Continuous-Time Derivatives Pricing

I have hardly met a mathematician
who was capable of reasoning.
— Plato (428 B.C.–347 B.C.)

Fischer [Black] is the only real genius
I've ever met in finance. Other people,
like Robert Merton or Stephen Ross,
are just very smart and quick,
but they think like me.
Fischer came from someplace else entirely.
— John C. Cox, quoted in Mehrling (2005)

Toward the Black-Scholes Differential Equation

- The price of any derivative on a non-dividend-paying stock must satisfy a partial differential equation (PDE).
- The key step is recognizing that the same random process drives both securities.
  - Their prices are perfectly correlated.
- We then figure out the amount of stock such that the gain from it offsets exactly the loss from the derivative.
- The removal of uncertainty forces the portfolio's return to be the riskless rate.
- PDEs make many numerical methods applicable.

Assumptions[a] and Notations

- The stock price follows $dS = \mu S\, dt + \sigma S\, dW$.
- There are no dividends.
- Trading is continuous, and short selling is allowed.
- There are no transactions costs or taxes.
- All securities are infinitely divisible.
- The term structure of riskless rates is flat at r.
- There is unlimited riskless borrowing and lending.
- t is the current time, T is the expiration time, and $\tau \stackrel{\Delta}{=} T - t$.

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[a] Derman & Taleb (2005) summarizes criticisms on these assumptions and the replication argument.

Black-Scholes Differential Equation

- Let C be the price of a *simple* derivative[a] on S.
- From Ito's lemma (p. 613),

$$dC = \left(\mu S \, \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \, \sigma^2 S^2 \, \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \, \frac{\partial C}{\partial S} \, dW.$$

  – The same W drives both C and S.

  – Unlike dS/S, the diffusion of dC/C is stochastic!

- Short one derivative and long $\partial C/\partial S$ shares of stock (call it Π).
- By construction,

$$\Pi = -C + S(\partial C/\partial S).$$

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[a]Recall p. 437.

Black-Scholes Differential Equation (continued)

- The change in the value of the portfolio at time dt is[a]

$$d\Pi = -dC + \frac{\partial C}{\partial S}\, dS. \tag{93}$$

- Substitute the formulas for dC and dS into the above to yield

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} \right) dt.$$

- As this equation does not involve dW, the portfolio is riskless during dt time: $d\Pi = r\Pi\, dt$.

[a] Bergman (1982) and Bartels (1995) argue this is not quite right. But see Macdonald (1997). Mathematically, it is wrong (Bingham & Kiesel, 2004).

Black-Scholes Differential Equation (continued)

- So

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 C}{\partial S^2}\right) dt = r\left(C - S\,\frac{\partial C}{\partial S}\right) dt.$$

- Equate the terms to finally obtain[a]

$$\frac{\partial C}{\partial t} + rS\,\frac{\partial C}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 C}{\partial S^2} = rC.$$

- This is a backward equation, which describes the dynamics of a derivative's price *forward* in physical time.

[a]Known as the Feynman-Kac stochastic representation formula.

Black-Scholes Differential Equation (concluded)

- When there is a dividend yield q,

$$\frac{\partial C}{\partial t} + (r - q)\, S\, \frac{\partial C}{\partial S} + \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} = rC. \tag{94}$$

- Dupire's formula[a] (90) for the local-volatility model is simply its dual:[b]

$$\frac{\partial C}{\partial T} + (r_T - q_T) X \frac{\partial C}{\partial X} - \frac{1}{2}\, \sigma(X, T)^2 X^2 \frac{\partial^2 C}{\partial X^2} = -q_T C.$$

- This is a forward equation, which describes the dynamics of a derivative's price *backward* in maturity time.

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[a] Recall p. 643.
[b] Derman & Kani (1997).

Rephrase

- The Black-Scholes differential equation can be expressed in terms of sensitivity numbers,

$$\Theta + rS\Delta + \frac{1}{2}\,\sigma^2 S^2 \Gamma = rC. \qquad (95)$$

- Identity (95) leads to an alternative way of computing Θ numerically from Δ and Γ.
- When a portfolio is delta-neutral,

$$\Theta + \frac{1}{2}\,\sigma^2 S^2 \Gamma = rC.$$

  – A definite relation thus exists between Γ and Θ.

[Black] got the equation [in 1969] but then
was unable to solve it. Had he been a better
physicist he would have recognized it as a form
of the familiar heat exchange equation,
and applied the known solution. Had he been
a better mathematician, he could have
solved the equation from first principles.
Certainly Merton would have known exactly
what to do with the equation
had he ever seen it.
— Perry Mehrling (2005)

Black-Scholes Differential Equation: An Alternative

- Perform the change of variable $V \stackrel{\Delta}{=} \ln S$.
- The option value becomes $U(V, t) \stackrel{\Delta}{=} C(e^V, t)$.
- Furthermore,

$$
\begin{aligned}
\frac{\partial C}{\partial t} &= \frac{\partial U}{\partial t}, \\
\frac{\partial C}{\partial S} &= \frac{1}{S}\,\frac{\partial U}{\partial V}, && (96) \\
\frac{\partial^2 C}{\partial^2 S} &= \frac{1}{S^2}\,\frac{\partial^2 U}{\partial V^2} - \frac{1}{S^2}\,\frac{\partial U}{\partial V}. && (97)
\end{aligned}
$$

Black-Scholes Differential Equation: An Alternative (concluded)

- Equations (96) and (97) are alternative ways to calculate delta and gamma.[a]
- They are very useful for trees of *logarithmic* prices.
- The Black-Scholes differential equation (94) on p. 687 becomes

$$\frac{1}{2}\,\sigma^2\,\frac{\partial^2 U}{\partial V^2} + \left(r - q - \frac{\sigma^2}{2}\right)\frac{\partial U}{\partial V} - rU + \frac{\partial U}{\partial t} = 0$$

  subject to $U(V, T)$ being the payoff such as $\max(X - e^V, 0)$.

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[a]Recall Eqs. (52) on p. 367 and (53) on p. 369.

PDEs for Asian Options

- Add the new variable $A(t) \stackrel{\Delta}{=} \int_0^t S(u)\, du$.
- Then the value V of the Asian option satisfies this two-dimensional PDE:[a]

$$\frac{\partial V}{\partial t} + rS\, \frac{\partial V}{\partial S} + \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 V}{\partial S^2} + S\, \frac{\partial V}{\partial A} = rV.$$

- The terminal conditions are

$$\begin{aligned} V(T, S, A) &= \max\left(\frac{A}{T} - X, 0\right) \quad \text{for call,} \\ V(T, S, A) &= \max\left(X - \frac{A}{T}, 0\right) \quad \text{for put.} \end{aligned}$$

[a]Kemna & Vorst (1990).

PDEs for Asian Options (continued)

- The two-dimensional PDE produces algorithms similar to that on pp. 449ff.[a]
- But one-dimensional PDEs are available for Asian options.[b]
- For example, Večeř (2001) derives the following PDE for Asian calls:

$$\frac{\partial u}{\partial t} + r\left(1 - \frac{t}{T} - z\right)\frac{\partial u}{\partial z} + \frac{\left(1 - \frac{t}{T} - z\right)^2 \sigma^2}{2}\frac{\partial^2 u}{\partial z^2} = 0$$

with the terminal condition $u(T, z) = \max(z, 0)$.

[a]Barraquand & Pudet (1996).

[b]Rogers & Shi (1995); Večeř (2001); Dubois & Lelièvre (2005).

PDEs for Asian Options (concluded)

- For Asian puts:

$$\frac{\partial u}{\partial t} + r\left(\frac{t}{T} - 1 - z\right)\frac{\partial u}{\partial z} + \frac{\left(\frac{t}{T} - 1 - z\right)^2 \sigma^2}{2}\frac{\partial^2 u}{\partial z^2} = 0$$

  with the same terminal condition.

- One-dimensional PDEs yield highly efficient numerical algorithms.

Hedging

When Professors Scholes and Merton and I
invested in warrants,
Professor Merton lost the most money.
And I lost the least.
— Fischer Black (1938–1995)

Delta Hedge

- Recall the delta (hedge ratio) of a derivative f:

$$\Delta \triangleq \frac{\partial f}{\partial S}.$$

- Thus

$$\Delta f \approx \Delta \times \Delta S$$

  for relatively small changes in the stock price, ΔS.

- A delta-neutral portfolio is hedged as it is immunized against small changes in the stock price.

Delta Hedge (concluded)

- A trading strategy that dynamically maintains a delta-neutral portfolio is called delta hedge.
  - Trading strategies can also be static (or constant).[a]
- Delta changes with the stock price.
- A delta hedge needs to be rebalanced periodically in order to maintain delta neutrality.
- In the limit where the portfolio is adjusted continuously, "perfect" hedge is achieved and the strategy becomes "self-financing."

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[a]Recall p. 496 for one in hedging the short forward contract with the underlying asset and loans.

Implementing Delta Hedge

- We want to hedge N *short* derivatives.
- Assume the stock pays no dividends.
- The delta-neutral portfolio maintains $N \times \Delta$ shares of stock plus B borrowed dollars such that

$$-N \times f + N \times \Delta \times S - B = 0.$$

- At next rebalancing point when the delta is Δ', buy $N \times (\Delta' - \Delta)$ shares to maintain $N \times \Delta'$ shares.
- Delta hedge is the discrete-time analog of the continuous-time limit.
- It will rarely be self-financing however small Δt is.

Example

- A hedger is *short* 10,000 European calls.
- $S = 50$, $\sigma = 30\%$, and $r = 6\%$.
- This call's expiration is four weeks away, its strike price is \$50, and each call has a current value of $f = 1.76791$.
- As an option covers 100 shares of stock, $N = 1{,}000{,}000$.
- The trader adjusts the portfolio weekly.
- The calls are replicated well if the cumulative cost of trading *stock* is close to the call premium's FV.[a]

[a]This takes the replication viewpoint: One starts with zero dollar.

Example (continued)

- As $\Delta = 0.538560$

$$N \times \Delta = 538,560$$

shares are purchased for a total cost of

$$538{,}560 \times 50 = 26{,}928{,}000$$

dollars to make the portfolio delta-neutral.

- The trader finances the purchase by borrowing

$$B = N \times \Delta \times S - N \times f = 25{,}160{,}090$$

dollars net.[a]

[a]This takes the hedging viewpoint: One starts with the option premium. The two viewpoints are equivalent. See Exercise 16.3.2 of the text.

Example (continued)

- At 3 weeks to expiration, the stock price rises to \$51.
- The new call value is $f' = 2.10580$.
- So before rebalancing, the portfolio is worth

$$-N \times f' + 538{,}560 \times 51 - Be^{0.06/52} = 171,622. \qquad (98)$$

- The delta hedge is not self-financing as \$171,622 can be withdrawn.
  - It does *not* replicate the calls perfectly.

Example (continued)

- The magnitude of the tracking error[a] can be mitigated if adjustments are made more frequently.
- The tracking error over *one* rebalancing act is positive about 68% of the time.
- Its expected value is ~ 0 under the *risk-neutral* probability measure.[b]
  - But the stock price should be sampled under the *real-world* probability measure.[c]

[a]The variation in the net portfolio value.

[b]Boyle & Emanuel (1980).

[c]Recall Eq. (93) on p. 685 or see p. 713.

Example (continued)

- The tracking error at maturity is proportional to vega.[a]
- In practice tracking errors will cease to decrease beyond a certain rebalancing frequency.
- With a higher delta $\Delta' = 0.640355$, the trader buys

$$N \times (\Delta' - \Delta) = 101,795$$

shares for \$5,191,545.

- The number of shares is increased to $N \times \Delta' = 640,355$.

[a] Kamal & Derman (1999).

Example (continued)

- The cumulative cost is[a]

$$26{,}928{,}000 \times e^{0.06/52} + 5{,}191{,}545 = 32{,}150{,}634.$$

- The portfolio is again delta-neutral.

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[a] We take the replication viewpoint again. Under the BOPM, the replicating strategy is self-financing and matches the payoff perfectly.

		Option value	Delta	Change in delta	No. shares bought	Cost of shares	Cumulative cost
τ	S	f	Δ		$N \times (5)$	$(1) \times (6)$	$\text{FV}(8') + (7)$
	(1)	(2)	(3)	(5)	(6)	(7)	(8)
4	50	1.7679	0.53856	—	538,560	26,928,000	26,928,000
3	51	2.1058	0.64036	0.10180	101,795	5,191,545	32,150,634
2	53	3.3509	0.85578	0.21542	215,425	11,417,525	43,605,277
1	52	2.2427	0.83983	−0.01595	−15,955	−829,660	42,825,960
0	54	4.0000	1.00000	0.16017	160,175	8,649,450	51,524,853

- We take the replication viewpoint.
- The total number of shares is 1,000,000 at expiration (trading takes place at expiration, too).

Example (continued)

- At expiration, the trader has 1,000,000 shares.
- They are exercised against by the in-the-money calls for \$50,000,000.
- The trader is left with an obligation of

$$51{,}524{,}853 - 50{,}000{,}000 = 1{,}524{,}853,$$

  which represents the replication cost.
- So if we had started with the PV of \$1,524,853, we would have replicated 10,000 such calls in *this* scenario.

Example (concluded)

- The FV of the call premium equals

$$1{,}767{,}910 \times e^{0.06 \times 4/52} = 1{,}776{,}088.$$

- That means the net gain in this scenario is

$$1{,}776{,}088 - 1{,}524{,}853 = 251{,}235$$

if we are hedging 10,000 short European calls.

Tracking Error Revisited

- Define the dollar gamma as $S^2\Gamma$.
- The change in value of a delta-hedged *long* option position after a duration of Δt is proportional to the dollar gamma.
- It is about

$$(1/2)S^2\Gamma[\,(\Delta S/S)^2 - \sigma^2\Delta t\,].$$

  - $(\Delta S/S)^2$ is called the daily realized variance.

Tracking Error Revisited (continued)

- In our particular case,

$$S = 50, \Gamma = 0.0957074, \Delta S = 1, \sigma = 0.3, \Delta t = 1/52.$$

- The estimated tracking error is

$$-(1/2)\times 50^2 \times 0.0957074 \times \left[\, (1/50)^2 - (0.09/52) \,\right] = 159,205.$$

- It is very close to our earlier number of 171,622.[a]
- Delta hedge is also called gamma scalping.[b]

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[a]Recall Eq. (98) on p. 702.

[b]Bennett (2014).

Tracking Error Revisited (continued)

- Let the rebalancing times be $t_1, t_2, \ldots, t_n$.
- Let $\Delta S_i = S_{i+1} - S_i$.
- The total tracking error at expiration is about

$$\sum_{i=0}^{n-1} e^{r(T-t_i)} \frac{S_i^2 \Gamma_i}{2} \left[\left(\frac{\Delta S_i}{S_i} \right)^2 - \sigma^2 \Delta t \right].$$

- The tracking error is clearly path dependent.
- Mathematically,[a]

$$\sum_{i=0}^{n-1} \left(\frac{\Delta S_i}{S_i} \right)^2 \to \sigma^2 T.$$

[a] Protter (2005).

Tracking Error Revisited (concluded)[a]

- The tracking error[b] ϵ_n *over* n *rebalancing acts* has about the same probability of being positive as being negative.
- Subject to certain regularity conditions, the root-mean-square tracking error $\sqrt{E[\,\epsilon_n^2\,]}$ is $O(1/\sqrt{n}\,)$.[c]
- The root-mean-square tracking error increases with σ at first and then decreases.

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[a] Bertsimas, Kogan, & Lo (2000).

[b] Such as 251,235 on p. 708.

[c] Grannan & Swindle (1996).

Which Probability Measure?[a]

- The profit and loss (i.e., tracking error) of a hedging strategy should be calculated under the *real-world* probability measure.
- But the deltas and option prices should be calculated under the *risk-neutral* probability measure.
- If whenever we sample the next stock price, backward induction is performed for the delta, it will take a long time to obtain the distribution of the profit and loss.
- How to do it efficiently?

[a] Contributed by Mr. Chiu, Tzu-Hsuan (R08723061) on April 9, 2021.

Delta-Gamma Hedge

- Delta hedge is based on the first-order approximation to changes in the derivative price, Δf, due to changes in the stock price, ΔS.
- When ΔS is not small, the second-order term, gamma $\Gamma \stackrel{\Delta}{=} \partial^2 f / \partial S^2$, helps.
- A delta-gamma hedge is a delta hedge that maintains zero portfolio gamma; it is gamma neutral.
- To meet this extra condition, one more security needs to be brought in.

Delta-Gamma Hedge (concluded)

- Suppose we want to hedge short calls as before.
- A hedging call f_2 is brought in.
- To set up a delta-gamma hedge, we solve

$$\begin{aligned} -N \times f + n_1 \times S + n_2 \times f_2 - B &= 0 \quad \text{(self-financing)}, \\ -N \times \Delta + n_1 + n_2 \times \Delta_2 - 0 &= 0 \quad \text{(delta neutrality)}, \\ -N \times \Gamma + 0 + n_2 \times \Gamma_2 - 0 &= 0 \quad \text{(gamma neutrality)}, \end{aligned}$$

  for n_1, n_2, and B.

  - The gammas of the stock and bond are 0.

- See the numerical example on pp. 231–232 of the text.

Other Hedges

- If volatility changes, delta-gamma hedge may not work well.
- An enhancement is the delta-gamma-vega hedge, which also maintains vega zero portfolio vega.
- To accomplish this, still one more security has to be brought into the process.
- In practice, delta-vega hedge, which may not maintain gamma neutrality, performs better than delta hedge.

Trees

I love a tree more than a man.
— Ludwig van Beethoven (1770–1827)

All those holes and pebbles.
Who could count them?
— James Joyce, *Ulysses* (1922)

And though the holes were rather small,
they had to count them all.
— The Beatles, *A Day in the Life* (1967)

The Combinatorial Method

- The combinatorial method can often cut the running time by an order of magnitude.
- The basic paradigm is to count the number of admissible paths that lead from the root to any terminal node.
- We first used this method in the linear-time algorithm for standard European option pricing.[a]
- We will now apply it to price barrier options.

[a]Recall p. 289.

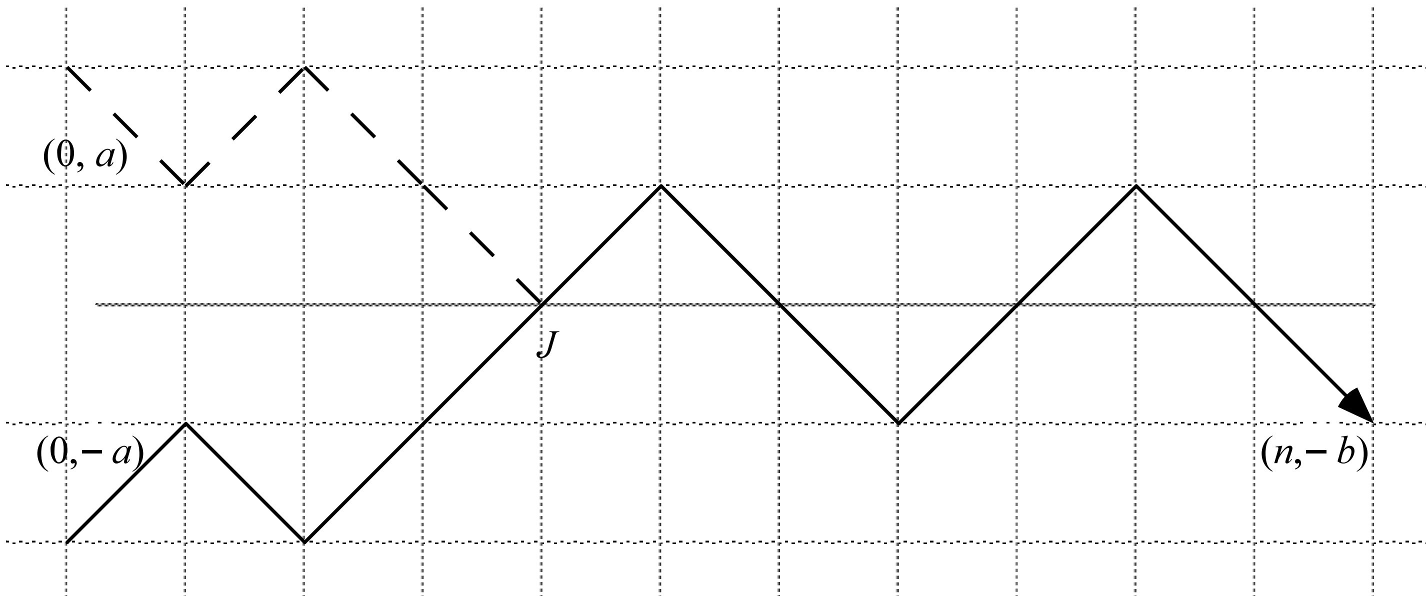
The Reflection Principle[a]

- Imagine a particle at position $(0, -\boldsymbol{a})$ on the integral lattice that is to reach $(n, -\boldsymbol{b})$.
- Without loss of generality, assume $\boldsymbol{a} > 0$ and $\boldsymbol{b} \geq 0$.
- This particle's movement:

$$(i, j) \nearrow (i+1, j+1) \quad \text{up move } S \to Su$$
$$(i, j) \searrow (i+1, j-1) \quad \text{down move } S \to Sd$$

- How many paths touch the x axis?

[a] André (1887).


(0, a)
J
(0,− a)
(n,− b)

The Reflection Principle (continued)

- For a path from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touches the x axis, let J denote the first point this happens.
- Reflect the portion of the path from $(0, -\boldsymbol{a})$ to J.
- A path from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$ is constructed.
- It also hits the x axis at J for the first time.
- The one-to-one mapping shows the number of paths from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touch the x axis equals the number of paths from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$.

The Reflection Principle (concluded)

- A path of this kind has $(n + \boldsymbol{b} + \boldsymbol{a})/2$ down moves and $(n - \boldsymbol{b} - \boldsymbol{a})/2$ up moves.[a]
- Hence there are

$$\binom{n}{\frac{n+\boldsymbol{a}+\boldsymbol{b}}{2}} = \binom{n}{\frac{n-\boldsymbol{a}-\boldsymbol{b}}{2}} \tag{99}$$

such paths for *even* $n + \boldsymbol{a} + \boldsymbol{b}$.

  – Convention: $\binom{n}{k} = 0$ for $k < 0$ or $k > n$.

[a] Verify it!

Pricing Barrier Options (Lyuu, 1998)

- Focus on the down-and-in call with barrier $H < X$.
- So $H < S$.
- Define

$$a \triangleq \left\lceil \frac{\ln\left(X/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rceil = \left\lceil \frac{\ln(X/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rceil,$$

$$h \triangleq \left\lfloor \frac{\ln\left(H/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rfloor = \left\lfloor \frac{\ln(H/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rfloor.$$

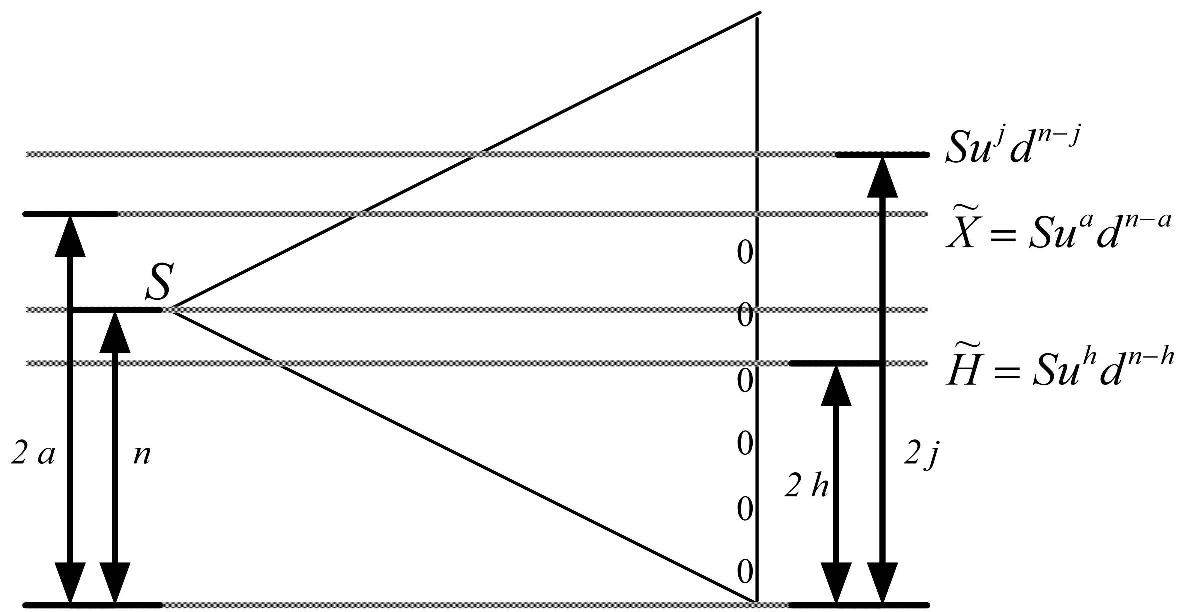
  - a is such that $\tilde{X} \triangleq Su^a d^{n-a}$ is the *terminal* price that is closest to X from above.
  - h is such that $\tilde{H} \triangleq Su^h d^{n-h}$ is the *terminal* price that is closest to H from below.[a]

[a] So we *under*estimate the option price.

Pricing Barrier Options (continued)

- The true barrier is replaced by the effective barrier $\tilde{H}$ in the binomial model.
- A process with n moves hence ends up in the money if and only if the number of up moves is at least a.
- The price $Su^k d^{n-k}$ is at a distance of $2k$ from the lowest possible price Sd^n on the binomial tree as

$$Su^k d^{n-k} = Sd^{-k} d^{n-k} = Sd^{n-\boxed{2k}}. \tag{100}$$



$Su^j d^{n-j}$
$\widetilde{X} = Su^a d^{n-a}$
S
$\widetilde{H} = Su^h d^{n-h}$
0
0
0
0
0
0
2 a
n
2 h
2 j

Pricing Barrier Options (continued)

- A path from S to the terminal price $Su^j d^{n-j}$ has probability $p^j (1-p)^{n-j}$ of being taken.
- With reference to p. 726, the reflection principle (p. 721) can be applied with

$$\begin{aligned} \boldsymbol{a} &= n - 2h, \\ \boldsymbol{b} &= 2j - 2h, \end{aligned}$$

in Eq. (99) on p. 723 by treating the $\tilde{H}$ line as the x axis.

Pricing Barrier Options (continued)

- Therefore,

$$\binom{n}{\frac{n+(n-2h)+(2j-2h)}{2}} = \binom{n}{n-2h+j}$$

paths hit $\tilde{H}$ in the process for $h \leq n/2$.

- The terminal price $Su^j d^{n-j}$ is reached by a path that hits the effective barrier with probability

$$\binom{n}{n-2h+j} p^j (1-p)^{n-j}, \quad j \leq 2h.$$

Pricing Barrier Options (concluded)

- The option value equals

$$\frac{\sum_{j=a}^{2h} \binom{n}{n-2h+j} p^j (1-p)^{n-j} \left(Su^j d^{n-j} - X\right)}{R^n}. \tag{101}$$

  - $R \stackrel{\Delta}{=} e^{r\tau/n}$ is the riskless return per period.

- It yields a linear-time algorithm.[a]

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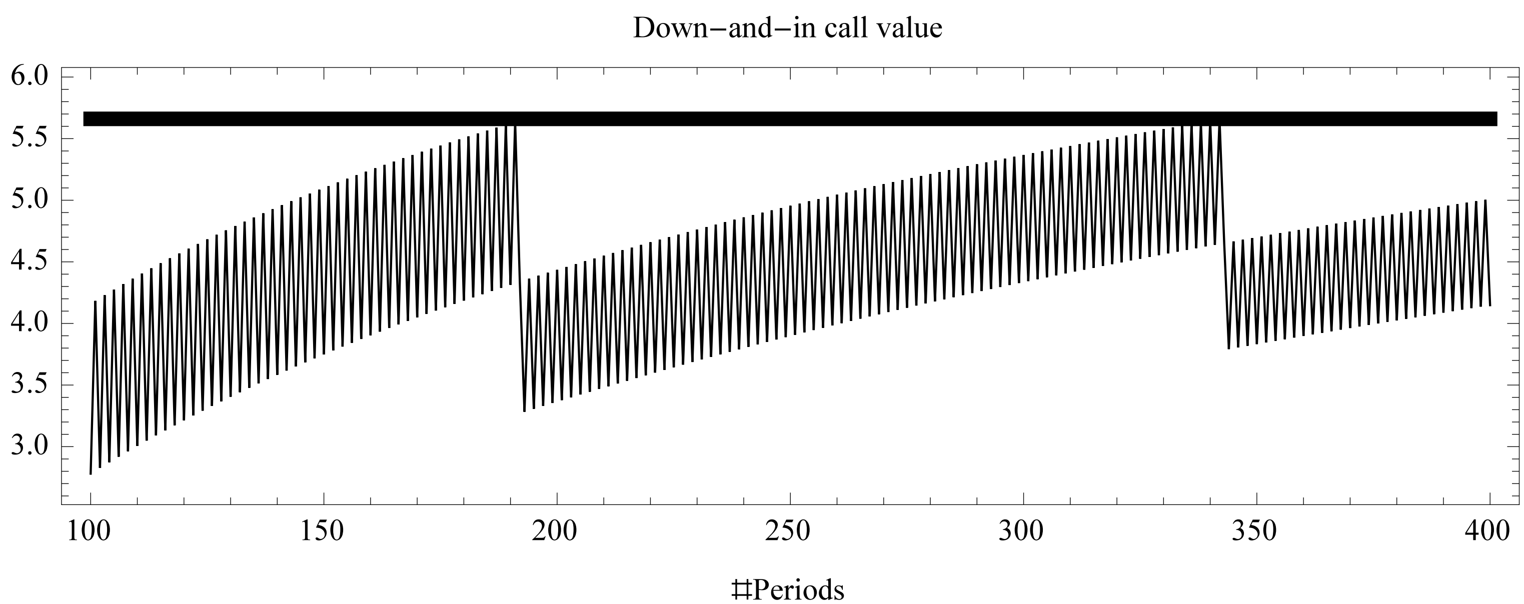
[a]Lyuu (1998).

Convergence of BOPM

- Equation (101) results in the same sawtooth-like convergence shown on p. 410 (repeated on next page).
- The reasons are not hard to see.
- The effective barrier $\tilde{H}$ rarely equals the true barrier H.

Convergence of BOPM (continued)



Convergence of BOPM (continued)

- Convergence is actually good if we limit n to certain values—191, for example.
- These values make the true barrier coincide with or just above one of the stock price levels, that is,

$$H \approx Sd^j = Se^{-j\sigma\sqrt{\tau/n}}$$

for some integer j.

- The preferred n's are thus

$$n = \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor, \quad j = 1, 2, 3, \ldots$$

Convergence of BOPM (continued)

- There is only one minor technicality left.
- We picked the effective barrier to be one of the $n + 1$ possible *terminal* stock prices.
- However, the effective barrier above, Sd^j, corresponds to a terminal stock price only when $n - j$ is even.[a]
- To close this gap, we decrement n by one, if necessary, to make $n - j$ an even number.

[a]This is because $j = n - 2k$ for some k by Eq. (100) on p. 725. Of course we could have adopted the more general form Sd^j $(-n \leq j \leq n)$ for the effective barrier. It makes a good exercise.

Convergence of BOPM (concluded)

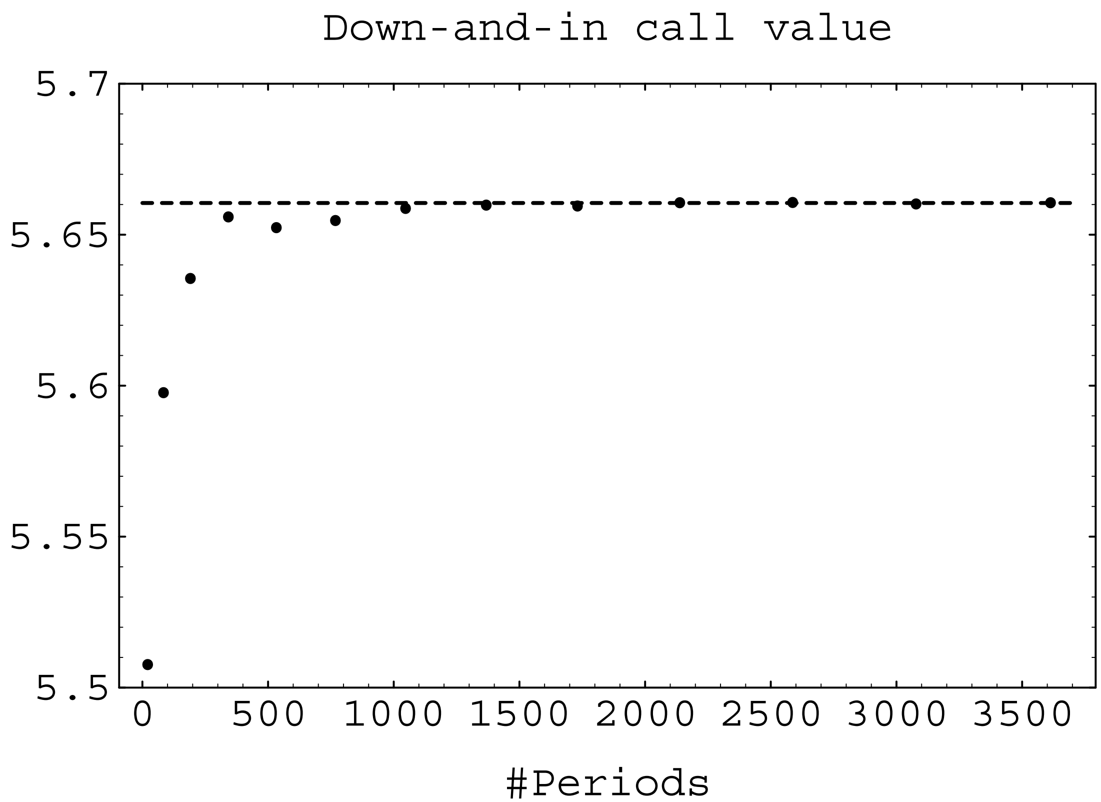
- The preferred n's are now

$$n = \begin{cases} \ell, & \text{if } \ell - j \text{ is even}, \\ \ell - 1, & \text{otherwise}, \end{cases} \tag{102}$$

  $j = 1, 2, 3, \ldots$, where

$$\ell \triangleq \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor.$$

- Evaluate pricing formula (101) on p. 729 only with the n above.


Down-and-in call value
5.7
5.65
5.6
5.55
5.5
0 500 1000 1500 2000 2500 3000 3500
#Periods

Practical Implications[a]

- This binomial model is $O(1/\sqrt{n})$ convergent in general but $O(1/n)$ convergent when the barrier is matched.[b]
- Now that barrier options can be efficiently priced, we can afford to pick very large n (see next page).
- This has profound consequences.[c]

---

[a] Lyuu (1998).
[b] J. Lin (R95221010) (2008); J. Lin (R95221010) & Palmer (2013).
[c] See pp. 751ff.

n	Combinatorial method	
	Value	Time (milliseconds)
21	5.507548	0.30
84	5.597597	0.90
191	5.635415	2.00
342	5.655812	3.60
533	5.652253	5.60
768	5.654609	8.00
1047	5.658622	11.10
1368	5.659711	15.00
1731	5.659416	19.40
2138	5.660511	24.70
2587	5.660592	30.20
3078	5.660099	36.70
3613	5.660498	43.70
4190	5.660388	44.10
4809	5.659955	51.60
5472	5.660122	68.70
6177	5.659981	76.70
6926	5.660263	86.90
7717	5.660272	97.20

Practical Implications (concluded)

- Pricing is prohibitively time consuming when $S \approx H$ because

$$n \sim 1/\ln^2(S/H)$$

  by formula (102) on 734.
  - This is called the barrier-too-close problem.
- This observation is indeed true of standard quadratic-time binomial tree algorithms.
- But it no longer applies to linear-time algorithms (see next page).

	Barrier at 95.0			Barrier at 99.5			Barrier at 99.9	
n	Value	Time	n	Value	Time	n	Value	Time
	⋮		795	7.47761	8	19979	8.11304	253
2743	2.56095	31.1	3184	7.47626	38	79920	8.11297	1013
3040	2.56065	35.5	7163	7.47682	88	179819	8.11300	2200
3351	2.56098	40.1	12736	7.47661	166	319680	8.11299	4100
3678	2.56055	43.8	19899	7.47676	253	499499	8.11299	6300
4021	2.56152	48.1	28656	7.47667	368	719280	8.11299	8500
True	2.5615			7.4767			8.1130	

(All times in milliseconds.)

Trinomial Tree

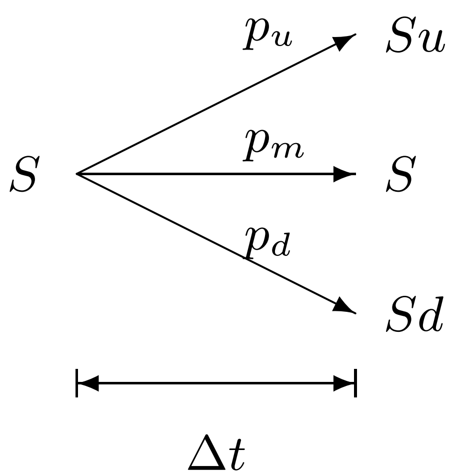
- Set up a trinomial approximation to the geometric Brownian motion[a]

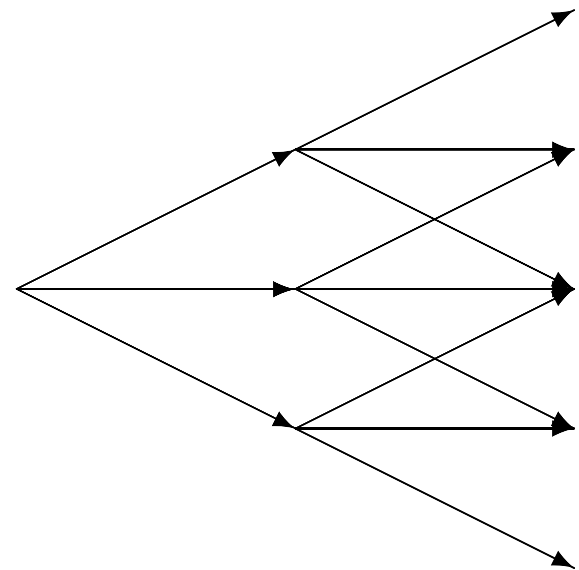
$$\frac{dS}{S} = r\,dt + \sigma\,dW.$$

- The three stock prices at time Δt are S, Su, and Sd, where $ud = 1$.
- Let the mean and variance of the stock price be SM and S^2V, respectively.

---

[a]Parkinson (1977); Boyle (1986).


p_u
Su
p_m
S
S
p_d
Sd
Δt



Trinomial Tree (continued)

- By Eqs. (29) on p. 181,

$$
\begin{aligned}
M &\stackrel{\Delta}{=} e^{r\Delta t}, \\
V &\stackrel{\Delta}{=} M^2(e^{\sigma^2 \Delta t} - 1).
\end{aligned}
$$

- Impose the matching of mean and that of variance:

$$
\begin{aligned}
1 &= p_u + p_m + p_d, \\
SM &= [\, p_u u + p_m + (p_d/u) \,]\, S, \\
S^2 V &= p_u(Su - SM)^2 + p_m(S - SM)^2 + p_d(Sd - SM)^2.
\end{aligned}
$$

Trinomial Tree (continued)

- Use linear algebra to verify that

$$p_u = \frac{u\left(V + M^2 - M\right) - (M - 1)}{(u - 1)\left(u^2 - 1\right)},$$

$$p_d = \frac{u^2\left(V + M^2 - M\right) - u^3(M - 1)}{(u - 1)\left(u^2 - 1\right)}.$$

- We must also make sure the probabilities lie between 0 and 1.

Trinomial Tree (concluded)

- There are countless variations.
- But all converge to the Black-Scholes option pricing model.[a]
- Like the binomial model,[b] the trinomial model has a linear-time algorithm for European options.[c]

---

[a] Madan, Milne, & Shefrin (1989).

[b] Recall p. 289 and p. 729.

[c] T. Chen (R94922003) (2007); J. Wang (R85526003), C. Wang (F95922018), T. Dai (B82506025, R86526008, D8852600), T. Chen (R94922003), L. Liu, & Zhou (2022).

A Trinomial Tree

- Use $u = e^{\lambda\sigma\sqrt{\Delta t}}$, where $\lambda \geq 1$ is a tunable parameter.
- Then

$$p_u \to \frac{1}{2\lambda^2} + \frac{\left(r + \sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma},$$

$$p_d \to \frac{1}{2\lambda^2} - \frac{\left(r - 2\sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}.$$

- A nice choice for λ is $\sqrt{\pi/2}$.[a]

---

[a] Omberg (1988).

Barrier Options Revisited

- BOPM introduces a specification error by replacing the barrier with a nonidentical effective barrier.
- The trinomial model solves the problem by adjusting λ so that the barrier is hit exactly.[a]
- When
$$Se^{-h\lambda\sigma\sqrt{\Delta t}} = H,$$
it takes h down moves to go from S to H, if h is an integer.
- Then
$$h = \frac{\ln(S/H)}{\lambda\sigma\sqrt{\Delta t}}.$$

[a] Ritchken (1995).

Barrier Options Revisited (continued)

- This is easy to achieve by adjusting λ.
- Typically, we find the smallest $\lambda \geq 1$ such that h is an integer.[a]
  - Such a λ may not exist for very small n's.[b]
- Toward that end, we find the *largest* integer $j \geq 1$ that satisfies $\frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}} \geq 1$ to be the h.
- Then let

$$\lambda = \frac{\ln(S/H)}{h\sigma\sqrt{\Delta t}}.$$

[a] Why must $\lambda \geq 1$?

[b] This is not hard to check.

Barrier Options Revisited (continued)

- Alternatively, simply pick

$$h = \left\lfloor \frac{\ln(S/H)}{\sigma\sqrt{\Delta t}} \right\rfloor.$$

- Make sure $h \geq 1$.
- Then let

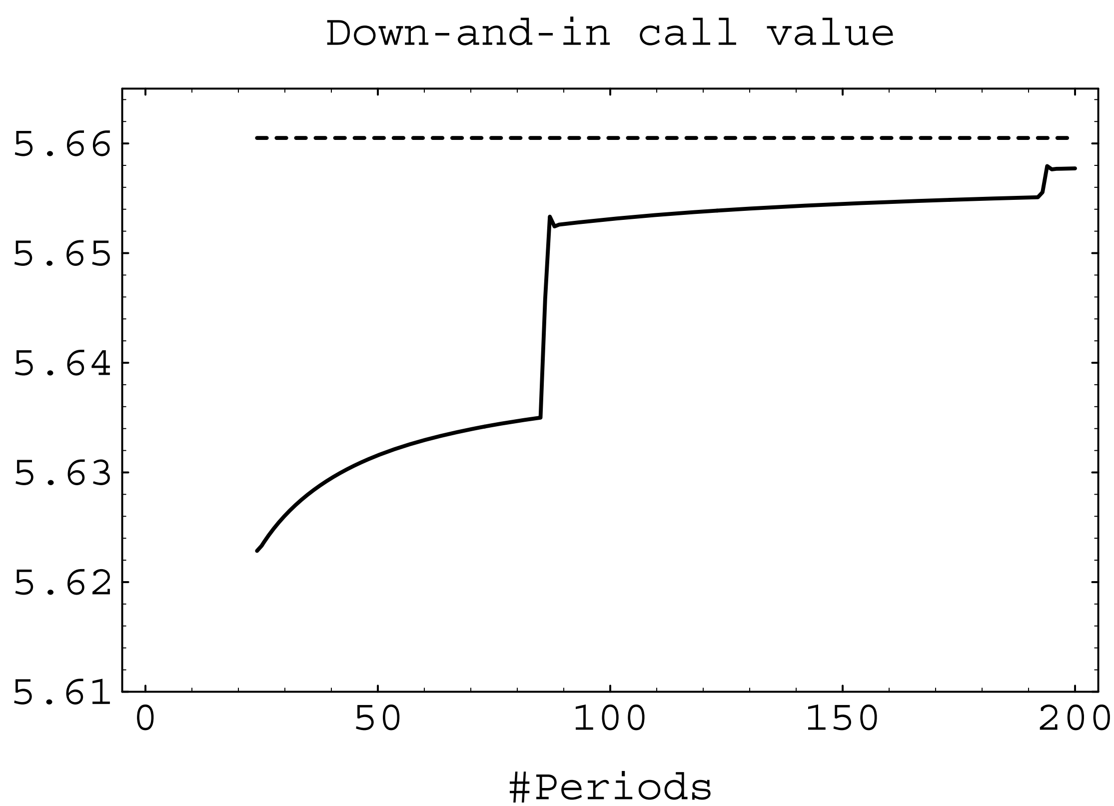
$$\lambda = \frac{\ln(S/H)}{h\sigma\sqrt{\Delta t}}.$$

Barrier Options Revisited (concluded)

- This done, one of the layers of the trinomial tree coincides with the barrier.
- The following probabilities may be used,

$$\begin{aligned} p_u &= \frac{1}{2\lambda^2} + \frac{\mu' \sqrt{\Delta t}}{2\lambda\sigma}, \\ p_m &= 1 - \frac{1}{\lambda^2}, \\ p_d &= \frac{1}{2\lambda^2} - \frac{\mu' \sqrt{\Delta t}}{2\lambda\sigma}. \end{aligned}$$

  – $\mu' \stackrel{\Delta}{=} r - (\sigma^2/2)$.


Down-and-in call value
5.66
5.65
5.64
5.63
5.62
5.61
0
50
100
150
200
#Periods

Algorithms Comparison[a]

- So which algorithm is better, binomial or trinomial?
- Algorithms are often compared based on the n value at which they "converge."
  - The one with the smallest n wins.
- So giraffes are faster than cheetahs because they take fewer strides to travel the same distance!
- Performance must be based on actual running times, not n.[b]

---

[a]Lyuu (1998).
[b]Patterson & Hennessy (1994).

Algorithms Comparison (continued)

- Pages 731 and 750 seem to show the trinomial model converges at a smaller n than BOPM.
- It is in this sense when people say trinomial models converge faster than binomial ones.
- But does it make the trinomial model better then?

Algorithms Comparison (concluded)

- The linear-time binomial tree algorithm actually performs better than the trinomial one.
- See the next page, expanded from p. 737.
- The barrier-too-close problem is also too hard for a quadratic-time trinomial tree algorithm.[a]
  - See pp. 764ff for an alternative solution.

---

[a] Lyuu (1998).

n	Combinatorial method		Trinomial tree algorithm	
	Value	Time	Value	Time
21	5.507548	0.30		
84	5.597597	0.90	5.634936	35.0
191	5.635415	2.00	5.655082	185.0
342	5.655812	3.60	5.658590	590.0
533	5.652253	5.60	5.659692	1440.0
768	5.654609	8.00	5.660137	3080.0
1047	5.658622	11.10	5.660338	5700.0
1368	5.659711	15.00	5.660432	9500.0
1731	5.659416	19.40	5.660474	15400.0
2138	5.660511	24.70	5.660491	23400.0
2587	5.660592	30.20	5.660493	34800.0
3078	5.660099	36.70	5.660488	48800.0
3613	5.660498	43.70	5.660478	67500.0
4190	5.660388	44.10	5.660466	92000.0
4809	5.659955	51.60	5.660454	130000.0
5472	5.660122	68.70		
6177	5.659981	76.70		

(All times in milliseconds.)

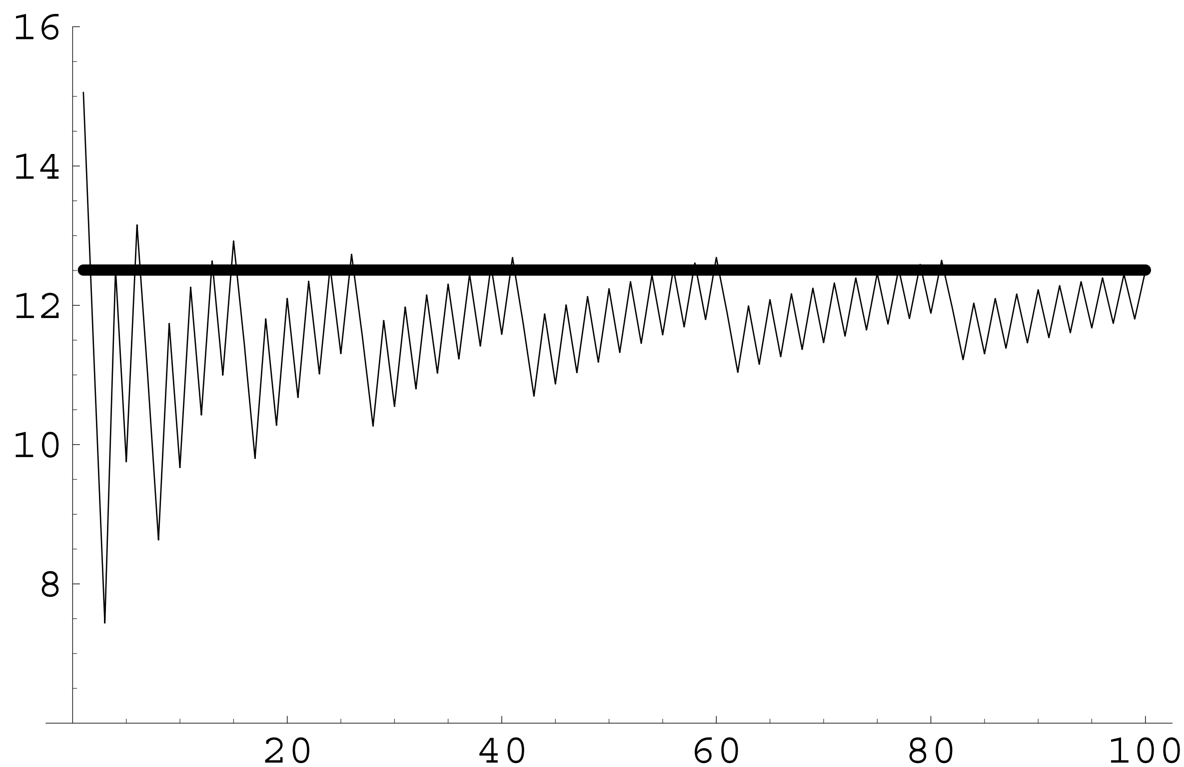
Double-Barrier Options

- Double-barrier options are barrier options with two barriers $L < H$.
  - They make up "less than 5% of the light exotic market."[a]
- Assume $L < S < H$.
- The binomial model produces oscillating option values (see plot on next page).[b]

---

[a] Bennett (2014).

[b] Chao (R86526053) (1999); T. Dai (B82506025, R86526008, D8852600) & Lyuu (2005).


16
14
12
10
8
20
40
60
80
100

Double-Barrier Options (concluded)

- The combinatorial method yields a linear-time algorithm.[a]
- This binomial model is $O(1/\sqrt{n})$ convergent in general.[b]
- If the barriers L and H depend on time, we have moving-barrier options.[c]

---

[a] See p. 241 of the textbook.
[b] Gobet (1999).
[c] Rogers & Zane (1998).

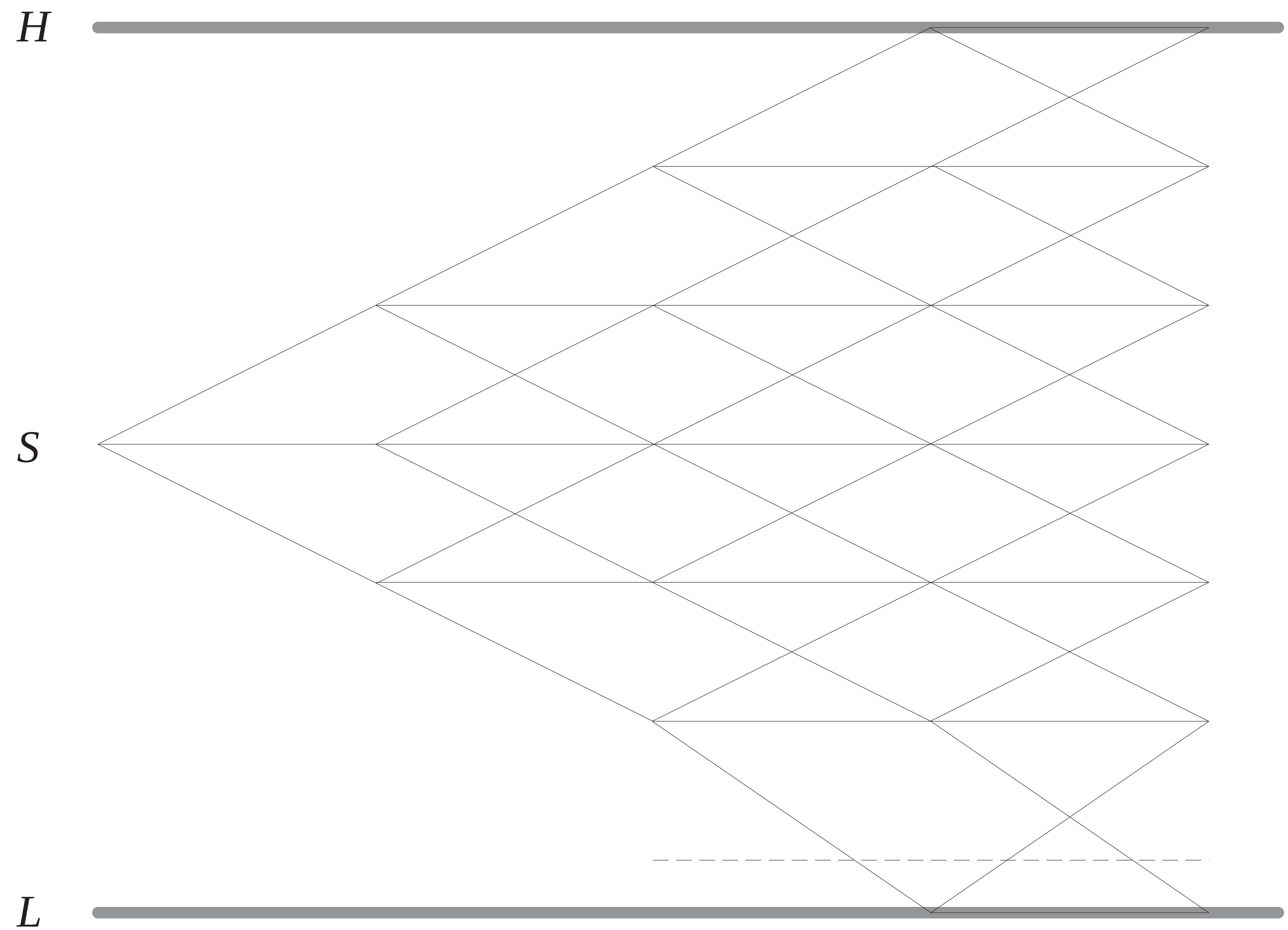
Double-Barrier Knock-Out Options

- We knew how to pick the λ so that one of the layers of the trinomial tree coincides with one barrier, say H.
- This choice, however, does not guarantee that the other barrier, L, is also hit.
- One way to handle this problem is to *lower* the layer of the tree just above L to coincide with L.[a]
  - More general ways to make the trinomial model hit both barriers are available.[b]

---

[a] Ritchken (1995); Hull (1999).

[b] Hsu (R7526001, D89922012) & Lyuu (2006). T. Dai (B82506025, R86526008, D8852600) & Lyuu (2006) combine binomial and trinomial trees to derive an $O(n)$-time algorithm for double-barrier options (see pp. 764ff).


H
S
L

Double-Barrier Knock-Out Options (continued)

- The probabilities of the nodes on the layer above L must be adjusted.
- Let ℓ be the positive integer such that

$$Sd^{\ell+1} < L < Sd^{\ell}.$$

- Hence the layer of the tree just above L has price Sd^{ℓ}.[a]

[a] You probably cannot do the same thing for binomial models (why?). Thanks to a lively discussion on April 25, 2012.

Double-Barrier Knock-Out Options (concluded)

- Define $\gamma > 1$ as the number satisfying

$$L = Sd^{\ell-1} e^{-\gamma\lambda\sigma\sqrt{\Delta t}}.$$

  – The prices between the barriers are (from low to high)

$$L, Sd^{\ell-1}, \ldots, Sd^2, Sd, S, Su, Su^2, \ldots, Su^{h-1}, Su^h = H.$$

- The probabilities for the nodes with price equal to $Sd^{\ell-1}$ are

$$p'_u = \frac{b + a\gamma}{1+\gamma}, \quad p'_d = \frac{b-a}{\gamma+\gamma^2}, \quad \text{and} \quad p'_m = 1 - p'_u - p'_d,$$

where $a \stackrel{\Delta}{=} \mu'\sqrt{\Delta t}/(\lambda\sigma)$ and $b \stackrel{\Delta}{=} 1/\lambda^2$.

Convergence: Binomial vs. Trinomial