Numerical Examples: Futures and Forward Prices

- A one-year futures contract on the one-year rate has a payoff of $100 - r$, where r is the one-year rate at maturity:

$$
F \begin{array}{l} \nearrow \; 92 \; (= 100 - 8) \\[1em] \searrow \; 98 \; (= 100 - 2) \end{array}
$$

- As the futures price F is the expected future payoff,[a]

$$F = (1 - p) \times 92 + p \times 98 = 93.914.$$

---

[a]See Exercise 13.2.11 of the textbook or p. 568.

Numerical Examples: Futures and Forward Prices (concluded)

- The forward price for a one-year forward contract on a one-year zero-coupon bond is[a]

$$90.703/96.154 = 94.331\%.$$

- The forward price exceeds the futures price.[b]

---

[a]By Eq. (145) on p. 1090.

[b]Unlike the nonstochastic case on p. 510.

Equilibrium Term Structure Models

The nature of modern trade
is to give to those who have much
and take from those who have little.
— Walter Bagehot (1867),
The English Constitution

8. What's your problem? Any moron
can understand bond pricing models.
— *Top Ten Lies Finance Professors
Tell Their Students*

Introduction

- We now survey equilibrium models.
- Recall that the spot rates satisfy

$$r(t,T) = -\frac{\ln P(t,T)}{T-t}$$

by Eq. (144) on p. 1089.

- Hence the discount function $P(t,T)$ suffices to establish the spot rate curve.
- All models to follow are short rate models.
- Unless stated otherwise, the processes are risk-neutral.

The Vasicek Model[a]

- The short rate follows

$$dr = \beta(\mu - r)\,dt + \sigma\,dW.$$

- The short rate is pulled to the long-term mean level μ at rate β.
- Superimposed on this "pull" is a normally distributed stochastic term $\sigma\,dW$.
- Since the process is an Ornstein-Uhlenbeck process,

$$E[\,r(T)\,|\,r(t) = r\,] = \mu + (r - \mu)\,e^{-\beta(T-t)}$$

from Eq. (89) on p. 635.

[a]Vasicek (1977).

The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

$$P(t,T) = A(t,T)\, e^{-B(t,T)\, r(t)}, \tag{158}$$

where

$$A(t,T) = \begin{cases} \exp\left[\frac{(B(t,T)-T+t)(\beta^2\mu-\sigma^2/2)}{\beta^2} - \frac{\sigma^2 B(t,T)^2}{4\beta} \right], & \text{if } \beta \neq 0, \\ \exp\left[\frac{\sigma^2 (T-t)^3}{6} \right], & \text{if } \beta = 0, \end{cases}$$

and

$$B(t,T) = \begin{cases} \frac{1-e^{-\beta(T-t)}}{\beta}, & \text{if } \beta \neq 0, \\ T-t, & \text{if } \beta = 0. \end{cases}$$

The Vasicek Model (continued)

- If $\beta = 0$, then P goes to infinity as $T \to \infty$.
- Sensibly, P goes to zero as $T \to \infty$ if $\beta \neq 0$.
- But even if $\beta \neq 0$, P may exceed one for a finite T.
- The long rate $r(t, \infty)$ is the constant

$$\mu - \frac{\sigma^2}{2\beta^2},$$

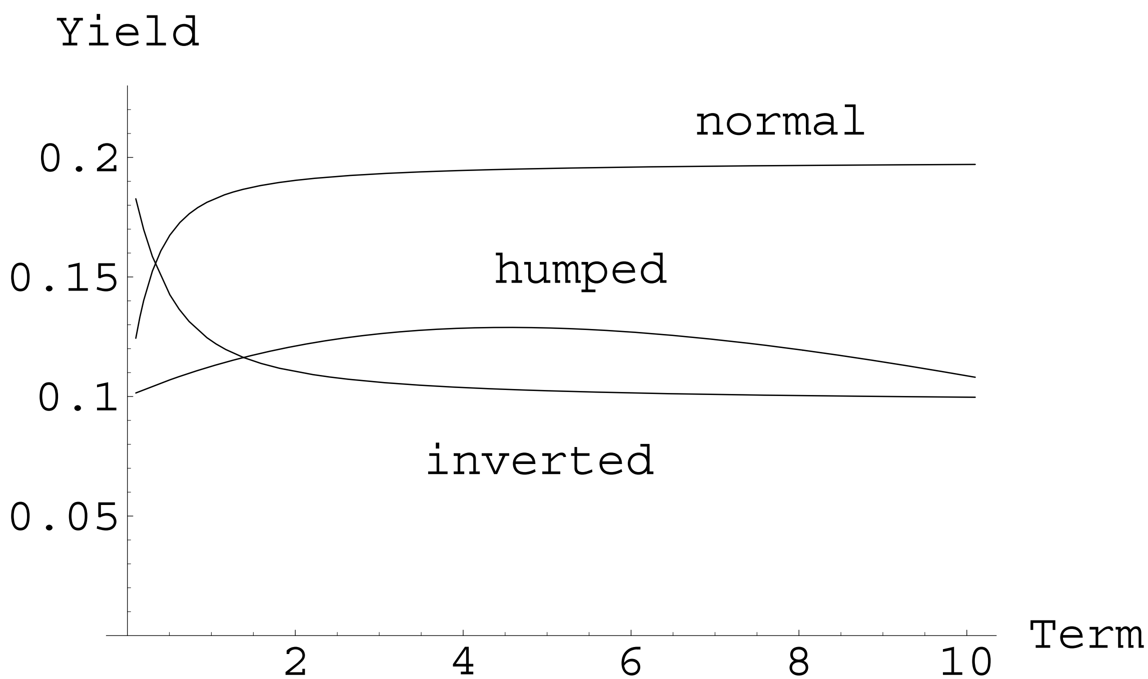
independent of the current short rate.

The Vasicek Model (concluded)

- The spot rate volatility structure is the curve

$$\sigma \frac{\partial r(t,T)}{\partial r} = \frac{\sigma B(t,T)}{T-t}.$$

- As it depends only on $T-t$ not on t by itself, the same curve is maintained for any future time t.
- When $\beta > 0$, the curve tends to decline with maturity.
  - The long rate's volatility is zero unless $\beta = 0$.
- The speed of mean reversion, β, controls the shape of the curve.
- Higher β leads to greater attenuation of volatility with maturity.


Yield
normal
humped
inverted
0.2
0.15
0.1
0.05
2
4
6
8
10
Term

The Vasicek Model: Options on Zeros[a]

- Consider a European call with strike price X expiring at time T on a zero-coupon bond with par value \$1 and maturing at time $s > T$.
- Its price is given by

$$P(t, s)\, N(x) - X P(t, T)\, N(x - \sigma_v).$$

[a] Jamshidian (1989).

The Vasicek Model: Options on Zeros (concluded)

- Above

$$
\begin{aligned}
x &\stackrel{\Delta}{=} \frac{1}{\sigma_v} \ln\left(\frac{P(t,s)}{P(t,T)\,X}\right) + \frac{\sigma_v}{2}, \\
\sigma_v &\equiv v(t,T)\,B(T,s), \\
v(t,T)^2 &\stackrel{\Delta}{=} \begin{cases} \frac{\sigma^2\left[1-e^{-2\beta(T-t)}\right]}{2\beta}, & \text{if } \beta \neq 0 \\ \sigma^2(T-t), & \text{if } \beta = 0 \end{cases}.
\end{aligned}
$$

- By the put-call parity, the price of a European put is

$$
XP(t,T)\,N(-x+\sigma_v) - P(t,s)\,N(-x).
$$

Binomial Vasicek[a]

- Consider a binomial model for the short rate in the time interval $[\,0, T\,]$ divided into n identical pieces.
- Let $\Delta t \stackrel{\Delta}{=} T/n$ and[b]

$$p(r) \stackrel{\Delta}{=} \frac{1}{2} + \frac{\beta(\mu - r)\sqrt{\Delta t}}{2\sigma}.$$

- The following binomial model converges to the Vasicek model,[c]

$$r(k+1) = r(k) + \sigma\sqrt{\Delta t}\,\xi(k), \quad 0 \leq k < n.$$

---

[a] Nelson & Ramaswamy (1990).

[b] The same form as Eq. (42) on p. 299 for the BOPM.

[c] Same as the CRR tree except that the probabilities vary here.

Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with

$$\text{Prob}[\,\xi(k) = 1\,] = \begin{cases} p(r(k)), & \text{if } 0 \leq p(r(k)) \leq 1 \\ 0, & \text{if } p(r(k)) < 0, \\ 1, & \text{if } 1 < p(r(k)). \end{cases}$$

- Observe that the probability of an up move, p, is a decreasing function of the interest rate r.
- This is consistent with mean reversion.

Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its *constant* volatility, σ.

The Cox-Ingersoll-Ross Model[a]

- It is the following square-root short rate model:

$$dr = \beta(\mu - r)\,dt + \sigma\sqrt{r}\,dW. \tag{159}$$

- The diffusion differs from the Vasicek model by a multiplicative factor $\sqrt{r}$.
- The parameter β determines the speed of adjustment.
- If $r(0) > 0$, then the short rate can reach zero *only if*

$$2\beta\mu < \sigma^2.$$

  – This is called the Feller (1951) condition.
- See text for the bond pricing formula.

[a]Cox, Ingersoll, & Ross (1985).

Binomial CIR

- We want to approximate the short rate process in the time interval $[\,0, T\,]$.
- Divide it into n periods of duration $\Delta t \stackrel{\Delta}{=} T/n$.
- Assume $\mu, \beta \geq 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will *not* combine.

Binomial CIR (continued)

- Instead, consider the transformed process[a]

$$x(r) \stackrel{\Delta}{=} 2\sqrt{r}/\sigma.$$

- By Ito's lemma (p. 610),

$$dx = m(x)\, dt + dW,$$

where

$$m(x) \stackrel{\Delta}{=} 2\beta\mu/(\sigma^2 x) - (\beta x/2) - 1/(2x).$$

- This new process has a *constant* volatility.
- Thus its binomial tree combines.

[a]See pp. 1147ff for justification.

Binomial CIR (continued)

- Construct the combining tree for r as follows.
- First, construct a tree for x.
- Then transform each node of the tree into one for r via the inverse transformation (see next page)

$$r = f(x) \triangleq \frac{x^2 \sigma^2}{4}.$$

- But when $x \approx 0$ (so $r \approx 0$), the moments may not be matched well.[a]

[a] Nawalkha & Beliaeva (2007).

$$
\begin{array}{ccccccccccc}
 & & & & x + 2\sqrt{\Delta t} & \qquad & & & & & f(x + 2\sqrt{\Delta t}) \\
 & & & \nearrow & & & & & & \nearrow & \\
 & & x + \sqrt{\Delta t} & & & & & & f(x + \sqrt{\Delta t}) & & \\
 & \nearrow & & \searrow & & & & \nearrow & & \searrow & \\
x & & & & x & & f(x) & & & & f(x) \\
 & \searrow & & \nearrow & & & & \searrow & & \nearrow & \\
 & & x - \sqrt{\Delta t} & & & & & & f(x - \sqrt{\Delta t}) & & \\
 & & & \searrow & & & & & & \searrow & \\
 & & & & x - 2\sqrt{\Delta t} & & & & & & f(x - 2\sqrt{\Delta t})
\end{array}
$$

Binomial CIR (concluded)

- The probability of an up move at each node r is
$$p(r) \triangleq \frac{\beta(\mu - r)\,\Delta t + r - r^-}{r^+ - r^-}.$$
  - $r^+ \triangleq f(x + \sqrt{\Delta t})$ denotes the result of an up move from r.
  - $r^- \triangleq f(x - \sqrt{\Delta t})$ the result of a down move.
- Finally, set the probability $p(r)$ to one as r goes to zero to make the probability stay between zero and one.

Numerical Examples

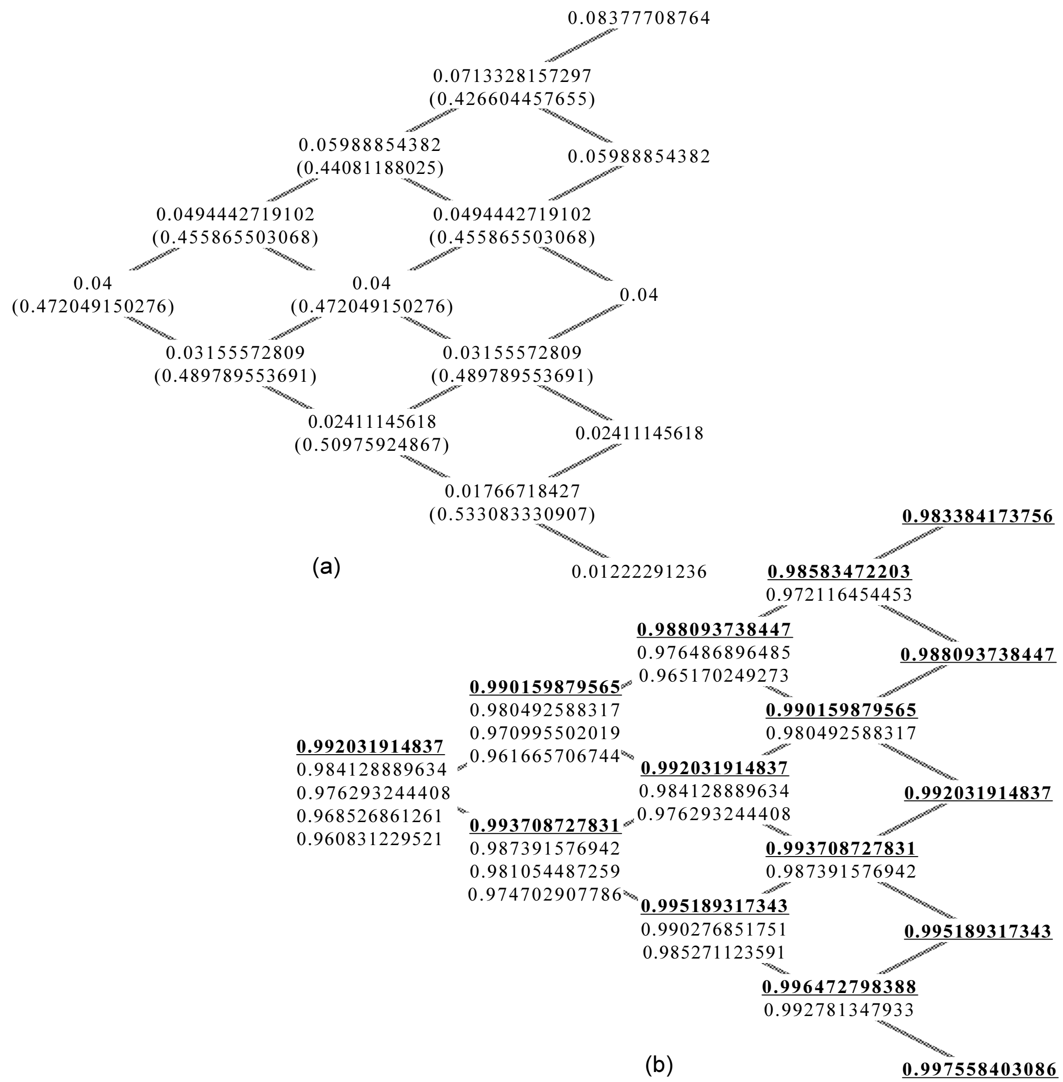
- Consider the process,

$$0.2\,(0.04 - r)\,dt + 0.1\sqrt{r}\,dW,$$

  for the time interval $[\,0, 1\,]$ given the initial rate $r(0) = 0.04$.

- We shall use $\Delta t = 0.2$ (year) for the binomial approximation.

- See p. 1143(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.


0.08377708764
0.0713328157297
(0.426604457655)
0.05988854382
(0.44081188025)
0.05988854382
0.0494442719102
(0.455865503068)
0.0494442719102
(0.455865503068)
0.04
(0.472049150276)
0.04
(0.472049150276)
0.04
0.03155572809
(0.489789553691)
0.03155572809
(0.489789553691)
0.02411145618
(0.50975924867)
0.02411145618
0.01766718427
(0.533083330907)
0.01222291236
(a)
0.983384173756
0.98583472203
0.972116454453
0.988093738447
0.976486896485
0.965170249273
0.988093738447
0.990159879565
0.980492588317
0.970995502019
0.961665706744
0.990159879565
0.980492588317
0.992031914837
0.984128889634
0.976293244408
0.968526861261
0.960831229521
0.992031914837
0.984128889634
0.976293244408
0.992031914837
0.993708727831
0.987391576942
0.981054487259
0.974702907786
0.993708727831
0.987391576942
0.995189317343
0.990276851751
0.985271123591
0.995189317343
0.996472798388
0.992781347933
0.997558403086
(b)

Numerical Examples (concluded)

- Consider the node which is the result of an up move from the root.
- Since the root has $x = 2\sqrt{r(0)}/\sigma = 4$, this particular node's x value equals $4 + \sqrt{\Delta t} = 4.4472135955$.
- Use the inverse transformation to obtain the short rate

$$\frac{x^2 \times (0.1)^2}{4} \approx 0.0494442719102.$$

- Once the short rates are in place, computing the probabilities is easy.
- Convergence is quite good.[a]

[a]See p. 369 of the textbook.

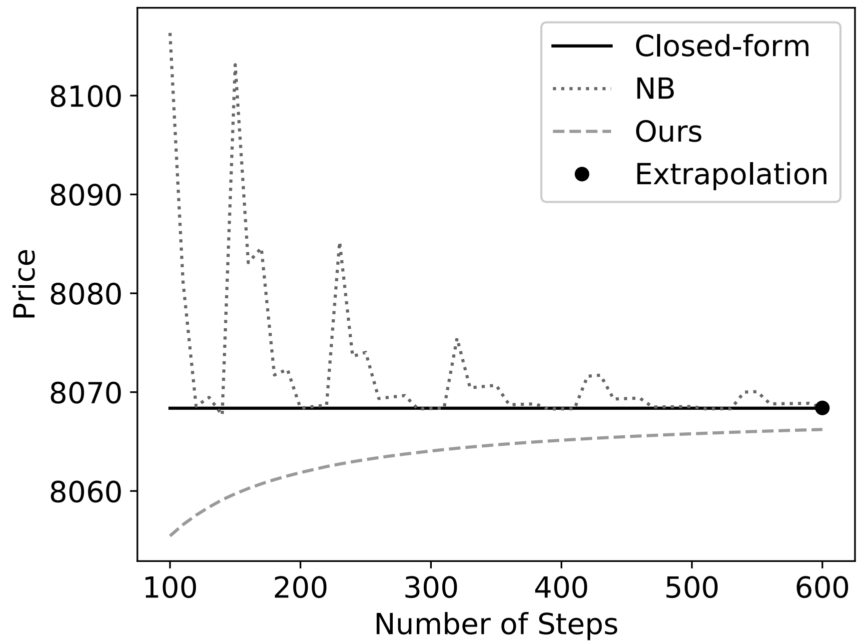
Trinomial CIR

- The binomial CIR tree does not have the degree of freedom to match the mean and variance exactly.
- It actually fails to match them at very low x.
- A trinomial tree for the CIR model with $O(n^{1.5})$ nodes that matches the mean and variance exactly is available.[a]

---

[a] Z. Lu (`D00922011`) & Lyuu (2018); H. Huang (`R03922103`) (2019).

A Comparison[a]



$r(0) = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\beta = 1.2$, $T = 5$, principal is 10,000.

[a] Plot from H. Huang (R03922103) (2019).

A General Method for Constructing Binomial Models[a]

- We are given a continuous-time process,

$$dy = \alpha(y, t)\, dt + \sigma(y, t)\, dW.$$

- Need to make sure the binomial model's drift and diffusion converge to the above process.
- Set the probability of an up move to

$$\frac{\alpha(y, t)\, \Delta t + y - y_{\mathrm{d}}}{y_{\mathrm{u}} - y_{\mathrm{d}}}.$$

- Here $y_{\mathrm{u}} \stackrel{\Delta}{=} y + \sigma(y, t)\sqrt{\Delta t}$ and $y_{\mathrm{d}} \stackrel{\Delta}{=} y - \sigma(y, t)\sqrt{\Delta t}$ represent the two rates that follow the current rate y.

[a]Nelson & Ramaswamy (1990).

A General Method (continued)

- The displacements are identical, at $\sigma(y, t)\sqrt{\Delta t}$.
- But the binomial tree may not combine as

$$\begin{aligned} & \sigma(y, t)\sqrt{\Delta t} - \sigma(y_{\mathrm{u}}, t + \Delta t)\sqrt{\Delta t} \\ \neq \quad & -\sigma(y, t)\sqrt{\Delta t} + \sigma(y_{\mathrm{d}}, t + \Delta t)\sqrt{\Delta t} \end{aligned}$$

  in general.
- When $\sigma(y, t)$ is a constant independent of y, equality holds and the tree combines.

A General Method (continued)

- To achieve this, define the transformation

$$x(y,t) \stackrel{\Delta}{=} \int^{y} \sigma(z,t)^{-1}\, dz.$$

- Then x follows

$$dx = m(y,t)\, dt + dW$$

  for some $m(y,t)$.[a]

- The diffusion term is now a constant, and the binomial tree for x combines.

---

[a]See Exercise 25.2.13 of the textbook.

A General Method (concluded)

- The transformation is unique.[a]
- The probability of an up move remains

$$\frac{\alpha(y(x,t),t)\,\Delta t + y(x,t) - y_{\mathrm{d}}(x,t)}{y_{\mathrm{u}}(x,t) - y_{\mathrm{d}}(x,t)},$$

  where $y(x,t)$ is the inverse transformation of $x(y,t)$ from x back to y.

- Note that

$$\begin{aligned} y_{\mathrm{u}}(x,t) &\stackrel{\Delta}{=} y(x+\sqrt{\Delta t},t+\Delta t), \\ y_{\mathrm{d}}(x,t) &\stackrel{\Delta}{=} y(x-\sqrt{\Delta t},t+\Delta t). \end{aligned}$$

[a] H. Chiu (R98723059) (2012).

Examples

- The transformation is

$$\int^{r} (\sigma \sqrt{z})^{-1} \, dz = \frac{2\sqrt{r}}{\sigma}$$

  for the CIR model.

- The transformation is

$$\int^{S} (\sigma z)^{-1} \, dz = \frac{\ln S}{\sigma}$$

  for the Black-Scholes model $dS = \mu S \, dt + \sigma S \, dW$.

- The familiar BOPM and CRR discretize $\ln S$ not S.

No-Arbitrage Term Structure Models

How much of the structure of our theories
really tells us about things in nature,
and how much do we contribute ourselves?
— Arthur Eddington (1882–1944)

How can I apply this model
if I don't understand it?
— Edward I. Altman (2019)

Motivations

- Recall the difficulties facing equilibrium models mentioned earlier.
  - They usually require the estimation of the market price of risk.[a]
  - They cannot fit the market term structure.
  - But consistency with the market is often mandatory in practice.

---

[a] Recall p. 1109.

No-Arbitrage Models

- No-arbitrage models utilize the full information of the term structure.
- They accept the observed term structure as consistent with an unobserved and unspecified equilibrium.
- From there, arbitrage-free movements of interest rates or bond prices over time are modeled.
- By definition, the market price of risk must be reflected in the current term structure; hence the resulting interest rate process is risk-neutral.

No-Arbitrage Models (concluded)

- No-arbitrage models can specify the dynamics of zero-coupon bond prices, forward rates, or the short rate.
- Bond price and forward rate models are usually non-Markovian (path dependent).
- In contrast, short rate models are generally constructed to be explicitly Markovian (path independent).
- Markovian models are easier to handle computationally.

The Ho-Lee Model[a]

- The short rates at any given time are evenly spaced.
- Let p denote the risk-neutral probability that the short rate makes an up move.
- We shall adopt continuous compounding.

[a] T. Ho & S. B. Lee (1986).

$$
\begin{array}{cccccc}
 & & & & & \nearrow \\
 & & & & r_3 & \\
 & & & \nearrow & & \searrow \\
 & & r_2 & & & \\
 & \nearrow & & \searrow & & \nearrow \\
r_1 & & & & r_3 + v_3 & \\
 & \searrow & & \nearrow & & \searrow \\
 & & r_2 + v_2 & & & \\
 & & & \searrow & & \nearrow \\
 & & & & r_3 + 2v_3 & \\
 & & & & & \searrow
\end{array}
$$

The Ho-Lee Model (continued)

- The Ho-Lee model starts with zero-coupon bond prices $P(t, t+1), P(t, t+2), \ldots$ at time t identified with the root of the tree.
- Let the discount factors in the next period be

$$\begin{array}{ll} P_{\mathrm{d}}(t+1, t+2), P_{\mathrm{d}}(t+1, t+3), \ldots, & \text{if short rate moves down,} \\ P_{\mathrm{u}}(t+1, t+2), P_{\mathrm{u}}(t+1, t+3), \ldots, & \text{if short rate moves up.} \end{array}$$

- By backward induction, it is not hard to see that for $n \geq 2$,[a]

$$P_{\mathrm{u}}(t+1, t+n) = P_{\mathrm{d}}(t+1, t+n)\, e^{-(v_2 + \cdots + v_n)}. \quad (160)$$

[a] See p. 376 of the textbook.

The Ho-Lee Model (continued)

- It is also not hard to check that the n-period zero-coupon bond has yields

$$\begin{aligned} y_{\mathrm{d}}(n) &\stackrel{\Delta}{=} -\frac{\ln P_{\mathrm{d}}(t+1,t+n)}{n-1} \\ y_{\mathrm{u}}(n) &\stackrel{\Delta}{=} -\frac{\ln P_{\mathrm{u}}(t+1,t+n)}{n-1} = y_{\mathrm{d}}(n) + \frac{v_2+\cdots+v_n}{n-1} \end{aligned}$$

- The volatility of the yield to maturity for this bond is therefore

$$\begin{aligned} \kappa_n &\stackrel{\Delta}{=} \sqrt{p y_{\mathrm{u}}(n)^2 + (1-p)\, y_{\mathrm{d}}(n)^2 - [\, p y_{\mathrm{u}}(n) + (1-p)\, y_{\mathrm{d}}(n)\,]^2} \\ &= \sqrt{p(1-p)}\; (y_{\mathrm{u}}(n) - y_{\mathrm{d}}(n)) \\ &= \sqrt{p(1-p)}\; \frac{v_2+\cdots+v_n}{n-1}. \end{aligned}$$

The Ho-Lee Model (concluded)

- In particular, the short rate volatility is determined by taking $n = 2$:

$$\sigma = \sqrt{p(1-p)}\, v_2. \tag{161}$$

- The volatility of the short rate therefore equals

$$\sqrt{p(1-p)}\,(r_{\mathrm{u}} - r_{\mathrm{d}}),$$

where r_{u} and r_{d} are the two successor rates.[a]

---

[a] Contrast this with the lognormal model (137) of the binomial interest rate tree on p. 1028.

The Ho-Lee Model: Volatility Term Structure

- The volatility term structure is composed of

$$\kappa_2, \kappa_3, \ldots .$$

  - The volatility structure is supplied by the market.
  - For the Ho-Lee model, it is independent of

$$r_2, r_3, \ldots .$$

- It is easy to compute the v_is from the volatility structure, and vice versa.[a]
- The r_is can be computed by forward induction.

[a] Review p. 1160.

The Ho-Lee Model: Bond Price Process

- In a risk-neutral economy, the initial discount factors satisfy[a]

$$P(t, t+n) = [\, p P_{\mathrm{u}}(t+1, t+n) + (1-p)\, P_{\mathrm{d}}(t+1, t+n)\,]\, P(t, t+1).$$

- Combine the above with Eq. (160) on p. 1159 and assume $p = 1/2$ to obtain[b]

$$P_{\mathrm{d}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)}\, \frac{2 \times \exp[\, v_2 + \cdots + v_n\,]}{1 + \exp[\, v_2 + \cdots + v_n\,]},$$

$$P_{\mathrm{u}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)}\, \frac{2}{1 + \exp[\, v_2 + \cdots + v_n\,]}.$$

---

[a] Recall Eq. (151) on p. 1097.

[b] In the limit, only the volatility matters; the first formula is similar to multiple logistic regression.

The Ho-Lee Model: Bond Price Process (concluded)

- The bond price tree combines.[a]
- Suppose all v_i equal some constant v and $\delta \stackrel{\Delta}{=} e^v > 0$.
- Then

$$\begin{aligned}
P_{\mathrm{d}}(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2\delta^{n-1}}{1+\delta^{n-1}}, \\
P_{\mathrm{u}}(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1+\delta^{n-1}}.
\end{aligned}$$

- Short rate volatility $\sigma = v/2$ by Eq. (161) on p. 1161.
- Price derivatives by taking expectations under the risk-neutral probability.

[a]See Exercise 26.2.3 of the textbook.

Calibration

- The Ho-Lee model can be calibrated in $O(n^2)$ time using state prices.
- But it can actually be calibrated in $O(n)$ time.[a]
  - Derive the v_i's in linear time.
  - Derive the r_i's in linear time.

[a] See Programming Assignment 26.2.6 of the textbook.

The Ho-Lee Model: Yields and Their Covariances

- The one-period rate of return of an n-period zero-coupon bond is[a]

$$r(t, t+n) \triangleq \ln\left(\frac{P(t+1, t+n)}{P(t, t+n)}\right).$$

- Its two possible value are

$$\ln \frac{P_{\mathrm{d}}(t+1, t+n)}{P(t, t+n)} \quad \text{and} \quad \ln \frac{P_{\mathrm{u}}(t+1, t+n)}{P(t, t+n)}.$$

- Thus the variance of return is[b]

$$\mathrm{Var}[\, r(t, t+n)\,] = p(1-p)\,[\,(n-1)\,v\,]^2 = (n-1)^2\sigma^2.$$

---

[a] So $r(t, t+n)$ does not mean the n-period spot rate at time t here.

[b] Recall that σ is the short rate volatility by Eq. (161) on p. 1161.

The Ho-Lee Model: Yields and Their Covariances (concluded)

- The covariance between $r(t, t+n)$ and $r(t, t+m)$ is[a]

$$(n-1)(m-1)\,\sigma^2.$$

- As a result, the correlation between any two one-period rates of return is one.
- Strong correlation between rates is inherent in all one-factor Markovian models.

[a]See Exercise 26.2.7 of the textbook.

The Ho-Lee Model: Short Rate Process

- The continuous-time limit of the Ho-Lee model is[a]

$$dr = \theta(t)\,dt + \sigma\,dW. \tag{162}$$

- This is Vasicek's model with the mean-reverting drift replaced by a deterministic, time-dependent drift.
- A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying,

$$dr = \theta(t)\,dt + \sigma(t)\,dW.$$

- This corresponds to the discrete-time model in which v_i are not all identical.

[a]See Exercise 26.2.10 of the textbook.

The Ho-Lee Model: Some Problems

- Future (nominal) interest rates may be negative.
- The short rate volatility is independent of the rate level.
- It has all the problems associated with a one-factor model.[a]

[a] See T. Ho & S. B. Lee (2004) for a multifactor Ho-Lee model.

Problems with No-Arbitrage Models in General

- Interest rate movements should reflect shifts in the model's state variables (factors) not its parameters.
- Model *parameters*, such as the drift $\theta(t)$ in the continuous-time Ho-Lee model, should be stable over time.
- But in practice, no-arbitrage models capture yield curve shifts through the recalibration of parameters.
  - A new model is thus born every day.

Problems with No-Arbitrage Models in General (concluded)

- This in effect says the model estimated at some time does not describe the term structure of interest rates and their volatilities at future times.
- So a model's intertemporal behavior is suspect, and using it for hedging and risk management may be unreliable.

The Black-Derman-Toy Model[a]

- This model is extensively used by practitioners.
- The BDT short rate process is the lognormal binomial interest rate process described on pp. 1024ff.[b]
- The volatility structure[c] is given by the market.
- From it, the short rate volatilities (thus v_i) are determined together with the baseline rates r_i.

---

[a] Black, Derman, & Toy (BDT) (1990), but essentially finished in 1986 according to Mehrling (2005).

[b] Repeated on next page.

[c] Recall Eq. (143) on p. 1075.

$$
\begin{array}{ccccccc}
 & & & & & & r_4 \\
 & & & & & \nearrow & \\
 & & & & r_3 & & \\
 & & & \nearrow & & \searrow & \\
 & & r_2 & & & & r_4 v_4 \\
 & \nearrow & & \searrow & & \nearrow & \\
r_1 & & & & r_3 v_3 & & \\
 & \searrow & & \nearrow & & \searrow & \\
 & & r_2 v_2 & & & & r_4 v_4^2 \\
 & & & \searrow & & \nearrow & \\
 & & & & r_3 v_3^2 & & \\
 & & & & & \searrow & \\
 & & & & & & r_4 v_4^3
\end{array}
$$

The Black-Derman-Toy Model (concluded)

- Our earlier binomial interest rate tree, in contrast, assumes v_i are given a priori.
- Lognormal models preclude negative short rates.

The BDT Model: Volatility Structure

- The volatility structure defines the yield volatilities of zero-coupon bonds of various maturities.
- Let the yield volatility of the i-period zero-coupon bond be denoted by κ_i.
- P_{u} is the price of the i-period zero-coupon bond one period from now if the short rate makes an up move.
- P_{d} is the price of the i-period zero-coupon bond one period from now if the short rate makes a down move.

The BDT Model: Volatility Structure (concluded)

- Corresponding to these two prices are the following yields to maturity,

$$\begin{aligned} y_{\mathrm{u}} &\stackrel{\Delta}{=} P_{\mathrm{u}}^{-1/(i-1)} - 1, \\ y_{\mathrm{d}} &\stackrel{\Delta}{=} P_{\mathrm{d}}^{-1/(i-1)} - 1. \end{aligned}$$

- The yield volatility is defined as[a]

$$\kappa_i \stackrel{\Delta}{=} \frac{\ln(y_{\mathrm{u}}/y_{\mathrm{d}})}{2}.$$

---

[a]Recall Eq. (143) on p. 1075.

The BDT Model: Calibration

- The inputs to the BDT model are riskless zero-coupon bond yields and their volatilities.
- For economy of expression, all numbers are period based.
- Suppose inductively that we have calculated

$$(r_1, v_1), (r_2, v_2), \ldots, (r_{i-1}, v_{i-1}).$$

  - They define the binomial tree up to time $i-2$ (thus period $i-1$).[a]
  - Thus the spot rates up to time $i-1$ have been matched.

---

[a] Recall that (r_{i-1}, v_{i-1}) defines $i-1$ short rates at time $i-2$, which are applicable to period $i-1$: The subscript refers to the period.

The BDT Model: Calibration (continued)

- We now proceed to calculate r_i and v_i to extend the tree to cover period i.
- Assume the price of the i-period zero can move to P_{u} or P_{d} one period from now.
- Let y denote the current i-period spot rate, which is known.
- In a risk-neutral economy,

$$\frac{P_{\mathrm{u}} + P_{\mathrm{d}}}{2(1 + r_i)} = \frac{1}{(1 + y)^i}. \tag{163}$$

- Obviously, P_{u} and P_{d} are functions of the unknown r_i and v_i.

The BDT Model: Calibration (continued)

- Viewed from now, the future $(i-1)$-period spot rate at time 1 is uncertain.
- Recall that y_{u} and y_{d} represent the spot rates at the up node and the down node, respectively.[a]
- With κ_i^2 denoting their variance, we have

$$\kappa_i = \frac{1}{2} \ln \left(\frac{{P_{\mathrm{u}}}^{-1/(i-1)} - 1}{{P_{\mathrm{d}}}^{-1/(i-1)} - 1} \right). \tag{164}$$

[a] Recall p. 1176.

The BDT Model: Calibration (continued)

- Solving Eqs. (163)–(164) for r_i and v_i with backward induction takes $O(i^2)$ time.
  - That leads to a cubic-time algorithm.
- We next employ forward induction to derive a quadratic-time calibration algorithm.[a]
- Forward induction figures out, by moving *forward* in time, how much \$1 at a node contributes to the price.[b]
- This number is called the state price and is the price of the claim that pays \$1 at that node and zero elsewhere.

[a] W. J. Chen (R84526007) & Lyuu (1997); Lyuu (1999).
[b] Review p. 1052(a).

The BDT Model: Calibration (continued)

- Let the unknown baseline rate for period i be $r_i = r$.
- Let the unknown multiplicative ratio be $v_i = v$.
- Let the state prices at time $i-1$ be

$$P_1, P_2, \ldots, P_i.$$

- The rates *from* them are

$$r, rv, \ldots, rv^{i-1}$$

  for period i, respectively.
- One dollar at time i has a present value of

$$f(r, v) \triangleq \frac{P_1}{1+r} + \frac{P_2}{1+rv} + \frac{P_3}{1+rv^2} + \cdots + \frac{P_i}{1+rv^{i-1}}.$$

The BDT Model: Calibration (continued)

- By Eq. (164) on p. 1179, the yield volatility is

$$g(r,v) \triangleq \frac{1}{2}\,\ln\left[\frac{\left(\frac{P_{\mathrm{u},1}}{1+rv}+\frac{P_{\mathrm{u},2}}{1+rv^2}+\cdots+\frac{P_{\mathrm{u},i-1}}{1+rv^{i-1}}\right)^{-1/(i-1)}-1}{\left(\frac{P_{\mathrm{d},1}}{1+r}+\frac{P_{\mathrm{d},2}}{1+rv}+\cdots+\frac{P_{\mathrm{d},i-1}}{1+rv^{i-2}}\right)^{-1/(i-1)}-1}\right].$$

- Above, $P_{\mathrm{u},1}, P_{\mathrm{u},2}, \ldots$ denote the state prices at time $i-1$ *of* the subtree rooted at the up node.[a]

- And $P_{\mathrm{d},1}, P_{\mathrm{d},2}, \ldots$ denote the state prices at time $i-1$ *of* the subtree rooted at the down node.[b]

---

[a]Like $r_2 v_2$ on p. 1173.

[b]Like r_2 on p. 1173.

The BDT Model: Calibration (concluded)

- Note that every node maintains *three* state prices: $P_*, P_{\mathrm{u},*}, P_{\mathrm{d},*}$.
- Now solve

$$
\begin{aligned}
f(r, v) &= \frac{1}{(1+y)^i}, \\
g(r, v) &= \kappa_i,
\end{aligned}
$$

  for $r = r_i$ and $v = v_i$.
- Finally, calculate the state prices at time i.
- This $O(n^2)$-time algorithm appears on p. 382 of the textbook.

Calibrating the BDT Model with the Differential Tree (in seconds)[a]

Number of years	Running time	Number of years	Running time	Number of years	Running time
3000	398.880	39000	8562.640	75000	26182.080
6000	1697.680	42000	9579.780	78000	28138.140
9000	2539.040	45000	10785.850	81000	30230.260
12000	2803.890	48000	11905.290	84000	32317.050
15000	3149.330	51000	13199.470	87000	34487.320
18000	3549.100	54000	14411.790	90000	36795.430
21000	3990.050	57000	15932.370	120000	63767.690
24000	4470.320	60000	17360.670	150000	98339.710
27000	5211.830	63000	19037.910	180000	140484.180
30000	5944.330	66000	20751.100	210000	190557.420
33000	6639.480	69000	22435.050	240000	249138.210
36000	7611.630	72000	24292.740	270000	313480.390

75MHz Sun SPARCstation 20, one period per year.

[a]Lyuu (1999).

The BDT Model: Continuous-Time Limit

- The continuous-time limit of the BDT model is[a]

$$d \ln r = \left[\theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln r \right] dt + \sigma(t)\, dW.$$

- The short rate volatility $\sigma(t)$ should be a *declining* function of time for the model to display mean reversion.
  - That makes $\sigma'(t) < 0$.
- In particular, constant $\sigma(t)$ will *not* attain mean reversion.

[a] Jamshidian (1991).

Problems with Lognormal Models in General

- Lognormal models such as BDT share the problem that $E^{\pi}[\, M(t) \,] = \infty$ for any finite t if they model the *continuously* compounded rate.[a]
- So *periodically* compounded rates should be modeled.[b]
- Another issue is computational.
- Lognormal models usually do not admit of analytical solutions to even basic fixed-income securities.
- As a result, to price short-dated derivatives on long-term bonds, the tree has to be built over the life of the underlying asset instead of the life of the derivative.

---

[a] Hogan & Weintraub (1993).
[b] Sandmann & Sondermann (1993).

Problems with Lognormal Models in General (concluded)

- This problem can be somewhat mitigated by adopting variable-duration time steps.[a]
  - Use a fine time step up to the maturity of the short-dated derivative.
  - Use a coarse time step beyond the maturity.
- A down side of this procedure is that it has to be tailor-made for each derivative.
- Finally, empirically, interest rates do not follow the lognormal distribution.

[a]Hull & White (1993).

The Extended Vasicek Model[a]

- Hull and White proposed models that extend the Vasicek model and the CIR model.
- They are called the extended Vasicek model and the extended CIR model.
- The extended Vasicek model adds time dependence to the original Vasicek model,
$$dr = [\,\theta(t) - a(t)\, r\,]\, dt + \sigma(t)\, dW.$$
- Like the Ho-Lee model, this is a normal model.
- The inclusion of $\theta(t)$ allows for an exact fit to the current spot rate curve.

[a] Hull & White (1990).

The Extended Vasicek Model (concluded)

- Function $\sigma(t)$ defines the short rate volatility, and $a(t)$ determines the shape of the volatility structure.
- Many European-style securities can be evaluated analytically.
- Efficient numerical procedures can be developed for American-style securities.

The Hull-White Model

- The Hull-White model is the following special case,

$$dr = (\theta(t) - ar)\, dt + \sigma\, dW. \tag{165}$$

- When the current term structure is matched,[a]

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

  – Recall that $f(0,t)$ defines the forward rate curve.

[a] Hull & White (1993).

The Extended CIR Model

- In the extended CIR model the short rate follows

$$dr = [\,\theta(t) - a(t)\, r\,]\, dt + \sigma(t) \sqrt{r}\, dW.$$

- The functions $\theta(t)$, $a(t)$, and $\sigma(t)$ are implied from market observables.
- With constant parameters, there exist analytical solutions to a small set of interest rate-sensitive securities.

The Hull-White Model: Calibration[a]

- We describe a trinomial forward induction scheme to calibrate the Hull-White model given a and σ.
- As with the Ho-Lee model, the set of achievable short rates is evenly spaced.
- Let r_0 be the annualized, continuously compounded short rate at time zero.
- Every short rate on the tree takes on a value

$$r_0 + j\Delta r$$

for some integer j.

---

[a]Hull & White (1993).

The Hull-White Model: Calibration (continued)

- Time increments on the tree are also equally spaced at Δt apart.
- Hence nodes are located at times $i\Delta t$ for $i = 0, 1, 2, \ldots$.
- We shall refer to the node on the tree with

$$\begin{aligned} t_i &\stackrel{\Delta}{=} i\Delta t, \\ r_j &\stackrel{\Delta}{=} r_0 + j\Delta r, \end{aligned} \tag{166}$$

  as the (i, j) node.
- The short rate at node (i, j), which equals r_j, is effective for the time period $[\,t_i, t_{i+1})$.

The Hull-White Model: Calibration (continued)

- Use

$$\mu_{i,j} \stackrel{\Delta}{=} \theta(t_i) - ar_j \tag{167}$$

  to denote the drift rate[a] of the short rate as seen from node (i, j).

- The three distinct possibilities for node (i, j) with three branches incident from it are displayed on p. 1195.

- The middle branch may be an increase of Δr, no change, or a decrease of Δr.

[a]Or, the annualized expected change.

The Hull-White Model: Calibration (continued)

$$
\begin{tikzcd}
 & (i+1, j+2) \\
 & (i+1, j+1) \\
(i, j) \arrow[uur] \arrow[ur] \arrow[r] & (i+1, j)
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & (i+1, j+1) \\
(i, j) \arrow[ur] \arrow[r] \arrow[dr] & (i+1, j) \\
 & (i+1, j-1)
\end{tikzcd}
$$

$$
\begin{tikzcd}
(i, j) \arrow[r] \arrow[dr] \arrow[ddr] & (i+1, j) \\
 & (i+1, j-1) \\
 & (i+1, j-2)
\end{tikzcd}
$$

The Hull-White Model: Calibration (continued)

- The upper and the lower branches bracket the middle branch.
- Define

$$
\begin{aligned}
p_1(i,j) &\stackrel{\Delta}{=} \text{the probability of following the upper branch from node } (i,j),\\
p_2(i,j) &\stackrel{\Delta}{=} \text{the probability of following the middle branch from node } (i,j),\\
p_3(i,j) &\stackrel{\Delta}{=} \text{the probability of following the lower branch from node } (i,j).
\end{aligned}
$$

- The root of the tree is set to the current short rate r_0.
- Inductively, the drift $\mu_{i,j}$ at node (i,j) is a function of (the still unknown) $\theta(t_i)$.

The Hull-White Model: Calibration (continued)

- Once $\theta(t_i)$ is available, $\mu_{i,j}$ can be derived via Eq. (167) on p. 1194.
- This in turn determines the branching scheme at every node (i, j) for each j, as we will see shortly.
- The value of $\theta(t_i)$ must thus be made consistent with the spot rate $r(0, t_{i+2})$.[a]

[a] Not $r(0, t_{i+1})$!

The Hull-White Model: Calibration (continued)

- The branches emanating from node (i, j) with their probabilities[a] must be chosen to be consistent with $\mu_{i,j}$ and σ.
- This is done by selecting the middle node to be as closest to the current short rate r_j plus the drift $\mu_{i,j} \Delta t$.[b]

---

[a] That is, $p_1(i, j)$, $p_2(i, j)$, and $p_3(i, j)$.

[b] A precursor of Lyuu and C. Wu's (R90723065) (2003, 2005) mean-tracking idea, which in turn is the precursor of the binomial-trinomial tree of T. Dai (B82506025, R86526008, D8852600) & Lyuu (2006, 2008, 2010).

The Hull-White Model: Calibration (continued)

- Let k be the number among $\{ j-1, j, j+1 \}$ that makes the short rate reached by the middle branch, r_k, closest to

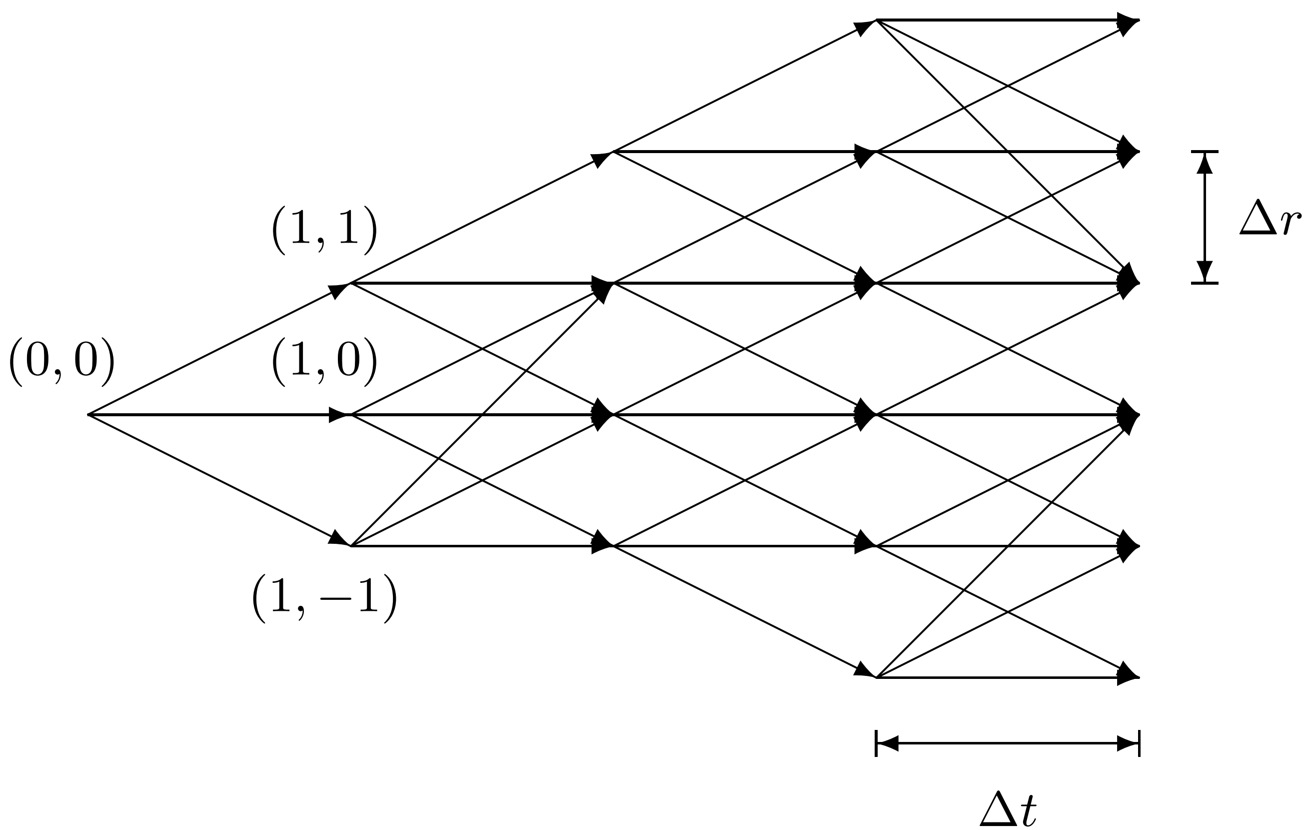
$$r_j + \mu_{i,j} \Delta t.$$

  - But note that $\mu_{i,j}$ is still *not* computed yet.

- Then the three nodes following node (i, j) are nodes

$$(i+1, k+1), (i+1, k), (i+1, k-1).$$

- See p. 1200 for a possible geometry.
- The resulting tree "combines."


(0, 0)
(1, 1)
(1, 0)
(1, −1)
Δr
Δt

The Hull-White Model: Calibration (continued)

- The probabilities for moving along these branches are functions of $\mu_{i,j}$, σ, j, and k:

$$p_1(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} + \frac{\eta}{2\Delta r}, \tag{168}$$

$$p_2(i,j) = 1 - \frac{\sigma^2 \Delta t + \eta^2}{(\Delta r)^2}, \tag{168$'$}$$

$$p_3(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} - \frac{\eta}{2\Delta r}, \tag{168$''$}$$

where

$$\eta \stackrel{\Delta}{=} \mu_{i,j} \Delta t + (j-k)\, \Delta r.$$

The Hull-White Model: Calibration (continued)

- As trinomial tree algorithms are but explicit methods in disguise,[a] certain relations must hold for Δr and Δt to guarantee stability.
- It can be shown that their values must satisfy

$$\frac{\sigma\sqrt{3\Delta t}}{2} \leq \Delta r \leq 2\sigma\sqrt{\Delta t}$$

for the probabilities to lie between zero and one.

  – For example, Δr can be set to $\sigma\sqrt{3\Delta t}$.[b]

- Now it only remains to determine $\theta(t_i)$.

---

[a] Recall p. 845.

[b] Hull & White (1988).

The Hull-White Model: Calibration (continued)

- At this point at time t_i,

$$r(0, t_1), r(0, t_2), \ldots, r(0, t_{i+1})$$

have already been matched.

- Let $Q(i, j)$ be the state price at node (i, j).
- By construction, the state prices $Q(i, j)$ for all j are known by now.
- We begin with state price $Q(0, 0) = 1$.

The Hull-White Model: Calibration (continued)

- Let $\hat{r}(i)$ refer to the short rate value at time t_i.
- The value at time zero of a zero-coupon bond maturing at time t_{i+2} is then

$$
\begin{aligned}
& e^{-r(0,t_{i+2})(i+2)\,\Delta t} \\
= \;& \sum_j Q(i,j)\, e^{-r_j\,\Delta t}\, E^{\pi}\left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i)=r_j \right]. \quad (169)
\end{aligned}
$$

- The right-hand side represents the value of \$1 at time t_{i+2} as seen at node (i,j) at time[a] t_i before being discounted by $Q(i,j)$.

---

[a]Thus $\hat{r}(i+1)$ is stochastic.

The Hull-White Model: Calibration (continued)

- The expectation in Eq. (169) can be approximated by[a]

$$E^{\pi}\left[\, e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right] \approx e^{-r_j \Delta t}\left(1 - \mu_{i,j}(\Delta t)^2 + \frac{\sigma^2(\Delta t)^3}{2}\right). \quad (170)$$

  - This solves the chicken-egg problem!

- Substitute Eq. (170) into Eq. (169) and replace $\mu_{i,j}$ with $\theta(t_i) - a r_j$ to obtain

$$\theta(t_i) \approx \frac{\sum_j Q(i,j)\, e^{-2r_j \Delta t}\left(1 + a r_j (\Delta t)^2 + \sigma^2 (\Delta t)^3/2\right) - e^{-r(0,t_{i+2})(i+2)\,\Delta t}}{(\Delta t)^2 \sum_j Q(i,j)\, e^{-2r_j \Delta t}}.$$

[a] See Exercise 26.4.2 of the textbook.

The Hull-White Model: Calibration (continued)

- For the Hull-White model, the expectation in Eq. (170) is actually known analytically by Eq. (29) on p. 181:

$$\begin{aligned} & E^{\pi}\left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right] \\ = \quad & e^{-r_j \Delta t + (-\theta(t_i) + a r_j + \sigma^2 \Delta t/2)(\Delta t)^2}. \end{aligned}$$

- Therefore, alternatively,

$$\theta(t_i) = \frac{r(0, t_{i+2})(i+2)}{\Delta t} + \frac{\sigma^2 \Delta t}{2} + \frac{\ln \sum_j Q(i,j)\, e^{-2 r_j \Delta t + a r_j (\Delta t)^2}}{(\Delta t)^2}.$$

- With $\theta(t_i)$ in hand, we can compute $\mu_{i,j}$.[a]

---

[a] See Eq. (167) on p. 1194.

The Hull-White Model: Calibration (concluded)

- With $\mu_{i,j}$ available, we compute the probabilities.[a]
- Finally the state prices at time t_{i+1}:

$$Q(i+1, j) = \sum_{(i, j^*) \text{ is connected to } (i+1, j) \text{ with probability } p_{j^*}} p_{j^*} e^{-r_{j^*} \Delta t} Q(i, j^*).$$

- There are at most 5 choices for j^* (why?).
- The total running time is $O(n^2)$.
- The space requirement is $O(n)$ (why?).

[a] See probabilities (168) on p. 1201.

Comments on the Hull-White Model

- One can try different values of a and σ for each option.
- Or have an a value common to all options but use a different σ value for each option.
- Either approach can match all the option prices exactly.
- But suppose the demand is for a single set of parameters to apply to *all* option prices.
- Then the Hull-White model can be calibrated to all the observed option prices by choosing a and σ that minimize the mean-squared pricing error.[a]

[a] Hull & White (1995).

The Hull-White Model: Calibration with Irregular Trinomial Trees

- The previous calibration algorithm is quite general.
- For example, it can be modified to apply to cases where the diffusion term has the form σr^b.
- But it has at least two shortcomings.
- First, the resulting trinomial tree is irregular.[a]
  - So it is harder to program.
- The second shortcoming is a consequence of the tree's irregular shape.

[a] Recall p. 1200.

The Hull-White Model: Calibration with Irregular Trinomial Trees (concluded)

- Recall that the algorithm figured out $\theta(t_i)$ that matches the spot rate $r(0, t_{i+2})$ in order to determine the branching schemes for the nodes at time t_i.
- But without those branches, the tree was not specified, and backward induction on the tree was not possible.
- To avoid this chicken-egg dilemma, the algorithm turned to the continuous-time model to evaluate Eq. (169) on p. 1204 that helps derive $\theta(t_i)$.
- The resulting $\theta(t_i)$ might not yield a tree that matches the spot rates exactly.

The Hull-White Model: Calibration with Regular Trinomial Trees[a]

- The next, simpler algorithm exploits the fact that the Hull-White model has a constant diffusion term σ.
- The resulting trinomial tree will be regular.
- All the $\theta(t_i)$ terms can be chosen by backward induction to match the spot rates exactly.
- The tree is constructed in two phases.

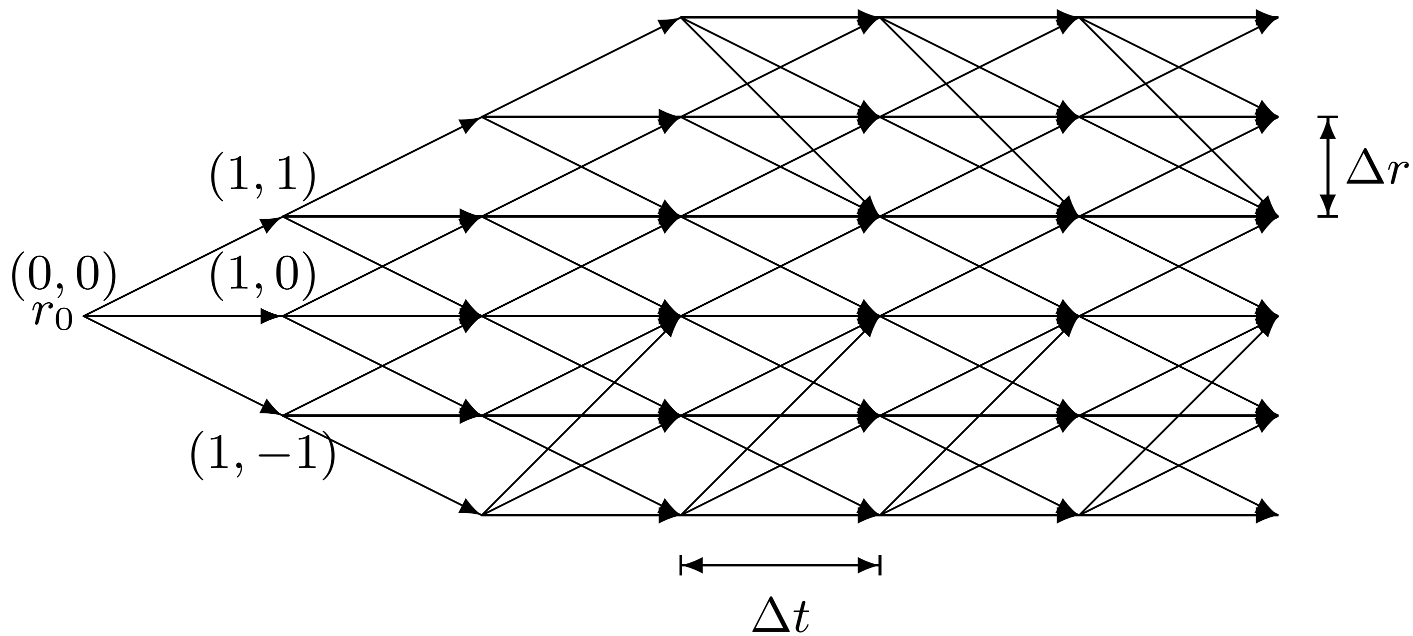
[a]Hull & White (1994).

The Hull-White Model: Calibration with Regular Trinomial Trees (continued)

- In the first phase, a tree is built for the $\theta(t) = 0$ case, which is an Ornstein-Uhlenbeck process:

$$dr = -ar\,dt + \sigma\,dW, \quad r(0) = 0.$$

  - The tree is dagger-shaped (see p. 1213).
  - The number of nodes above the r_0-line is $j_{\max}$, and that below the line is $j_{\min}$.
  - They will be picked so that the probabilities (168) on p. 1201 are positive for all nodes.



The short rate at node $(0,0)$ equals $r_0 = 0$; here $j_{\max} = 3$ and $j_{\min} = 2$.

The Hull-White Model: Calibration with Regular Trinomial Trees (concluded)

- The tree's branches and probabilities are now in place.
- Phase two fits the term structure.
  - Backward induction is applied to calculate the β_i to *add* to the short rates on the tree at time t_i so that the spot rate $r(0, t_{i+1})$ is matched exactly.[a]

---

[a]Contrast this with the previous algorithm, where it was $r(0, t_{i+2})$ that was being matched!

The Hull-White Model: Calibration

- Assume that $a > 0$.
- Set $\Delta r = \sigma \sqrt{3 \Delta t}$.[a]
- Node (i, j) is a top node if $j = j_{\max}$ and a bottom node if $j = -j_{\min}$.
- Because the root has a short rate of $r_0 = 0$, phase one sets $r_j = j \Delta r$.[b]
- Hence the probabilities (168) on p. 1201 use

$$\eta \stackrel{\Delta}{=} -a j \Delta r \Delta t + (j - k) \Delta r.$$

- Recall that k tracks the middle branch.

[a] Recall p. 1202.

[b] Similar to formula (166) on p. 1193.

The Hull-White Model: Calibration (continued)

- The probabilities become

$$
\begin{aligned}
& p_1(i,j) \\
= \quad & \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 - aj\Delta t + (j-k)}{2}, && (171) \\
& p_2(i,j) \\
= \quad & \frac{2}{3} - \left[a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 \right], && (172) \\
& p_3(i,j) \\
= \quad & \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 + aj\Delta t - (j-k)}{2}. && (173)
\end{aligned}
$$

- p_1: up move; p_2: flat move; p_3: down move.

The Hull-White Model: Calibration (continued)

- The dagger shape dictates this:
  - Let $k = j - 1$ if node (i, j) is a top node.
  - Let $k = j + 1$ if node (i, j) is a bottom node.
  - Let $k = j$ for the rest of the nodes.
- Note that the probabilities are identical for nodes (i, j) with the *same* j.
- Note also the symmetry,

$$p_1(i, j) = p_3(i, -j).$$

The Hull-White Model: Calibration (continued)

- The inequalities

$$\frac{3-\sqrt{6}}{3} < ja\Delta t < \sqrt{\frac{2}{3}} \tag{174}$$

ensure that all the branching probabilities are positive in the upper half of the tree, that is, $j > 0$ (verify this).

- The inequalities

$$-\sqrt{\frac{2}{3}} < ja\Delta t < -\frac{3-\sqrt{6}}{3}$$

ensure that the probabilities are positive in the lower half of the tree, that is, $j < 0$.

The Hull-White Model: Calibration (continued)

- To further make the tree symmetric across the r_0-line, we let $j_{\min} = j_{\max}$.
- As

$$\frac{3-\sqrt{6}}{3} \approx 0.184,$$

a good choice is

$$j_{\max} = \left\lceil \frac{0.184}{a\Delta t} \right\rceil = O(n).$$

The Hull-White Model: Calibration (continued)

- Phase two computes the β_is to fit the spot rates.
- We begin with state price $Q(0, 0) = 1$.
- Inductively, suppose that spot rates

$$r(0, t_1), r(0, t_2), \ldots, r(0, t_i)$$

  have already been matched.
- By construction, the state prices $Q(i, j)$ for all j are known by now.

The Hull-White Model: Calibration (continued)

- The value of a zero-coupon bond maturing at time t_{i+1} equals

$$e^{-r(0,t_{i+1})(i+1)\,\Delta t} = \sum_j Q(i,j)\, e^{-(\beta_i + r_j)\Delta t}$$

by risk-neutral valuation.

- Hence

$$\beta_i = \frac{r(0,t_{i+1})(i+1)\,\Delta t + \ln \sum_j Q(i,j)\, e^{-r_j \Delta t}}{\Delta t}. \qquad (175)$$

The Hull-White Model: Calibration (concluded)

- The short rate at node (i, j) now equals $\beta_i + r_j$.
- The state prices at time t_{i+1},

$$Q(i+1, j)$$

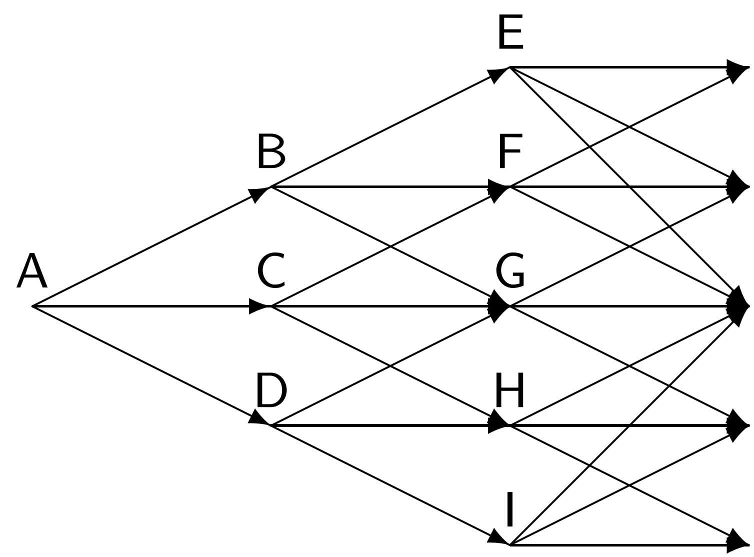
  for $-\min(i+1, j_{\max}) \le j \le \min(i+1, j_{\max})$, can now be calculated as before.[a]
- The total running time is $O(n j_{\max})$.
- The space requirement is $O(n)$.

---

[a]Recall p. 1207.

A Numerical Example

- Assume $a = 0.1$, $\sigma = 0.01$, and $\Delta t = 1$ (year).
- Immediately, $\Delta r = 1.73205\%$ and $j_{\max} = 2$.
- The plot on p. 1224 illustrates the 3-period trinomial tree after phase one.
- For example, the branching probabilities for node E are calculated by Eqs. (171)–(173) on p. 1216 with $j = 2$ and $k = 1$.



Node	A, C, G	B, F	E	D, H	I
r (%)	0.00000	1.73205	3.46410	−1.73205	−3.46410
p_1	0.16667	0.12167	0.88667	0.22167	0.08667
p_2	0.66667	0.65667	0.02667	0.65667	0.02667
p_3	0.16667	0.22167	0.08667	0.12167	0.88667

A Numerical Example (continued)

- Suppose that phase two is to fit the spot rate curve

$$0.08 - 0.05 \times e^{-0.18 \times t}.$$

- The annualized continuously compounded spot rates are

$$r(0,1) = 3.82365\%, r(0,2) = 4.51162\%, r(0,3) = 5.08626\%.$$

- Start with state price $Q(0,0) = 1$ at node A.

A Numerical Example (continued)

- Now, by Eq. (175) on p. 1221,

$$\beta_0 = r(0,1) + \ln Q(0,0)\, e^{-r_0} = r(0,1) = 3.82365\%.$$

- Hence the short rate at node A equals

$$\beta_0 + r_0 = 3.82365\%.$$

- The state prices at year one are calculated as

$$\begin{aligned} Q(1,1) &= p_1(0,0)\, e^{-(\beta_0 + r_0)} = 0.160414, \\ Q(1,0) &= p_2(0,0)\, e^{-(\beta_0 + r_0)} = 0.641657, \\ Q(1,-1) &= p_3(0,0)\, e^{-(\beta_0 + r_0)} = 0.160414. \end{aligned}$$

A Numerical Example (continued)

- The 2-year rate spot rate $r(0,2)$ is matched by picking

$$\beta_1 = r(0,2) \times 2 + \ln \left[Q(1,1)\, e^{-\Delta r} + Q(1,0) + Q(1,-1)\, e^{\Delta r} \right] = 5.20459\%.$$

- Hence the short rates at nodes B, C, and D equal

$$\beta_1 + r_j,$$

  where $j = 1, 0, -1$, respectively.

- They are found to be 6.93664%, 5.20459%, and 3.47254%.

A Numerical Example (continued)

- The state prices at year two are calculated as

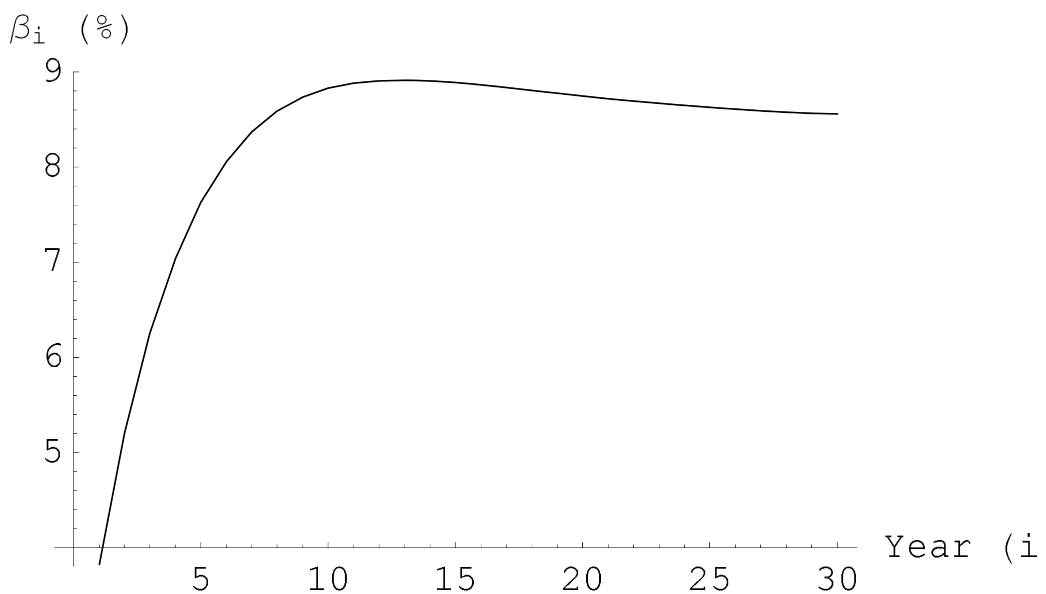
$$
\begin{aligned}
Q(2,2) &= p_1(1,1)\, e^{-(\beta_1+r_1)} Q(1,1) = 0.018209, \\
Q(2,1) &= p_2(1,1)\, e^{-(\beta_1+r_1)} Q(1,1) + p_1(1,0)\, e^{-(\beta_1+r_0)} Q(1,0) \\
&= 0.199799, \\
Q(2,0) &= p_3(1,1)\, e^{-(\beta_1+r_1)} Q(1,1) + p_2(1,0)\, e^{-(\beta_1+r_0)} Q(1,0) \\
&\quad + p_1(1,-1)\, e^{-(\beta_1+r_{-1})} Q(1,-1) = 0.473597, \\
Q(2,-1) &= p_3(1,0)\, e^{-(\beta_1+r_0)} Q(1,0) + p_2(1,-1)\, e^{-(\beta_1+r_{-1})} Q(1,-1) \\
&= 0.203263, \\
Q(2,-2) &= p_3(1,-1)\, e^{-(\beta_1+r_{-1})} Q(1,-1) = 0.018851.
\end{aligned}
$$

A Numerical Example (concluded)

- The 3-year rate spot rate $r(0,3)$ is matched by picking

$$\begin{aligned}\beta_2 \;&=\; r(0,3)\times 3+\ln\Big[\,Q(2,2)\,e^{-2\times\Delta r}+Q(2,1)\,e^{-\Delta r}+Q(2,0)\\ &\quad +Q(2,-1)\,e^{\Delta r}+Q(2,-2)\,e^{2\times\Delta r}\Big]=6.25359\%.\end{aligned}$$

- Hence the short rates at nodes E, F, G, H, and I equal $\beta_2 + r_j$, where $j = 2, 1, 0, -1, -2$, respectively.
- They are found to be 9.71769%, 7.98564%, 6.25359%, 4.52154%, and 2.78949%.
- The figure on p. 1230 plots β_i for $i = 0, 1, \ldots, 29$.



β_i (%)
9
8
7
6
5
5
10
15
20
25
30
Year (i)