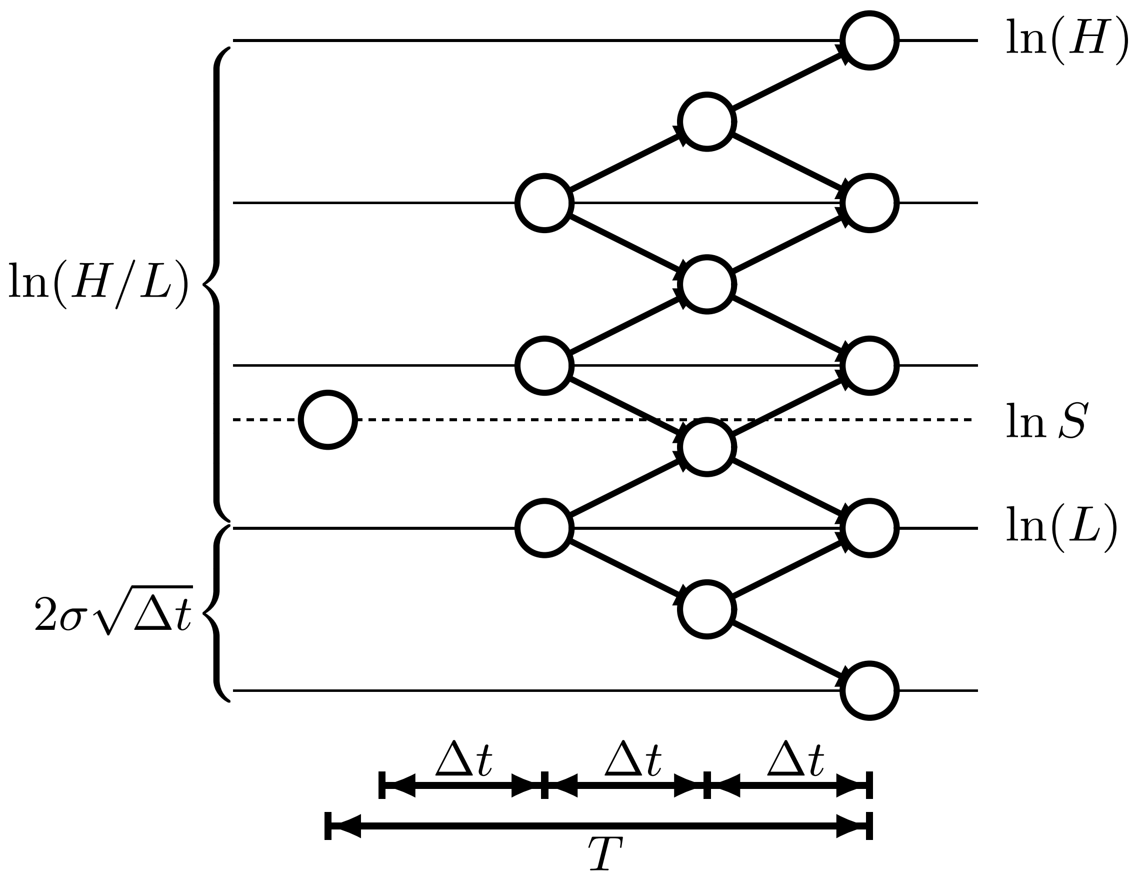
Ideas for Binomial Trees To Handle Two Barriers



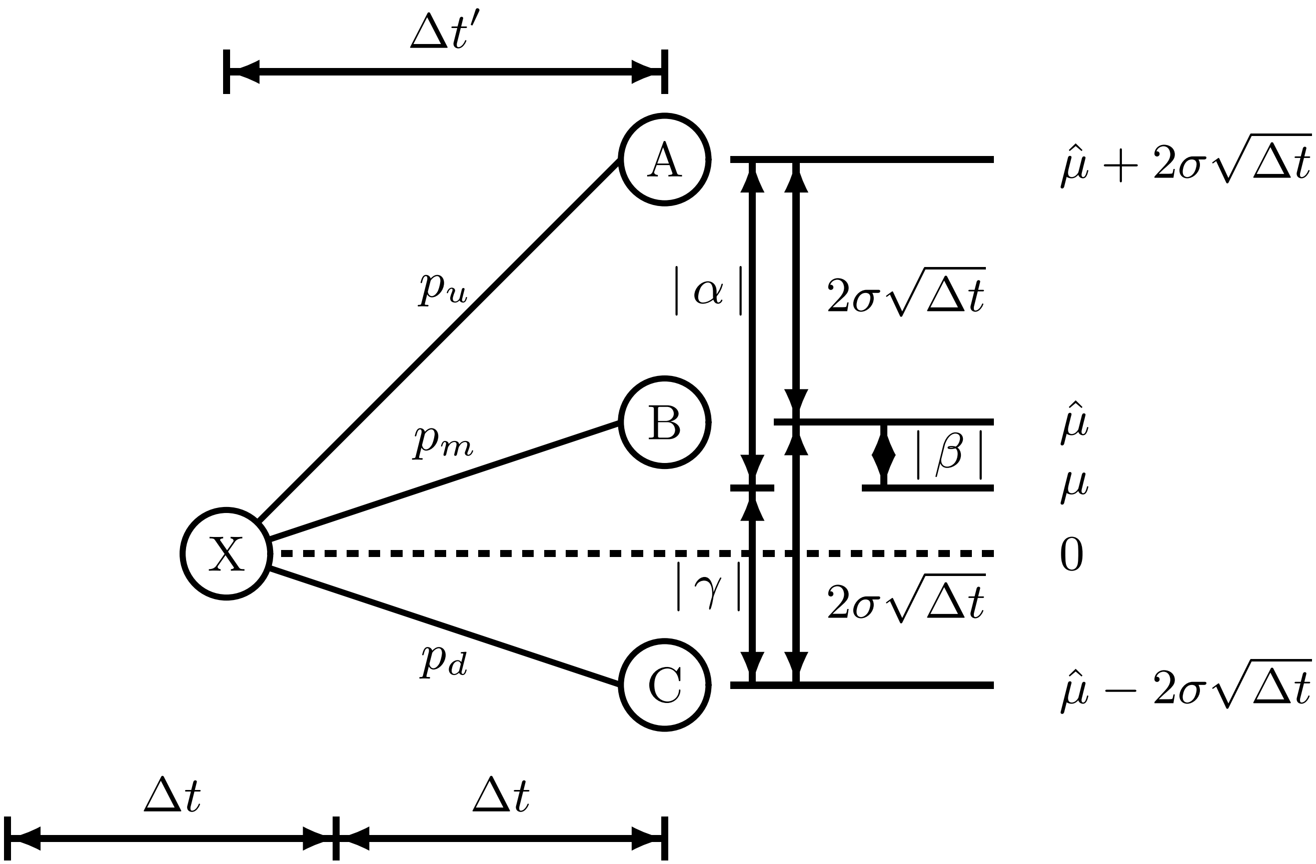
The Binomial-Trinomial Tree

- Replace the first step of the binomial tree with a trinomial structure for convergence and efficiency.[a]
- The resulting tree is called the binomial-trinomial tree.[b]
- Suppose a *binomial* tree will be built with Δt as the duration of one period.
- Node X at time t needs to pick three nodes on the binomial tree at time $t + \Delta t'$ as its successor nodes.
  - Later, $\Delta t \leq \Delta t' < 2\Delta t$.

[a]T. Dai (B82506025, R86526008, D8852600) & Lyuu (2006, 2008, 2010).

[b]The idea first emerged in a hotel in Muroran, Hokkaido, Japan, in May of 2005.

The Binomial-Trinomial Tree (continued)



The Binomial-Trinomial Tree (continued)

- These three nodes should guarantee:
  1. The mean and variance of the stock price are matched.
  2. The branching probabilities are between 0 and 1.
- Let S be the stock price at node X.
- Use $s(z)$ to denote the stock price at node z.

The Binomial-Trinomial Tree (continued)

- Recall that the expected value of the logarithmic return $\ln(S_{t+\Delta t'}/S)$ at time $t + \Delta t'$ equals[a]

$$\mu \stackrel{\Delta}{=} \left(r - \frac{\sigma^2}{2}\right) \Delta t'. \tag{103}$$

- Its variance equals

$$\text{Var} \stackrel{\Delta}{=} \sigma^2 \Delta t'. \tag{104}$$

- Let node B be the node whose logarithmic return $\hat{\mu} \stackrel{\Delta}{=} \ln(s(\text{B})/S)$ is closest to μ among all the nodes at time $t + \Delta t'$.

---

[a]Recall p. 304.

The Binomial-Trinomial Tree (continued)

- The middle branch from node X will end at node B.
- The two nodes A and C, which bracket node B, are the destinations of the other two branches from node X.
- Recall that adjacent nodes on the binomial tree are spaced at $2\sigma\sqrt{\Delta t}$ apart.
- Review the illustration on p. 765.

The Binomial-Trinomial Tree (continued)

- The three branching probabilities from node X are obtained through matching the mean and variance of the logarithmic return $\ln(S_{t+\Delta t'}/S)$.
- Recall that

$$\hat{\mu} \stackrel{\Delta}{=} \ln\left(s(B)/S\right)$$

is the logarithmic return of the middle node B.

- Let α, β, and γ be the differences between μ and the three logarithmic returns

$$\ln(s(\mathrm{A})/S), \ln(s(\mathrm{B})/S), \ln(s(\mathrm{C})/S),$$

in that order.

The Binomial-Trinomial Tree (continued)

- In other words,

$$\alpha \stackrel{\Delta}{=} \hat{\mu} + 2\sigma\sqrt{\Delta t} - \mu = \beta + 2\sigma\sqrt{\Delta t}\,, \tag{105}$$

$$\beta \stackrel{\Delta}{=} \hat{\mu} - \mu, \tag{106}$$

$$\gamma \stackrel{\Delta}{=} \hat{\mu} - 2\sigma\sqrt{\Delta t} - \mu = \beta - 2\sigma\sqrt{\Delta t}\,. \tag{107}$$

- The three branching probabilities p_u, p_m, p_d then satisfy

$$p_u\alpha + p_m\beta + p_d\gamma = 0, \tag{108}$$

$$p_u\alpha^2 + p_m\beta^2 + p_d\gamma^2 = \text{Var}, \tag{109}$$

$$p_u + p_m + p_d = 1. \tag{110}$$

The Binomial-Trinomial Tree (concluded)

- Equation (108) matches the mean (103) of the logarithmic return $\ln(S_{t+\Delta t'}/S)$ on p. 767.
- Equation (109) matches its variance (104) on p. 767.
- The three probabilities can be proved to lie between 0 and 1 by Cramer's rule.

Pricing Double-Barrier Options

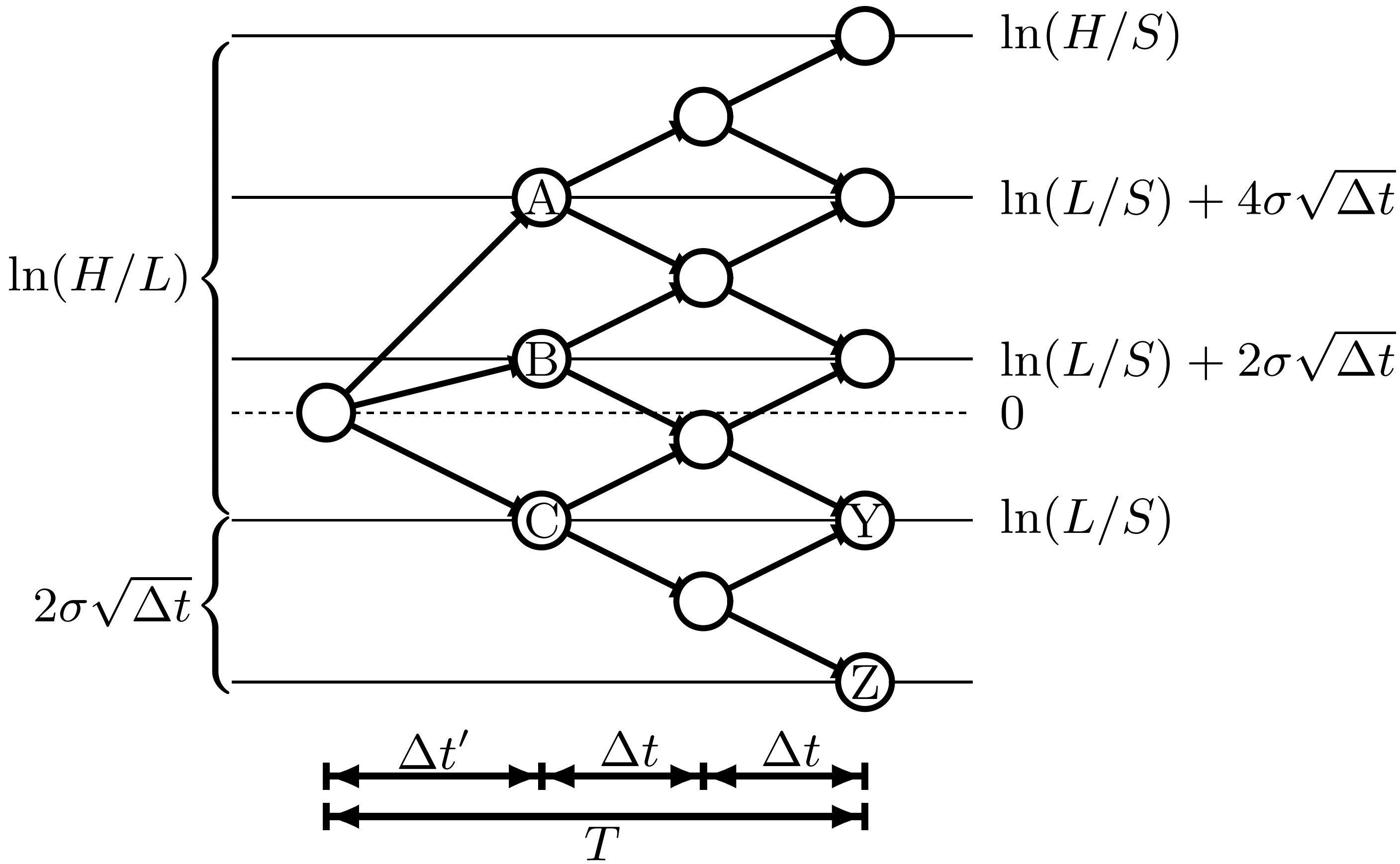
- Consider a double-barrier option with two barriers L and H, where $L < S < H$.
- We need to make each barrier coincide with a layer of the binomial tree for better convergence.
- The idea is to choose a Δt such that

$$\frac{\ln(H/L)}{2\sigma\sqrt{\Delta t}} \tag{111}$$

is a positive integer.

  - The distance between two adjacent nodes such as nodes Y and Z in the figure on p. 773 is $2\sigma\sqrt{\Delta t}$.

Pricing Double-Barrier Options (continued)



Pricing Double-Barrier Options (continued)

- Suppose that the goal is a tree with $\sim m$ periods.
- Suppose we pick $\Delta\tau \triangleq T/m$ for the length of each period.
- There is no guarantee that $\frac{\ln(H/L)}{2\sigma\sqrt{\Delta\tau}}$ is an integer.
- Pick the largest $\Delta t \leq \Delta\tau$ which makes $\frac{\ln(H/L)}{2\sigma\sqrt{\Delta t}}$ some integer κ.
- In other words, we select

$$\Delta t = \left(\frac{\ln(H/L)}{2\kappa\sigma} \right)^2,$$

where $\kappa = \left\lceil \frac{\ln(H/L)}{2\sigma\sqrt{\Delta\tau}} \right\rceil$.

Pricing Double-Barrier Options (continued)

- We now proceed to build the binomial-trinomial tree.
- Start with the binomial part.
- Lay out the nodes from the low barrier L upward.
- Automatically, a layer coincides with the high barrier H.
- It is unlikely that Δt divides T.
- The initial stock price is also unlikely to be on a layer of the binomial tree.[a]

---

[a] Recall p. 773.

Pricing Double-Barrier Options (continued)

- The binomial-trinomial structure can address this problem as follows.
- Between time 0 and time T, the binomial tree spans $\lfloor T/\Delta t \rfloor - 1$ periods.
- Let the first period have a duration equal to

$$\Delta t' \triangleq T - \left(\left\lfloor \frac{T}{\Delta t} \right\rfloor - 1 \right) \Delta t.$$

- Then these $\lfloor T/\Delta t \rfloor$ periods span T years.
- It is easy to verify that $\Delta t \leq \Delta t' < 2\Delta t$.

Pricing Double-Barrier Options (continued)

- Start with the root node at time 0 and at a price with logarithmic return $\ln(S/S) = 0$.
- Find the three nodes on the binomial tree at time $\Delta t'$ as described earlier.
- Calculate the three branching probabilities to them.
- Grow the binomial tree from these three nodes until time T to obtain a binomial-trinomial tree with $\lfloor T/\Delta t \rfloor$ periods.
- Review the illustration on p. 773.

Pricing Double-Barrier Options (continued)

- Now the binomial-trinomial tree can be used to price double-barrier options by backward induction.
- That takes quadratic time.
- But a linear-time algorithm exists for double-barrier options on the *binomial* tree.[a]
- Apply that algorithm to price the double-barrier option's prices at the three nodes at time $\Delta t'$.
  - That is, nodes A, B, and C on p. 773.
- Then calculate their expected discounted value at the root node.

---

[a]See p. 241 of the textbook; Chao (`R86526053`) (1999); T. Dai (`B82506025`, `R86526008`, `D8852600`) & Lyuu (2008).

Pricing Double-Barrier Options (continued)

- The overall running time is only linear!
- Binomial trees have troubles pricing barrier options.[a]
- Even pit against the trinomial tree, the binomial-trinomial tree converges faster and smoother.[b]
- In fact, the binomial-trinomial tree has an error of $O(1/n)$ for single-barrier options.[c]
- Binomial trees with interpolation have an error of $O(1/n^{1-a})$ for any $0 < a < 1$ for double-barrier options.[d]

---

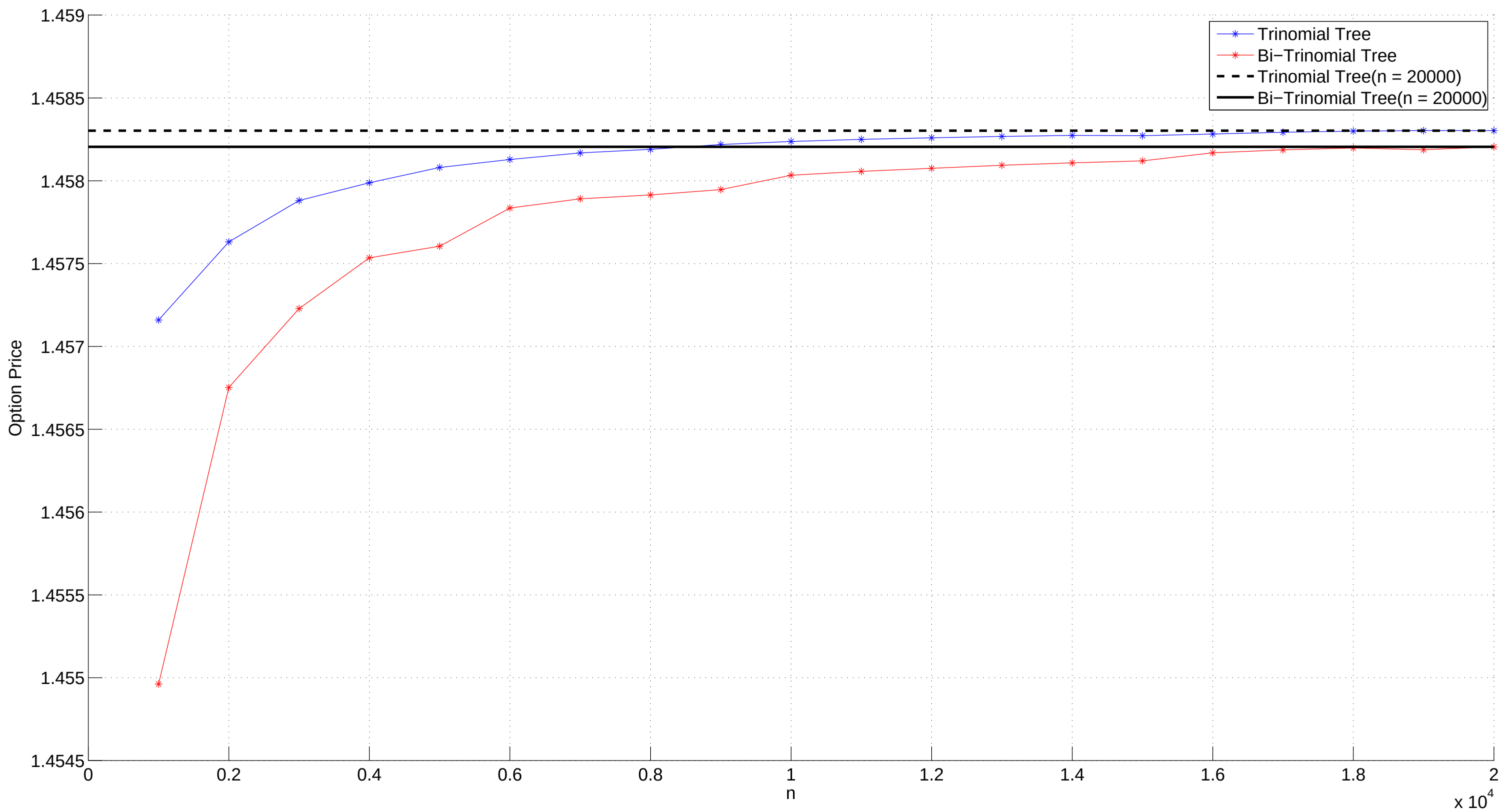
[a] See p. 410, p. 756, and p. 762.

[b] See p. 780 and p. 781.

[c] Lyuu & Palmer (2010); Y. Lu (R06723032, D08922008) & Lyuu (2023).

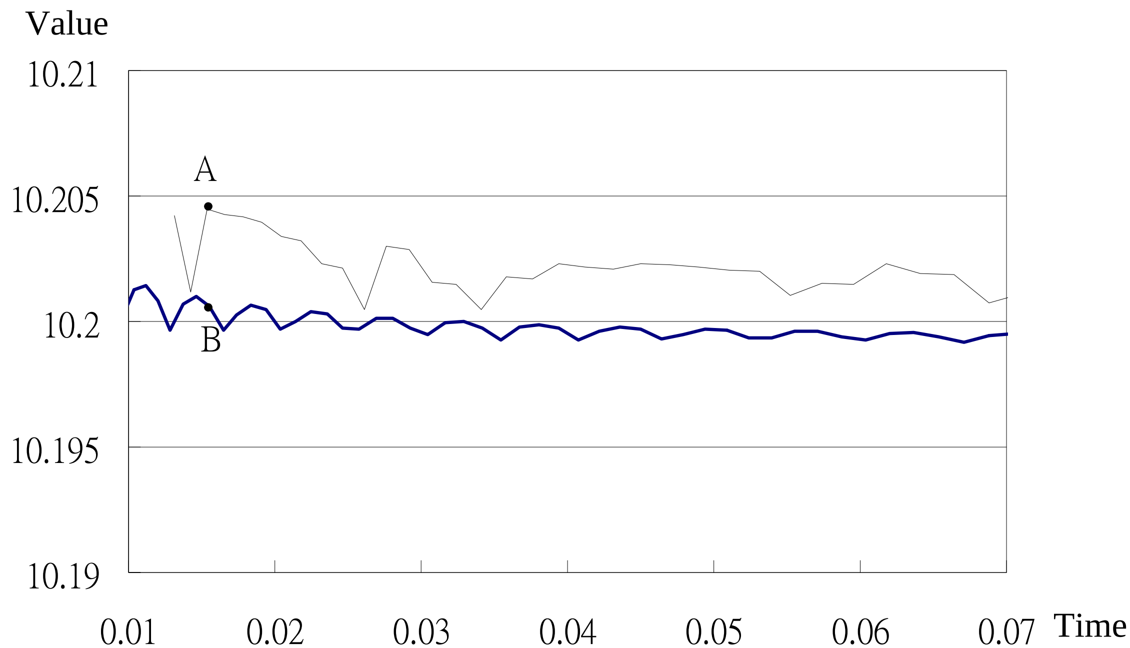
[d] Appolloni, Gaudenziy, & Zanette (2014).

Pricing Double-Barrier Options[a] (continued)



[a] Generated by Mr. Lin, Ying-Hung (`R01723029`) on June 6, 2014.

Pricing Double-Barrier Options (concluded)



The thin line denotes the double-barrier option prices computed by the trinomial tree against the running time in seconds (such as point A). The thick line denotes those computed by the binomial-trinomial tree (such as point B).

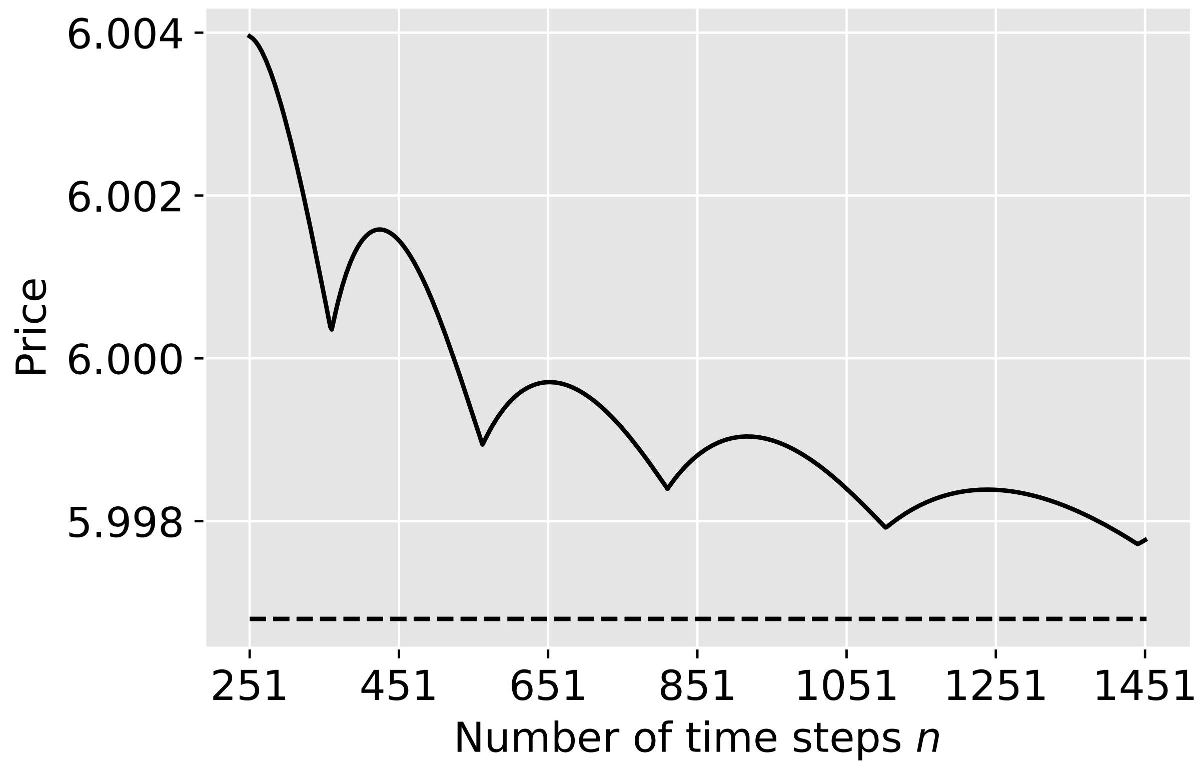
The Barrier-Too-Close Problem (p. 738) Revisited

- Our idea solves it even if one barrier is very close to S.
  - It runs in linear time, unlike an earlier quadratic-time solution with trinomial trees (pp. 746ff).
  - Unlike an earlier solution using combinatorics (p. 729), now the choice of n is not that restricted.
- So it combines the strengths of binomial and trinomial trees.
- This holds for single-barrier options too.
- Here is how.

The Barrier-Too-Close Problem Revisited (continued)

- We can build the tree treating S as if it were a second barrier.
- So both H and S are matched.
- Alternatively, we can pick $\Delta\tau \stackrel{\Delta}{=} T/m$ as our length of a period Δt without the subsequent adjustment.[a]
- Then build the tree from the price H down.
- So H is matched.
- The initial price S is matched by the *trinomial* structure.

[a] There is no second barrier to match!



Plot of convergence of a down-and-out call to 5.9968 supplied by Mr. Lu, Yu-Ming (R06723032, D08922008) on February 26. 2021.

The Barrier-Too-Close Problem Revisited (concluded)

- The earlier trinomial tree is impractical as it needs a very large n when the barrier H is very close to S.[a]
  - It needs at least one *up* move to connect S to H as its middle branch is flat.
  - But when $S \approx H$, that up move must take a very small step, necessitating a small Δt.
- Our trinomial structure's middle branch is *not* required to be flat.
- So S can be connected to H via the middle branch, and the need of a very large n disappears completely!

[a]Recall the table on p. 739.

Pricing Discrete and Moving Barrier Options

- Barrier options whose barrier is monitored only at discrete times are called discrete barrier options.
- They are less common than the continuously monitored versions for single stocks.[a]
- The main difficulty with pricing discrete barrier options lies in matching the monitored *times*.
- Here is why.

---

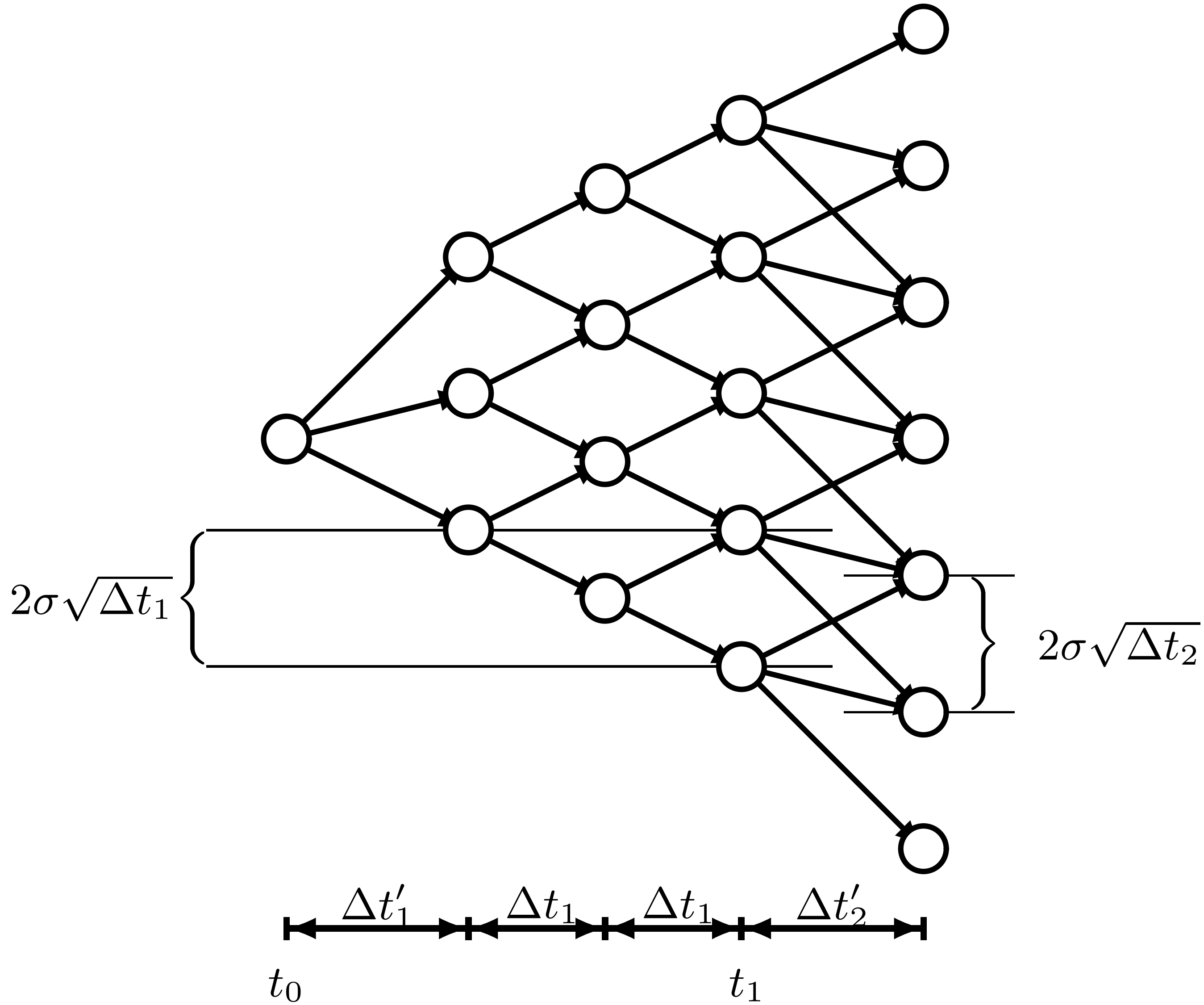
[a] Bennett (2014).

Pricing Discrete and Moving Barrier Options (continued)

- Suppose each period has a duration of Δt and the $\ell > 1$ monitored times are

$$t_0 = 0, t_1, t_2, \ldots, t_\ell = T.$$

- It is unlikely that *all* monitored times coincide with the end of a period on the tree, or Δt divides t_i for all i.
- The binomial-trinomial tree can handle discrete options with ease, however.
- Simply build a binomial-trinomial tree from time 0 to time t_1, followed by one from time t_1 to time t_2, and so on until time t_ℓ.


$2\sigma\sqrt{\Delta t_1}$
$2\sigma\sqrt{\Delta t_2}$
$\Delta t_1'$
Δt_1
Δt_1
$\Delta t_2'$
t_0
t_1

Pricing Discrete and Moving Barrier Options (concluded)

- This procedure works even if each t_i is associated with a distinct barrier or if each window $[\,t_i, t_{i+1})$ has its own continuously monitored barrier or double barriers.
- Pricing in both scenarios can actually be done in time $O[\ell n \ln(n/\ell)\,]$.[a]
- For typical discrete barriers, placing barriers midway between two price levels on the tree may increase accuracy.[b]

---

[a] Y. Lu (R06723032, D08922008) & Lyuu (2021).
[b] Steiner & Wallmeier (1999); Tavella & Randall (2000).

Options on a Stock That Pays Known Dividends

- Many ad hoc assumptions have been postulated for option pricing with known dividends.[a]
  1. The one we saw earlier[b] models the stock price minus the present value of the anticipated dividends as following geometric Brownian motion.
  2. One can also model the stock price plus the forward values of the dividends as following geometric Brownian motion.

---

[a]Frishling (2002).
[b]On p. 329.

Options on a Stock That Pays Known Dividends (continued)

- Realistic models assume:
  - The stock price decreases by the amount of the dividend paid at the ex-dividend date.
  - The dividend is part cash and part yield (i.e., $\alpha(t) S_0 + \beta(t) S_t$), for practitioners.[a]
- The stock price follows geometric Brownian motion between adjacent ex-dividend dates.
- But they result in exponential-sized binomial trees.[b]
- The binomial-trinomial tree can avoid this problem in most cases.

[a] Henry-Labordère (2009).
[b] Recall p. 328.

Options on a Stock That Pays Known Dividends (continued)

- Suppose that the known dividend is D dollars and the ex-dividend date is at time t.
- So there are $m \stackrel{\Delta}{=} t/\Delta t$ periods between time 0 and the ex-dividend date.[a]
- To avoid negative stock prices, we need to make sure the lowest stock price at time t is at least D, i.e.,

$$Se^{-(t/\Delta t)\sigma\sqrt{\Delta t}} \geq D.$$

  - Or,

$$\Delta t \geq \left[\frac{t\sigma}{\ln(S/D)}\right]^2.$$

[a] That is, m is an integer input and $\Delta t \stackrel{\Delta}{=} t/m$.

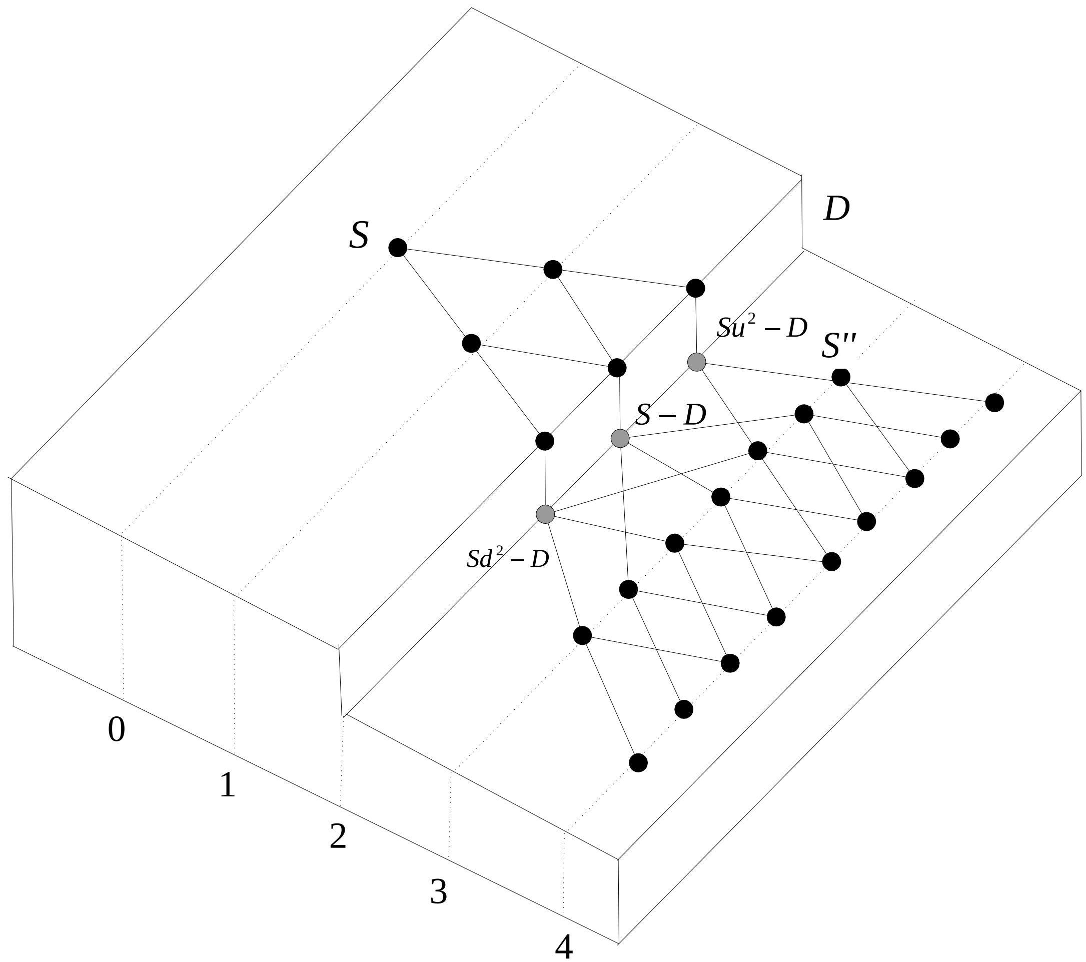
Options on a Stock That Pays Known Dividends (continued)

- Build a CRR tree from time 0 to time t as before.
- Subtract D from all the stock prices on the tree at time t to represent the price drop on the ex-dividend date.
- Assume the top node's price equals S'.
  - As usual, its two successor nodes will have prices $S'u$ and $S'u^{-1}$.
- The remaining nodes' successor nodes at time $t + \Delta t$ will choose from prices

$$S'u, S', S'u^{-1}, S'u^{-2}, S'u^{-3}, \ldots,$$

same as the CRR tree.

A Stair Tree



Options on a Stock That Pays Known Dividends (continued)

- For each node at time t below the top node, we build the trinomial connection.
- Note that the binomial-trinomial structure remains valid in the special case when $\Delta t' = \Delta t$ on p. 765.

Options on a Stock That Pays Known Dividends (concluded)

- Hence the construction can be completed.
- From time $t + \Delta t$ onward, the standard binomial tree will be used until the maturity date or the next ex-dividend date when the procedure can be repeated.
- The resulting tree is called the stair tree.[a]

[a]T. Dai (B82506025, R86526008, D8852600) & Lyuu (2004); T. Dai (B82506025, R86526008, D8852600) (2009).

Other Applications of Binomial-Trinomial Trees

- Pricing guaranteed minimum withdrawal benefits.[a]
- Option pricing with stochastic volatilities.[b]
- Efficient Parisian option pricing.[c]
- Defaultable bond pricing.[d]
- Implied barrier.[e]

---

[a] H. Wu (R96723058) (2009).
[b] C. Huang (R97922073) (2010).
[c] Y. Huang (R97922081) (2010).
[d] T. Dai (B82506025, R86526008, D8852600), Lyuu, & C. Wang (F95922018) (2009, 2010, 2014).
[e] Y. Lu (R06723032, D08922008) (2019); Y. Lu (R06723032, D08922008) & Lyuu (2021, 2023).

Mean Tracking[a]

- The general idea behind the binomial-trinomial tree on pp. 764ff is very powerful.
- One finds the sucessor middle node as the one closest to the mean.
- The two flanking successor nodes are then spaced at $c\sigma\sqrt{\Delta t}$ from the middle node for a suitably large $c > 0$.
- The resulting trinomial structure are then guaranteed to have valid branching probabilities.

[a]Lyuu & C. Wu (R90723065) (2003, 2005).

Default Boundary as Implied Barrier

- Under the structural model,[a] the default boundary is modeled as a barrier.[b]
- The constant barrier can be inferred from the closed-form formula given the firm's market capitalization, etc.[c]
- More generally, the moving barrier can be inferred from the term structure of default probabilities with the binomial-trinomial tree.[d]

---

[a] Recall p. 375.

[b] Black & Cox (1976).

[c] Brockman & Turtle (2003).

[d] Y. Lu (R06723032, D08922008) (2019); Y. Lu (R06723032, D08922008) & Lyuu (2021, 2023).

Default Boundary as Implied Barrier (continued)

- This barrier is called the implied barrier.[a]
- If the barrier is a step function, the implied barrier can be obtained in $O(n \ln n)$ time.[b]
- The next plot shows the convergence of the implied barrier (as a percentage of the initial stock price).[c]
  - The implied barrier is already very good with $n = 1$!

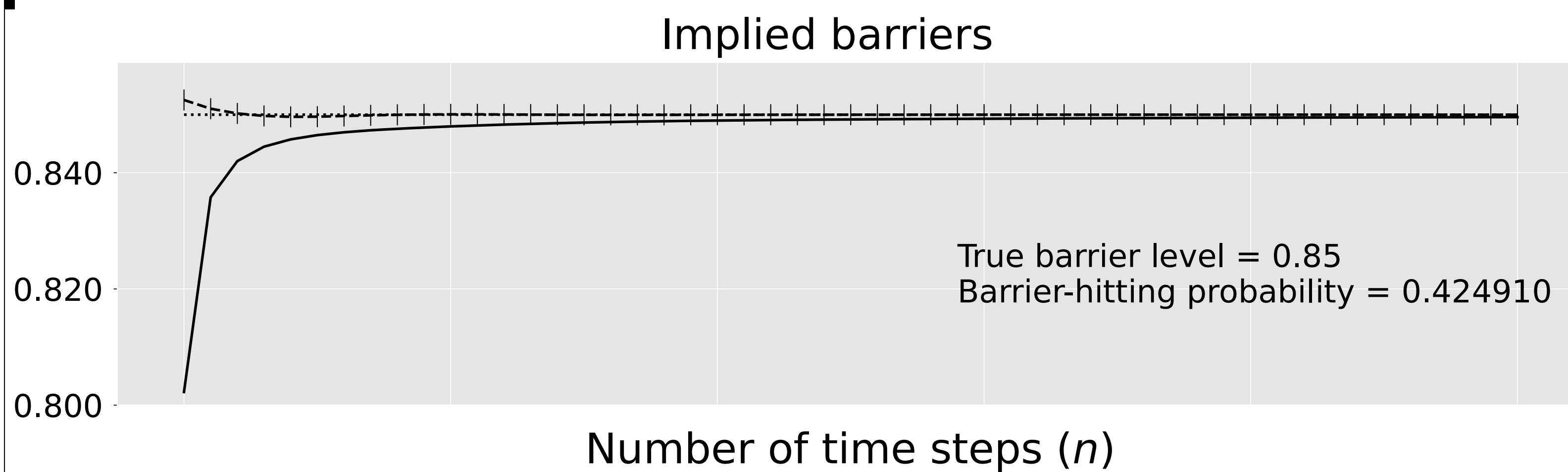
---

[a] Brockman & Turtle (2003).

[b] Y. Lu (`R06723032`, `D08922008`) & Lyuu (2021, 2023). The error is $O(1/n)$: This is linear convergence.

[c] Plot supplied by Mr. Lu, Yu-Ming (`R06723032`, `D08922008`) on November 20. 2021.

Default Boundary as Implied Barrier (continued)

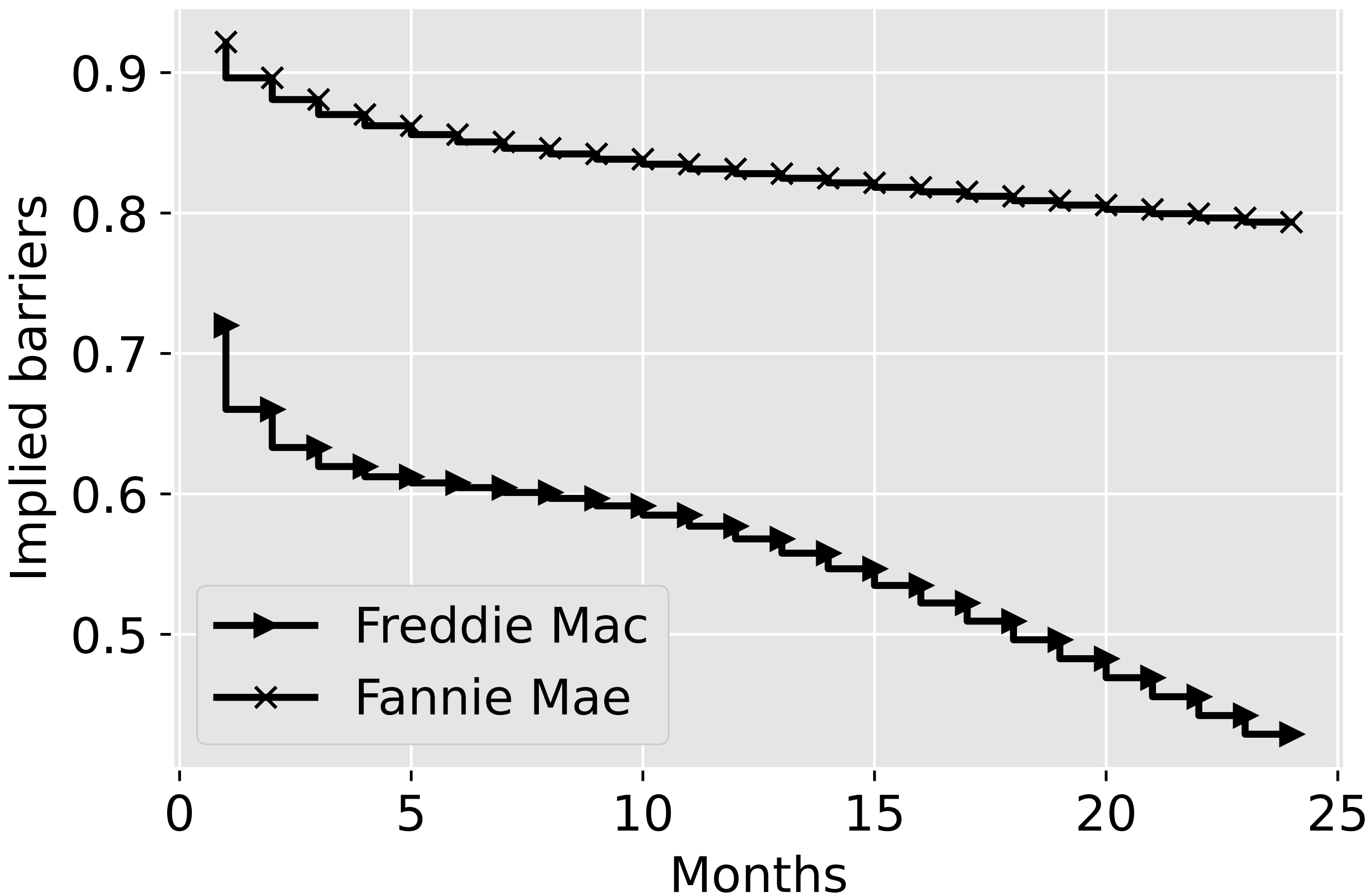


$1 \leq n \leq 101.$

Default Boundary as Implied Barrier (concluded)

- The next plot shows the implied barriers of Freddie Mac and Fannie Mae as of February 2008 (as percentages of the initial asset values).[a]

[a]Plot supplied by Mr. Lu, Yu-Ming (`R06723032`, `D08922008`) on February 26. 2021.


Implied barriers
0.9
0.8
0.7
0.6
0.5
0 5 10 15 20 25
Months
Freddie Mac
Fannie Mae

Time-Varying Double Barriers under Time-Dependent Volatility[a]

- More general models allow a time-varying $\sigma(t)$ (p. 320).
- Let the two barriers $L(t)$ and $H(t)$ be functions of time.[b]
  - Exponential functions are popular.[c]
- Still, we can price double-barrier options in $O(n^2)$ time or less with trinomial trees.
- Continuously monitored double-barrier knock-out options with time-varying barriers are called hot dog options.[d]

---

[a] Y. Zhang (R05922052) (2019).
[b] So the barriers are continuously monitored.
[c] C. Chou (R97944012) (2010); C. I. Chen (R98922127) (2011).
[d] El Babsiri & Noel (1998).

General Local-Volatility Models and Their Trees

- Consider the general local-volatility model

$$\frac{dS}{S} = (r_t - q_t)\, dt + \sigma(S, t)\, dW,$$

where $L \leq \sigma(S, t) \leq U$ for some positive L and U.

- This model has a unique (weak) solution.[a]
- The positive lower bound is justifiable because prices fluctuate.

[a] Achdou & Pironneau (2005).

General Local-Volatility Models and Their Trees (continued)

- The upper-bound assumption is also reasonable.
- Even on October 19, 1987, the CBOE S&P 100 Volatility Index (VXO) was about 150%, the highest ever.[a]
- An efficient quadratic-sized tree for this range-bounded model is easy.[b]
- Pick any $\sigma' \geq U$.
- Grow the trinomial tree with the node spacing $\sigma' \sqrt{\Delta t}$.[c]
- The branching probabilities are valid for small Δt.

[a] Caprio (2012).
[b] Lok (D99922028) & Lyuu (2016, 2017, 2020).
[c] Haahtela (2010).

General Local-Volatility Models and Their Trees (continued)

- They are

$$\begin{aligned}
p_u &= \frac{\sigma^2(S,t)}{2\sigma'^2} + \frac{r-q-\sigma^2(S,t)/2}{2\sigma'}\sqrt{\Delta t} \\
&\quad + \frac{\big[\, r-q-\sigma^2(S,t)/2 \,\big]^2}{2\sigma'^2}\,\Delta t, \\
p_d &= \frac{\sigma^2(S,t)}{2\sigma'^2} - \frac{r-q-\sigma^2(S,t)/2}{2\sigma'}\sqrt{\Delta t} \\
&\quad + \frac{\big[\, r-q-\sigma^2(S,t)/2 \,\big]^2}{2\sigma'^2}\,\Delta t, \\
p_m &= 1 - \frac{\sigma^2(S,t)}{\sigma'^2} - \frac{\big[\, r-q-\sigma^2(S,t)/2 \,\big]^2}{\sigma'^2}\,\Delta t.
\end{aligned}$$

General Local-Volatility Models and Their Trees (concluded)

- The same idea can be applied to price double-barrier options.
- Pick any

$$\sigma' \geq \max\left[\max_{S, 0 \leq t \leq T} \sigma(S, t), \sqrt{2}\,\sigma(S_0, 0)\right].$$

- Grow the trinomial tree with the node spacing $\sigma' \sqrt{\Delta t}$.
- Apply the mean-tracking idea to the first period and Eqs. (105)–(110) on p. 770 to obtain the probabilities

Merton's Jump-Diffusion Model

- Empirically, stock returns tend to have fat tails, inconsistent with the Black-Scholes model's assumptions.
- Stochastic volatility and jump processes have been proposed to address this problem.
- Merton's (1976) jump-diffusion model is our focus here.

Merton's Jump-Diffusion Model (continued)

- This model superimposes a jump component on a diffusion component.
- The diffusion component is the familiar geometric Brownian motion.
- The jump component is composed of lognormal jumps driven by a Poisson process.
  - It models the *rare* but *large* changes in the stock price because of the arrival of important news.[a]

[a]Derman & M. B. Miller (2016), "There is no precise, universally accepted definition of a jump, but it usually comes down to magnitude, duration, and frequency."

Merton's Jump-Diffusion Model (continued)

- Let S_t be the stock price at time t.
- The risk-neutral jump-diffusion process for the stock price follows[a]

$$\frac{dS_t}{S_t} = (r - \lambda \bar{k})\, dt + \sigma\, dW_t + k\, dq_t. \tag{112}$$

- Above, σ denotes the volatility of the diffusion component.

---

[a] Derman & M. B. Miller (2016), "[M]ost jump-diffusion models simply assume risk-neutral pricing without convincing justification."

Merton's Jump-Diffusion Model (continued)

- The jump event is governed by a compound Poisson process q_t with intensity λ, where k denotes the magnitude of the *random* jump.
  - The distribution of k obeys

    $$\ln(1+k) \sim N\left(\gamma, \delta^2\right)$$

    with mean $\bar{k} \stackrel{\Delta}{=} E\left(k\right) = e^{\gamma+\delta^2/2} - 1$.
  - Note that $k > -1$.
  - Note also that k is not related to dt.
- The model with $\lambda = 0$ reduces to the Black-Scholes model.

Merton's Jump-Diffusion Model (continued)

- The solution to Eq. (112) on p. 811 is

$$S_t = S_0 e^{(r-\lambda\bar{k}-\sigma^2/2)\,t+\sigma W_t} U(n(t)), \tag{113}$$

where

$$U(n(t)) = \prod_{i=0}^{n(t)} (1+k_i)\,.$$

  - k_i is the magnitude of the ith jump with $\ln(1+k_i) \sim N(\gamma, \delta^2)$.
  - $k_0 = 0$.
  - $n(t)$ is a Poisson process with intensity λ.

Merton's Jump-Diffusion Model (concluded)

- Recall that $n(t)$ denotes the number of jumps that occur up to time t.
- It is known that $E[\, n(t)\,] = \text{Var}[\, n(t)\,] = \lambda t$.
- As $k_i > -1$, stock prices will stay positive.
- The geometric Brownian motion, the lognormal jumps, and the Poisson process are assumed to be independent.

Tree for Merton's Jump-Diffusion Model[a]

- Define the S-logarithmic return of the stock price S' as

$$\ln(S'/S).$$

- Define the logarithmic distance between stock prices S' and S as

$$|\ln(S') - \ln(S)| = |\ln(S'/S)|.$$

[a]T. Dai (B82506025, R86526008, D8852600), C. Wang (F95922018), Lyuu, & Y. Liu (2010).

Tree for Merton's Jump-Diffusion Model (continued)

- Take the logarithm of Eq. (113) on p. 813:

$$M_t \triangleq \ln\left(\frac{S_t}{S_0}\right) = X_t + Y_t, \tag{114}$$

where

$$X_t \triangleq \left(r - \lambda\bar{k} - \frac{\sigma^2}{2}\right) t + \sigma W_t, \tag{115}$$

$$Y_t \triangleq \sum_{i=0}^{n(t)} \ln\left(1 + k_i\right). \tag{116}$$

- It decomposes the S_0-logarithmic return of S_t into the diffusion component X_t and the jump component Y_t.

Tree for Merton's Jump-Diffusion Model (continued)

- Motivated by decomposition (114) on p. 816, the tree construction divides each period into a diffusion phase followed by a jump phase.
- In the diffusion phase, X_t is approximated by the BOPM.
- So X_t moves up to $X_t + \sigma\sqrt{\Delta t}$ with probability p_u and down to $X_t - \sigma\sqrt{\Delta t}$ with probability p_d.

Tree for Merton's Jump-Diffusion Model (continued)

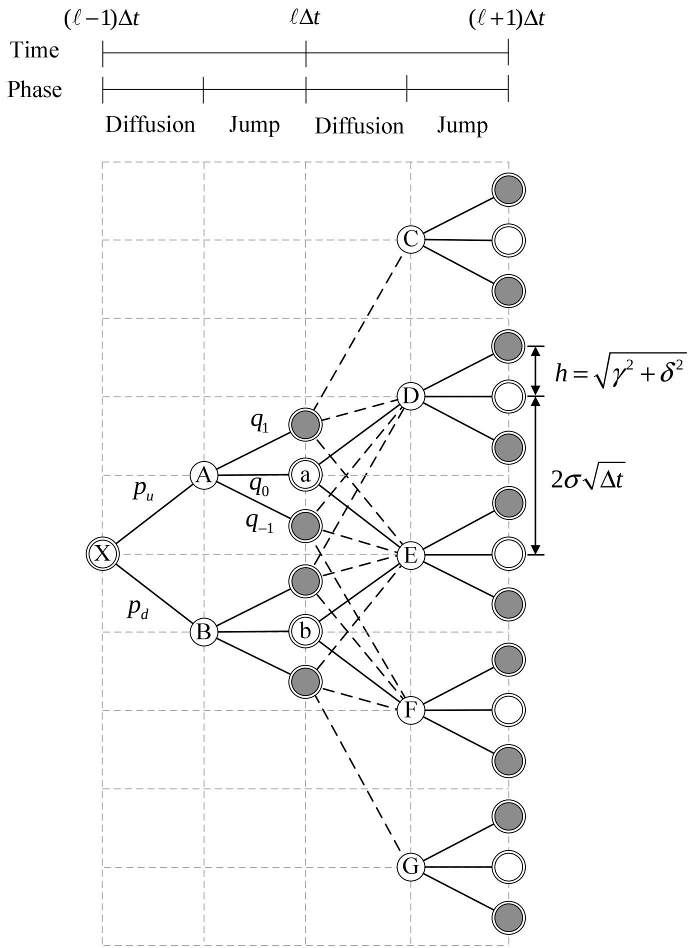
- According to BOPM,

$$\begin{aligned} p_u &= \frac{e^{\mu \Delta t} - d}{u - d}, \\ p_d &= 1 - p_u, \end{aligned}$$

  except that $\mu = r - \lambda \bar{k}$ here.

- The diffusion component gives rise to diffusion nodes.

- They are spaced at $2\sigma\sqrt{\Delta t}$ apart such as the white nodes A, B, C, D, E, F, and G on p. 819.



White nodes are *diffusion nodes*. Gray nodes are *jump nodes*. In the diffusion phase, the solid black lines denote the binomial structure of BOPM; the dashed lines denote the trinomial structure. Only the double-circled nodes will remain after the construction. Note that a and b are diffusion nodes because no jump occurs in the jump phase.

Tree for Merton's Jump-Diffusion Model (continued)

- In the jump phase, $Y_{t+\Delta t}$ is approximated by moves from *each* diffusion node to $2m$ jump nodes that match the first $2m$ moments of the lognormal jump.
- The m jump nodes above the diffusion node are spaced at $h \stackrel{\Delta}{=} \sqrt{\gamma^2 + \delta^2}$ apart.
- Note that h is independent of Δt.

Tree for Merton's Jump-Diffusion Model (concluded)

- The same holds for the m jump nodes below the diffusion node.
- The gray nodes at time $\ell \Delta t$ on p. 819 are jump nodes.
  - We set $m = 1$ on p. 819.
- The size of the tree is $O(n^{2.5})$.

Multivariate Contingent Claims

- They depend on two or more underlying assets.
- The basket call on m assets has the terminal payoff

$$\max\left(\sum_{i=1}^{m} \alpha_i S_i(\tau) - X, 0\right),$$

  where α_i is the percentage of asset i.

- Basket options are essentially options on a portfolio of stocks (or index options).[a]
- Option on the best of two risky assets and cash has a terminal payoff of $\max(S_1(\tau), S_2(\tau), X)$.

[a] Except that membership and weights do *not* change for basket options (Bennett, 2014).

Multivariate Contingent Claims (concluded)[a]

Name	Payoff
Exchange option	$\max(S_1(\tau) - S_2(\tau), 0)$
Better-off option	$\max(S_1(\tau), \ldots, S_k(\tau), 0)$
Worst-off option	$\min(S_1(\tau), \ldots, S_k(\tau), 0)$
Binary maximum option	$I\{\max(S_1(\tau), \ldots, S_k(\tau)) > X\}$
Maximum option	$\max(\max(S_1(\tau), \ldots, S_k(\tau)) - X, 0)$
Minimum option	$\max(\min(S_1(\tau), \ldots, S_k(\tau)) - X, 0)$
Spread option	$\max(S_1(\tau) - S_2(\tau) - X, 0)$
Basket average option	$\max((S_1(\tau) + \cdots + S_k(\tau))/k - X, 0)$
Multi-strike option	$\max(S_1(\tau) - X_1, \ldots, S_k(\tau) - X_k, 0)$
Pyramid rainbow option	$\max(\mid S_1(\tau) - X_1 \mid + \cdots + \mid S_k(\tau) - X_k \mid - X, 0)$
Madonna option	$\max(\sqrt{(S_1(\tau) - X_1)^2 + \cdots + (S_k(\tau) - X_k)^2} - X, 0)$

[a] Lyuu & Teng (R91723054) (2011).

Correlated Trinomial Model[a]

- Two risky assets S_1 and S_2 follow

$$\frac{dS_i}{S_i} = r\,dt + \sigma_i\,dW_i$$

  in a risk-neutral economy, $i = 1, 2$.

- Let

$$\begin{aligned} M_i &\stackrel{\Delta}{=} e^{r\Delta t}, \\ V_i &\stackrel{\Delta}{=} M_i^2 (e^{\sigma_i^2 \Delta t} - 1). \end{aligned}$$

  – $S_i M_i$ is the mean of S_i at time Δt.
  – $S_i^2 V_i$ the variance of S_i at time Δt.

[a]Boyle, Evnine, & Gibbs (1989).

Correlated Trinomial Model (continued)

- The value of $S_1 S_2$ at time Δt has a joint lognormal distribution with mean $S_1 S_2 M_1 M_2 e^{\rho \sigma_1 \sigma_2 \Delta t}$, where ρ is the correlation between dW_1 and dW_2.
- Next match the 1st and 2nd moments of the approximating discrete distribution to those of the continuous counterpart.
- At time Δt from now, there are 5 distinct outcomes.

Correlated Trinomial Model (continued)

- The five-point probability distribution of the asset prices is

Probability	Asset 1	Asset 2
p_1	$S_1 u_1$	$S_2 u_2$
p_2	$S_1 u_1$	$S_2 d_2$
p_3	$S_1 d_1$	$S_2 d_2$
p_4	$S_1 d_1$	$S_2 u_2$
p_5	S_1	S_2

- As usual, impose $u_i d_i = 1$.

Correlated Trinomial Model (continued)

- The probabilities must sum to one, and the means must be matched:

$$
\begin{aligned}
1 &= p_1 + p_2 + p_3 + p_4 + p_5, \\
S_1 M_1 &= (p_1 + p_2)\, S_1 u_1 + p_5 S_1 + (p_3 + p_4)\, S_1 d_1, \\
S_2 M_2 &= (p_1 + p_4)\, S_2 u_2 + p_5 S_2 + (p_2 + p_3)\, S_2 d_2.
\end{aligned}
$$

Correlated Trinomial Model (concluded)

- Let $R \stackrel{\Delta}{=} M_1 M_2 e^{\rho\sigma_1\sigma_2\Delta t}$.
- Match the variances and covariance:

$$\begin{aligned}
S_1^2 V_1 &= (p_1+p_2)\left[(S_1u_1)^2-(S_1M_1)^2\right]+p_5\left[S_1^2-(S_1M_1)^2\right]\\
&\quad +(p_3+p_4)\left[(S_1d_1)^2-(S_1M_1)^2\right],\\
S_2^2 V_2 &= (p_1+p_4)\left[(S_2u_2)^2-(S_2M_2)^2\right]+p_5\left[S_2^2-(S_2M_2)^2\right]\\
&\quad +(p_2+p_3)\left[(S_2d_2)^2-(S_2M_2)^2\right],\\
S_1S_2R &= (p_1u_1u_2+p_2u_1d_2+p_3d_1d_2+p_4d_1u_2+p_5)\,S_1S_2.
\end{aligned}$$

- The solutions appear on p. 246 of the textbook.

Correlated Trinomial Model Simplified[a]

- Let $\mu_i' \stackrel{\Delta}{=} r - \sigma_i^2/2$ and $u_i \stackrel{\Delta}{=} e^{\lambda\sigma_i\sqrt{\Delta t}}$ for $i = 1, 2$.
- The following simpler scheme is often good enough:

$$
\begin{aligned}
p_1 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{\mu_1'}{\sigma_1} + \frac{\mu_2'}{\sigma_2}\right) + \frac{\rho}{\lambda^2}\right],\\
p_2 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{\mu_1'}{\sigma_1} - \frac{\mu_2'}{\sigma_2}\right) - \frac{\rho}{\lambda^2}\right],\\
p_3 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(-\frac{\mu_1'}{\sigma_1} - \frac{\mu_2'}{\sigma_2}\right) + \frac{\rho}{\lambda^2}\right],\\
p_4 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(-\frac{\mu_1'}{\sigma_1} + \frac{\mu_2'}{\sigma_2}\right) - \frac{\rho}{\lambda^2}\right],\\
p_5 &= 1 - \frac{1}{\lambda^2}.
\end{aligned}
$$

[a] Madan, Milne, & Shefrin (1989).

Correlated Trinomial Model Simplified (continued)

- All of the probabilities lie between 0 and 1 if and only if

$$-1+\lambda\sqrt{\Delta t}\left|\frac{\mu_1'}{\sigma_1}+\frac{\mu_2'}{\sigma_2}\right| \leq \rho \leq 1-\lambda\sqrt{\Delta t}\left|\frac{\mu_1'}{\sigma_1}-\frac{\mu_2'}{\sigma_2}\right|, \tag{117}$$

$$1 \leq \lambda. \tag{118}$$

- We call a multivariate tree (correlation-) optimal if it guarantees valid probabilities as long as

$$-1+O(\sqrt{\Delta t})<\rho<1-O(\sqrt{\Delta t}),$$

such as the above one.[a]

[a] W. Kao (R98922093) (2011); W. Kao (R98922093), Lyuu, & Wen (D94922003) (2014).

Correlated Trinomial Model Simplified (continued)

- But this model cannot price 2-asset 2-barrier options accurately.[a]
- Few multivariate trees are both optimal and able to handle multiple barriers.[b]
- An alternative is to use orthogonalization.[c]

---

[a]See Y. Chang (B89704039, R93922034), Hsu (R7526001, D89922012), & Lyuu (2006); W. Kao (R98922093), Lyuu, & Wen (D94922003) (2014) for solutions.

[b]See W. Kao (R98922093), Lyuu, & Wen (D94922003) (2014) for an exception.

[c]Hull & White (1990); T. Dai (B82506025, R86526008, D8852600), C. Wang (F95922018), & Lyuu (2013).

Correlated Trinomial Model Simplified (concluded)

- Suppose we allow each asset's volatility to be a function of time.[a]
- There are k assets.
- One can build an optimal multivariate tree that handles two barriers on each asset in time $O(n^{k+1})$.[b]

---

[a] Recall p. 319.

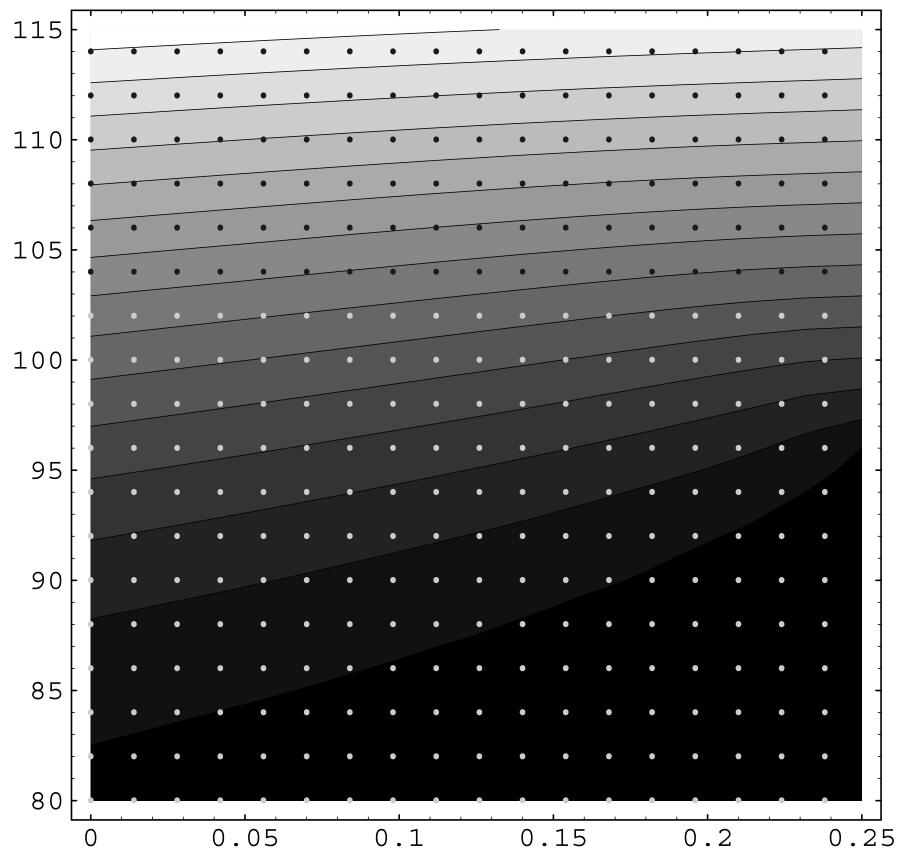
[b] Y. Zhang (R05922052) (2019); Y. Zhang (R05922052) & Lyuu (2023).

Numerical Methods

All science is dominated
by the idea of approximation.
— Bertrand Russell

Finite-Difference Methods

- Place a grid of points on the space over which the desired function takes value.
- Then approximate the function value at each of these points (p. 836).
- Solve the equation numerically by introducing difference equations in place of derivatives.


115
110
105
100
95
90
85
80
0
0.05
0.1
0.15
0.2
0.25

Example: Poisson's Equation

- It is $\partial^2\theta/\partial x^2 + \partial^2\theta/\partial y^2 = -\rho(x, y)$, which describes the electrostatic field.
- Replace second derivatives with finite differences through central difference.
- Introduce evenly spaced grid points with distance of Δx along the x axis and Δy along the y axis.
- The finite-difference form is

$$\begin{aligned} -\rho(x_i, y_j) = {} & \frac{\theta(x_{i+1}, y_j) - 2\theta(x_i, y_j) + \theta(x_{i-1}, y_j)}{(\Delta x)^2} \\ & + \frac{\theta(x_i, y_{j+1}) - 2\theta(x_i, y_j) + \theta(x_i, y_{j-1})}{(\Delta y)^2}. \end{aligned}$$

Example: Poisson's Equation (concluded)

- In the above, $\Delta x \stackrel{\Delta}{=} x_i - x_{i-1}$ and $\Delta y \stackrel{\Delta}{=} y_j - y_{j-1}$ for $i, j = 1, 2, \ldots$.
- When the grid points are evenly spaced in both axes so that $\Delta x = \Delta y = h$, the difference equation becomes

$$\begin{aligned} -h^2 \rho(x_i, y_j) &= \theta(x_{i+1}, y_j) + \theta(x_{i-1}, y_j) \\ &+ \theta(x_i, y_{j+1}) + \theta(x_i, y_{j-1}) - 4\theta(x_i, y_j). \end{aligned}$$

- Given boundary values, we can solve for the x_is and the y_js within the square $[\pm L, \pm L]$.
- From now on, $\theta_{i,j}$ will denote the finite-difference approximation to the exact $\theta(x_i, y_j)$.

Explicit Methods

- Consider the diffusion equation[a]
  $D(\partial^2\theta/\partial x^2) - (\partial\theta/\partial t) = 0$, $D > 0$.
- Use evenly spaced grid points (x_i, t_j) with distances Δx and Δt, where $\Delta x \stackrel{\Delta}{=} x_{i+1} - x_i$ and $\Delta t \stackrel{\Delta}{=} t_{j+1} - t_j$.
- Employ central difference for the second derivative and forward difference for the time derivative to obtain

$$\left.\frac{\partial\theta(x,t)}{\partial t}\right|_{t=t_j} = \frac{\theta(x, \boxed{t_{j+1}}) - \theta(x, \boxed{t_j})}{\Delta t} + \cdots, \tag{119}$$

$$\left.\frac{\partial^2\theta(x,t)}{\partial x^2}\right|_{x=x_i} = \frac{\theta(\boxed{x_{i+1}}, t) - 2\theta(\boxed{x_i}, t) + \theta(\boxed{x_{i-1}}, t)}{(\Delta x)^2} + \cdots \tag{120}$$

[a]It is a parabolic partial differential equation.

Explicit Methods (continued)

- Next, assemble Eqs. (119) and (120) into a single equation at (x_i, t_j).
- But we need to decide how to evaluate x in the first equation and t in the second.
- Since central difference around x_i is used in Eq. (120), we might as well use x_i for x in Eq. (119).
- Two choices are possible for t in Eq. (120).
- The first choice uses $t = t_j$ to yield the following finite-difference equation,

$$\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = D \, \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta x)^2}. \tag{121}$$

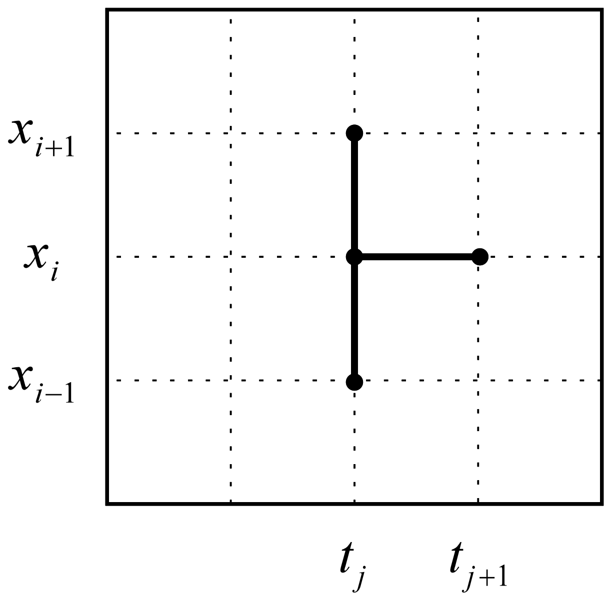
Explicit Methods (continued)

- The stencil of grid points involves four values, $\theta_{i,j+1}$, $\theta_{i,j}$, $\theta_{i+1,j}$, and $\theta_{i-1,j}$.
- Rearrange Eq. (121) on p. 840 as

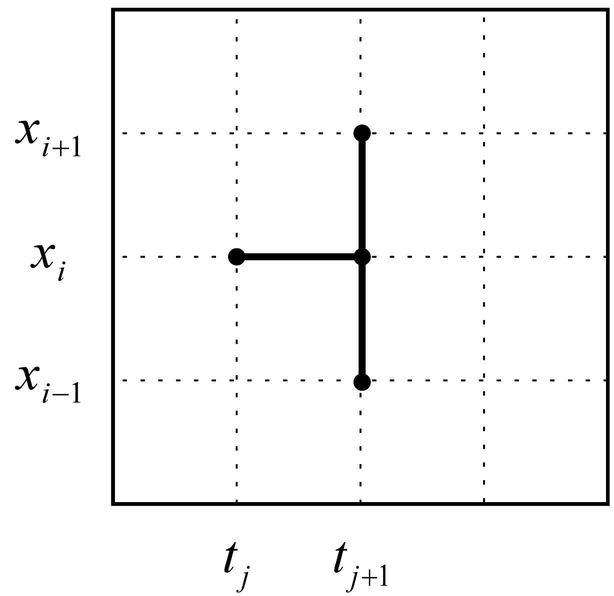
$$\theta_{i,j+1} = \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i+1,j} + \left(1 - \frac{2D\Delta t}{(\Delta x)^2}\right)\theta_{i,j} + \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i-1,j}. \tag{122}$$

- We can calculate $\theta_{i,j+1}$ from $\theta_{i,j}, \theta_{i+1,j}, \theta_{i-1,j}$, at the previous time t_j (see exhibit (a) on next page).

Stencils



(a)



(b)

Explicit Methods (concluded)

- Starting from the initial conditions at t_0, that is, $\theta_{i,0} = \theta(x_i, t_0)$, $i = 1, 2, \ldots$, we calculate

$$\theta_{i,1}, \quad i = 1, 2, \ldots .$$

- And then

$$\theta_{i,2}, \quad i = 1, 2, \ldots .$$

- And so on.

Stability

- The explicit method is numerically unstable unless

$$\Delta t \leq (\Delta x)^2/(2D).$$

  - A numerical method is unstable if the solution is highly sensitive to changes in initial conditions.

- The stability condition may lead to high running times and memory requirements.

- For instance, halving Δx would imply quadrupling $(\Delta t)^{-1}$, resulting in a running time 8 times as much.

Explicit Method and Trinomial Tree

- Recall Eq. (122) on p. 841:

$$\theta_{i,j+1} = \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i+1,j} + \left(1 - \frac{2D\Delta t}{(\Delta x)^2}\right)\theta_{i,j} + \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i-1,j}.$$

- When the stability condition is satisfied, the three coefficients for $\theta_{i+1,j}$, $\theta_{i,j}$, and $\theta_{i-1,j}$ all lie between zero and one and sum to one.
- They can be interpreted as probabilities.
- So the finite-difference equation becomes identical to backward induction on trinomial trees!

Explicit Method and Trinomial Tree (concluded)

- The freedom in choosing Δx corresponds to similar freedom in the construction of trinomial trees.
- The explicit finite-difference equation is also identical to backward induction on a binomial tree.[a]
  - Let the binomial tree take 2 steps each of length $\Delta t/2$.
  - It is now a trinomial tree.

[a]Hilliard (2014).

Implicit Methods

- Suppose we use $t = t_{j+1}$ in Eq. (120) on p. 839 instead.
- The finite-difference equation becomes

$$\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = D\, \frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{(\Delta x)^2}. \tag{123}$$

- The stencil involves $\theta_{i,j}$, $\theta_{i,j+1}$, $\theta_{i+1,j+1}$, and $\theta_{i-1,j+1}$.
- This method is now implicit:
  - The value of any one of the three quantities at t_{j+1} cannot be calculated unless the other two are known.
  - See exhibit (b) on p. 842.

Implicit Methods (continued)

- Equation (123) can be rearranged as

$$\theta_{i-1,j+1} - (2+\gamma)\,\theta_{i,j+1} + \theta_{i+1,j+1} = -\gamma\theta_{i,j},$$

where $\gamma \stackrel{\Delta}{=} (\Delta x)^2/(D\Delta t)$.

- This equation is unconditionally stable.
- Suppose the boundary conditions are given at $x = x_0$ and $x = x_{N+1}$.
- After $\theta_{i,j}$ has been calculated for $i = 1, 2, \ldots, N$, the values of $\theta_{i,j+1}$ at time t_{j+1} can be computed as the solution to the following tridiagonal linear system,

Implicit Methods (continued)

$$\begin{bmatrix} a & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & a & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & a & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & a & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & a \end{bmatrix} \begin{bmatrix} \theta_{1,j+1} \\ \theta_{2,j+1} \\ \theta_{3,j+1} \\ \vdots \\ \vdots \\ \vdots \\ \theta_{N,j+1} \end{bmatrix} = \begin{bmatrix} -\gamma\theta_{1,j} - \theta_{0,j+1} \\ -\gamma\theta_{2,j} \\ -\gamma\theta_{3,j} \\ \vdots \\ \vdots \\ -\gamma\theta_{N-1,j} \\ -\gamma\theta_{N,j} - \theta_{N+1,j+1} \end{bmatrix},$$

where $a \stackrel{\Delta}{=} -2 - \gamma$.

Implicit Methods (concluded)

- Tridiagonal systems can be solved in $O(N)$ time and $O(N)$ space.
  - Never invert a matrix to solve a tridiagonal system.
- The matrix above is nonsingular when $\gamma \geq 0$.
  - A square matrix is nonsingular if its inverse exists.

Crank-Nicolson Method

- Take the average of explicit method (121) on p. 840 and implicit method (123) on p. 847:

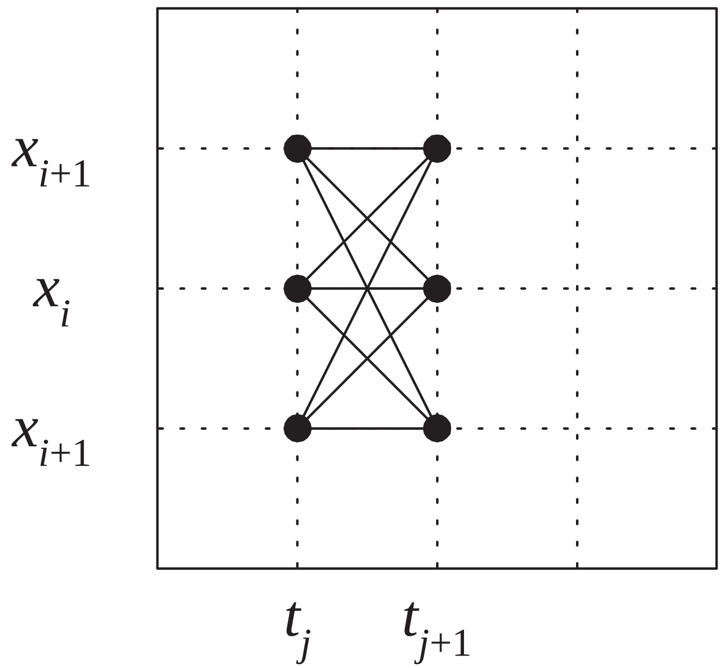
$$\begin{aligned}
&\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} \\
= \quad & \frac{1}{2}\left(D\,\frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta x)^2} + D\,\frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{(\Delta x)^2} \right).
\end{aligned}$$

- After rearrangement,

$$\gamma\theta_{i,j+1} - \frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{2} = \gamma\theta_{i,j} + \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{2}.$$

- This is an unconditionally stable implicit method with excellent rates of convergence.

Stencil



Numerically Solving the Black-Scholes PDE (94) on p. 687

- See text.
- Brennan and Schwartz (1978) analyze the stability of the implicit method.