
- The set of yields to maturity for bonds form the term structure.
  - The bonds must be of equal quality.
  - They differ solely in their terms to maturity.
- The term structure is fundamental to the valuation of fixed-income securities.

Term Structure of Interest Rates (concluded)

- The term "term structure" often refers exclusively to the yields of zero-coupon bonds.
- A yield curve plots the yields to maturity of coupon bonds against maturity.
- A par yield curve is constructed from bonds trading near par.

Yield Curve of U.S. Treasuries as of July 24, 2015



Four Typical Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.

Spot Rates

- The i-period spot rate $S(i)$ is the yield to maturity of an i-period zero-coupon bond.
- The PV of one dollar i periods from now is by definition

$$[\,1 + S(i)\,]^{-i}.$$

  - It is the price of an i-period zero-coupon bond.[a]
- The one-period spot rate is called the short rate.
- Spot rate curve:[b] Plot of spot rates against maturity:

$$S(1), S(2), \ldots, S(n).$$

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[a] Recall Eq. (9) on p. 70.

[b] That is, term structure, per our convention.

Problems with the PV Formula

- In the bond price formula (4) on p. 41,

$$\sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n},$$

every cash flow is discounted at the same yield y.

- Consider two riskless bonds with different yields to maturity because of their different cash flows:

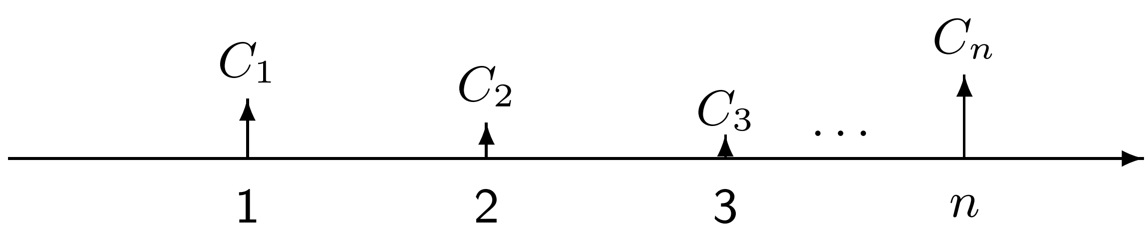
$$\begin{aligned} \text{PV}_1 &= \sum_{i=1}^{n_1} \frac{C}{(1+y_1)^i} + \frac{F}{(1+y_1)^{n_1}}, \\ \text{PV}_2 &= \sum_{i=1}^{n_2} \frac{C}{(1+y_2)^i} + \frac{F}{(1+y_2)^{n_2}}. \end{aligned}$$

Problems with the PV Formula (concluded)

- The yield-to-maturity methodology discounts their *contemporaneous* cash flows with *different* rates.
- But shouldn't they be discounted at the *same* rate?

Spot Rate Discount Methodology

- A cash flow $C_1, C_2, \ldots, C_n$ is equivalent to a package of zero-coupon bonds with the ith bond paying C_i dollars at time i.



Spot Rate Discount Methodology (concluded)

- So a level-coupon bond has the price

$$P = \sum_{i=1}^{n} \frac{C}{[\,1 + S(i)\,]^{i}} + \frac{F}{[\,1 + S(n)\,]^{n}}. \qquad (18)$$

- This pricing method incorporates information from the term structure.
- It discounts each cash flow at the matching spot rate.

Discount Factors

- In general, any riskless security having a cash flow $C_1, C_2, \ldots, C_n$ should have a market price of

$$P = \sum_{i=1}^{n} C_i d(i).$$

  - Above, $d(i) \stackrel{\Delta}{=} [\,1 + S(i)\,]^{-i}$, $i = 1, 2, \ldots, n$, are called the discount factors.
  - $d(i)$ is the PV of one dollar i periods from now.
  - The above formula will be justified on p. 224.
- The discount factors are often interpolated to form a continuous function called the discount function.

Extracting Spot Rates from Yield Curve

- Start with the short rate $S(1)$.
  - Note that short-term Treasuries are zero-coupon bonds.
- Compute $S(2)$ from the two-period coupon bond price P by solving

$$P = \frac{C}{1+S(1)} + \frac{C+100}{[\,1+S(2)\,]^2}.$$

Extracting Spot Rates from Yield Curve (concluded)

- Inductively, we are given the market price P of the n-period coupon bond and

$$S(1), S(2), \ldots, S(n-1).$$

- Then $S(n)$ can be computed from Eq. (18) on p. 129, repeated below,

$$P = \sum_{i=1}^{n} \frac{C}{[\,1 + S(i)\,]^{i}} + \frac{F}{[\,1 + S(n)\,]^{n}}.$$

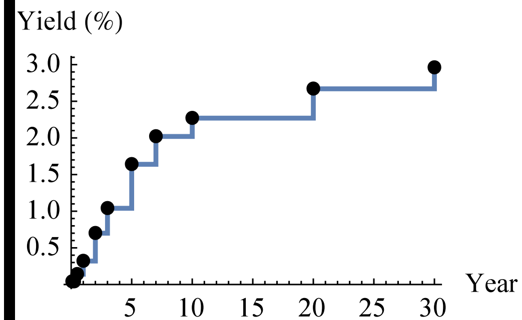
- The running time can be made to be $O(n)$ (see text).
- The procedure is called bootstrapping.

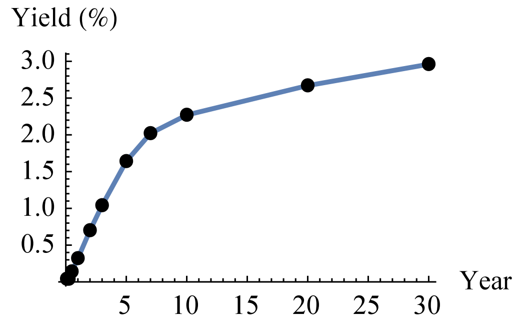
Some Problems

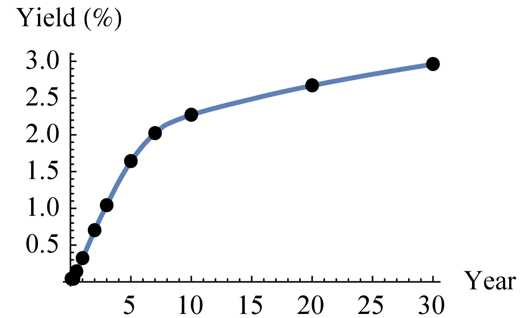
- Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).
- Some maturities might be missing from the data points (the incompleteness problem).
- Treasuries might not be of the same quality.
- Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.[a]

[a] Often without economic justifications.

Which One (from P. 123)?







Yield Spread

- Consider a *risky* bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for P.
- Calculate the IRR of the risky bond.
- Calculate the IRR of a riskless bond with comparable maturity.
- Yield spread is their difference.

Static Spread

- Were the risky bond riskless, it would fetch

$$P^* = \sum_{t=1}^{n} \frac{C_t}{[\,1 + S(t)\,]^t}.$$

- But as risk must be compensated, in reality $P < P^*$.
- The static spread is the amount s by which the spot rate curve has to shift *in parallel* to price the risky bond:

$$P = \sum_{t=1}^{n} \frac{C_t}{[\,1 + s + S(t)\,]^t}.$$

- Unlike the yield spread, the static spread explicitly incorporates information from the term structure.

Of Spot Rate Curve and Yield Curve

- y_i: yield to maturity for the i-period coupon bond.
- $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
- $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
- $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
- $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).
- If the yield curve is flat, the spot rate curve coincides with the yield curve.

Shapes

- The spot rate curve often has the same shape as the yield curve.
  - If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).
- But this is only a trend not a mathematical truth.[a]

[a] See a counterexample in the text.

Forward Rates

- The yield curve contains information regarding future interest rates currently "expected" by the market.
- Invest \$1 for j periods to end up with $[\,1 + S(j)\,]^j$ dollars at time j.
  - The maturity strategy.
- Invest \$1 in bonds for i periods and at time i invest the proceeds in bonds for another $j - i$ periods where $j > i$.
- Will have $[\,1 + S(i)\,]^i[\,1 + S(i, j)\,]^{j-i}$ dollars at time j.
  - $S(i, j)$: $(j - i)$-period spot rate i periods from now.
  - The rollover strategy.

Forward Rates (concluded)

- When $S(i, j)$ equals

$$f(i, j) \triangleq \left[\frac{(1 + S(j))^j}{(1 + S(i))^i} \right]^{1/(j-i)} - 1, \tag{19}$$

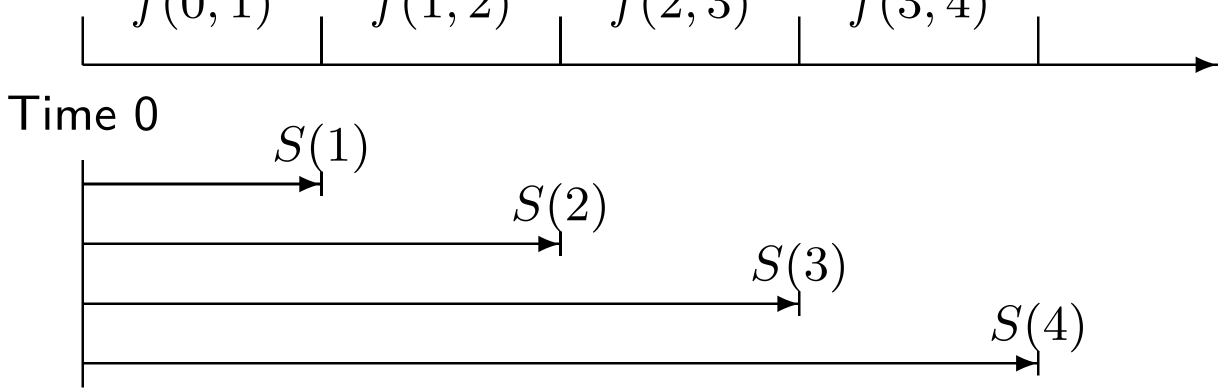
  we will end up with the same $[\,1 + S(j)\,]^j$ dollars.

- As expected,

$$f(0, j) = S(j).$$

- The $f(i, j)$ are the (implied) forward (interest) rates.
  - More precisely, the $(j - i)$-period forward rate i periods from now.

Time Line



Forward Rates and Future Spot Rates

- We did not assume any a priori relation between $f(i, j)$ and future spot rate $S(i, j)$.
  - This is the subject of the term structure theories.
- We merely looked for the future spot rate that, *if realized*, will equate the two investment strategies.
- The $f(i, i+1)$ are the *instantaneous* forward rates or one-period forward rates.

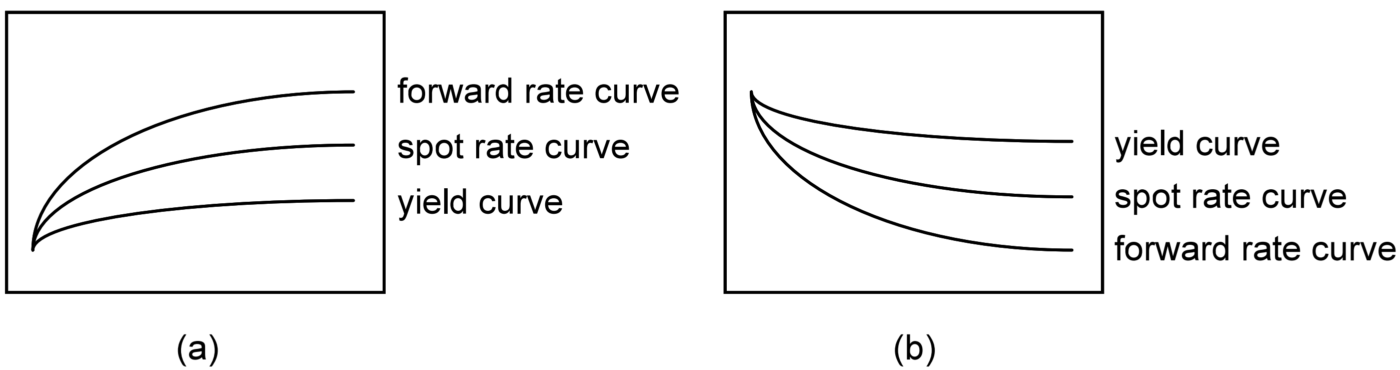
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,

$$f(i, j) > S(j) > \cdots > S(i).$$

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

$$f(i, j) < S(j) < \cdots < S(i).$$


forward rate curve
spot rate curve
yield curve
(a)
yield curve
spot rate curve
forward rate curve
(b)

Forward Rates ≡ Spot Rates ≡ Yield Curve

- The FV of \$1 at time n can be derived in two ways.
- Buy n-period zero-coupon bonds and receive

$$[\,1 + S(n)\,]^n.$$

- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is

$$[\,1 + S(1)\,][\,1 + f(1,2)\,]\cdots[\,1 + f(n-1,n)\,].$$

Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curves (concluded)

- Since they are identical,

$$S(n) = \{[\,1 + S(1)\,][\,1 + f(1,2)\,] \cdots [\,1 + f(n-1,n)\,]\}^{1/n} - 1. \tag{20}$$

- Hence, the forward rates (specifically the one-period forward rates) determine the spot rate curve.
- Other equivalencies can be derived similarly, such as

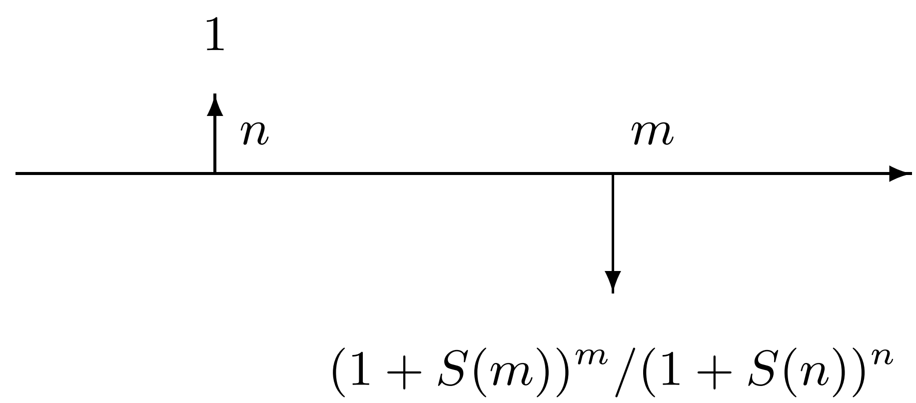
$$f(T, T+1) = \frac{d(T)}{d(T+1)} - 1. \tag{21}$$

Locking in the Forward Rate $f(n, m)$

- Buy one n-period zero-coupon bond for $1/(1 + S(n))^n$ dollars.
- Sell $(1 + S(m))^m/(1 + S(n))^n$ m-period zero-coupon bonds.
- No net initial investment because the cash inflow equals the cash outflow: $1/(1 + S(n))^n$.
- At time n there will be a cash inflow of \$1.
- At time m there will be a cash outflow of $(1 + S(m))^m/(1 + S(n))^n$ dollars.

Locking in the Forward Rate $f(n, m)$ (concluded)

- This implies the interest rate between times n and m equals $f(n, m)$ by formula (19) on p. 140.[a]



[a]Note that $(1 + S(m))^m/(1 + S(n))^n = (1 + f(n, m))^{m-n}$ by that formula.

Forward Loans

- We had generated the cash flow of a type of forward contract called the forward loan.
- Agreed upon today, it enables one to
  - Borrow money at time n in the future, and
  - Repay the loan at time $m > n$ with an interest rate equal to the known forward rate

$$f(n, m).$$

- Can the spot rate curve be arbitrarily drawn?[a]

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[a] Contributed by Mr. Dai, Tian-Shyr (B82506025, R86526008, D88526006) in 1998.

Synthetic Bonds

- We had seen that

$$\begin{aligned} & \text{forward loan} \\ = \; & n\text{-period zero} - [\,1 + f(n,m)\,]^{m-n} \times m\text{-period zero.} \end{aligned}$$

- Thus

$$\begin{aligned} & n\text{-period zero} \\ = \; & \text{forward loan} + [\,1 + f(n,m)\,]^{m-n} \times m\text{-period zero.} \end{aligned}$$

- We have created a *synthetic* zero-coupon bond with forward loans and other zero-coupon bonds.
- Useful if the n-period zero is unavailable or illiquid.

Spot and Forward Rates under Continuous Compounding

- The pricing formula:

$$P = \sum_{i=1}^{n} C e^{-iS(i)} + F e^{-nS(n)}.$$

- The market discount function:

$$d(n) = e^{-nS(n)}.$$

- The spot rate is an arithmetic average of forward rates,[a]

$$S(n) = \frac{f(0,1) + f(1,2) + \cdots + f(n-1,n)}{n}.$$

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[a]Compare it with formula (20) on p. 146.

Spot and Forward Rates under Continuous Compounding (continued)

- The formula for the forward rate:[a]

$$f(i,j) = \frac{jS(j) - iS(i)}{j - i}. \tag{22}$$

- The one-period forward rate:[b]

$$f(j,j+1) = -\ln \frac{d(j+1)}{d(j)}.$$

---

[a]Compare it with formula (19) on p. 140.

[b]Compare it with formula (21) on p. 146.

Spot and Forward Rates under Continuous Compounding (concluded)

- Now, the (instantaneous) forward rate curve is:

$$\begin{aligned} f(T) &\stackrel{\Delta}{=} \lim_{\Delta T \to 0} f(T, T + \Delta T) \\ &= S(T) + T \frac{\partial S}{\partial T}. \end{aligned} \tag{23}$$

- So $f(T) > S(T)$ if and only if $\partial S / \partial T > 0$ (i.e., a normal spot rate curve).
- If $S(T) < -T(\partial S / \partial T)$, then $f(T) < 0$.[a]

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[a] Consistent with the plot on p. 144. Contributed by Mr. Huang, Hsien-Chun (R03922103) on March 11, 2015.

An Example

- Let the interest rates be continuously compounded.
- Suppose the spot rate curve is[a]

$$S(T) \stackrel{\Delta}{=} 0.08 - 0.05\, e^{-0.18T}.$$

- Then by Eq. (23) on p. 153, the forward rate curve is

$$\begin{aligned} & f(T) \\ = \quad & S(T) + T S'(T) \\ = \quad & 0.08 - 0.05\, e^{-0.18T} + 0.009T\, e^{-0.18T}. \end{aligned}$$

[a]Hull & White (1994).

Unbiased Expectations Theory

- Forward rate equals the average future spot rate,

$$f(a,b) = E[\, S(a,b)\,]. \tag{24}$$

- It does not imply that the forward rate is an accurate predictor for the future spot rate.
- It implies the maturity strategy and the rollover strategy produce the same result at the horizon "on average."

Unbiased Expectations Theory and Spot Rate Curve

- It implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  - $f(j, j+1) > S(j+1)$ if and only if $S(j+1) > S(j)$ from formula (19) on p. 140.
  - So
$$E[\, S(j, j+1)\,] > S(j+1) > \cdots > S(1)$$
    if and only if $S(j+1) > \cdots > S(1)$.
- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.

A "Bad" Expectations Theory

- The expected returns[a] on all possible riskless bond strategies are equal for *all* holding periods.
- So

$$(1 + S(2))^2 = (1 + S(1))\, E[\,1 + S(1,2)\,] \qquad (25)$$

because of the equivalency between buying a two-period bond and rolling over one-period bonds.

- After rearrangement,

$$\frac{1}{E[\,1 + S(1,2)\,]} = \frac{1 + S(1)}{(1 + S(2))^2}.$$

[a] More precisely, the one-plus returns.

A "Bad" Expectations Theory (continued)

- Now consider two one-period strategies.
  - Strategy one buys a two-period bond for $(1 + S(2))^{-2}$ dollars and sells it after one period.
  - The expected return is

$$E[\,(1 + S(1, 2))^{-1}\,]/(1 + S(2))^{-2}.$$

  - Strategy two buys a one-period bond with a return of $1 + S(1)$.

A "Bad" Expectations Theory (continued)

- The theory says the returns are equal:

$$\frac{1+S(1)}{(1+S(2))^2} = E\left[\frac{1}{1+S(1,2)}\right].$$

- Combine this with Eq. (25) on p. 157 to obtain

$$E\left[\frac{1}{1+S(1,2)}\right] = \frac{1}{E[\,1+S(1,2)\,]}.$$

A "Bad" Expectations Theory (concluded)

- But this is impossible save for a certain economy.
  - Jensen's inequality states that $E[\,g(X)\,] > g(E[\,X\,])$ for any nondegenerate random variable X and strictly convex function g (i.e., $g''(x) > 0$).
  - Use
$$g(x) \stackrel{\Delta}{=} (1+x)^{-1}$$
to prove our point.

Local Expectations Theory

- The expected rate of return of any bond over *a single period* equals the prevailing one-period spot rate:

$$\frac{E\left[\,(1+S(1,n))^{-(n-1)}\,\right]}{(1+S(n))^{-n}} = 1 + S(1) \;\text{ for } \textit{all } n > 1.$$

- This theory is the basis of many interest rate models.

Duration, in Practice

- We had assumed parallel shifts in the spot rate curve.
- To handle more general shifts, define a vector $[\, c_1, c_2, \ldots, c_n \,]$ that characterizes the shift.
  - Parallel shift: $[\, 1, 1, \ldots, 1 \,]$.
  - Twist: $[\, 1, 1, \ldots, 1, -1, \ldots, -1 \,]$, $[\, 1.8, 1.6, 1.4, 1, 0, -1, -1.4, \ldots \,]$, etc.
  - ….
- At least one c_i should be 1 as the reference point.

Duration in Practice (concluded)

- Let
$$P(y) \stackrel{\Delta}{=} \sum_i C_i/(1+S(i)+yc_i)^i$$
be the price associated with the cash flow $C_1, C_2, \ldots$.

- Define duration as
$$-\frac{\partial P(y)/P(0)}{\partial y}\bigg|_{y=0} \quad \text{or} \quad -\frac{P(\Delta y)-P(-\Delta y)}{2P(0)\Delta y}.$$

- Modified duration equals the above when
$$[\, c_1, c_2, \ldots, c_n \,] = [\, 1, 1, \ldots, 1 \,],$$
$$S(1) = S(2) = \cdots = S(n).$$

Some Loose Ends on Dates

- Holidays.
- Weekends.
- Business days ($T + 2$, etc.).
- Shall we treat a year as 1 year whether it has 365 or 366 days?

Fundamental Statistical Concepts

There are three kinds of lies:
lies, damn lies, and statistics.
— Misattributed to Benjamin Disraeli
(1804–1881)

If 50 million people believe a foolish thing,
it's still a foolish thing.
— George Bernard Shaw (1856–1950)

One death is a tragedy,
but a million deaths are a statistic.
— Josef Stalin (1879–1953)

Moments

- The variance of a random variable X is defined as

$$\text{Var}[\,X\,] \stackrel{\Delta}{=} E\left[\,(X - E[\,X\,])^2\,\right].$$

- The covariance between random variables X and Y is

$$\text{Cov}[\,X, Y\,] \stackrel{\Delta}{=} E\left[\,(X - \mu_X)(Y - \mu_Y)\,\right],$$

  where μ_X and μ_Y are the means of X and Y, respectively.

- Random variables X and Y are uncorrelated if

$$\text{Cov}[\,X, Y\,] = 0.$$

Correlation

- The standard deviation of X is the square root of the variance,

$$\sigma_X \triangleq \sqrt{\mathrm{Var}[\,X\,]}\,.$$

- The correlation (or correlation coefficient) between X and Y is

$$\rho_{X,Y} \triangleq \frac{\mathrm{Cov}[\,X,Y\,]}{\sigma_X \sigma_Y},$$

provided both have nonzero standard deviations.[a]

[a] Wilmott (2009), "the correlations between financial quantities are notoriously unstable." It may even break down "at high-frequency time intervals" (Budish, Cramton, & Shim, 2015).

Variance of Sum

- Variance of a weighted sum of random variables equals

$$\mathrm{Var}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \,\mathrm{Cov}[\,X_i, X_j\,].$$

- It becomes

$$\sum_{i=1}^{n} a_i^2 \,\mathrm{Var}[\,X_i\,]$$

when X_i are uncorrelated.[a]

[a] Bienaymé (1853).

Conditional Expectation

- "$X \mid I$" denotes X conditional on the information set I.
- The information set can be another random variable's value or the past values of X, say.
- The conditional expectation

$$E[\,X \mid I\,]$$

  is the expected value of X conditional on I.
  - It is a random variable.
- The law of iterated conditional expectations[a] says

$$E[\,X\,] = E[\,E[\,X \mid I\,]\,].$$

[a]Or the tower law.

Conditional Expectation (concluded)

- If I_2 contains at least as much information as I_1, then

$$E[\,X \mid I_1\,] = E[\,E[\,X \mid I_2\,] \mid I_1\,]. \tag{26}$$

  - I_1 contains price information up to time t_1, and I_2 contains price information up to a later time $t_2 > t_1$.

- In general,

$$I_1 \subseteq I_2 \subseteq \cdots$$

  means the players never forget past data so the information sets are increasing over time.[a]

[a] Hirsa & Neftci (2014). This idea is used in sigma fields and filtration in probability theory.

The Normal Distribution

- A random variable X has the normal distribution with mean μ and variance σ^2 if its probability density function is
$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}.$$
- This is expressed by $X \sim N(\mu, \sigma^2)$.
- The standard normal distribution has zero mean, unit variance, and the following distribution function
$$\text{Prob}[\, X \leq z\,] = N(z) \stackrel{\Delta}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx.$$

Moment Generating Function

- The moment generating function of random variable X is defined as

$$\theta_X(t) \stackrel{\Delta}{=} E[\, e^{tX} \,].$$

- The moment generating function of $X \sim N(\mu, \sigma^2)$ is

$$\theta_X(t) = \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]. \tag{27}$$

The Multivariate Normal Distribution

- If $X_i \sim N(\mu_i, \sigma_i^2)$ are independent, then

$$\sum_i X_i \sim N\left(\sum_i \mu_i, \sum_i \sigma_i^2\right).$$

- Let $X_i \sim N(\mu_i, \sigma_i^2)$, which may not be independent.
- Suppose

$$\sum_{i=1}^n t_i X_i \sim N\left(\sum_{i=1}^n t_i\, \mu_i, \sum_{i=1}^n \sum_{j=1}^n t_i t_j \,\mathrm{Cov}[\, X_i, X_j\,]\right)$$

for every linear combination $\sum_{i=1}^n t_i X_i$ with $\sum_{i=1}^n \sum_{j=1}^n t_i t_j \,\mathrm{Cov}[\, X_i, X_j\,] \neq 0$.

The Multivariate Normal Distribution (concluded)

- Then X_i are said to have a multivariate normal distribution.[a]
- With $M \equiv C^{-1}$ and the (i, j)th entry of the matrix M being $M_{i,j}$, the probability density function for the X_i is

$$\frac{1}{\sqrt{(2\pi)^n \det(C)}} \exp\left[-\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (X_i - \mu_i)\, M_{ij}\, (X_j - \mu_j) \right],$$

with a positive-definite covariance matrix

$$C \stackrel{\Delta}{=} [\,\mathrm{Cov}[\,X_i, X_j\,]\,]_{1 \leq i,j \leq n}.$$

[a] Corrected by Mr. Huang, Guo-Hua (R98922107) on March 10, 2010.

Generation of Univariate Normal Distributions

- Let X be uniformly distributed over $(0, 1\,]$ so that

$$\text{Prob}[\,X \leq x\,] = x, \quad 0 < x \leq 1.$$

- Repeatedly draw two samples x_1 and x_2 from X until

$$\omega \stackrel{\Delta}{=} (2x_1 - 1)^2 + (2x_2 - 1)^2 < 1.$$

- Then $c(2x_1 - 1)$ and $c(2x_2 - 1)$ are independent standard normal variables where[a]

$$c \stackrel{\Delta}{=} \sqrt{-2(\ln \omega)/\omega}\,.$$

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[a]As they are normally distributed, to prove independence, it suffices to prove that they are uncorrelated, which is easy. Thanks to a lively class discussion on March 5, 2014.

A Dirty Trick and a Right Attitude

- Let ξ_i are independent and uniformly distributed over $(0, 1)$.
- A simple method to generate the standard normal variable is to calculate[a]

$$\left(\sum_{i=1}^{12} \xi_i\right) - 6.$$

- But why use 12?
- Recall the mean and variance of ξ_i are $1/2$ and $1/12$, respectively.

[a] Jäckel (2002), "this is not a highly accurate approximation and should only be used to establish ballpark estimates."

A Dirty Trick and a Right Attitude (concluded)

- The general formula is

$$\frac{(\sum_{i=1}^{n} \xi_i) - (n/2)}{\sqrt{n/12}}.$$

- Choosing $n = 12$ yields a formula without the need of division and square-root operations.[a]
- Always blame your random number generator last.[b]
- Instead, check your programs first.

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[a] Contributed by Mr. Chen, Shih-Hang (`R02723031`) on March 5, 2014.

[b] "The fault, dear Brutus, lies not in the stars but in ourselves that we are underlings." William Shakespeare (1564–1616), *Julius Caesar*.

Generation of Bivariate Normal Distributions

- Pairs of normally distributed variables with correlation ρ can be generated as follows.
- Let X_1 and X_2 be independent standard normal variables.
- Set

$$
\begin{aligned}
U &\stackrel{\Delta}{=} aX_1, \\
V &\stackrel{\Delta}{=} a\rho X_1 + a\sqrt{1-\rho^2}\, X_2.
\end{aligned}
$$

Generation of Bivariate Normal Distributions (continued)

- U and V are the desired random variables with

$$\begin{aligned} \text{Var}[\,U\,] &= \text{Var}[\,V\,] = a^2, \\ \text{Cov}[\,U, V\,] &= \rho a^2. \end{aligned}$$

- Note that the mapping from (X_1, X_2) to (U, V) is a one-to-one correspondence for $a \neq 0$.

Generation of Bivariate Normal Distributions (concluded)

- The mapping in matrix form is

$$\begin{bmatrix} U \\ V \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}. \tag{28}$$

The Lognormal Distribution

- A random variable Y is said to have a lognormal distribution if $\ln Y$ has a normal distribution.
- Let $X \sim N(\mu, \sigma^2)$ and $Y \stackrel{\Delta}{=} e^X$.
- The mean and variance of Y are

$$\begin{aligned} \mu_Y &= e^{\mu+\sigma^2/2}, \\ \sigma_Y^2 &= e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right), \end{aligned} \tag{29}$$

respectively.[a]

[a] They follow from $E[\,Y^n\,] = e^{n\mu+n^2\sigma^2/2}$.

The Lognormal Distribution (continued)

- Conversely, suppose Y is lognormally distributed with mean μ and variance σ^2.
- Then $\ln Y$ has a normal distribution with

$$\begin{aligned} E[\,\ln Y\,] &= \ln\left[\,\mu/\sqrt{1+(\sigma/\mu)^2}\,\right], \\ \mathrm{Var}[\,\ln Y\,] &= \ln\left[\,1+(\sigma/\mu)^2\,\right]. \end{aligned}$$

- If X and Y are joint-lognormally distributed, then

$$\begin{aligned} E[\,XY\,] &= E[\,X\,]\,E[\,Y\,]\,e^{\mathrm{Cov}[\,\ln X, \ln Y\,]}, \\ \mathrm{Cov}[\,X,Y\,] &= E[\,X\,]\,E[\,Y\,]\left(e^{\mathrm{Cov}[\,\ln X, \ln Y\,]}-1\right). \end{aligned}$$

The Lognormal Distribution (concluded)

- Let Y be lognormally distributed such that $\ln Y \sim N(\mu, \sigma^2)$.
- Then

$$\int_a^\infty y f(y)\, dy = e^{\mu+\sigma^2/2}\, N\left(\frac{\mu - \ln a}{\sigma} + \sigma\right). \qquad (30)$$

Option Basics

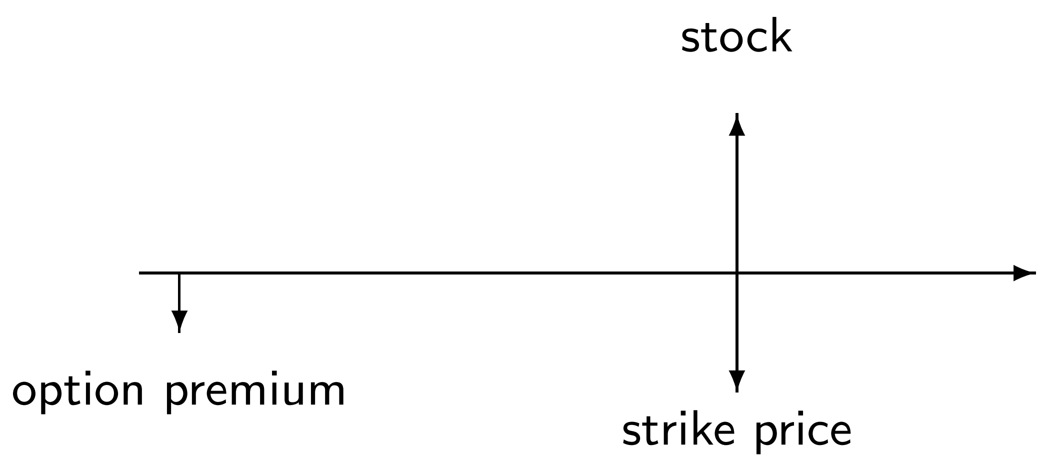
The shift toward options as
the center of gravity of finance [...]
— Merton H. Miller[a] (1923–2000)

Too many potential physicists and engineers
spend their careers shifting money around
in the financial sector,
instead of applying their talents to
innovating in the real economy.
— Barack Obama (2016)

[a] Co-winner of the 1990 Nobel Prize in Economic Sciences.

Calls and Puts

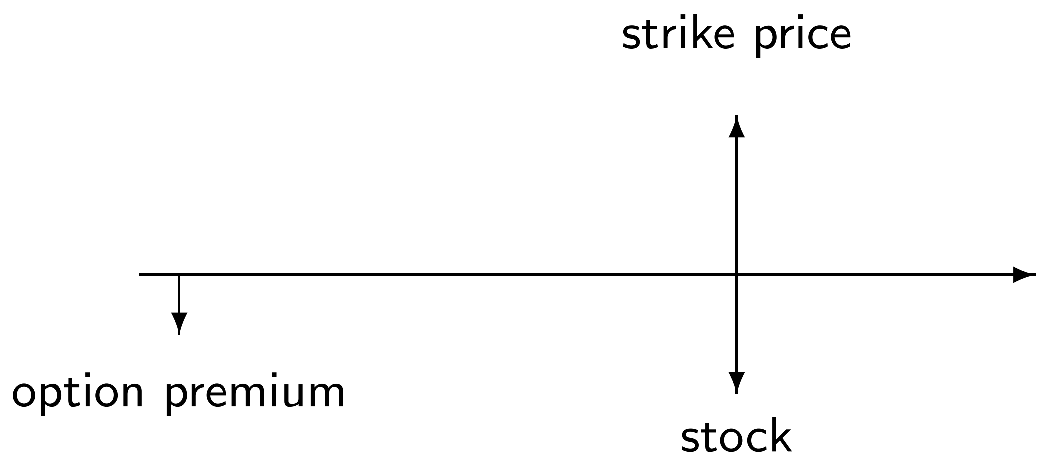
- A call gives its holder the right to *buy* a unit of the underlying asset by paying a strike price.[a]



[a] The cash flow at expiration is contingent.

Calls and Puts (continued)

- A put gives its holder the right to *sell* a unit of the underlying asset for the strike price.



Calls and Puts (concluded)

- An embedded option has to be traded along with the underlying asset.
- How to price options?
  - It can be traced to Aristotle's (384 B.C.–322 B.C.) *Politics*, if not earlier.

Exercise

- When a call is exercised, the holder pays the strike price in exchange for the stock.
- When a put is exercised, the holder receives from the writer the strike price in exchange for the stock.
- Some options can be exercised prior to the expiration date.
  - This is called early exercise.

American and European

- American options can be exercised at any time up to the expiration date.
- European options can only be exercised at expiration.
- An American option is worth at least as much as an otherwise identical European option.

Convenient Conventions

- C: call value.
- P: put value.
- X: strike price.
- S: stock price.[a]
- D: dividend.

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[a] Assume $S \geq 0$. Contributed by Mr. Tang, Bert (`B08902102`) on March 10, 2021.

Payoff, Mathematically Speaking

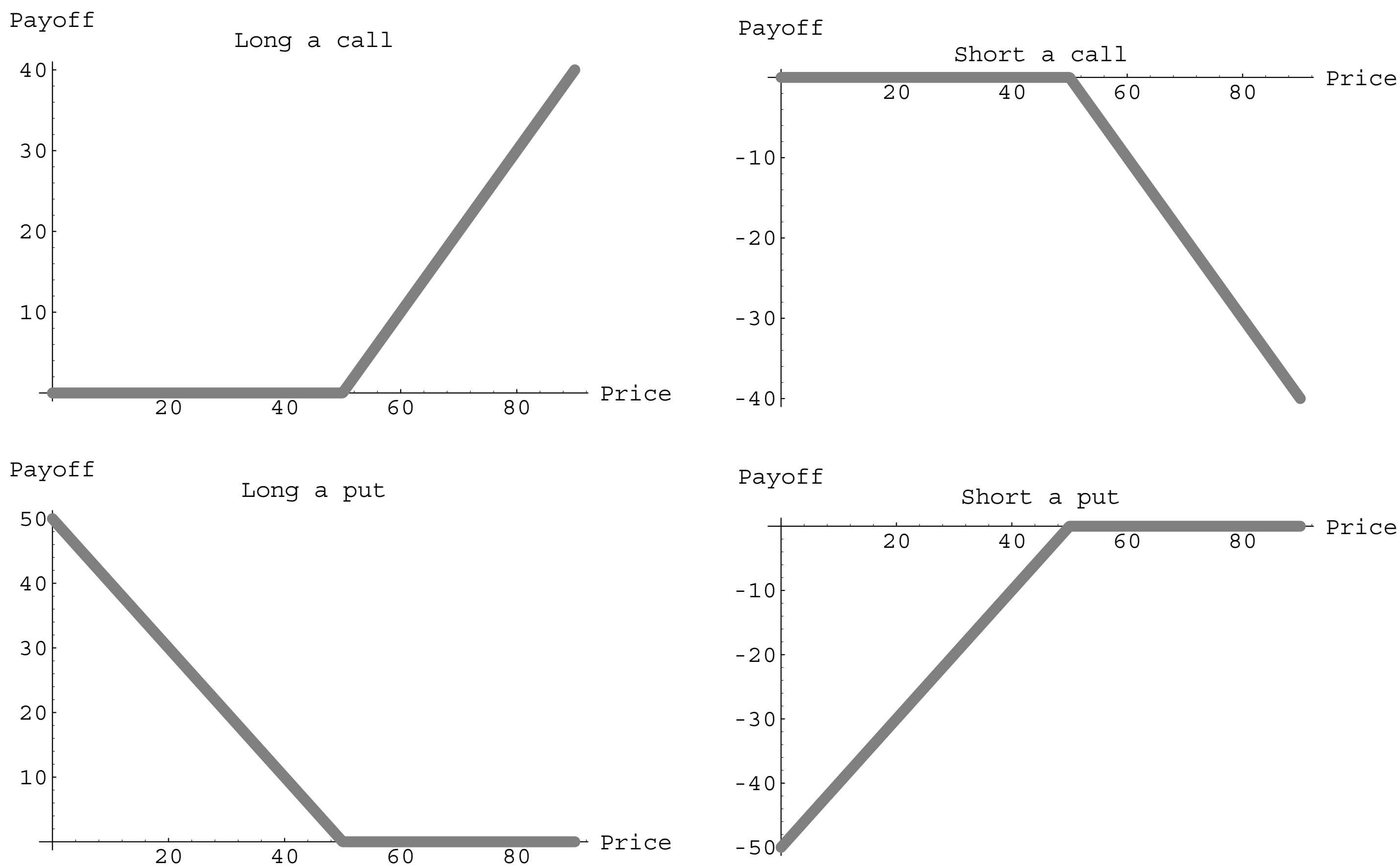
- The payoff of a call at expiration is

$$C = \max(0, S - X).$$

- The payoff of a put at expiration is

$$P = \max(0, X - S).$$

- A call will be exercised only if the stock price is higher than the strike price.
- A put will be exercised only if the stock price is less than the strike price.


Payoff
Long a call
40
30
20
10
20
40
60
80
Price
Payoff
Short a call
20
40
60
80
Price
-10
-20
-30
-40
Payoff
Long a put
50
40
30
20
10
20
40
60
80
Price
Payoff
Short a put
20
40
60
80
Price
-10
-20
-30
-40
-50

Payoff, Mathematically Speaking (continued)

- At any time t before the expiration date, we call

$$\max(0, S_t - X)$$

  the intrinsic value of a call.

- At any time t before the expiration date, we call

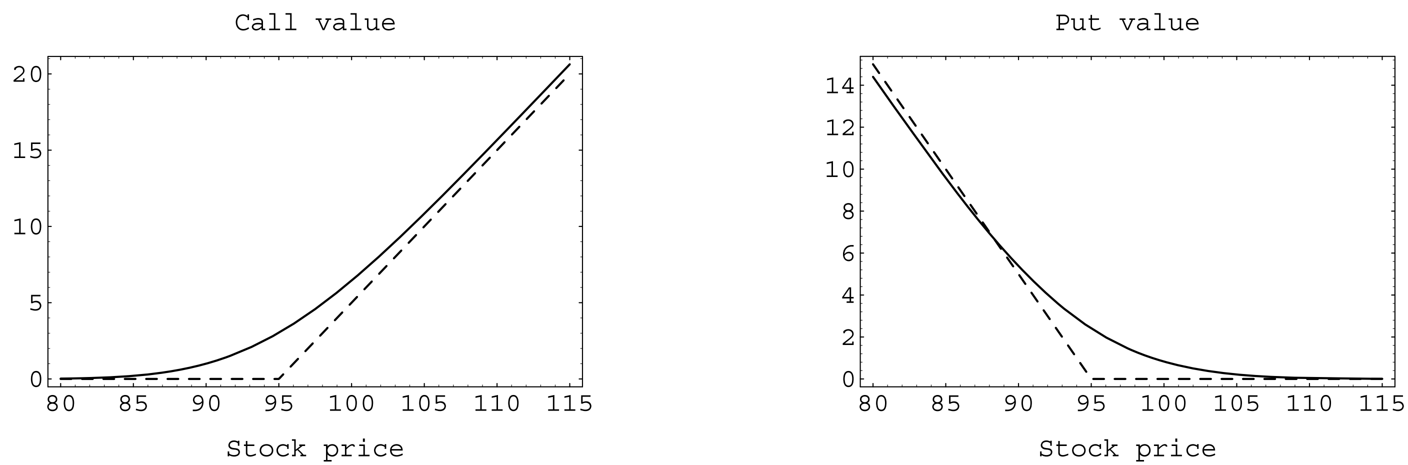
$$\max(0, X - S_t)$$

  the intrinsic value of a put.

Payoff, Mathematically Speaking (concluded)

- A call is in the money if $S > X$, at the money if $S = X$, and out of the money if $S < X$.
- A put is in the money if $S < X$, at the money if $S = X$, and out of the money if $S > X$.
- Options that are in the money at expiration should be exercised.[a]
- Finding an option's value at any time *before* expiration is a major intellectual breakthrough.

[a] About 11% of option holders let in-the-money options expire worthless.


Call value
20
15
10
5
0
80 85 90 95 100 105 110 115
Stock price
Put value
14
12
10
8
6
4
2
0
80 85 90 95 100 105 110 115
Stock price