Principles of Financial Computing

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Introduction

You must go into finance, Amory.
— F. Scott Fitzgerald (1896–1940),
This Side of Paradise (1920)

The two most dangerous words in Wall Street
vocabulary are "financial engineering."
— Wilbur Ross (2007)

Class Information

- Yuh-Dauh Lyuu. *Financial Engineering & Computation: Principles, Mathematics, Algorithms.* Cambridge University Press, 2002.
- Official Web page is

  `www.csie.ntu.edu.tw/~lyuu/finance1.html`

  - Lecture notes will be uploaded before class.
  - Homeworks and teaching assistants will also be announced there.
  - Do not mistake last year's homeworks for this year's!

Class Information (continued)

- Check

  `www.csie.ntu.edu.tw/~lyuu/capitals.html`

  for some of the software.

Class Information (concluded)

- Please ask many questions in class.
  - This is the best way for me to remember you in a large class.[a]

[a] "[A] science concentrator [...] said that in his eighth semester of [Harvard] college, there was not a single science professor who could identify him by name." (*New York Times*, September 3, 2003.)

Grading

- Programming assignments.
- Treat each homework as an examination.
- You are expected to write your own codes and turn in your source code.
- Do not copy or collaborate with fellow students.
- Never ask your friends to write programs for you.
- Never give your codes to other students or publish your codes.

What This Course Is About

- Financial theories in pricing.
- Mathematical backgrounds.
- Derivative securities.
- Pricing models.
- Efficient algorithms in pricing financial instruments.
- Research problems.
- Help in finding your thesis directions.

What This Course Is *Not* About

- How to program.[a]
  - A software bug cost Knight Capital Group, Inc. US$457.6 million on August 1, 2012.[b]
- Basic calculus, probability, combinatorics, and algebra.
- Details of the financial markets.
- How to be rich.
- How the markets will perform tomorrow.
- Professional behavior.

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[a] `http://www.csie.ntu.edu.tw/train/`

[b] Kirilenko & Lo (2013).

Useful Journals

- *Applied Mathematical Finance.*
- *Applied Mathematics and Computations.*
- *Communications on Pure and Applied Methematics.*
- *Computers & Mathematics with Applications.*
- *European Journal of Finance.*
- *European Journal of Operational Research.*
- *Finance and Stochastics.*
- *Finance Research Letters.*
- *Financial Analysts Journal.*
- *Financial Management.*

Useful Journals (continued)

- *Insurance: Mathematics and Economics.*
- *International Journal of Finance & Economics.*
- *International Journal of Theoretical and Applied Finance.*
- *Journal of Banking & Finance.*
- *Journal of Computational Finance.*
- *Journal of Derivatives.*
- *Journal of Economic Dynamics & Control.*
- *Journal of Empirical Finance.*

Useful Journals (continued)

- *Journal of Finance.*
- *Journal of Financial Economics.*
- *Journal of Financial Markets.*
- *Journal of Fixed Income.*
- *Journal of Futures Markets.*
- *Journal of Financial and Quantitative Analysis.*
- *Journal of Portfolio Management.*
- *Journal of Real Estate Finance and Economics.*

Useful Journals (concluded)

- *Journal of Risk and Uncertainty.*
- *Management Science.*
- *Mathematical Finance.*
- *Quantitative Finance.*
- *Review of Financial Studies.*
- *Review of Derivatives Research.*
- *Review of Finance.*
- *Risk Magazine.*
- *SIAM Journal on Financial Mathematics.*
- *Stochastics and Stochastics Reports.*

The Modelers' Hippocratic Oath[a]

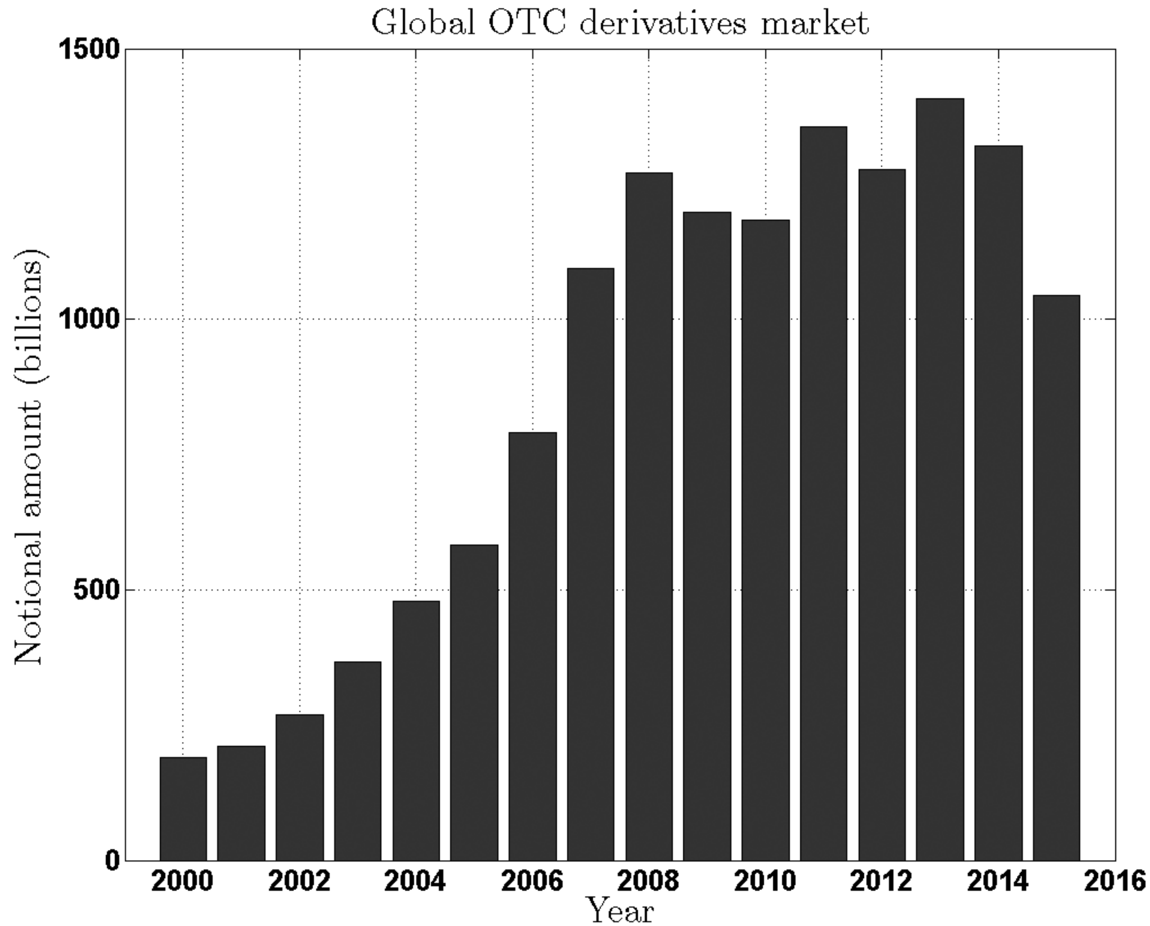
- I will remember that I didn't make the world, and it doesn't satisfy my equations.
- Though I will use models boldly to estimate value, I will not be overly impressed by mathematics.
- I will never sacrifice reality for elegance without explaining why I have done so.
- Nor will I give the people who use my model false comfort about its accuracy. Instead, I will make explicit its assumptions and oversights.
- I understand that my work may have enormous effects on society and the economy, many of them beyond my comprehension.

[a]Emanuel Derman & Paul Wilmott, January 7, 2009.

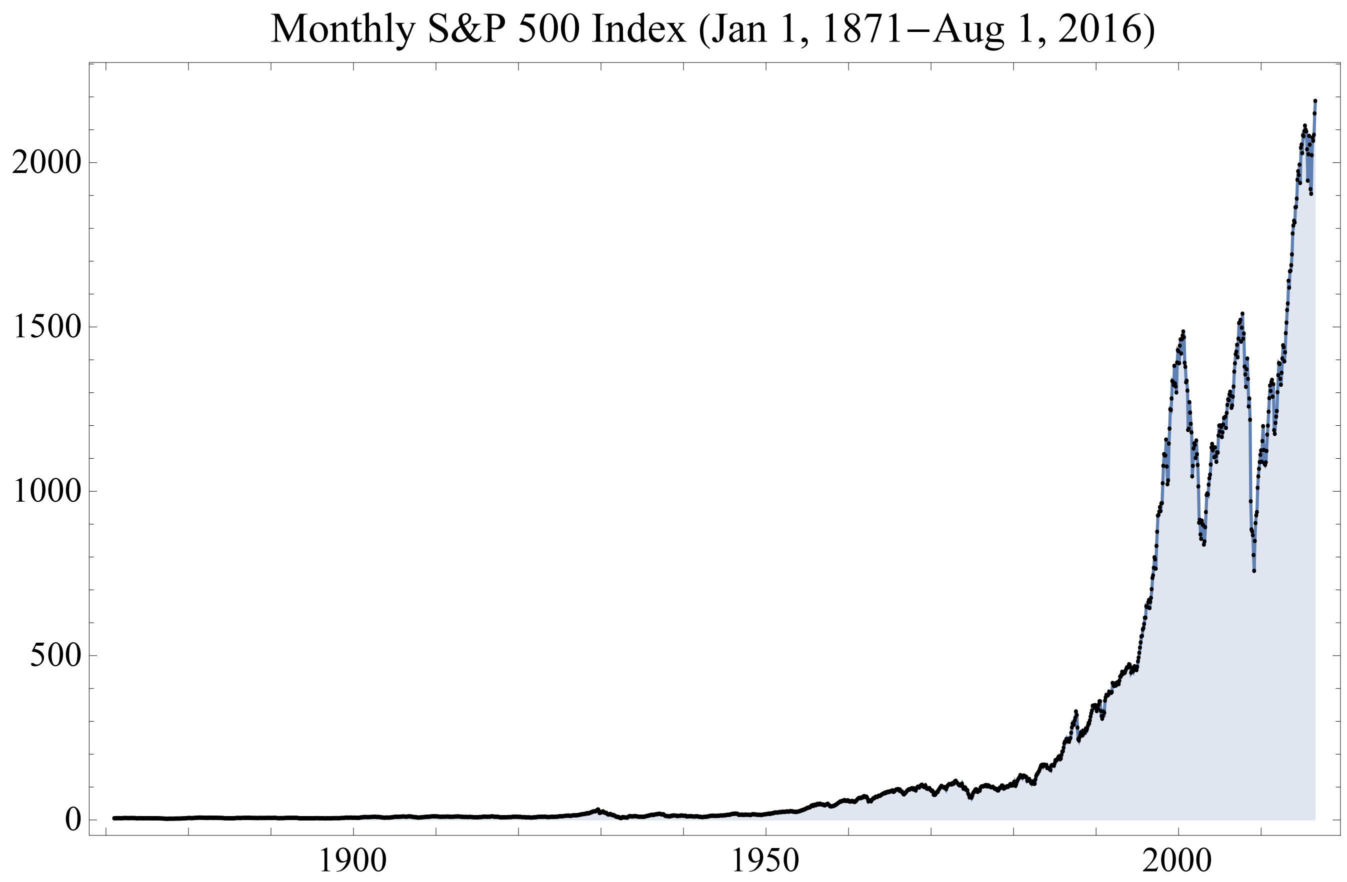
Outstanding U.S. Debts (bln)

Year	Municipal	Treasury	Mortgage—related	U.S. corporate	Fed agencies	Money market	Asset—backed	Total
85	859.5	1,437.7	372.1	776.5	293.9	847.0	0.9	4,587.6
86	920.4	1,619.0	534.4	959.6	307.4	877.0	7.2	5,225.0
87	1,010.4	1,724.7	672.1	1,074.9	341.4	979.8	12.9	5,816.2
88	1,082.3	1,821.3	772.4	1,195.7	381.5	1,108.5	29.3	6,391.0
89	1,135.2	1,945.4	971.5	1,292.5	411.8	1,192.3	51.3	7,000.0
90	1,184.4	2,195.8	1,333.4	1,350.4	434.7	1,156.8	89.9	7,745.4
91	1,272.2	2,471.6	1,636.9	1,454.7	442.8	1,054.3	129.9	8,462.4
92	1,302.8	2,754.1	1,937.0	1,557.0	484.0	994.2	163.7	9,192.8
93	1,377.5	2,989.5	2,144.7	1,674.7	570.7	971.8	199.9	9,928.8
94	1,341.7	3,126.0	2,251.6	1,755.6	738.9	1,034.7	257.3	10,505.8
95	1,293.5	3,307.2	2,352.1	1,937.5	844.6	1,177.3	316.3	11,228.5
96	1,296.0	3,459.7	2,486.1	2,122.2	925.8	1,393.9	404.4	12,088.1
97	1,367.5	3,456.8	2,680.2	2,346.3	1,022.6	1,692.8	535.8	13,102.0
98	1,464.3	3,355.5	2,955.2	2,666.2	1,296.5	1,978.0	731.5	14,447.2
99	1,532.5	3,281.0	3,334.2	3,022.9	1,616.5	2,338.2	900.8	16,026.4
00	1,567.8	2,966.9	3,564.7	3,372.0	1,851.9	2,661.0	1,071.8	17,056.1
01	1,688.4	2,967.5	4,125.5	3,817.4	2,143.0	2,542.4	1,281.1	18,565.3
02	1,783.8	3,204.9	4,704.9	3,997.2	2,358.5	2,577.5	1,543.3	20,170.1

Global OTC Derivatives Market

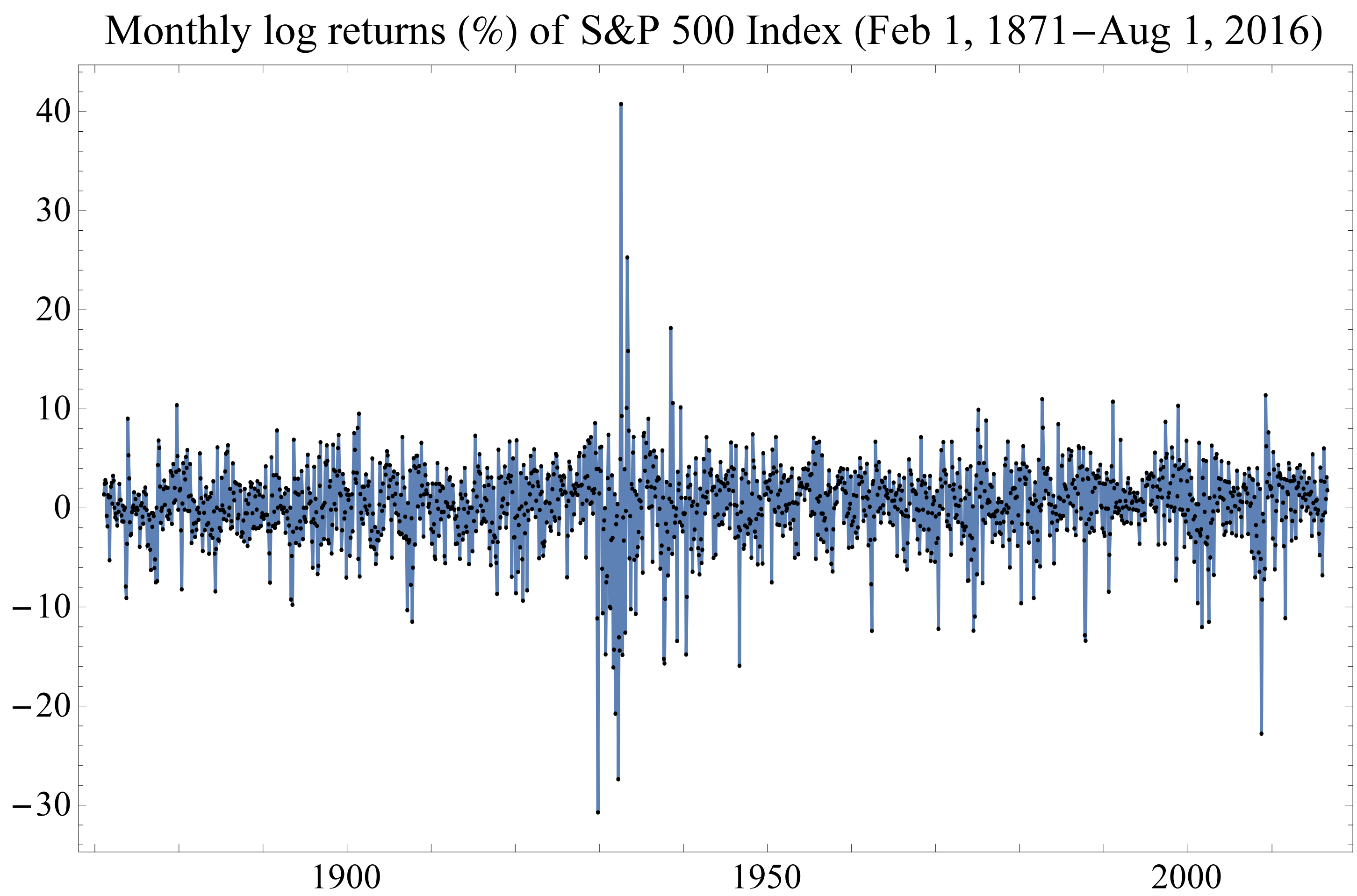


Standard and Poor's (S&P) 500 Index (by Robert Shiller[a])

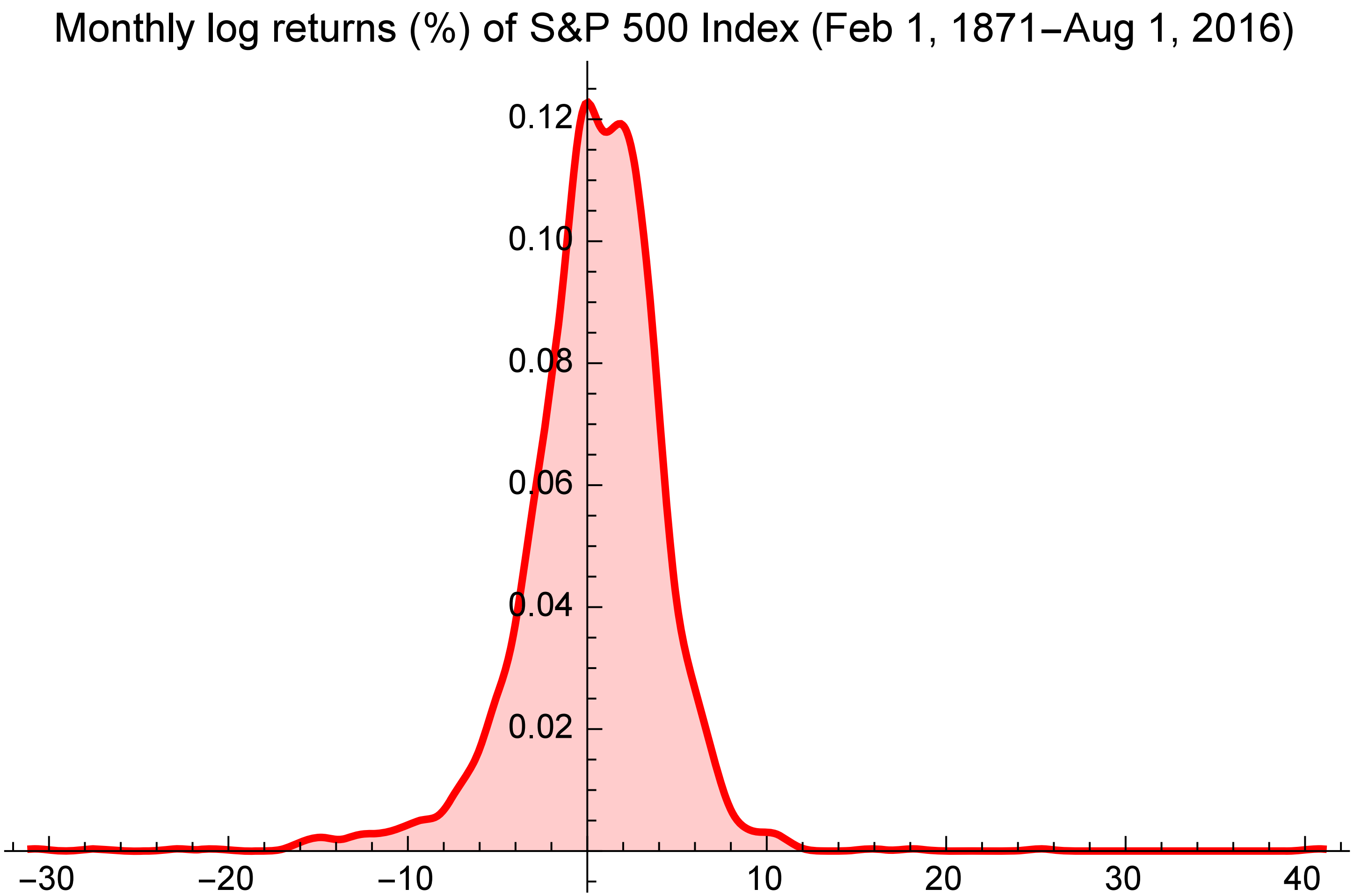


[a]Co-winner of the 2013 Nobel Prize in Economic Sciences.

Returns of S&P 500 Index



Distribution of Returns of S&P 500 Index



Analysis of Algorithms

I can calculate the motions
of the heavenly bodies,
but not the madness of people.
— Isaac Newton (1642–1727)

It is unworthy of excellent men
to lose hours like slaves
in the labor of computation.
— Gottfried Wilhelm Leibniz (1646–1716)

Computability and Algorithms

- Algorithms are precise procedures that can be turned into computer programs.
- Uncomputable problems.
  - Does this program have infinite loops?
  - Is this program bug free?
- Computable problems.
  - Intractable problems.
  - Tractable problems.

Complexity

- A set of basic operations are assumed to take one unit of time $(+, -, \times, /, \log, x^y, e^x, \ldots)$.
- The total number of these operations is the total work done by an algorithm (its computational complexity).
- The space complexity is the amount of memory space used by an algorithm.
- Concentrate on the abstract complexity of an algorithm instead of its detailed implementation.
- Complexity is a good guide to an algorithm's *actual* running time.

Common (Asymptotic) Complexities

- Let n stand for the "size" of the problem.
  - Number of elements, number of cash flows, number of time periods, etc.
- Linear time if the complexity is $O(n)$.
- Quadratic time if the complexity is $O(n^2)$.
- Cubic time if the complexity is $O(n^3)$.
- Superpolynomial if the complexity is higher than polynomials, say $2^{O(\sqrt{n})}$.[a]
- Exponential time if the complexity is $2^{O(n)}$.

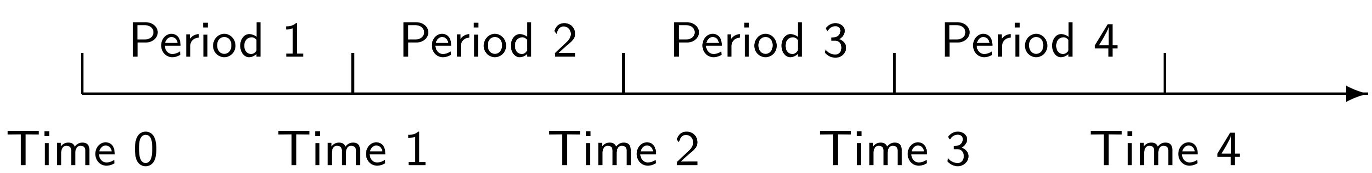
[a] E.g., T. Dai (B82506025, R86526008, D8852600) & Lyuu (2007); Lyuu & C. Wang (F95922018) (2011); H. Chiu (R98723059) (2012).

Basic Financial Mathematics

In the fifteenth century
mathematics was mainly concerned with
questions of commercial arithmetic and
the problems of the architect.
— Joseph Alois Schumpeter (1883–1950)

I'm more concerned about
the return of my money than
the return on my money.
— Will Rogers (1879–1935)

The Time Line



Time Value of Money[a]

$$
\begin{aligned}
\text{FV} &= \text{PV}(1+r)^n, \qquad (1)\\
\text{PV} &= \text{FV} \times (1+r)^{-n}.
\end{aligned}
$$

- FV (future value).
- PV (present value).
- r: interest rate.

[a] Fibonacci (1170–1240); Irving Fisher (1867–1947).

Periodic Compounding

- Suppose the *annual* interest rate r is compounded m times per annum.
- Then

$$1 \to \left(1 + \frac{r}{m}\right) \to \left(1 + \frac{r}{m}\right)^2 \to \left(1 + \frac{r}{m}\right)^3 \to \cdots$$

- Hence, after n years,

$$\text{FV} = \text{PV}\left(1 + \frac{r}{m}\right)^{nm}. \tag{2}$$

Common Compounding Methods

- Annual compounding: $m = 1$.
- Semiannual compounding: $m = 2$.
- Quarterly compounding: $m = 4$.
- Monthly compounding: $m = 12$.
- Weekly compounding: $m = 52$.
- Daily compounding: $m = 365$.
- Continuous compounding: $m = \infty$.

Easy Translations

- An annual interest rate of r compounded m times a year is "equivalent to" an interest rate of r/m per $1/m$ year.
- If a loan asks for a return of 1% *per month*, the *annual* interest rate will be 12% with monthly compounding.

Example

- Annual interest rate is 10% compounded twice per annum.
- Each dollar will grow to be

$$[\,1 + (0.1/2)\,]^2 = 1.1025$$

  one year from now.
- The rate is "equivalent to" an interest rate of 10.25% compounded once *per annum*,

$$1 + 0.1025 = 1.1025$$

- So an interest rate is meaningless without the compounding frequency.

Rule of 72

- Let the annual interest rate be r with annual compounding.
- How many years T will it take for your money to double?
- The identity to solve is

$$(1+r)^T = 2.$$

- So

$$T = \frac{\ln 2}{\ln(1+r)}.$$

- Is there an easier way?

Rule of 72 (concluded)

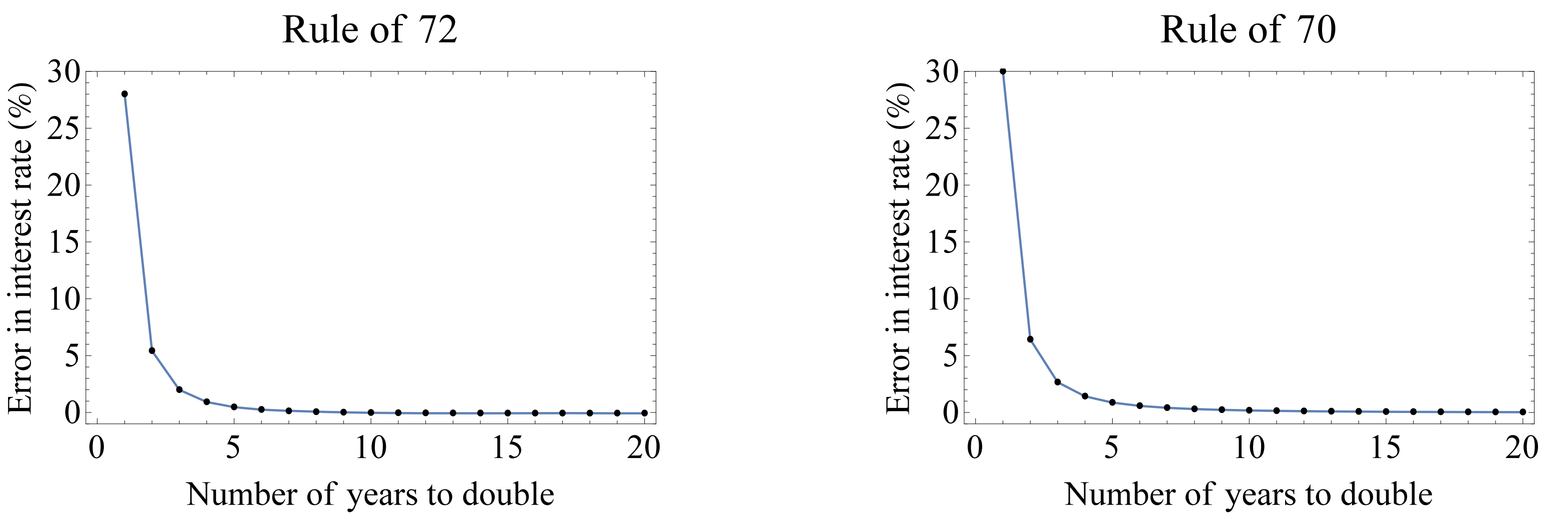
- The rule of 72 is a heuristic to estimate T.
- It says

$$T \approx \frac{72}{r\ (\%)}.$$

- So it takes about $72/12 = 6$ years to double the GDP if the annual growth rate is 12%.
- Reason:

$$\frac{\ln 2}{\ln(1+r)} \approx \frac{0.693}{r} + 0.3466.$$

How Good Is the Rule of 72?[a]



[a]True interest rate subtracted by the approximation (in %).

Continuous Compounding[a]

- Let $m \to \infty$ so that

$$\left(1 + \frac{r}{m}\right)^m \to e^r$$

in Eq. (2) on p. 29.

- Then

$$\text{FV} = \text{PV} \times e^{rn},$$

where $e = 2.71828\ldots$.

[a] Jacob Bernoulli (1654–1705) in 1685.

Continuous Compounding (concluded)

- Continuous compounding is easier to work with.
- Suppose the annual interest rate is r_1 for n_1 years and r_2 for the following n_2 years.
- Then the FV of one dollar will be

$$e^{r_1 n_1 + r_2 n_2}$$

after $n_1 + n_2$ years.

Conversion between Compounding Methods

- Let r_1 be the annual rate with continuous compounding.
- Let r_2 be the *equivalent* rate compounded m times per annum.
- How are they related?

Conversion between Compounding Methods (continued)

- Principle: Both interest rates must produce the same amount of money after one year.
- That is,

$$\left(1+\frac{r_2}{m}\right)^m = e^{r_1}.$$

- Therefore,[a]

$$\begin{aligned} r_1 &= m \ln\left(1+\frac{r_2}{m}\right), \\ r_2 &= m\left(e^{r_1/m}-1\right). \end{aligned}$$

[a] Are they really equivalent? In what sense are they equivalent? Contributed by Mr. Chen, Tung-Li (D09922014) on February 24, 2023.

Conversion between Compounding Methods (concluded)

- Suppose r_1 is the annual rate compounded m_1 times per annum.
- Suppose r_2 is the *equivalent* rate compounded m_2 times per annum.
- Then
$$\left(1+\frac{r_1}{m_1}\right)^{m_1}=\left(1+\frac{r_2}{m_2}\right)^{m_2}. \tag{3}$$

The PV Formula

- The PV of the cash flow $C_1, C_2, \ldots, C_n$ at times $1, 2, \ldots, n$ is

$$\text{PV} = \frac{C_1}{1+y} + \frac{C_2}{(1+y)^2} + \cdots + \frac{C_n}{(1+y)^n}. \tag{4}$$

- This formula and its variations are the engine behind most of financial calculations.[a]
  - What is y?
  - What are C_i?
  - What is n?
- It will be justified on p. 224 based on "sound" principles.

[a]Cochrane (2005), "Asset pricing theory all stems from one simple concept [...]: price equals expected discounted payoff."

An Algorithm for Evaluating PV in Eq. (4)

```
1: x := 0;
2: for i = 1, 2, ..., n do
3:     x := x + C_i / (1 + y)^i;
4: end for
5: return x;
```

- The algorithm takes time proportional to $\sum_{i=1}^{n} i = O(n^2)$.[a]
- Can improve it to $O(n)$ if you apply $a^b = e^{b \ln a}$ in step 3.[b]

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[a] If only $+, -, \times$, and / are allowed.

[b] Recall that we count x^y as taking one unit of time.

Another Algorithm for Evaluating PV

$x := 0;$

$d := 1 + y;$

for $i = n, n-1, \ldots, 1$ **do**

  $x := (x + C_i)/d;$

end for

`return` x;

Horner's Rule: The Idea Behind p. 43

- The underlying idea is

$$\left(\cdots \left(\left(\frac{C_n}{1+y} + C_{n-1} \right) \frac{1}{1+y} + C_{n-2} \right) \frac{1}{1+y} + \cdots \right) \frac{1}{1+y}.$$

  – Due to Horner (1786–1837) in 1819.

- The algorithm takes $O(n)$ time.

- It is the most efficient possible in terms of the absolute number of arithmetic operations.[a]

- It can also be computed in parallel in $O(\log n)$ time using prefix sums.[b]

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[a] A. Borodin & Munro (1975).

[b] Contributed by Mr. Chen, Yi-Feng (R08922077) on February 24, 2021.

Annuities[a] (Certain)

- An annuity pays out the same C dollars at the end of each year for n years.
- With a rate of r, the PV is

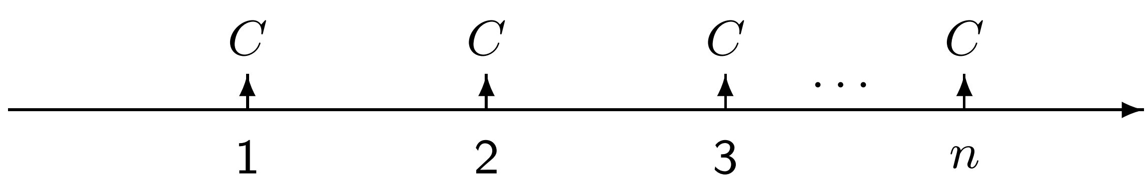
$$\sum_{i=1}^{n} C(1+r)^{-i} = C\,\frac{1-(1+r)^{-n}}{r}.$$

- The FV at the end of the nth year is

$$\sum_{i=0}^{n-1} C(1+r)^{i} = C\,\frac{(1+r)^{n}-1}{r}.$$

[a] Jan de Witt (1625–1672) in 1671; Nicholas Bernoulli (1687–1759) in 1709.

Cash flow:



General Annuities

- Suppose that m payments of C dollars each are received per year (the general annuity).
- Let r be compounded m times per annum.[a]
- Then

$$\text{PV} = \sum_{i=1}^{nm} C \left(1+\frac{r}{m}\right)^{-i} = C\,\frac{1-\left(1+\frac{r}{m}\right)^{-nm}}{\frac{r}{m}}, \quad (5)$$

$$\text{FV} = \sum_{i=0}^{nm-1} C \left(1+\frac{r}{m}\right)^{i} = C\,\frac{\left(1+\frac{r}{m}\right)^{nm}-1}{\frac{r}{m}}.$$

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[a]If your r has a compounding frequency different from m, convert it with Eq. (3) on p. 40 first.

Amortization

- It is a method of repaying a loan through regular payments of interest *and* principal.
- The size of the loan (the original balance) is reduced by the principal part of each payment.
- The interest part of each payment pays the interest incurred on the *remaining* principal balance.
- As the principal gets paid down over the term of the loan, the interest part of the payment diminishes.

Amortization (concluded)

- From Eq. (5) on p. 47, the regular payment equals

$$C = \text{loan amount} \times \frac{\frac{r}{m}}{1 - \left(1 + \frac{r}{m}\right)^{-nm}}. \tag{6}$$

  – For monthly payments, take $m = 12$.

Example: Home Mortgage

- By paying down the principal consistently, the risk to the lender is lowered.
- When the borrower sells the house, only the remaining principal is due the lender.
- Consider the equal-payment case, i.e., fixed-rate, level-payment, fully amortized mortgages.
  - They are called traditional mortgages in the U.S.[a]

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[a] *The Economist* (2016), "In most countries banks minimize their risk by offering short-term or floating-rate mortgages. American borrowers get a better deal: cheap 30-year fixed-rate mortgages that can be repaid early free."

A Numerical Example

- Consider a 15-year, $250,000 loan at 8.0% interest rate.
- Solve Eq. (5) on p. 47 with $\text{PV} = 250000$, $n = 15$, $m = 12$, and $r = 0.08$.
- That is,

$$250000 = C \, \frac{1 - \left(1 + \frac{0.08}{12}\right)^{-15\times 12}}{\frac{0.08}{12}}.$$

- This gives a monthly payment of $C = 2389.13$.

The Amortization Schedule

Month	Payment	Interest	Principal	Remaining principal
				250,000.000
1	2,389.13	1,666.667	722.464	249,277.536
2	2,389.13	1,661.850	727.280	248,550.256
3	2,389.13	1,657.002	732.129	247,818.128
		...		
178	2,389.13	47.153	2,341.980	4,730.899
179	2,389.13	31.539	2,357.591	2,373.308
180	2,389.13	15.822	2,373.308	0.000
Total	430,043.438	180,043.438	250,000.000	

A Numerical Example (continued)

In every month:

- The principal and interest parts add up to \$2,389.13.
- The remaining principal is reduced by the amount indicated under the Principal heading.[a]
  - The Principal column forms a geometric sequence.[b]
- The interest is computed by multiplying the remaining balance of the previous month by 0.08/12.

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[a]This column varies with r. Thanks to a lively class discussion on Feb 24, 2010. In fact, every column varies with r. Contributed by Ms. Wu, Japie (R01921056) on February 20, 2013.

[b]See p. 1281. Contributed by Mr. Sun, Ao (R05922147) on February 22, 2017.

A Numerical Example (concluded)

Note that:

- The Principal column adds up to $250,000.
- The Payment column adds up to $430,043.438!
- If the borrower plans to pay off the mortgage right *after* the 178th month's monthly payment, he needs to pay another $4,730.899.[a]

[a]Contributed by Ms. Wu, Japie (R01921056) on February 20, 2013.

Method 1 of Calculating the Remaining Principal

- A month's principal payment = monthly payment − (previous month's remaining principal) × (monthly interest rate).
- A month's remaining principal = previous month's remaining principal − principal payment calculated above.
- Generate the amortization schedule until the particular month you are interested in.

Method 1 of Calculating the Remaining Principal (concluded)

- This method is relatively slow but is universal in its applicability.
- It can, for example, accommodate prepayments and floating interest rates.

Method 2 of Calculating the Remaining Principal

- Right after the kth payment, the remaining principal is the PV of the future $nm - k$ cash flows,

$$\sum_{i=1}^{nm-k} C \left(1 + \frac{r}{m}\right)^{-i} = C \, \frac{1 - \left(1 + \frac{r}{m}\right)^{-nm+k}}{\frac{r}{m}}. \qquad (7)$$

- This method is much faster.[a]
- But it is limited in applications because it makes more assumptions.

[a]The above formula together with Eq. (6) on p. 49 can be used to prove that the Principal column forms a geometric sequence (try it).

Yields

- The term yield denotes the return of investment.
- Two widely used yields are the bond equivalent yield (BEY) and the mortgage equivalent yield (MEY).
- Recall Eq. (2) on p. 29: $\text{FV} = \text{PV}\left(1 + \frac{r}{m}\right)^{nm}$.
- BEY corresponds to the r above that equates PV with FV when $m = 2$.
- MEY corresponds to the r above that equates PV with FV when $m = 12$.
- BEY and MEY may differ, but they refer to the same yield.

Internal Rate of Return (IRR)

- It is the yield y which equates an investment's PV with its price P,

$$P = \frac{C_1}{(1+y)} + \frac{C_2}{(1+y)^2} + \cdots + \frac{C_n}{(1+y)^n}.$$

- IRR assumes all cash flows are reinvested at the *same* rate as the internal rate of return because:

$$\text{FV} = C_1(1+y)^{n-1} + C_2(1+y)^{n-2} + \cdots + C_n.$$

- So it must be used with caution.
- This issue disappears when there are no intermediate cash flows.

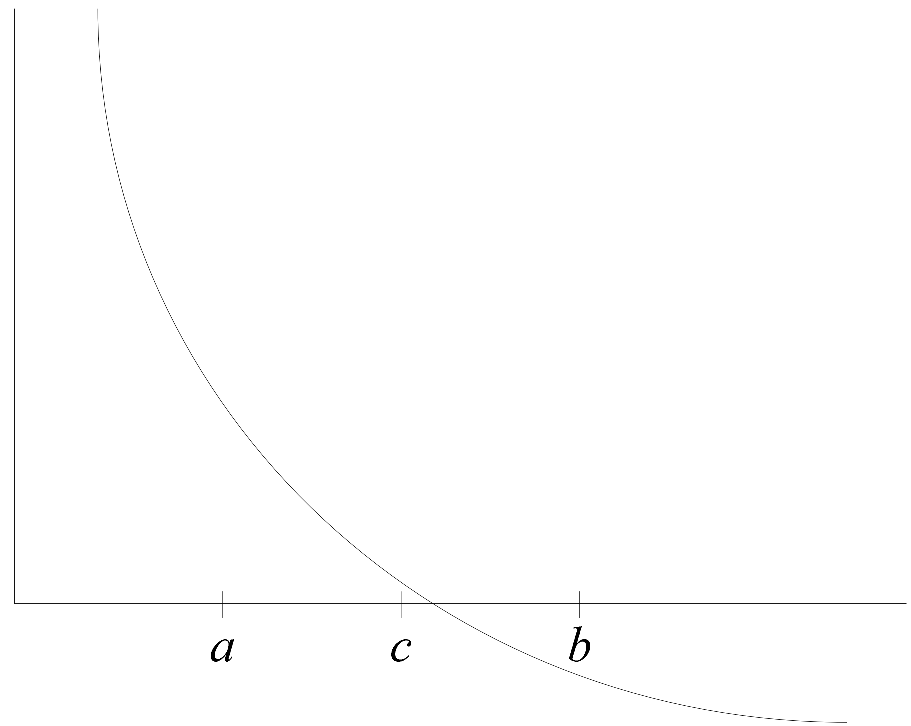
Numerical Methods for IRRs

- Define
$$f(y) \triangleq \sum_{t=1}^{n} \frac{C_t}{(1+y)^t} - P.$$
  - P is the market price.
- Solve $f(y) = 0$ for a real $y \geq -1$.[a]
- $f(y)$ is monotonically decreasing in y if $C_t > 0$ for all t.
- So a unique real-number solution exists.

[a]Negative interest rates became a reality for German and Swiss bonds in 2015. In 2016, Sweden, Denmark, and Japan imposed negative interest rates on excess reserves. As many as 355 corporate bonds were issued with negative yields as of June of 2016. Even so, −100% should be a natural lower bound because why would anyone or financial institutions want to have every cent confiscated?

The Bisection Method

- Start with a and b where $a < b$ and $f(a)\, f(b) < 0$.
- Then $f(\xi)$ must be zero for some $\xi \in [\, a, b\,]$.
- If we evaluate f at the midpoint $c \stackrel{\Delta}{=} (a+b)/2$, either (1) $f(c) = 0$, (2) $f(a)\, f(c) < 0$, or (3) $f(c)\, f(b) < 0$.
- In the first case we are done, in the second case we continue the process with the new bracket $[\, a, c\,]$, and in the third case we continue with $[\, c, b\,]$.
- The bracket is halved in the latter two cases.
- After n steps, ξ lies within a bracket of length $(b-a)/2^n$.


a
c
b

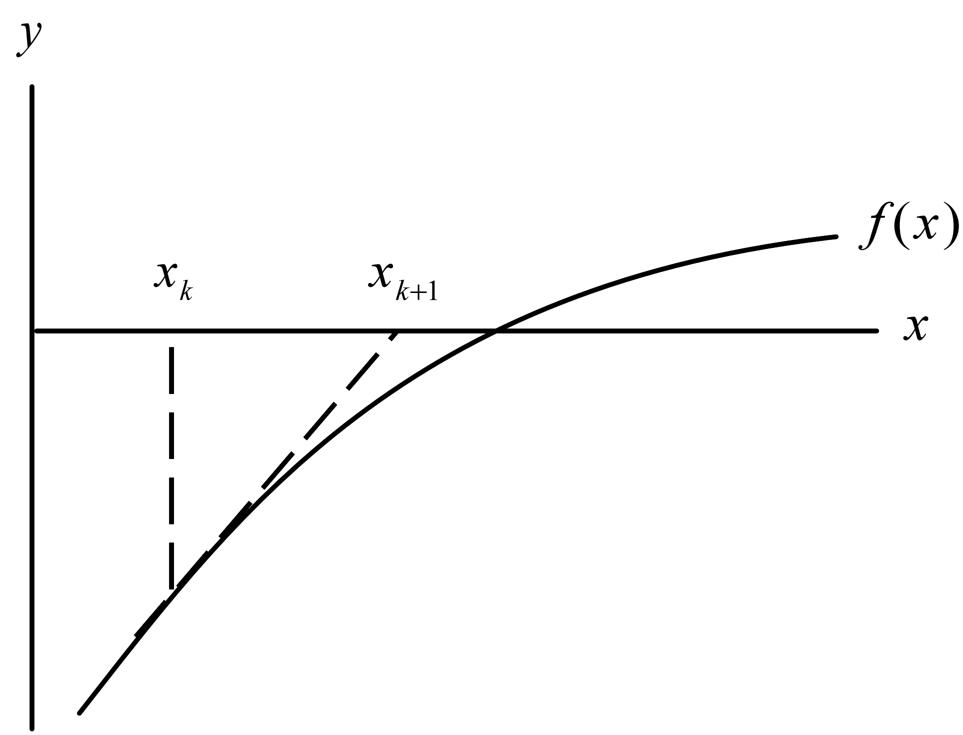
The Newton-Raphson Method

- It converges faster than the bisection method.
- Start with a first approximation x_0 to a root of $f(x) = 0$.
- Then

$$x_{k+1} \stackrel{\Delta}{=} x_k - \frac{f(x_k)}{f'(x_k)}.$$

- When computing yields,

$$f'(x) = -\sum_{t=1}^{n} \frac{tC_t}{(1+x)^{t+1}}. \tag{8}$$


y
f(x)
x
x_k
x_{k+1}

The Secant Method

- A variant of the Newton-Raphson method.
- Replace differentiation with difference.
- Start with two approximations x_0 and x_1.
- Then compute the $(k+1)$st approximation with

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}.$$

- So $[\, f(x_k) - f(x_{k-1})\,]/(x_k - x_{k-1})$ replaces $f'(x_k)$.

The Secant Method (concluded)

- Its convergence rate is 1.618.
- This is slightly worse than the Newton-Raphson method's 2.
- The secant method does not evaluate $f'(x_k)$.
- This saves about 50% in computation efforts per iteration.
- The convergence rate of the bisection method is 1.

Solving Systems of Nonlinear Equations

- It is not easy to extend the bisection method to higher dimensions.
- But the Newton-Raphson method can be extended to higher dimensions.
- Let (x_k, y_k) be the kth approximation to the solution of the two simultaneous equations,

$$\begin{aligned} f(x,y) &= 0, \\ g(x,y) &= 0. \end{aligned}$$

Solving Systems of Nonlinear Equations (continued)

- The $(k+1)$st approximation (x_{k+1}, y_{k+1}) satisfies the following linear equations,

$$\begin{bmatrix} \frac{\partial f(x_k, y_k)}{\partial x} & \frac{\partial f(x_k, y_k)}{\partial y} \\ \frac{\partial g(x_k, y_k)}{\partial x} & \frac{\partial g(x_k, y_k)}{\partial y} \end{bmatrix} \begin{bmatrix} \Delta x_{k+1} \\ \Delta y_{k+1} \end{bmatrix} = - \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix},$$

with unknowns

$$\begin{aligned} \Delta x_{k+1} &\stackrel{\Delta}{=} x_{k+1} - x_k, \\ \Delta y_{k+1} &\stackrel{\Delta}{=} y_{k+1} - y_k. \end{aligned}$$

Solving Systems of Nonlinear Equations (concluded)

- The above has a unique solution for $(\Delta x_{k+1}, \Delta y_{k+1})$ when the 2×2 matrix is invertible.
- Finally, set

$$
\begin{aligned}
x_{k+1} &= x_k + \Delta x_{k+1}, \\
y_{k+1} &= y_k + \Delta y_{k+1}.
\end{aligned}
$$

Zero-Coupon Bonds (Pure Discount Bonds)

- By Eq. (1) on p. 28, the price of a zero-coupon bond that pays F dollars in n periods is

$$F/(1+r)^n, \tag{9}$$

where r is the interest rate per period.

- Can be used to meet future obligations as there is no reinvestment risk.[a]

---

[a]Recall p. 59.

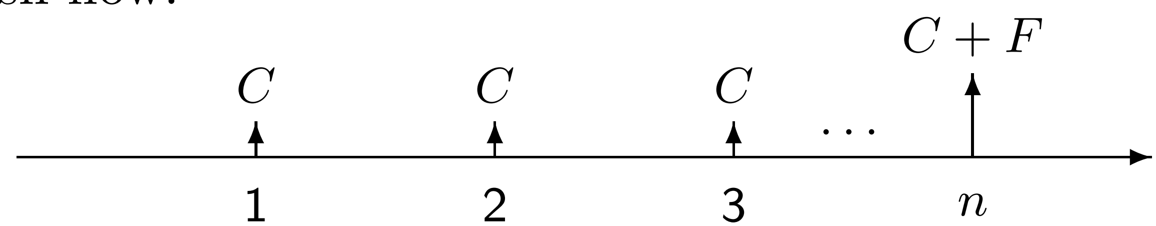
Example

- The interest rate is 8% compounded semiannually.
- A zero-coupon bond that pays the par value 20 years from now will be priced at $1/(1.04)^{40}$.
- That is 20.83% of its par value.[a]
- It will be quoted as 20.83.
- If the bond matures in 10 years instead of 20, its price would be 45.64.

[a] One fifth!

Level-Coupon Bonds

- Coupon rate.
- Par value, paid at maturity.
- F denotes the par value, and C denotes the coupon.
- Cash flow:



- Coupon bonds can be thought of as a matching package of zero-coupon bonds, at least theoretically.[a]

---

[a] "You see, Daddy didn't bake the cake, and Daddy isn't the one who gets to eat it. But he gets to slice the cake and hand it out. And when he does, little golden crumbs fall off the cake. And Daddy gets to eat those," wrote Tom Wolfe (1931–2018) in *Bonfire of the Vanities* (1987).

Pricing Formula

$$\begin{aligned} P &= \sum_{i=1}^{n} \frac{C}{\left(1+\frac{r}{m}\right)^{i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{n}} \\ &= C\,\frac{1-\left(1+\frac{r}{m}\right)^{-n}}{\frac{r}{m}} + \frac{F}{\left(1+\frac{r}{m}\right)^{n}}. \end{aligned} \tag{10}$$

- n: number of cash flows.
- m: number of payments per year.
- r: annual rate compounded m times per annum.
- Note $C = Fc/m$ when c is the annual coupon rate.
- Price P can be computed in $O(1)$ time.

Yields to Maturity

- It is the r that satisfies Eq. (10) on p. 73 with P being the bond price.
- For a 15% BEY, a 10-year bond with a coupon rate of 10% paid semiannually sells for

$$
\begin{aligned}
& 5 \times \frac{1 - [\,1 + (0.15/2)\,]^{-2\times 10}}{0.15/2} + \frac{100}{[\,1 + (0.15/2)\,]^{2\times 10}} \\
= \; & 74.5138
\end{aligned}
$$

percent of par.

- So 15% is the yield to maturity if the bond sells for 74.5138.[a]

[a]Note that the yield 15% exceeds the coupon rate 10%.

Price Behavior (1)

- Bond prices fall when interest rates rise, and vice versa.
- "Only 24 percent answered the question correctly."[a]

[a] CNN, December 21, 2001.

Price Behavior (2)[a]

- A level-coupon bond sells
  - at a premium (above its par value) when its coupon rate c is above the market interest rate r;
  - at par (at its par value) when its coupon rate is equal to the market interest rate;
  - at a discount (below its par value) when its coupon rate is below the market interest rate.

[a]Consult the text for proofs.

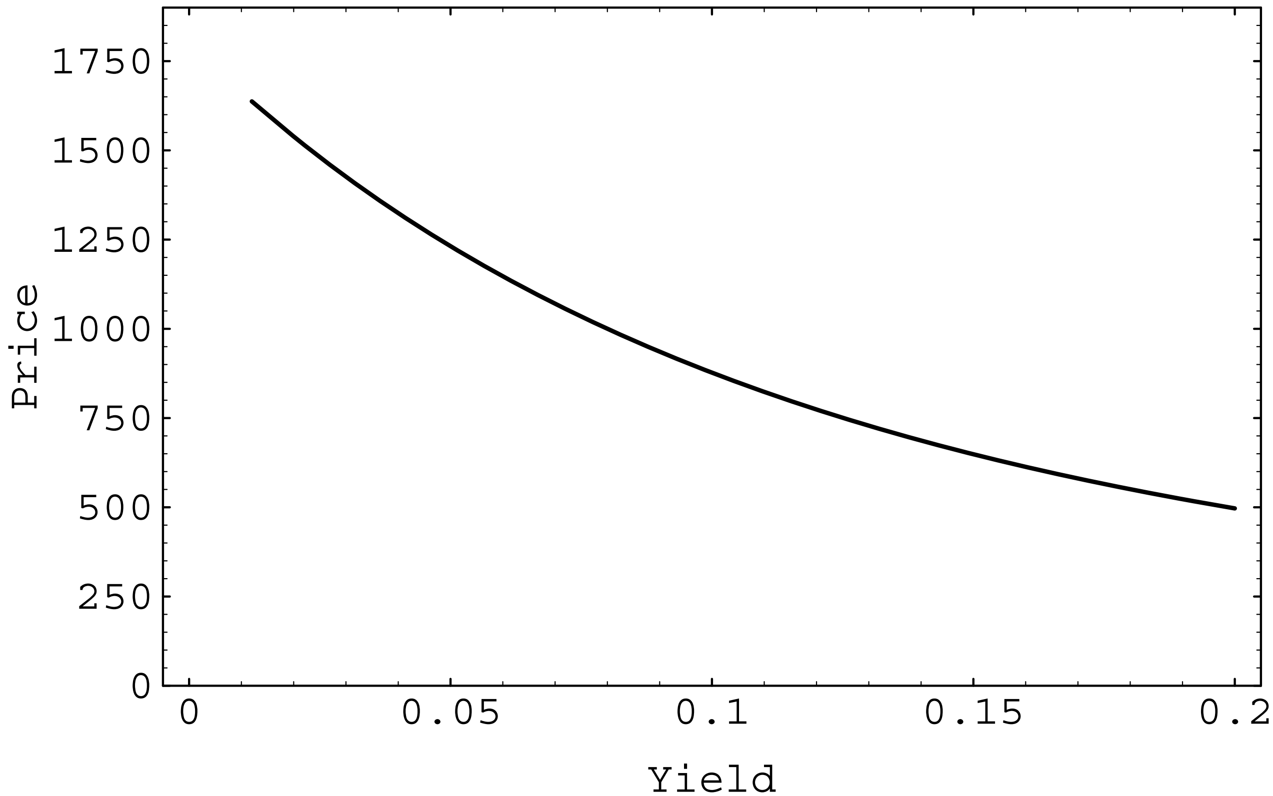
9% Coupon Bond

Yield (%)	Price (% of par)
7.5	113.37
8.0	108.65
8.5	104.19
9.0	100.00
9.5	96.04
10.0	92.31
10.5	88.79

Terminology

- Bonds selling at par are called par bonds.
- Bonds selling at a premium are called premium bonds.
- Bonds selling at a discount are called discount bonds.

Price Behavior (3): Convexity



Day Count Conventions: Actual/Actual

- The first "actual" refers to the actual number of days in a month.
- The second refers to the actual number of days in a year
- The number of days between June 17, 1992, and October 1, 1992, is 106.
  - 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.

Day Count Conventions: 30/360

- Each month has 30 days and each year 360 days.
- The number of days between June 17, 1992, and October 1, 1992, is 104.
  - 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October.
- In general, the number of days from date (y_1, m_1, d_1) to date (y_2, m_2, d_2) is

$$360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1). \tag{11}$$

Day Count Conventions: 30/360 (continued)

- If d_1 or d_2 is 31, we must change it to 30 *before* applying formula (11).[a]
- Hence:
  - There are 3 days between February 28 and March 1.
  - There are 2 days between February 29 and March 1.
  - There are 29 days between March 1 and March 31.

---

[a]This is the simplest of all the "30/360" variations (called the "30E/360" convention), used mainly in the Eurobond market (Kosowski & Neftci, 2015).

Day Count Conventions: 30/360 (concluded)

- An equivalent formula to (11) on p. 81 without any adjustment is (check it)

$$360 \times (y_2 - y_1) + 30 \times (m_2 - m_1 - 1)$$
$$+ \max(30 - d_1, 0) + \min(d_2, 30).$$

- There are many variations on the "30/360" convention regarding 31, February 28, and February 29.[a]

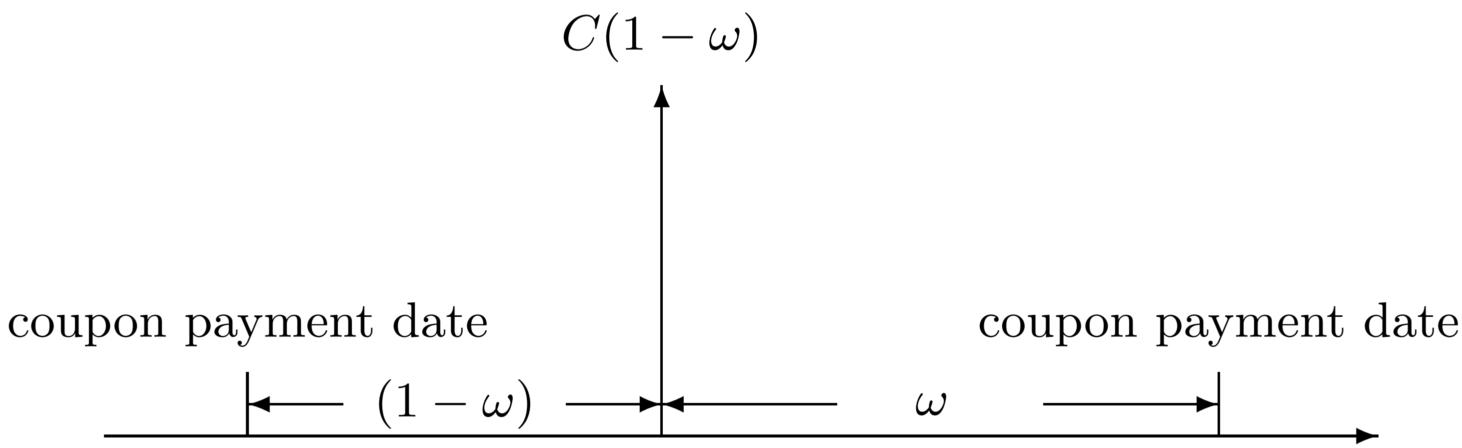
[a] Kosowski & Neftci (2015).

Full Price (Dirty Price, Invoice Price)

- In reality, the settlement date may fall on any day between two coupon payment dates.
- Let

$$\omega \stackrel{\Delta}{=} \frac{\text{number of days between the settlement and the next coupon payment date}}{\text{number of days in the coupon period}}. \qquad (12)$$

Full Price (continued)



Full Price (concluded)

- The price is now calculated by

$$\begin{aligned} \text{PV} &= \frac{C}{\left(1+\frac{r}{m}\right)^{\omega}} + \frac{C}{\left(1+\frac{r}{m}\right)^{\omega+1}} \cdots \\ &= \sum_{i=0}^{n-1} \frac{C}{\left(1+\frac{r}{m}\right)^{\omega+i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{\omega+n-1}}. \end{aligned} \tag{13}$$

Accrued Interest

- The quoted price in the U.S./U.K. does not include the accrued interest; it is called the clean price or flat price.
- The buyer pays the invoice price: the quoted price *plus* the accrued interest (AI).
- AI equals

$$C \times \frac{\begin{array}{c}\textsf{number of days from the last}\\ \textsf{coupon payment to the settlement date}\end{array}}{\textsf{number of days in the coupon period}} = C \times (1 - \omega).$$

Accrued Interest (concluded)

- The yield to maturity is the r satisfying Eq. (13) on p. 86 when PV is the invoice price:

$$\text{clean price} + \text{AI} = \sum_{i=0}^{n-1} \frac{C}{\left(1+\frac{r}{m}\right)^{\omega+i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{\omega+n-1}}.$$

Example ("30/360")

- A bond with a 10% coupon rate and paying interest semiannually, with clean price 111.2891.
- The settlement date is July 1, 1993, and the maturity date is March 1, 1995.
- There are 60 days between July 1, 1993, and the next coupon date, September 1, 1993.
- The accrued interest is $(10/2) \times (1 - \frac{60}{180}) = 3.3333$ per \$100 of par value.

Example ("30/360") (concluded)

- The yield to maturity is 3%.
- This can be verified by Eq. (13) on p. 86 with
  - $\omega = 60/180$,
  - $n = 4$,
  - $m = 2$,
  - $F = 100$,
  - $C = 5$,
  - PV$= 111.2891 + 3.3333$,
  - $r = 0.03$.

Price Behavior (2) Revisited

- Previously, a bond selling at par if the yield to maturity equals the coupon rate.
- But it assumed that the settlement date is on a coupon payment date.
- Suppose the settlement date for a bond selling at par[a] falls between two coupon payment dates.
- Then its yield to maturity is *less* than the coupon rate.[b]
  - The reason: Exponential growth *to* C is replaced by linear growth, hence overpaying the accrued interest.

---

[a] The *quoted price* equals the par value.

[b] See Exercise 3.5.6 of the textbook for proof.

Bond Price Volatility

"Well, Beethoven, what is this?[a]"

— Attributed to Prince Anton Esterházy

[a]Mass in C major, Op. 86 (1808).

Price Volatility

- Volatility measures how bond prices respond to interest rate changes.
- It is key to the risk management of interest rate-sensitive securities.

Price Volatility (concluded)

- What is the sensitivity of the percentage price change to changes in interest rates?
- Define price volatility by

$$-\frac{\frac{\partial P}{\partial y}}{P}. \tag{14}$$

- Intuitively,

$$\frac{\Delta P}{P} \approx -\text{price volatility} \times \Delta y.$$

Price Volatility of Bonds

- The price volatility of a level-coupon bond is

$$-\frac{(C/y)\,n - \left(C/y^2\right)\left((1+y)^{n+1} - (1+y)\right) - nF}{(C/y)\,((1+y)^{n+1} - (1+y)) + F(1+y)}.$$

  - F is the par value.
  - C is the coupon payment per period.
  - Formula can be simplified a bit with $C = Fc/m$.

- For the above bond,

$$-\frac{\frac{\partial P}{\partial y}}{P} > 0.$$

Macaulay Duration[a]

- The Macaulay duration (MD) is a weighted average of the times to an asset's cash flows.
- The weights are the cash flows' PVs divided by the asset's price.
- Formally,

$$\text{MD} \triangleq \frac{1}{P} \sum_{i=1}^{n} \frac{C_i}{(1+y)^i}\, i.$$

  – What if $C_i = (1+c)^i$ for some constant c and letting $n \to \infty$ and assuming $c > y$?[b]

---

[a]Frederick Macaulay (1882–1970) in 1938.

[b]Contributed by Mr. Chen, Yu-Hsing (B06901048, R11922045) on March 3, 2023.

Macaulay Duration (concluded)

- The Macaulay duration, in periods, is equal to

$$\text{MD} = -(1+y)\,\frac{\partial P}{\partial y}\,\frac{1}{P}. \tag{15}$$

MD of Bonds

- The MD of a level-coupon bond is

$$\mathrm{MD} = \frac{1}{P}\left[\sum_{i=1}^{n} \frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n}\right]. \tag{16}$$

- It can be simplified to

$$\mathrm{MD} = \frac{c(1+y)\left[\,(1+y)^n - 1\,\right] + ny(y-c)}{cy\left[\,(1+y)^n - 1\,\right] + y^2},$$

  where c is the period coupon rate.

- The MD of a level-coupon bond ($c > 0$) is less than n.
- The MD of a zero-coupon bond ($c = 0$) equals n, its term to maturity.

Remarks

- Formulas (15) on p. 98 and (16) on p. 99 hold only if the coupon C, the par value F, and the maturity n are all independent of the yield y.
  - That is, if the cash flow is independent of yields.
- To see this point, suppose the market yield declines.
- The MD will be lengthened.
- But for securities whose maturity actually decreases as a result, the price volatility[a] may decrease.

---

[a] As originally defined in formula (14) on p. 95.

How *Not* To Think about MD

- The MD has its origin in measuring the length of time a bond investment is outstanding.
- But it should be seen mainly as measuring *price volatility*.
- Duration of a security can be longer than its maturity or negative!
- Neither makes sense under the maturity interpretation.
- Many, if not most, duration-related terminology can only be comprehended as measuring volatility.

Conversion

- For the MD to be year-based, modify formula (16) on p. 99 to

$$\frac{1}{P}\left[\sum_{i=1}^{n}\frac{i}{k}\,\frac{C}{\left(1+\frac{y}{k}\right)^{i}}+\frac{n}{k}\,\frac{F}{\left(1+\frac{y}{k}\right)^{n}}\right],$$

where y is the *annual* yield and k is the compounding frequency per annum.

- Formula (15) on p. 98 also becomes

$$\text{MD}=-\left(1+\frac{y}{k}\right)\frac{\partial P}{\partial y}\,\frac{1}{P}.$$

- By definition, MD (in years) $= \dfrac{\text{MD (in periods)}}{k}$.

Modified Duration

- Modified duration is defined as

$$\text{modified duration} \stackrel{\Delta}{=} -\frac{\partial P}{\partial y}\frac{1}{P} = \frac{\text{MD}}{(1+y)}. \tag{17}$$

  - Modified duration equals MD under continuous compounding.

- By the Taylor expansion,

$$\text{percent price change} \approx -\text{modified duration} \times \text{yield change}.$$

Example

- Consider a bond whose modified duration is 11.54 with a yield of 10%.
- If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

$$-11.54 \times 0.001 = -0.01154 = -1.154\%.$$

Modified Duration of a Portfolio

- By calculus, the modified duration of a portfolio equals

$$\sum_i \omega_i D_i.$$

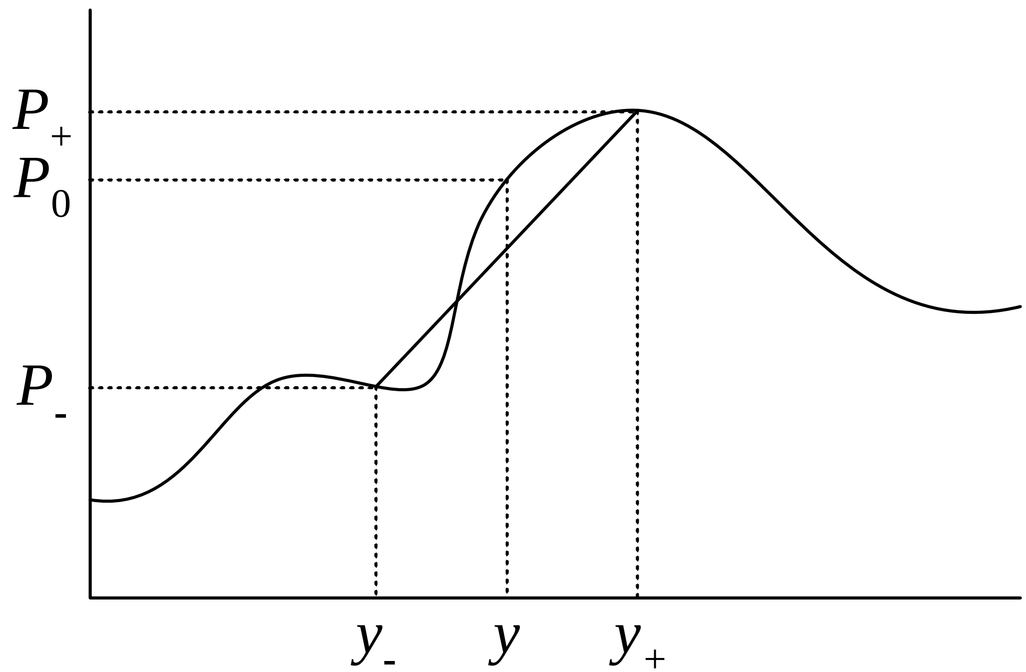
  - D_i is the modified duration of the ith asset.
  - ω_i is the market value of that asset expressed as a percentage of the market value of the portfolio.

Effective Duration

- Yield changes may alter the cash flow or the cash flow may be too complex for simple formulas.
- We need a general numerical formula for volatility.
- The effective duration is defined as

$$\frac{P_- - P_+}{P_0(y_+ - y_-)}.$$

  - P_- is the price if the yield is decreased by Δy.
  - P_+ is the price if the yield is increased by Δy.
  - P_0 is the initial price, y is the initial yield.
  - Δy is small.


P_+
P_0
P_-
y_-
y
y_+

Effective Duration (concluded)

- One can compute the effective duration of just about any financial instrument.
- An alternative is to use

$$\frac{P_0 - P_+}{P_0\,\Delta y}.$$

  - More economical but theoretically less accurate.

The Practices

- Duration is usually expressed in percentage terms — call it $D_{\%}$ — for quick mental calculation.[a]
- The percentage price change expressed in percentage terms is then approximated by

$$-D_{\%} \times \Delta r$$

  when the yield increases instantaneously by $\Delta r\%$.
  - Suppose $D_{\%} = 10$ and $\Delta r = 2$.
  - Price will drop by 20% as $10 \times 2 = 20$.
- $D_{\%}$ in fact equals the original duration (prove it!).

[a]Neftci (2008), "Market professionals do not like to use decimal points."

Hedging

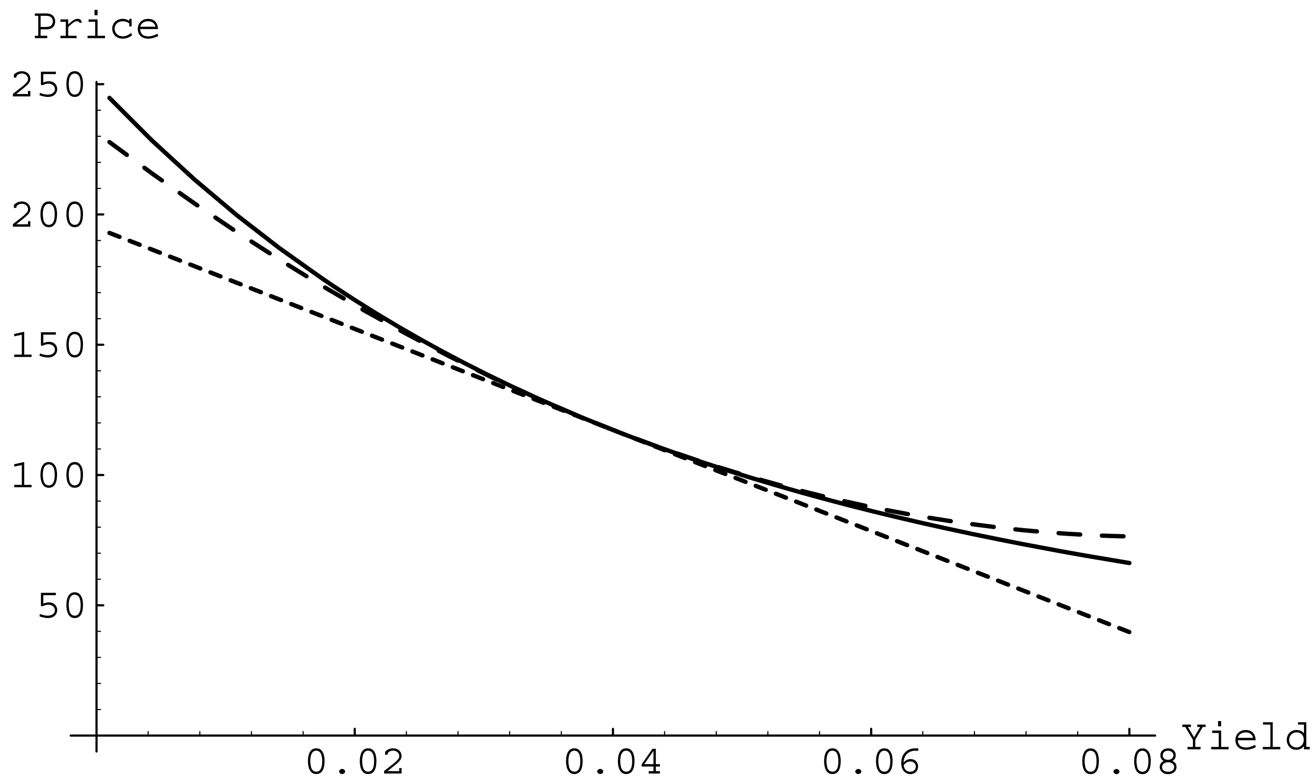
- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.
- Define dollar duration as
$$\text{modified duration} \times \text{price} = -\frac{\partial P}{\partial y}.$$
- The approximate *dollar* price change is
$$\text{price change} \approx -\text{dollar duration} \times \text{yield change}.$$
- One can hedge a bond portfolio with a dollar duration D by bonds with a dollar duration $-D$.

Convexity

- Convexity is defined as

$$\textsf{convexity} \text{ (in periods)} \stackrel{\Delta}{=} \frac{\partial^2 P}{\partial y^2} \frac{1}{P}.$$

- The convexity of a level-coupon bond is positive (prove it!).
- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude (see plot next page).


Price
250
200
150
100
50
Yield
0.02
0.04
0.06
0.08

Convexity (concluded)

- So between two bonds with the same price and duration, the one with a higher convexity is more valuable.[a]
- Suppose there are k periods per annum.
- Convexity measured in periods and convexity measured in years are related by

$$\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}.$$

---

[a] Do you spot a problem here (Christensen & Sørensen, 1994)?

Use of Convexity

- The approximation $\Delta P/P \approx -\,\text{duration} \times \text{yield change}$ works for small yield changes.
- For larger yield changes, use

$$
\begin{aligned}
\frac{\Delta P}{P} &\approx \frac{\partial P}{\partial y}\,\frac{1}{P}\,\Delta y + \frac{1}{2}\,\frac{\partial^2 P}{\partial y^2}\,\frac{1}{P}\,(\Delta y)^2 \\
&= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.
\end{aligned}
$$

- Recall the figure on p. 112.

The Practices

- Convexity is usually expressed in percentage terms — call it $C_{\%}$ — for quick mental calculation.
- The percentage price change expressed in percentage terms is approximated by
$$-D_{\%} \times \Delta r + C_{\%} \times (\Delta r)^2/2$$
when the yield increases instantaneously by $\Delta r\%$.
  - Price will drop by 17% if $D_{\%} = 10$, $C_{\%} = 1.5$, and $\Delta r = 2$ because
$$-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.$$
- $C_{\%}$ equals convexity divided by 100 (prove it!).

Effective Convexity

- The effective convexity is defined as

$$\frac{P_+ + P_- - 2P_0}{P_0\,(0.5 \times (y_+ - y_-))^2},$$

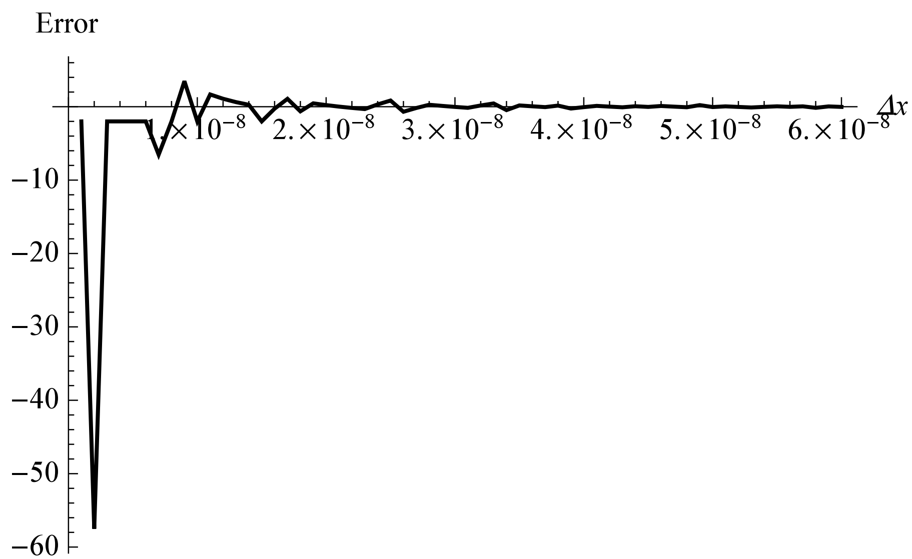
  - P_- is the price if the yield is decreased by Δy.
  - P_+ is the price if the yield is increased by Δy.
  - P_0 is the initial price, y is the initial yield.
  - Δy is small.

- Effective convexity is most relevant when a bond's cash flow is interest rate sensitive.

- How to choose the right Δy is a delicate matter.

Approximate $d^2 f(x)^2/dx^2$ at $x = 1$, Where $f(x) = x^2$

- The difference of $[\,(1+\Delta x)^2 + (1-\Delta x)^2 - 2\,]/(\Delta x)^2$ and 2:



- This numerical issue is common in financial engineering but does not have general solutions yet (see pp. 872ff).