

Forwards, Futures, Futures Options, Swaps

Summon the nations to come to the trial.
Which of their gods can predict the future?
— Isaiah 43:9

The sure fun of the evening
outweighed the uncertain treasure[.]
— Mark Twain (1835–1910),
The Adventures of Tom Sawyer

Terms

- r will denote the riskless interest rate.
- The current time is t .
- The maturity date is T .
- The remaining time to maturity is $\tau \triangleq T - t$ (years).
- The spot price is S .
- The spot price at maturity is S_T .
- The delivery price is X .

Terms (concluded)

- The forward or futures price is F for a newly written contract.
- The value of the contract is f .
- A price with a subscript t usually refers to the price at time t .
- Continuous compounding will be assumed.

Forward Contracts

- *Long* forward contracts are for the *purchase* of the underlying asset for a certain delivery price on a specific time.
 - Foreign currencies, bonds, corn, etc.
- Ideal for hedging purposes.
- A farmer enters into a forward contract with a food processor to deliver 100,000 bushels of corn for \$2.5 per bushel on September 27, 1995.^a
- The farmer is assured of a buyer at an acceptable price.
- The processor knows the cost of corn in advance.

^aThe farmer assumes a *short* position.

Forward Contracts (concluded)

- A forward agreement limits both risk and rewards.
 - If the spot price of corn rises on the delivery date, the farmer will miss the opportunity of extra profits.
 - If the price declines, the processor will be paying more than it would.
- Either side has an incentive to default.
- Other problems: The food processor may go bankrupt, the farmer can go bust, the farmer might not be able to harvest 100,000 bushels of corn because of bad weather, the cost of growing corn may skyrocket, etc.

Spot and Forward Exchange Rates

- Let S denote the spot exchange rate.
- Let F denote the forward exchange rate one year from now (both in domestic/foreign terms).
- r_f denotes the annual interest rate of the foreign currency.
- r_ℓ denotes the annual interest rate of the local currency.
- Arbitrage opportunities will arise unless these four numbers satisfy an equation.

Interest Rate Parity^a

$$\frac{F}{S} = e^{r_\ell - r_f}. \quad (59)$$

- A holder of the local currency can do either of:
 - Lend the money in the domestic market to receive e^{r_ℓ} one year from now.
 - Convert local currency for foreign currency, lend for 1 year in foreign market, and convert foreign currency into local currency at the fixed forward exchange rate, F , by selling forward foreign currency *now*.

^aKeynes (1923). John Maynard Keynes (1883–1946) was one of the greatest economists in history. The parity broke down in late 2008 (Mancini-Griffoli & Ranaldo, 2013).

Interest Rate Parity (concluded)

- No money changes hand in entering into a forward contract.
- One unit of local currency will hence become Fe^{r_f}/S one year from now in the 2nd case.
- If $Fe^{r_f}/S > e^{r_\ell}$, an arbitrage profit can result from borrowing money in the domestic market and lending it in the foreign market.
- If $Fe^{r_f}/S < e^{r_\ell}$, an arbitrage profit can result from borrowing money in the foreign market and lending it in the domestic market.

Forward Price

- The payoff of a forward contract at maturity is

$$S_T - X.$$

- Contrast that with call's payoff

$$\max(S_T - X, 0).$$

- Forward contracts do not involve any initial cash flow.
- The forward price F is the delivery price X which makes the forward contract zero valued.
 - That is,

$$f = 0 \quad \text{when} \quad X = F.$$

Forward Price (continued)



Forward Price (concluded)

- The delivery price cannot change because it is written in the contract.
- But the forward price may change after the contract comes into existence.
- So although the value of a forward contract, f , is 0 at the outset, it will fluctuate thereafter.
 - This value is enhanced when the spot price climbs.
 - It is depressed when the spot price declines.
- The forward price also varies with the maturity of the contract.

Forward Price: Underlying Pays No Income

Lemma 12 *For a forward contract on an underlying asset providing no income,*

$$F = Se^{r\tau}. \quad (60)$$

- If $F > Se^{r\tau}$:
 - Borrow S dollars for τ years.
 - Buy the underlying asset.
 - Short the forward contract with delivery price F .

Proof (concluded)

- At maturity:
 - Deliver the asset for F .
 - Use $Se^{r\tau}$ to repay the loan, leaving an arbitrage profit of

$$F - Se^{r\tau} > 0.$$

- If $F < Se^{r\tau}$, do the opposite.

Example: Zero-Coupon Bonds

- r is the annualized 3-month riskless interest rate.
- S is the spot price of the 6-month zero-coupon bond.
- A new 3-month forward contract on that 6-month zero-coupon bond should command a delivery price of $Se^{r/4}$.
- So if $r = 6\%$ and $S = 970.87$, then the delivery price is

$$970.87 \times e^{0.06/4} = 985.54.$$

Example: Options

- Suppose S is the spot price of the *European call* that expires at some time later than T .
- A τ -year forward contract on that call commands a delivery price of $Se^{r\tau}$.
- So it equals the future value of the Black-Scholes formula on p. 306.

Contract Value: The Underlying Pays No Income

The value of a forward contract is

$$f = S - Xe^{-r\tau}. \quad (61)$$

- Consider a portfolio consisting of:
 - One long forward contract;
 - Cash amount $Xe^{-r\tau}$;
 - One short position in the underlying asset.

Contract Value: The Underlying Pays No Income (concluded)

- The cash will grow to X at maturity, which can be used to take delivery of the forward contract.
- The delivered asset will then close out the short position.
- Since the value of the portfolio is zero at maturity, its PV must be zero.^a
- So a forward contract can be replicated by a long position in the underlying and a loan of $Xe^{-r\tau}$ dollars.

^aRecall p. 222.

Lemma 12 (p. 492) Revisited

- Set $f = 0$ in Eq. (61) on p. 496.
- Then $X = Se^{r\tau}$, the forward price.

Forward Price: Underlying Pays Predictable Income

Lemma 13 *For a forward contract on an underlying asset providing a predictable income with a PV of I ,*

$$F = (S - I) e^{r\tau}. \quad (62)$$

- If $F > (S - I) e^{r\tau}$, borrow S dollars for τ years, buy the underlying asset, and short the forward contract with delivery price F .
- Use the income to repay part of the loan.

The Proof (concluded)

- At maturity, the asset is delivered for F , and $(S - I) e^{r\tau}$ is used to repay the *remaining* loan.
- That leaves an arbitrage profit of

$$F - (S - I) e^{r\tau} > 0.$$

- If $F < (S - I) e^{r\tau}$, reverse the above.

Example

- Consider a 10-month forward contract on a \$50 stock.
- The stock pays a dividend of \$1 every 3 months.
- The forward price is

$$\left(50 - e^{-r_3/4} - e^{-r_6/2} - e^{-3 \times r_9/4}\right) e^{r_{10} \times (10/12)}.$$

– r_i is the annualized i -month interest rate.

Underlying Pays a Continuous Dividend Yield of q

- The value of a forward contract at any time prior to T is^a

$$f = Se^{-q\tau} - Xe^{-r\tau}. \quad (63)$$

- One consequence of Eq. (63) is that the forward price is

$$F = Se^{(r-q)\tau}. \quad (64)$$

^aSee p. 160 of the textbook for proof.

Futures Contracts vs. Forward Contracts

- They are traded on a central exchange.
- A clearinghouse.
 - Credit risk is minimized.
- Futures contracts are standardized instruments.
- Gains and losses are marked to market daily.
 - Adjusted at the end of each trading day based on the settlement price.

Size of a Futures Contract

- The amount of the underlying asset to be delivered under the contract.
 - 5,000 bushels for the corn futures on the Chicago Board of Trade (CBOT).
 - One million U.S. dollars for the Eurodollar futures on the Chicago Mercantile Exchange (CME).^a
- A position can be closed out (or offset) by entering into a reversing trade to the original one.
- Most futures contracts are closed out in this way rather than have the underlying asset delivered.
 - Forward contracts are meant for delivery.

^aCME and CBOT merged on July 12, 2007.

Daily Settlements

- Price changes in the futures contract are settled daily.
- Hence the spot price rather than the initial futures price is paid on the delivery date.
- Marking to market nullifies any financial incentive for not making delivery.
 - A farmer enters into a forward contract to sell 100,000 bushels of corn at \$2.00 per bushel in November.
 - Suppose the price of corn rises to \$2.5 by November.
 - The farmer has incentive to sell his harvest in the spot market for \$2.5.

Daily Settlements (concluded)

- (continued)
 - With marking to market, the farmer has transferred \$0.5 per bushel from his futures account to that of the clearinghouse by November.^a
 - When the farmer makes delivery, he is paid the spot price, \$2.5 per bushel.
 - The farmer has little incentive to default.
 - The *net* price remains $\$2.5 - 0.5 = 2$ per bushel, the original delivery price.

^aSee p. 507.

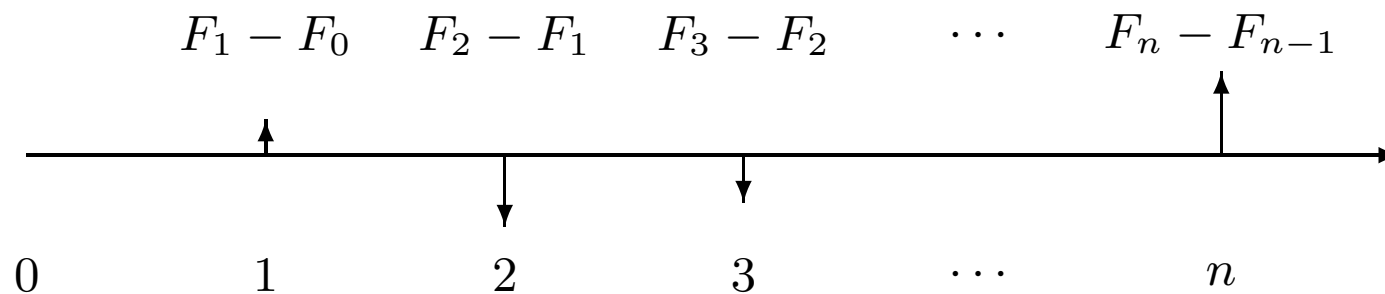
Daily Cash Flows

- Let F_i denote the futures price at the end of day i .
- The contract's cash flow on day i is $F_i - F_{i-1}$.
- The net cash flow over the life of the contract is

$$\begin{aligned}(F_1 - F_0) + (F_2 - F_1) + \cdots + (F_n - F_{n-1}) \\ = F_n - F_0 = S_T - F_0.\end{aligned}$$

- A futures contract has the same *accumulated* payoff $S_T - F_0$ as a forward contract.
- The actual payoff may vary because of the reinvestment of daily cash flows and how $S_T - F_0$ is distributed.

Daily Cash Flows (concluded)



Delivery and Hedging

- Futures price is the delivery price that makes the futures contract zero-valued.
- Delivery ties the futures price to the spot price.
- On the delivery date, the settlement price of the futures contract is determined by the spot price.
- Hence, when the delivery period is reached, the futures price should be very close to the spot price.^a
- Changes in futures prices usually track those in spot price, making hedging possible.

^aBut since early 2006, futures for corn, wheat, and soybeans occasionally expired at a price much higher than that day's spot price (Henriques, 2008).

Forward and Futures Prices

- Surprisingly, futures price equals forward price if interest rates are nonstochastic!^a
- This result “justifies” treating a futures contract as if it were a forward contract, ignoring its marking-to-market feature.
- The West Texas Intermediate (WTI) futures price was negative on April 20, 2020!^b

^aCox, Ingersoll, & Ross (1981); see p. 164 of the textbook for proof.

^bApril 21 was the last trading day for oil delivery in May to Cushing, Oklahoma.

Remarks

- When interest rates are stochastic, forward and futures prices are no longer theoretically identical.
 - Suppose interest rates are uncertain and futures prices move in the same direction as interest rates.
 - Then futures prices will exceed forward prices.
- For short-term contracts, the differences tend to be small.
- Unless stated otherwise, assume forward and futures prices are identical.

Futures Options

- The underlying of a futures option is a futures contract.
- Upon exercise, the option holder takes a position in the futures contract with a futures price equal to the option's strike price.
 - A call holder acquires a *long* futures position.
 - A put holder acquires a *short* futures position.
- The futures contract is then marked to market.
- And the futures positions of the two parties are at the prevailing futures price (thus zero-valued).

Futures Options (concluded)

- It works as if the *call* holder received a futures contract plus cash equivalent to the prevailing futures price F_t minus the strike price X :

$$F_t - X.$$

– This futures contract has zero value.

- It works as if the *put* holder sold a futures contract for

$$X - F_t$$

dollars.

Forward Options

- What is delivered is now a forward contract with a delivery price equal to the option's strike price.
 - Exercising a call forward option results in a *long* position in a forward contract.
 - Exercising a put forward option results in a *short* position in a forward contract.
- Exercising a forward option incurs no immediate cash flows as there is no marking to market.

Example

- Consider an American call with strike \$100 and an expiration date in September.
- The underlying asset is a *forward* contract with a delivery date in December.
- Suppose the forward price in July is \$110.
- Upon exercise, the call holder receives a forward contract with a delivery price of \$100.^a
- If an offsetting position is then taken in the forward market,^b a \$10 profit *in December* will be assured.
- A call on the futures would realize the \$10 profit *in July*.

^aRecall p. 484.

^bThe counterparty will pay you \$110 for the underlying asset.

Some Pricing Relations

- Let T be the delivery date, the current time be 0, and the option^a have expiration date t ($t \leq T$).
- Assume a constant, positive interest rate.
- Although forward price equals futures price, a forward option does *not* have the same value as a futures option.
- The payoffs of calls at time t are, respectively,^b

$$\text{futures option} = \max(F_t - X, 0), \quad (66)$$

$$\text{forward option} = \max(F_t - X, 0) e^{-r(T-t)}. \quad (67)$$

^aOn the futures or forward contract

^bRecall p. 513.

Some Pricing Relations (concluded)

- A European futures option is worth the same as a European option on the underlying asset if $T = t$.
 - Futures price equals spot price at maturity.
- This conclusion is model independent.

Put-Call Parity^a

The put-call parity is slightly different from the one in Eq. (31) on p. 229.

Theorem 14 (1) *For European options on futures contracts,*

$$C = P - (X - F) e^{-rt}.$$

(2) *For European options on forward contracts,*

$$C = P - (X - F) e^{-rT}.$$

^aSee Theorem 12.4.4 of the textbook for proof.

Early Exercise

The early exercise feature is not valuable for *forward* options.^a

Theorem 15 *American forward options should not be exercised before expiration as long as the probability of their ending up out of the money is positive.*

Early exercise may be optimal for American *futures* options even if the underlying asset generates no payouts.

Theorem 16 *American futures options may be exercised optimally before expiration.*

^aSee Theorem 12.4.5 of the textbook for proof.

Black's Model^a

- Formulas for European futures options:

$$C = Fe^{-rt}N(x) - Xe^{-rt}N(x - \sigma\sqrt{t}), \quad (68)$$

$$P = Xe^{-rt}N(-x + \sigma\sqrt{t}) - Fe^{-rt}N(-x), \quad (69)$$

where $x \triangleq \frac{\ln(F/X) + (\sigma^2/2)t}{\sigma\sqrt{t}}$.

- The above formulas are related to those for options on a stock paying a continuous dividend yield.
- They are Eqs. (44) on p. 337 with q set to r and S replaced by F .

^aBlack (1976). It is also called the Black-76 model.

Black's Model (concluded)

- Volatility σ is that of the stock price.^a
 - It is the same as the volatility of the futures price.^b
- This observation incidentally proves Theorem 16 (p. 519).
- For European forward options, just multiply the above formulas by $e^{-r(T-t)}$.
 - Forward options differ from futures options by a factor of $e^{-r(T-t)}$.^c

^aContributed by Mr. Lu, Yu-Ming (R06723032, D08922008) on April 7, 2021.

^bContributed by Mr. Chen, Tung-Li (D09922014) on April 7, 2023.
See also p. 632.

^cRecall Eqs. (66)–(67) on p. 516.

Binomial Model for Forward and Futures Options

- In a risk-neutral economy, futures price behaves *like* a stock paying a continuous dividend yield of r .
 - Let the futures price at time 0 be F .
 - From Lemma 10 (p. 304), the expected value of S at time Δt in a risk-neutral economy is

$$S e^{r\Delta t}.$$

- By Eq. (60) on p. 492, the expected futures price at time Δt is

$$S e^{r\Delta t} e^{r(T-\Delta t)} = S e^{rT} = F.$$

Binomial Model for Forward and Futures Options (continued)

- The above observation continues to hold even if S pays a dividend yield!^a
 - Let the futures price at time 0 be F .
 - From Lemma 10 (p. 304), the expected value of S at time Δt in a risk-neutral economy is

$$S e^{(r-q) \Delta t}.$$

- By Eq. (64) on p. 502, the expected futures price at time Δt is

$$S e^{(r-q) \Delta t} e^{(r-q)(T-\Delta t)} = S e^{(r-q) T} = F.$$

^aContributed by Mr. Liu, Yi-Wei (R02723084) on April 16, 2014.

Binomial Model for Forward and Futures Options (concluded)

- Now, under the BOPM, the risk-neutral probability for the futures price is

$$p_f \triangleq (1 - d)/(u - d) \quad (70)$$

by Eq. (45) on p. 339.

- The futures price moves from F to Fu with probability p_f and to Fd with probability $1 - p_f$.
 - The *original* u and d are used here.
- The binomial tree algorithm for *forward* options is identical except that Eq. (67) on p. 516 is the payoff.

Spot and Futures Prices under BOPM

- The futures price is related to the spot price via

$$F = Se^{rT}$$

if the underlying asset pays no dividends.^a

- Recall the futures price F moves to Fu with probability p_f per period.
- So the stock price moves from $S = Fe^{-rT}$ to

$$Fue^{-r(T-\Delta t)} = Sue^{r\Delta t}$$

with probability p_f per period.

^aRecall Lemma 12 (p. 492).

Spot and Futures Prices under BOPM (concluded)

- Similarly, the stock price moves from $S = Fe^{-rT}$ to

$$Sde^{r\Delta t}$$

with probability $1 - p_f$ per period.

- Note that

$$S(ue^{r\Delta t})(de^{r\Delta t}) = Se^{2r\Delta t} \neq S.$$

- This binomial model for S differs from the CRR tree.
- This model may not be suitable for pricing barrier options (why?).

Negative Probabilities Revisited

- As $0 < p_f < 1$, we have $0 < 1 - p_f < 1$ as well.
- The problem of negative risk-neutral probabilities is solved:
 - Build the tree for the futures price F of the futures contract expiring at the same time as the option.
 - Let the stock pay a continuous dividend yield of q .
 - By Eq. (64) on p. 502, recover S from F at each node via

$$S = F e^{-(r-q)(T-t)}.$$

Swaps

- Swaps are agreements between two counterparties to exchange cash flows in the future according to some predetermined formula.
- There are two basic types of swaps: interest rate and currency.
- An interest rate swap occurs when two parties exchange interest payments periodically.
- Currency swaps are agreements to deliver one currency against another (our focus here).
- There are theories about why swaps exist.^a

^aThanks to a lively discussion on April 16, 2014.