Issues in Calibration

- The model prices generated by the binomial interest rate tree should match the observed market prices.
- Perhaps the most crucial aspect of model building.
- Treat the backward induction for the model price of the \( m \)-period zero-coupon bond as computing some function \( f(r_m) \) of the unknown baseline rate \( r_m \) for period \( m \).
- A root-finding method is applied to solve \( f(r_m) = P \) for \( r_m \) given the zero’s price \( P \) and \( r_1, r_2, \ldots, r_{m-1} \).
- This procedure is carried out for \( m = 1, 2, \ldots, n \).
- It runs in \( O(n^3) \) time.
Binomial Interest Rate Tree Calibration

- Calibration can be accomplished in $O(n^2)$ time by the use of forward induction.\(^a\)

- The scheme records how much $1 at a node contributes to the model price.

- This number is called the state price.\(^b\)
  - It is the price of a state contingent claim that pays $1 at that particular node (state) and 0 elsewhere.

- The column of state prices will be established by moving forward from time 0 to time $n$.

\(^b\)Recall p. 213. Alternative names are the Arrow-Debreu price and Green’s function.
Binomial Interest Rate Tree Calibration (continued)

- Suppose we are at time $j$ and there are $j + 1$ nodes.
  - $P_1, P_2, \ldots, P_j$ are the known state prices at the earlier time $j - 1$.
  - The unknown baseline rate for period $j$ is $r \Delta = r_j$.
  - The known multiplicative ratio is $v \Delta = v_j$.
  - The rates for period $j$ are thus $r, rv, \ldots, rv^{j-1}$.

- By definition, $\sum_{i=1}^{j} P_i$ is the price of the $(j - 1)$-period zero-coupon bond.

- We want to find $r$ based on $P_1, P_2, \ldots, P_j$ and the price of the $j$-period zero-coupon bond.

\[\text{a Recall p. 1031, repeated on next page with } j = 3.\]
Binomial Interest Rate Tree Calibration (continued)
Binomial Interest Rate Tree Calibration (continued)

• One dollar at time $j$ has a known market value of $1/[1 + S(j)]^j$, where $S(j)$ is the $j$-period spot rate.

• Alternatively, this dollar has a present value of

$$g(r) \triangleq \frac{P_1}{(1 + r)} + \frac{P_2}{(1 + rv)} + \frac{P_3}{(1 + rv^2)} + \cdots + \frac{P_j}{(1 + rv^{j-1})}$$

(see the next plot).

• So we solve

$$g(r) = \frac{1}{[1 + S(j)]^j} \quad (141)$$

for $r$. 
$P_i \rightarrow rv^{i-1}$
Binomial Interest Rate Tree Calibration (continued)

- Given a decreasing market discount function, a unique positive real-number solution for $r$ is guaranteed.
- The state prices at time $j$ can now be calculated (see panel (a) of the next page with $j = 2$).
- We call a tree with these state prices a binomial state price tree (see panel (b) of the next page).
- The calibrated tree is depicted on p. 1053.
(a) 

\[ r \]

\[ P_1 \]

\[ \frac{P_1}{2(1+r)} \]

\[ P_2 \]

\[ \frac{P_2}{2(1+rv)} \]

\[ \frac{P_1}{2(1+r)} + \frac{P_2}{2(1+rv)} \]

(b) 

**Implied forward rates:**

<table>
<thead>
<tr>
<th>Period 1</th>
<th>Period 2</th>
<th>Period 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0%</td>
<td>4.4%</td>
<td>4.5%</td>
</tr>
</tbody>
</table>

\[ 4.0\% \]

\[ 4.4\% \]

\[ 4.5\% \]

\[ 0.112832 \]

\[ 0.333501 \]

\[ 0.107173 \]
Implied forward rates: 4.0% 4.4% 4.5%

period 1  period 2  period 3
Binomial Interest Rate Tree Calibration (concluded)

- Use the Newton-Raphson method to solve for the $r$ in Eq. (141) on p. 1049 as $g'(r)$ is easy to evaluate.

- The monotonicity and the convexity of $g(r)$ facilitates root finding.

- The total running time is $O(n^2)$ as each root-finding routine consumes $O(j)$ time.

- With a good initial guess, the Newton-Raphson method converges in only a few steps.\(^a\)

\(^a\)Such as $r_j = (\frac{2}{1+v_j})^{j-1} f_j$ on p. 1041.

\(^b\)Lyuu (1999).
A Numerical Example

- One dollar at the end of the second period should have a present value of 0.92101 by the sample term structure.

- The baseline rate for the second period, \( r_2 \), satisfies
  \[
  \frac{0.480769}{1 + r_2} + \frac{0.480769}{1 + 1.5 \times r_2} = 0.92101. 
  \]

- The result is \( r_2 = 3.526\% \).

- This is used to derive the next column of state prices shown in panel (b) on p. 1052 as 0.232197, 0.460505, and 0.228308.

- Their sum matches the market discount factor 0.92101.
A Numerical Example (concluded)

• The baseline rate for the third period, $r_3$, satisfies

$$\frac{0.232197}{1 + r_3} + \frac{0.460505}{1 + 1.5 \times r_3} + \frac{0.228308}{1 + (1.5)^2 \times r_3} = 0.88135.$$  

• The result is $r_3 = 2.895\%$.

• Now, redo the calculation on p. 1042 using the new rates:

$$\frac{1}{4} \times \frac{1}{1.04} \times \left[ \frac{1}{1.03526} \times \left( \frac{1}{1.02895} + \frac{1}{1.04343} \right) + \frac{1}{1.05289} \times \left( \frac{1}{1.04343} + \frac{1}{1.06514} \right) \right],$$

which equals $0.88135$, an exact match.

• The tree on p. 1053 prices without bias the benchmark securities.
Spread of Nonbenchmark Bonds

- Model prices by the calibrated tree seldom match the market prices of nonbenchmark bonds.
- The incremental return over the benchmark bonds is called spread.
- If we add the spread uniformly over the short rates in the tree, the model price will equal the market price.
- We will apply the spread concept to option-free bonds next.
Spread of Nonbenchmark Bonds (continued)

- We illustrate the idea with an example.
- Start with the tree on p. 1059.
- Consider a security with cash flow $C_i$ at time $i$ for $i = 1, 2, 3$.
- Its model price is $p(s)$, which is equal to

$$
\frac{1}{1.04 + s} \times \left[ C_1 + \frac{1}{2} \times \frac{1}{1.03526 + s} \times \left( C_2 + \frac{1}{2} \left( \frac{C_3}{1.02895 + s} + \frac{C_3}{1.04343 + s} \right) \right) \right] + \frac{1}{2} \times \frac{1}{1.05289 + s} \times \left( C_2 + \frac{1}{2} \left( \frac{C_3}{1.04343 + s} + \frac{C_3}{1.06514 + s} \right) \right].
$$

- Given a market price of $P$, the spread is the $s$ that solves $P = p(s)$.
Implied forward rates: 4.0%  4.4%  4.5%

period 1  period 2  period 3
Spread of Nonbenchmark Bonds (continued)

- The model price $p(s)$ is a monotonically decreasing, convex function of $s$.
- Employ any root-finding method to solve
  
  $$p(s) - P = 0$$

  for $s$.
- But a quick look at the equation for $p(s)$ reveals that evaluating $p'(s)$ directly is infeasible.
- Fortunately, the tree can be used to evaluate both $p(s)$ and $p'(s)$ during backward induction.
Spread of Nonbenchmark Bonds (continued)

- Consider an arbitrary node A in the tree associated with the short rate $r$.

- While computing the model price $p(s)$, a price $p_A(s)$ is computed at A.

- Prices computed at A’s two successor nodes B and C are discounted by $r + s$ to obtain $p_A(s)$ as follows,

$$p_A(s) = c + \frac{p_B(s) + p_C(s)}{2(1 + r + s)},$$

where $c$ denotes the cash flow at A.
Spread of Nonbenchmark Bonds (continued)

• To compute $p'_A(s)$ as well, node A calculates

$$p'_A(s) = \frac{p'_B(s) + p'_C(s)}{2(1 + r + s)} - \frac{p_B(s) + p_C(s)}{2(1 + r + s)^2}. \quad (142)$$

• This is easy if $p'_B(s)$ and $p'_C(s)$ are also computed at nodes B and C.

• When A is a terminal node, simply use the payoff function for $p_A(s)$.$^a$

---

$^a$Contributed by Mr. Chou, Ming-Hsin (R02723073) on May 28, 2014.
\[
p_A(s) = c + \frac{p_B(s) + p_C(s)}{2(1+r+s)}
\]

\[
p_A'(s) = \frac{p_B'(s) + p_C'(s)}{2(1+r+s)} - \frac{p_B(s) + p_C(s)}{2(1+r+s)^2}
\]
Spread of Nonbenchmark Bonds (continued)

- Apply the above procedure inductively to yield \( p(s) \) and \( p'(s) \) at the root (p. 1063).

- This is called the differential tree method.a
  
  - Similar ideas can be found in automatic differentiationb (AD) and backpropagationc in artificial neural networks.

- The total running time is \( O(n^2) \).

- The memory requirement is \( O(n) \).

---

bRall (1981).
cWerbos (1974); Rumelhart, Hinton, & Williams (1986).
## Spread of Nonbenchmark Bonds (continued)

<table>
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<th>Running time (s)</th>
<th>Number of iterations</th>
<th>Number of partitions</th>
<th>Running time (s)</th>
<th>Number of iterations</th>
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</tbody>
</table>

75MHz Sun SPARCstation 20.
Spread of Nonbenchmark Bonds (concluded)

- Consider a three-year, 5% bond with a market price of 100.569.
- Assume the bond pays annual interest.
- The spread is 50 basis points over the tree.\(^a\)
- Note that the idea of spread does not assume parallel shifts in the term structure.
- It also differs from the yield spread (p. 134) and static spread (p. 135) of the nonbenchmark bond over an otherwise identical benchmark bond.

\(^a\)See plot on the next page.
Cash flows:  

A: 5  
B: 5  
C: 105  
D: 105  

A: 4.50% 100.569  
B: 4.026% 106.754  
C: 5.789% 103.436  
D: 7.014% 103.118  

C: 3.395% 106.552  
D: 4.843% 105.150
More Applications of the Differential Tree: Calculating Implied Volatility (in seconds)$^a$

<table>
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<th>Number of partitions</th>
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<th>Number of iterations</th>
<th>Number of partitions</th>
<th>Running time</th>
<th>Number of iterations</th>
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<td>800</td>
<td>0.569605</td>
<td>2</td>
</tr>
</tbody>
</table>

Intel 166MHz Pentium, running on Microsoft Windows 95.

Fixed-Income Options

• Consider a 2-year 99 European call on the 3-year, 5% Treasury.

• Assume the Treasury pays annual interest.

• On p. 1070 the 3-year Treasury’s price minus the $5 interest at year 2 are $102.046, $100.630, and $98.579.
  – The accrued interest is not included as it belongs to the bond seller.

• Now compare the strike price against the bond prices.

• The call is in the money in the first two scenarios out of the money in the third.
Fixed-Income Options (continued)

- The option value is calculated to be $1.458 on p. 1070(a).

- European interest rate puts can be valued similarly.

- Consider a two-year 99 European put on the same security.

- At expiration, the put is in the money only when the Treasury is worth $98.579.

- The option value is computed to be $0.096 on p. 1070(b).
Fixed-Income Options (concluded)

- The present value of the strike price is 
  \[ PV(X) = 99 \times 0.92101 = 91.18 \].
- The Treasury is worth \( B = 101.955 \).
- The present value of the interest payments during the life of the options is\(^a\)
  \[ PV(I) = 5 \times 0.96154 + 5 \times 0.92101 = 9.41275 \].
- The call and the put are worth \( C = 1.458 \) and \( P = 0.096 \), respectively.
- The put-call parity is preserved:
  \[ C = P + B - PV(I) - PV(X). \]

\(^a\)There is no coupon today.
Delta or Hedge Ratio

• How much does the option price change in response to changes in the price of the underlying bond?

• This relation is called delta (or hedge ratio), defined as

\[
\frac{O_h - O_\ell}{P_h - P_\ell}.
\]

• In the above \( P_h \) and \( P_\ell \) denote the bond prices if the short rate moves up and down, respectively.

• Similarly, \( O_h \) and \( O_\ell \) denote the option values if the short rate moves up and down, respectively.
Delta or Hedge Ratio (concluded)

- Delta measures the sensitivity of the option value to changes in the underlying bond price.
- So it shows how to hedge one with the other.
- Take the call and put on p. 1070 as examples.
- Their deltas are

\[
\frac{0.774 - 2.258}{99.350 - 102.716} = 0.441, \\
\frac{0.200 - 0.000}{99.350 - 102.716} = -0.059,
\]

respectively.
Volatility Term Structures

• The binomial interest rate tree can be used to calculate the yield volatility of zero-coupon bonds.

• Consider an $n$-period zero-coupon bond.

• First find its yield to maturity $y_h$ ($y_{\ell}$, respectively) at the end of the initial period if the short rate rises (declines, respectively).

• The yield volatility for our model is defined as

$$\frac{1}{2} \ln \left( \frac{y_h}{y_{\ell}} \right).$$

(143)
Volatility Term Structures (continued)

- For example, take the tree on p. 1053 (repeated on next page).

- The two-year zero’s yield at the end of the first period is 5.289% if the rate rises and 3.526% if the rate declines.

- Its yield volatility is therefore

\[
\frac{1}{2} \ln \left( \frac{0.05289}{0.03526} \right) = 20.273\%.
\]
Volatility Term Structures (continued)

Implied forward rates: 4.0%  4.4%  4.5%

period 1  period 2  period 3
Volatility Term Structures (continued)

- Consider the three-year zero-coupon bond.

- If the short rate rises, the price of the zero one year from now will be
  \[
  \frac{1}{2} \times \frac{1}{1.05289} \times \left( \frac{1}{1.04343} + \frac{1}{1.06514} \right) = 0.90096.
  \]

- Thus its yield is \( \sqrt{\frac{1}{0.90096}} - 1 = 0.053531 \).

- If the short rate declines, the price of the zero one year from now will be
  \[
  \frac{1}{2} \times \frac{1}{1.03526} \times \left( \frac{1}{1.02895} + \frac{1}{1.04343} \right) = 0.93225.
  \]
Volatility Term Structures (continued)

- Thus its yield is \( \sqrt{\frac{1}{0.93225}} - 1 = 0.0357 \).
- The yield volatility is hence
  \[
  \frac{1}{2} \ln \left( \frac{0.053531}{0.0357} \right) = 20.256\%,
  \]
  slightly less than the one-year yield volatility.
- This is consistent with the reality that longer-term bonds typically have lower yield volatilities than shorter-term bonds.\(^a\)
- The procedure can be repeated for longer-term zeros to obtain their yield volatilities.

\(^a\)The relation is reversed for price volatilities (duration).
(Short rate volatility given a flat %10 volatility structure.)
Volatility Term Structures (concluded)

- We started with $v_i$ and then derived the volatility term structure.
- In practice, the steps are reversed.
- The volatility term structure is supplied by the user along with the term structure.
- The $v_i$—hence the short rate volatilities via Eq. (138) on p. 1030—and the $r_i$ are then simultaneously determined.
- The result is the Black-Derman-Toy (1990) model of Goldman Sachs.
Foundations of Term Structure Modeling
[Meriwether] scoring especially high marks in mathematics — an indispensable subject for a bond trader.

— Roger Lowenstein, 

*When Genius Failed* (2000)
[The] fixed-income traders I knew seemed smarter than the equity trader [...] there’s no competitive edge to being smart in the equities business.[.]


Bond market terminology was designed less to convey meaning than to bewilder outsiders.

Terminology

• A period denotes a unit of elapsed time.
  
  – Viewed at time $t$, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.

• Bonds will be assumed to have a par value of one — unless stated otherwise.

• The time unit for continuous-time models will usually be measured by the year.
Standard Notations

The following notation will be used throughout.

\( t \): a point in time.

\( r(t) \): the one-period riskless rate prevailing at time \( t \) for repayment one period later.\(^a\)

\( P(t, T) \): the present value at time \( t \) of one dollar at time \( T \).

\(^a\)Alternatively, the instantaneous spot rate, or short rate, at time \( t \).
Standard Notations (continued)

\( r(t, T) \): the \((T - t)\)-period interest rate prevailing at time \( t \) stated on a per-period basis and compounded once per period.\(^a\)

\( F(t, T, M) \): the forward price at time \( t \) of a forward contract that delivers at time \( T \) a zero-coupon bond maturing at time \( M \geq T \).

\(^a\)In other words, the \((T - t)\)-period spot rate at time \( t \).
Standard Notations (concluded)

$f(t, T, L)$: the $L$-period forward rate at time $T$ implied at time $t$ stated on a per-period basis and compounded once per period.

$f(t, T)$: the one-period or instantaneous forward rate at time $T$ as seen at time $t$ stated on a per period basis and compounded once per period.

- It is $f(t, T, 1)$ in the discrete-time model and $f(t, T, dt)$ in the continuous-time model.
- Note that $f(t, t)$ equals the short rate $r(t)$. 
Fundamental Relations

• The price of a zero-coupon bond equals

\[
P(t, T) = \begin{cases} 
(1 + r(t, T))^{-(T-t)}, & \text{in discrete time}, \\
e^{-r(t,T)(T-t)}, & \text{in continuous time}.
\end{cases}
\]  

\hspace{1cm} (144)

• \( r(t, T) \) as a function of \( T \) defines the spot rate curve at time \( t \).

• By definition,

\[
f(t, t) = \begin{cases} 
r(t, t + 1), & \text{in discrete time}, \\
r(t, t), & \text{in continuous time}.
\end{cases}
\]
Fundamental Relations (continued)

• Forward prices and zero-coupon bond prices are related:

\[ F(t, T, M) = \frac{P(t, M)}{P(t, T)}, \quad T \leq M. \]  \hspace{1cm} (145)

  – The forward price equals the future value at time \( T \)
    of the underlying asset.\(^a\)

• The above identity holds for discrete-time and
  continuous-time models.

\(^a\)See Exercise 24.2.1 of the textbook for proof.
Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by
  \[ f(t, T, L) = \left( \frac{1}{F(t, T, T + L)} \right)^{1/L} - 1 = \left( \frac{P(t, T)}{P(t, T + L)} \right)^{1/L} - 1 \]
  \[ \text{(146)} \]
  in discrete time.

- The analog under simple compounding is
  \[ f(t, T, L) = \frac{1}{L} \left( \frac{P(t, T)}{P(t, T + L) - 1} \right). \]
Fundamental Relations (continued)

• In continuous time,

\[
f(t, T, L) = -\frac{\ln F(t, T, T + L)}{L} = \frac{\ln(P(t, T)/P(t, T + L))}{L}
\]  (147)

by Eq. (145) on p. 1090.

• Furthermore,

\[
f(t, T, \Delta t) = \frac{\ln(P(t, T)/P(t, T + \Delta t))}{\Delta t} \rightarrow -\frac{\partial \ln P(t, T)}{\partial T}
\]

\[
= -\frac{\partial P(t, T)/\partial T}{P(t, T)}.
\]
Fundamental Relations (continued)

• So

\[ f(t, T) \triangleq - \frac{\partial \ln P(t, T)}{\partial T} = - \frac{\partial P(t, T)/\partial T}{P(t, T)}, \quad t \leq T. \]  

(148)

• Because the above identity is equivalent to

\[ P(t, T) = e^{- \int_t^T f(t, s) \, ds}, \]  

(149)

the spot rate curve is

\[ r(t, T) = \frac{\int_t^T f(t, s) \, ds}{T - t}. \]
Fundamental Relations (concluded)

• The discrete analog to Eq. (149) is

\[
P(t, T) = \frac{1}{(1 + r(t))(1 + f(t, t + 1)) \cdots (1 + f(t, T - 1))}.
\]

• The short rate and the market discount function are related by

\[
r(t) = - \frac{\partial P(t, T)}{\partial T} \bigg|_{T=t}.
\]
Risk-Neutral Pricing

- Assume the local expectations theory.
- The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  - For all $t + 1 < T$,
    \[
    \frac{E_t[P(t+1,T)]}{P(t,T)} = 1 + r(t).
    \] (150)
  - Relation (150) in fact follows from the risk-neutral valuation principle.\(^a\)

\(^a\)Recall Theorem 17 on p. 567.
Risk-Neutral Pricing (continued)

• The local expectations theory is thus a consequence of the existence of a risk-neutral probability $\pi$.

• Equation (150) on p. 1095 can also be expressed as

$$E_t[P(t+1,T)] = F(t,t+1,T).$$

– Verify that with, e.g., Eq. (145) on p. 1090.

• Hence the forward price for the next period is an unbiased estimator of the expected bond price.$^a$

– But the forward rate is not an unbiased estimator of the expected future short rate.$^b$

---

$^a$Under the local expectations theory.

$^b$Recall p. 1044.
Risk-Neutral Pricing (continued)

- Rewrite Eq. (150) on p. 1095 as

\[ E_t^\pi \left[ \frac{P(t + 1, T)}{1 + r(t)} \right] = P(t, T). \]  \hspace{1cm} (151)

- It says the current market discount function equals the expected market discount function one period from now discounted by the short rate.
Risk-Neutral Pricing (concluded)

• Apply the above equality iteratively to obtain

\[
P(t, T) = E_t^\pi \left[ \frac{P(t+1, T)}{1 + r(t)} \right] \\
= E_t^\pi \left[ E_{t+1}^\pi \left[ \frac{P(t+2, T)}{(1 + r(t))(1 + r(t+1))} \right] \right] = \ldots \\
= E_t^\pi \left[ \frac{1}{(1 + r(t))(1 + r(t+1)) \cdots (1 + r(T-1))} \right].
\]
Continuous-Time Risk-Neutral Pricing

• In continuous time, the local expectations theory implies

\[ P(t, T) = E_t \left[ e^{-\int_t^T r(s) \, ds} \right], \quad t < T. \quad (152) \]

• Note that \( e^{\int_t^T r(s) \, ds} \) is the bank account process, which denotes the rolled-over money market account.
Interest Rate Swaps

• Consider an interest rate swap made at time $t$ (now) with payments to be exchanged at times $t_1, t_2, \ldots, t_n$.

• For simplicity, assume $t_{i+1} - t_i$ is a fixed constant $\Delta t$ for all $i$, and the notional principal is one dollar.

• The fixed rate is $c$ per annum.

• The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.

• The payoff at time $t_{i+1}$ for the fixed-rate payer is $(f_i - c) \Delta t$. 
Interest Rate Swaps (continued)

\[
(f_0 - c) \Delta t
\]

\[
(f_1 - c) \Delta t
\]

\[
(f_{n-1} - c) \Delta t
\]

\[t_0 \quad t_1 \quad t_2 \quad t_n\]
Interest Rate Swaps (continued)

- Simple rates are adopted here.
- Hence \( f_i \) satisfies
  \[
P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.
  \]
- If \( t < t_0 \), we have a forward interest rate swap.
- The ordinary swap corresponds to \( t = t_0 \).
Interest Rate Swaps (continued)

- The value of the swap at time $t$ is thus

$$
\sum_{i=1}^{n} E_t^\pi \left[ e^{-\int_{t}^{t_i} r(s) \, ds} (f_{i-1} - c) \Delta t \right]
$$

$$
= \sum_{i=1}^{n} E_t^\pi \left[ e^{-\int_{t}^{t_i} r(s) \, ds} \left( \frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right) \right]
$$

$$
= \sum_{i=1}^{n} E_t^\pi \left[ e^{-\int_{t}^{t_i} r(s) \, ds} \left( e^{\int_{t_{i-1}}^{t_i} r(s) \, ds} - (1 + c\Delta t) \right) \right]
$$

$$
= \sum_{i=1}^{n} \left[ P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i) \right]
$$

$$
= P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).
$$
Interest Rate Swaps (concluded)

• So a swap can be replicated as a portfolio of bonds, statically.

• In fact, it can be priced by simple PV calculations.
Swap Rate

• The swap rate, which gives the swap zero value, equals

\[ S_n(t) \triangleq \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} P(t, t_i) \Delta t}. \] (153)

• The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.

• For an ordinary swap, \( P(t, t_0) = 1 \).

• The swap rate is called a forward swap rate if \( t_0 > t \).
The Term Structure Equation\textsuperscript{a}

- Let us start with the zero-coupon bonds and the money market account.

- Let the zero-coupon bond price $P(r, t, T)$ follow

$$\frac{dP}{P} = \mu_p \, dt + \sigma_p \, dW.$$ 

- At time $t$, short one unit of a bond maturing at time $s_1$ and buy $\alpha$ units of a bond maturing at time $s_2$.

\textsuperscript{a}Vasicek (1977). Vasicek co-founded KMV, which was sold to Moody’s for USD$210 million in 2002.
The Term Structure Equation (continued)

• The net wealth change follows

\[-dP(r, t, s_1) + \alpha dP(r, t, s_2)\]
\[= (-P(r, t, s_1) \mu_p(r, t, s_1) + \alpha P(r, t, s_2) \mu_p(r, t, s_2)) \, dt\]
\[+ (-P(r, t, s_1) \sigma_p(r, t, s_1) + \alpha P(r, t, s_2) \sigma_p(r, t, s_2)) \, dW.\]

• Pick

\[\alpha \triangleq \frac{P(r, t, s_1) \sigma_p(r, t, s_1)}{P(r, t, s_2) \sigma_p(r, t, s_2)}. \tag{154}\]
The Term Structure Equation (continued)

• Then the net wealth has no volatility and must earn the riskless return:

\[
- P(r, t, s_1) \mu_p(r, t, s_1) + \alpha P(r, t, s_2) \mu_p(r, t, s_2) = r.
\]

• Simplify the above with formula (154) to obtain

\[
\frac{\sigma_p(r, t, s_1) \mu_p(r, t, s_2) - \sigma_p(r, t, s_2) \mu_p(r, t, s_1)}{\sigma_p(r, t, s_1) - \sigma_p(r, t, s_2)} = r.
\]

• This becomes

\[
\frac{\mu_p(r, t, s_2) - r}{\sigma_p(r, t, s_2)} = \frac{\mu_p(r, t, s_1) - r}{\sigma_p(r, t, s_1)}
\]

after rearrangement.
The Term Structure Equation (continued)

- Since the above equality holds for any $s_1$ and $s_2$,

$$\frac{\mu_p(r, t, s) - r}{\sigma_p(r, t, s)} \triangleq \lambda(r, t)$$  \hspace{1cm} (155)

for some $\lambda$ independent of the bond maturity $s$.

- As $\mu_p = r + \lambda \sigma_p$, all assets are expected to appreciate at a rate equal to the sum of the short rate and a constant times the asset’s volatility.

- The term $\lambda(r, t)$ is called the market price of risk.

- The market price of risk must be the same for all bonds to preclude arbitrage opportunities.
The Term Structure Equation (continued)

• Assume a Markovian short rate model,

\[ dr = \mu(r, t) \, dt + \sigma(r, t) \, dW. \]

• Then the bond price process is also Markovian.

• By Eq. (14.15) on p. 202 of the textbook,

\[ \mu_p = \left[ -\frac{\partial P}{\partial T} + \mu(r, t) \frac{\partial P}{\partial r} + \frac{\sigma(r, t)^2}{2} \frac{\partial^2 P}{\partial r^2} \right]/P, \quad (156) \]

\[ \sigma_p = \sigma(r, t) \frac{\partial P}{\partial r}/P, \quad (156') \]

subject to \( P(\cdot, T, T) = 1. \)
The Term Structure Equation (concluded)

• Substitute $\mu_p$ and $\sigma_p$ into Eq. (155) on p. 1109 to obtain

$$\frac{-\partial P}{\partial T} + \left[ \mu(r,t) - \lambda(r,t) \sigma(r,t) \right] \frac{\partial P}{\partial r} + \frac{1}{2} \sigma(r,t)^2 \frac{\partial^2 P}{\partial r^2} = rP. \quad (157)$$

• This is called the term structure equation.

• It applies to all interest rate derivatives: The differences are the terminal and boundary conditions.

• Once $P$ is available, the spot rate curve emerges via

$$r(t,T) = -\frac{\ln P(t,T)}{T - t}.$$
Numerical Examples

- Assume this spot rate curve:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot rate</td>
<td>4%</td>
<td>5%</td>
</tr>
</tbody>
</table>

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:

4% ← 8% → 2%
Numerical Examples (continued)

- *No* real-world probabilities are given.
- The prices of one- and two-year zero-coupon bonds are, respectively,
  
  \[
  \frac{100}{1.04} = 96.154,
  \]
  
  \[
  \frac{100}{(1.05)^2} = 90.703.
  \]
- They follow the binomial processes on p. 1114.
Numerical Examples (continued)

90.703 \quad 92.593 \quad (= 100/1.08) \quad 96.154

98.039 \quad (= 100/1.02) \quad 100

The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.
Numerical Examples (continued)

- The pricing of derivatives can be simplified by assuming investors are risk-neutral.
- Suppose all securities have the same expected one-period rate of return, the riskless rate.
- Then

\[(1 - p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\% ,\]

where \(p\) denotes the risk-neutral probability of a down move in rates.
Numerical Examples (concluded)

- Solving the equation leads to $p = 0.319$.
- Interest rate contingent claims can be priced under this probability.
Numerical Examples: Fixed-Income Options

- A one-year European call on the two-year zero with a $95 strike price has the payoffs,
  \[ C \left\{ \begin{array}{c} 0.000 \\ 3.039 \; (= \; 98.039 \; - \; 95) \end{array} \right. \]

- To solve for the option value \( C \), we replicate the call by a portfolio of \( x \) one-year and \( y \) two-year zeros.
Numerical Examples: Fixed-Income Options (continued)

- This leads to the simultaneous equations,

\[ x \times 100 + y \times 92.593 = 0.000, \]
\[ x \times 100 + y \times 98.039 = 3.039. \]

- They give \( x = -0.5167 \) and \( y = 0.5580 \).

- Consequently,

\[ C = x \times 96.154 + y \times 90.703 \approx 0.93 \]

to prevent arbitrage.
Numerical Examples: Fixed-Income Options (continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.
Numerical Examples: Fixed-Income Options (concluded)

- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth

\[
C = \frac{(1 - p) \times 0 + p \times 3.039}{1.04} \approx 0.93, \]

the same as before.

- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.