Trading Days and Calendar Days

- Interest accrues based on the calendar day.
- But $\sigma$ is usually calculated based on trading days only.
  - Stock price seems to have lower volatilities when the exchange is closed.$^a$

- How to harmonize these two different times into the Black-Scholes formula and binomial tree algorithms?$^b$

---

$^a$Fama (1965); K. French (1980); K. French & Roll (1986).
$^b$Recall p. 163 about dating issues.
Trading Days and Calendar Days (continued)

- Think of $\sigma$ as measuring the *annualized* volatility of stock price *one year from now*.

- Suppose a year has $m$ (say 253) trading days.

- We can replace $\sigma$ in the Black-Scholes formula with\(^a\)

$$
\sigma \sqrt{\frac{365}{m}} \times \frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}.
$$

\(^a\)D. French (1984).
Trading Days and Calendar Days (concluded)

- This works only for European options.
- How about binomial tree algorithms?\textsuperscript{a}

\textsuperscript{a}Contributed by Mr. Lu, Zheng-Liang (D00922011) in 2015.
Options on a Stock That Pays Dividends

- Early exercise must be considered.
- Proportional dividend payout model is tractable (see text).
  - The dividend amount is a constant proportion of the \textit{prevailing} stock price.
- In general, the corporate dividend policy is a complex issue.
Known Dividends

- Constant dividends introduce complications.
- Use $D$ to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$.
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: $(Su - D)u$, $(Su - D)d$, $(Sd - D)u$, $(Sd - D)d$.
  - The binomial tree no longer combines.
\[(Su - D) u\]
\[Su - D\]
\[(Su - D) d\]
\[S\]
\[(Sd - D) u\]
\[Sd - D\]
\[(Sd - D) d\]
An Ad-Hoc Approximation

- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends.

- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.

- The riskless component at any time is the PV of future dividends during the life of the option.
  
  - Then, $\sigma$ is the volatility of the process followed by the risky component.

- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

---

*a*Roll (1977); Heath & Jarrow (1988).
An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all future dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.
The Ad-Hoc Approximation vs. P. 328 (Step 1)

\[(S - D/R)u^2\]

\[(S - D/R)u\]

\[(S - D/R)d\]

\[(S - D/R)ud\]

\[(S - D/R)d^2\]
The Ad-Hoc Approximation vs. P. 328 (Step 2)

\[(S - D/R)u^2\]

\[(S - D/R)u\]

\[(S - D/R) + D/R = S\]

\[(S - D/R)ud\]

\[(S - D/R)d\]

\[(S - D/R)d^2\]
The Ad-Hoc Approximation vs. P. 328

- The trees are different.
- The stock prices at maturity are also different.
  - \((Su - D)u, (Su - D)d, (Sd - D)u, (Sd - D)d\) (p. 328).
  - \((S - D/R)u^2, (S - D/R)ud, (S - D/R)d^2\) (ad hoc).
- Note that, as \(d < R < u\),
  - \((Su - D)u > (S - D/R)u^2\),
  - \((Sd - D)d < (S - D/R)d^2\),

---

\(^{a}\)Contributed by Mr. Yang, Jui-Chung (D97723002) on March 18, 2009.
The Ad-Hoc Approximation vs. P. 328 (concluded)

- So the ad hoc approximation has a smaller dynamic range.

- This explains why in practice the volatility is usually increased when using the ad hoc approximation.
A General Approach\textsuperscript{a}

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases (see pp. 790ff).
- Other approaches include adjusting $\sigma$ and approximating the known dividend with a dividend yield.\textsuperscript{b}

\textsuperscript{a}Dai (B82506025, R86526008, D8852600) & Lyuu (2004). Also Arealy & Rodrigues (2013).

\textsuperscript{b}Geske & Shastri (1985). It works well for American options but not European ones (Dai, 2009).
Continuous Dividend Yields

- Dividends are paid continuously.
  - Approximates a broad-based stock market portfolio.

- The payment of a continuous dividend yield at rate $q$ reduces the growth rate of the stock price by $q$.
  - A stock that grows from $S$ to $S_\tau$ with a continuous dividend yield of $q$ would have grown from $S$ to $S_\tau e^{q\tau}$ without the dividends.

- A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays no dividends.\(^a\)

\(^a\)In pricing European options, only the distribution of $S_\tau$ matters.
Continuous Dividend Yields (continued)

• So the Black-Scholes formulas hold with $S$ replaced by $Se^{-q\tau}$.

$$C = Se^{-q\tau}N(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}), \quad (44)$$

$$P = Xe^{-r\tau}N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau}N(-x), \quad (44')$$

where

$$x \equiv \frac{\ln(S/X) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$ 

• Formulas (44) and (44') remain valid as long as the dividend yield is predictable.

---

\(^{a}\)Merton (1973).
Continuous Dividend Yields (continued)

- To run binomial tree algorithms, replace $u$ with $ue^{-q\Delta t}$ and $d$ with $de^{-q\Delta t}$, where $\Delta t \triangleq \tau/n$.
  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.

- Other than the changes, binomial tree algorithms stay the same.
  - In particular, $p$ should use the original $u$ and $d$!{a}

---

{a}Contributed by Ms. Wang, Chuan-Ju (F95922018) on May 2, 2007.
Continuous Dividend Yields (concluded)

- Alternatively, pick the risk-neutral probability as

\[
\frac{e^{(r-q)\Delta t} - d}{u - d},
\]

(45)

where \( \Delta t \overset{\Delta}{=} \tau/n \).

- The reason: The stock price grows at an expected rate of \( r - q \) in a risk-neutral economy.

- The \( u \) and \( d \) remain unchanged.

- Except the change in Eq. (45), binomial tree algorithms stay the same \textit{as if there were no dividends}. 
Distribution of Logarithmic Returns of TAIEX

Exercise Boundaries of American Options (in the Continuous-Time Model)\textsuperscript{a}

- The exercise boundary is a nondecreasing function of $t$ for American puts (see the plot next page).

- The exercise boundary is a nonincreasing function of $t$ for American calls.

\textsuperscript{a}See Section 9.7 of the textbook for the tree analog.
Sensitivity Analysis of Options
Cleopatra’s nose, had it been shorter, the whole face of the world would have been changed.
— Blaise Pascal (1623–1662)
Sensitivity Measures ("The Greeks")

- How the value of a security changes relative to changes in a given parameter is key to hedging.
  - Duration, for instance.

- Let $x \triangleq \frac{\ln(S/X) + (r + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}$ (recall p. 306).

- Recall that

\[ N'(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} > 0, \]

the density function of standard normal distribution.
Delta

• Defined as

\[ \Delta \triangleq \frac{\partial f}{\partial S}. \]

- \( f \) is the price of the derivative.
- \( S \) is the price of the underlying asset.

• The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.\(^\text{a}\)

• The delta used in the BOPM (p. 250) is the discrete analog.

• The delta of a long stock is 1.

\(^\text{a}\)Elementary calculus.
Delta (continued)

- The delta of a European call on a non-dividend-paying stock equals
\[
\frac{\partial C}{\partial S} = N(x) > 0. \tag{46}
\]

- The delta of a European put equals
\[
\frac{\partial P}{\partial S} = N(x) - 1 = -N(-x) < 0. \tag{47}
\]

- So the deltas of a call and an otherwise identical put cancel each other when \( N(x) = 1/2 \), i.e., when\(^a\)
\[
X = Se^{(r+\sigma^2/2)\tau}. \tag{48}
\]

\(^a\)The straddle (p. 214) \( C + P \) then has zero delta!
Dotted curve: in-the-money call or out-of-the-money put.
Solid curves: at-the-money options \((X = 50)\).
Dashed curves: out-of-the-money calls or in-the-money puts.
Delta (continued)

- Suppose the stock pays a continuous dividend yield of $q$.
- Let

$$x \triangleq \frac{\ln(S/X) + (r - q + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}$$

(recall p. 337).

- Then

$$\frac{\partial C}{\partial S} = e^{-q\tau} N(x) > 0,$$
$$\frac{\partial P}{\partial S} = -e^{-q\tau} N(-x) < 0.$$
Delta (continued)

- Consider an $X_1$-strike call and an $X_2$-strike put, $X_1 \geq X_2$.
- They are otherwise identical.
- Let

$$x_i \equiv \frac{\ln(S/X_i) + (r - q + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}. \quad (50)$$

- Then their deltas sum to zero when $x_1 = -x_2$.\(^a\)
- That implies

$$\frac{S}{X_1} = \frac{X_2}{S} e^{-(2r-2q+\sigma^2) \tau}. \quad (51)$$

\(^a\)The strangle (p. 216) \(C + P\) then has zero delta!
Delta (concluded)

- Suppose we demand \( X_1 = X_2 = X \) and have a straddle.
- Then
  \[
  X = Se^{(r - q + \sigma^2 / 2) \tau}
  \]
  leads to a straddle with zero delta.
  - This generalizes Eq. (48) on p. 347.
- When \( C(X_1) \)'s delta and \( P(X_2) \)'s delta sum to zero,\(^a\) does the portfolio \( C(X_1) - P(X_2) \) have zero value?
- In general, no.

\(^a\)Meaning \( C(X_1) + P(X_2) \) has zero delta.
Delta Neutrality

• A position with a total delta equal to 0 is delta-neutral.
  – A delta-neutral portfolio is immune to small price changes in the underlying asset.

• Creating one serves for hedging purposes.
  – A portfolio consisting of a call and \(-\Delta\) shares of stock is delta-neutral.
  – Short \(\Delta\) shares of stock to hedge a long call.
  – Long \(\Delta\) shares of stock to hedge a short call.

• In general, hedge a position in a security with delta \(\Delta_1\) by shorting \(\Delta_1/\Delta_2\) units of a security with delta \(\Delta_2\).
Theta (Time Decay)

- Defined as the rate of change of a security’s value with respect to time, or \( \Theta \triangleq -\partial f / \partial \tau = \partial f / \partial t \).

- For a European call on a non-dividend-paying stock,
  \[
  \Theta = -\frac{S N'(x) \sigma}{2 \sqrt{\tau}} - r X e^{-r \tau} N(x - \sigma \sqrt{\tau}) < 0.
  \]
  - The call loses value with the passage of time.

- For a European put,
  \[
  \Theta = -\frac{S N'(x) \sigma}{2 \sqrt{\tau}} + r X e^{-r \tau} N(-x + \sigma \sqrt{\tau}).
  \]
  - Can be negative or positive.

- Both are consistent with the plots on p. 196.
Dotted curve: in-the-money call or out-of-the-money put.
Solid curves: at-the-money options.
Dashed curve: out-of-the-money call or in-the-money put.
Theta (concluded)

• Suppose the stock pays a continuous dividend yield of \( q \).

• Define \( x \) as in Eq. (49) on p. 349.

• For a European call, add an extra term to the earlier formula for the theta:

\[
\Theta = -\frac{SN'(x) \sigma}{2\sqrt{\tau}} - rX e^{-r\tau} N(x - \sigma \sqrt{\tau}) + qS e^{-q\tau} N(x).
\]

• For a European put, add an extra term to the earlier formula for the theta:

\[
\Theta = -\frac{SN'(x) \sigma}{2\sqrt{\tau}} + rX e^{-r\tau} N(-x + \sigma \sqrt{\tau}) - qS e^{-q\tau} N(-x).
\]
Gamma

- Defined as the rate of change of its delta with respect to the price of the underlying asset, or \( \Gamma \triangleq \frac{\partial^2 \Pi}{\partial S^2} \).

- Measures how sensitive delta is to changes in the price of the underlying asset.

- In practice, a portfolio with a high gamma needs to be rebalanced more often to maintain delta neutrality.

- Roughly, delta ~ duration, and gamma ~ convexity.

- The gamma of a European call or put on a non-dividend-paying stock is

\[
N'(x)/(S\sigma\sqrt{\tau}) > 0.
\]
Dotted lines: in-the-money call or out-of-the-money put.
Solid lines: at-the-money option.
Dashed lines: out-of-the-money call or in-the-money put.
Gamma (concluded)

• Gamma is maximized when the option is nearly at the money, i.e.,
  \[ S = X e^{-(r + 3\sigma^2 / 2) \tau}. \]

• As the at-the-money option approaches expiration, its gamma tends to rise.

• The gammas of other options, however, tend to zero.
Vega\(^\text{a}\) (Lambda, Kappa, Sigma, Zeta)

- Defined as the rate of change of a security’s value with respect to the volatility of the underlying asset
  \[ \Lambda \triangleq \frac{\partial f}{\partial \sigma}. \]

- Volatility often changes over time.

- A security with a high vega is very sensitive to changes to or estimation error in volatility.

- The vega of a European call or put on a non-dividend-paying stock is \( S \sqrt{\tau} N'(x) > 0. \)
  - So higher volatility raises the option value.

\(^a\)Vega is not Greek. Alexander (2001), “This is a term that was invented by Americans, and intended to sound like a Greek letter.”
Vega (continued)

• Note that

\[ \Lambda = \tau \sigma S^2 \Gamma. \]

• If the stock pays a continuous dividend yield of \( q \), then

\[ \Lambda = S e^{-q \tau} \sqrt{\tau} N'(x), \]

where \( x \) is defined in Eq. (49) on p. 349.

• Vega is maximized when \( x = 0 \), i.e., when

\[ S = X e^{-(r-q+\sigma^2/2) \tau}. \]

• Vega declines very fast as \( S \) moves away from that peak.

\(^a\)Reiss & Wystup (2001).
Vega (continued)

- Now consider a portfolio consisting of an \( X_1 \)-strike call \( C \) and a short \( X_2 \)-strike put \( P \), \( X_1 \geq X_2 \).

- The options’ vegas cancel out when

\[
x_1 = -x_2,
\]

where \( x_i \) are defined in Eq. (50) on p. 350.

- This also leads to Eq. (51) on p. 350.
  - Recall the same condition led to zero delta for the strangle \( C + P \) (p. 350).
Vega (concluded)

- Note that $\tau \to 0$ implies

$$\Lambda \to 0$$

(which answers the question on p. 311).

- The Black-Scholes formula (p. 306) implies

$$C \to S, \quad P \to X e^{-r\tau},$$

as $\sigma \to \infty$.\(^a\)

- These boundary conditions are handy for some numerical methods.

\(^a\)Recall that $C \geq \max (S - X e^{-r\tau}, 0)$ by Exercise 8.3.2 of the text and $P \geq \max (X e^{-r\tau} - S, 0)$ by Lemma 4 (p. 234).
Dotted curve: in-the-money call or out-of-the-money put.
Solid curves: at-the-money option.
Dashed curve: out-of-the-money call or in-the-money put.
Rho

- Defined as the rate of change in its value with respect to interest rates
  \[ \rho \triangleq \frac{\partial f}{\partial r} \, . \]

- The rho of a European call on a non-dividend-paying stock is
  \[ X \tau e^{-r \tau} N(x - \sigma \sqrt{\tau}) > 0. \]

- The rho of a European put on a non-dividend-paying stock is
  \[ -X \tau e^{-r \tau} N(-x + \sigma \sqrt{\tau}) < 0. \]
Dotted curves: in-the-money call or out-of-the-money put.
Solid curves: at-the-money option.
Dashed curves: out-of-the-money call or in-the-money put.
Numerical Greeks

• Needed when closed-form formulas do not exist.

• Take delta as an example.

• A standard method computes the finite difference,

\[
\frac{f(S + \Delta S) - f(S - \Delta S)}{2\Delta S}.
\]

• The computation time roughly doubles that for evaluating the derivative security itself.
An Alternative Numerical Delta\textsuperscript{a}

- Use intermediate results of the binomial tree algorithm.
- When the algorithm reaches the end of the first period, \( f_u \) and \( f_d \) are computed.
- These values correspond to derivative values at stock prices \( S_u \) and \( S_d \), respectively.
- Delta is approximated by
  \[
  \frac{f_u - f_d}{S_u - S_d}.
  \tag{52}
  \]
- Essentially zero extra cost.

\textsuperscript{a}Pelsser & Vorst (1994).
Numerical Gamma

• At the stock price \((Suu + Sud)/2\), delta is approximately \((f_{uu} - f_{ud})/(Suu - Sud)\).

• At the stock price \((Sud + Sdd)/2\), delta is approximately \((f_{ud} - f_{dd})/(Sud - Sdd)\).

• Gamma is the rate of change in deltas between \((Suu + Sud)/2\) and \((Sud + Sdd)/2\), that is,

\[
\frac{f_{uu} - f_{ud}}{Suu - Sud} - \frac{f_{ud} - f_{dd}}{Sud - Sdd} \quad \frac{f_{uu} - f_{ud}}{Suu - Sdd}/2. \tag{53}
\]
Alternative Numerical Delta and Gamma$^a$

• Let $\epsilon \equiv \ln u$.

• Think in terms of $\ln S$.

• Then

$$\left( \frac{f_u - f_d}{2\epsilon} \right) \frac{1}{S}$$

approximates the numerical delta.

• And

$$\left( \frac{f_{uu} - 2f_{ud} + f_{dd}}{\epsilon^2} - \frac{f_{uu} - f_{dd}}{2\epsilon} \right) \frac{1}{S^2}$$

approximates the numerical gamma.

$^a$See p. 690.
Finite Difference Fails for Numerical Gamma

• Numerical differentiation gives

\[
\frac{f(S + \Delta S) - 2f(S) + f(S - \Delta S)}{(\Delta S)^2}.
\]

• It does not work (see text for the reason).

• In general, calculating gamma is a hard problem numerically.\(^a\)

• But why did the binomial tree version work?

\(^a\)Recall p. 116.
Other Numerical Greeks

- The theta can be computed as

\[ \frac{f_{ud} - f}{2(\tau/n)} \] .

- In fact, the theta of a European option can be derived from delta and gamma.\(^a\)

- The vega of a European option can be derived from gamma.\(^b\)

- For rho, there seems no alternative but to run the binomial tree algorithm twice.\(^c\)

---

\(^a\)See p. 688.

\(^b\)Recall p. 360.

\(^c\)But see p. 870 and pp. 1064ff.
Extensions of Options Theory
As I never learnt mathematics, so I have had to think.
— Joan Robinson (1903–1983)
Pricing Corporate Securities\textsuperscript{a}

- Interpret the underlying asset as the firm’s total value.\textsuperscript{b}
- The option pricing methodology can be applied to price corporate securities.
- The result is called the structural model.
- Assumptions:
  - A firm can finance payouts by the sale of assets.
  - If a promised payment to an obligation other than stock is missed, the claim holders take ownership of the firm and the stockholders get nothing.

\textsuperscript{a}Black & Scholes (1973); Merton (1974).
\textsuperscript{b}More realistic models posit firm value = asset value + tax benefits − bankruptcy costs (Leland & Toft, 1996).
Risky Zero-Coupon Bonds and Stock

• Consider XYZ.com.

• Capital structure:
  – $n$ shares of its own common stock, $S$.
  – Zero-coupon bonds with an aggregate par value of $X$.

• What is the value of the bonds, $B$?

• What is the value of the XYZ.com stock, $S$?
Risky Zero-Coupon Bonds and Stock (continued)

- On the bonds’ maturity date, suppose the total value of the firm $V^*$ is less than the bondholders’ claim $X$.
- Then the firm declares bankruptcy, and the stock becomes worthless.
- If $V^* > X$, then the bondholders obtain $X$ and the stockholders $V^* - X$.

<table>
<thead>
<tr>
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<th>$V^* \leq X$</th>
<th>$V^* &gt; X$</th>
</tr>
</thead>
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<td>Bonds</td>
<td>$V^*$</td>
<td>$X$</td>
</tr>
<tr>
<td>Stock</td>
<td>0</td>
<td>$V^* - X$</td>
</tr>
</tbody>
</table>
Risky Zero-Coupon Bonds and Stock (continued)

• The stock has the same payoff as a call!

• It is a call on the total value of the firm with a strike price of $X$ and an expiration date equal to the bonds’.
  – This call provides the limited liability for the stockholders.

• The bonds are a covered call\(^a\) on the total value of the firm.

• Let $V$ stand for the total value of the firm.

• Let $C$ stand for a call on $V$.

\(^a\)Recall p. 203.
Risky Zero-Coupon Bonds and Stock (continued)

• Thus

\[ nS = C \text{ (market capitalization of XYZ.com)}, \]
\[ B = V - C. \]

• Knowing \( C \) amounts to knowing how the value of the firm is divided between stockholders and bondholders.

• Whatever the value of \( C \), the total value of the stock and bonds at maturity remains \( V^* \).

• The relative size of debt and equity is irrelevant to the firm’s current value \( V \).
Risky Zero-Coupon Bonds and Stock (continued)

- From Theorem 11 (p. 306) and the put-call parity,\textsuperscript{a}

\[ nS = VN(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}), \quad (54) \]
\[ B = VN(-x) + Xe^{-r\tau}N(x - \sigma\sqrt{\tau}). \quad (55) \]

- Above,

\[
x \triangleq \frac{\ln(V/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.
\]

- The continuously compounded yield to maturity of the firm’s bond is

\[
\frac{\ln(X/B)}{\tau}.
\]

\textsuperscript{a}This is sometimes called Merton’s (1974) structural model.
Risky Zero-Coupon Bonds and Stock (continued)

- Define the credit spread or default premium as the yield difference between risky and riskless bonds,

\[
\frac{\ln(X/B)}{\tau} - r = -\frac{1}{\tau} \ln \left[ N(-z) + \frac{1}{\omega} N(z - \sigma \sqrt{\tau}) \right].
\]

- \( \omega \triangleq X e^{-r\tau}/V. \)
- \( z \triangleq \ln \omega/(\sigma \sqrt{\tau}) + (1/2) \sigma \sqrt{\tau} = -x + \sigma \sqrt{\tau}. \)
- Note that \( \omega \) is the debt-to-total-value ratio.
Risky Zero-Coupon Bonds and Stock (concluded)

- In general, suppose the firm has a dividend yield at rate $q$ and the bankruptcy costs are a constant proportion $\alpha$ of the remaining firm value.

- Then Eqs. (54)–(55) on p. 380 become, respectively,

\[ nS = Ve^{-q\tau}N(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}), \]
\[ B = (1 - \alpha)Ve^{-q\tau}N(-x) + Xe^{-r\tau}N(x - \sigma\sqrt{\tau}). \]

- Above,

\[ x \triangleq \frac{\ln(V/X) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}. \]
A Numerical Example

- XYZ.com’s assets consist of 1,000 shares of Merck as of March 20, 1995.
  - Merck’s market value per share is $44.5.
- It issues 1,000 shares of XYZ.com common stock and 30 zero-coupon bonds maturing on July 21, 1995.
- Each bond promises to pay $1,000 at maturity.
- \( n = 1,000, \ V = 44.5 \times n = 44,500, \) and \( X = 30 \times 1,000 = 30,000. \)
- As Merck calls are being traded, we do not need formulas to price them.
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<th>Exp.</th>
<th>Vol.</th>
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<th>Vol.</th>
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<td>441/2</td>
<td>45</td>
<td>Apr</td>
<td>3050</td>
<td>7/8</td>
<td>100</td>
<td>11/8</td>
<td></td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>May</td>
<td>462</td>
<td>13  8/9</td>
<td>50</td>
<td>13/8</td>
<td></td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>Jul</td>
<td>883</td>
<td>115/16</td>
<td>147</td>
<td>13/4</td>
<td></td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>Oct</td>
<td>367</td>
<td>23  4/8</td>
<td>188</td>
<td>21/16</td>
<td></td>
</tr>
</tbody>
</table>
A Numerical Example (continued)

- The Merck option relevant for pricing is the July call with a strike price of $X/n = 30$ dollars.
- Such a call is selling for $15.25$.
- So XYZ.com’s stock is worth $15.25 \times n = 15,250$ dollars.
- The entire bond issue is worth

\[
B = 44,500 - 15,250 = 29,250
\]

dollars.

- Or $975 per bond.
A Numerical Example (continued)

- The XYZ.com bonds are equivalent to a default-free zero-coupon bond with $X$ par value plus $n$ written European puts on Merck at a strike price of $30$.
  - By the put-call parity.\(^a\)

- The difference between $B$ and the price of the default-free bond is the value of these puts.

- The next table shows the total market values of the XYZ.com stock and bonds under various debt amounts $X$.

\(^a\)Recall p. 229.
<table>
<thead>
<tr>
<th>Promised payment to bondholders</th>
<th>Current market value of bonds ( B )</th>
<th>Current market value of stock ( nS )</th>
<th>Current total value of firm ( V )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X )</td>
<td>( B )</td>
<td>( nS )</td>
<td>( V )</td>
</tr>
<tr>
<td>30,000</td>
<td>29,250.0</td>
<td>15,250.0</td>
<td>44,500</td>
</tr>
<tr>
<td>35,000</td>
<td>35,000.0</td>
<td>9,500.0</td>
<td>44,500</td>
</tr>
<tr>
<td>40,000</td>
<td>39,000.0</td>
<td>5,500.0</td>
<td>44,500</td>
</tr>
<tr>
<td>45,000</td>
<td>42,562.5</td>
<td>1,937.5</td>
<td>44,500</td>
</tr>
</tbody>
</table>
A Numerical Example (continued)

- Suppose the promised payment to bondholders is $45,000.
- Then the relevant option is the July call with a strike price of $45,000/n = 45$ dollars.
- Since that option is selling for $1\frac{15}{16}$, the market value of the XYZ.com stock is $(1 + 15/16) \times n = 1,937.5$ dollars.
- The market value of the stock decreases as the debt-to-equity ratio increases.
A Numerical Example (continued)

- There are conflicts between stockholders and bondholders.
- An option’s terms cannot be changed after issuance.
- But a firm can change its capital structure.
- There lies one key difference between options and corporate securities.
  - Parameters such volatility,\(^a\) dividend, and strike price are under partial control of the stockholders or boards.

\(^a\)This is called the asset substitution problem (Myers, 1977).
A Numerical Example (continued)

- Suppose XYZ.com issues 15 more bonds with the same terms to buy back stock.
- The total debt is now \( X = 45,000 \) dollars.
- The table on p. 387 says the total market value of the bonds should be $42,562.5.
- The new bondholders pay
  \[
  42,562.5 \times \left(\frac{15}{45}\right) = 14,187.5
  \]
  dollars.
- The remaining stock is worth $1,937.5.
A Numerical Example (continued)

- The stockholders therefore gain

\[ 14,187.5 + 1,937.5 - 15,250 = 875 \]
dollars.

- The *original* bondholders lose an equal amount,

\[ 29,250 - \frac{30}{45} \times 42,562.5 = 875. \]

- This is called claim dilution.\(^a\)

\(^a\)Fama & M. H. Miller (1972).
A Numerical Example (continued)

• Suppose the stockholders sell \( \frac{1}{3} \times n \) Merck shares to fund a $14,833.3 cash dividend.

• The stockholders now have $14,833.3 in cash plus a call on \( \frac{2}{3} \times n \) Merck shares.

• The strike price remains \( X = 30,000 \).

• This is equivalent to owning \( \frac{2}{3} \) of a call on \( n \) Merck shares with a strike price of $45,000.

• \( n \) such calls are worth $1,937.5 (p. 387).

• So the total market value of the XYZ.com stock is \( \frac{2}{3} \times 1,937.5 = 1,291.67 \) dollars.
A Numerical Example (concluded)

- The market value of the XYZ.com bonds is hence
  \[(2/3) \times n \times 44.5 - 1,291.67 = 28,375\]
dollars.

- Hence the stockholders gain
  \[14,833.3 + 1,291.67 - 15,250 \approx 875\]
dollars.

- The bondholders watch their value drop from $29,250 to $28,375, a loss of $875.
Further Topics

- Other examples:\textsuperscript{a}
  - Stock as compound call when company issues coupon bonds.
  - Subordinated debts as bull call spreads.
  - Warrants as calls.
  - Callable bonds as American calls with 2 strike prices.
  - Convertible bonds.
  - Bonds with safety covenants as barrier options.

\textsuperscript{a}Cox & Rubinstein (1985); Geske (1977).
Further Topics (concluded)

- Securities issued by firms with a complex capital structure must be solved by trees.\textsuperscript{a}

\textsuperscript{a}Dai (B82506025, R86526008, D8852600), Lyuu, & C. Wang (F95922018) (2010).
Distance to Default (DTD)$^a$

- Let $\mu$ be the total value $V$’s rate of expected return.$^b$
- From Eq. (54), on p. 380, the probability of default $\tau$ years from now equals

$$N(-\text{DTD}),$$

where

$$\text{DTD} \triangleq \frac{\ln(V/X) + (\mu - \sigma^2/2)\tau}{\sigma \sqrt{\tau}}.$$  

- $V/X$ is called the leverage ratio.

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$^a$Merton (1974).

$^b$This parameter is generally hard to estimate. Campbell, Hilscher, and Szilagyi use $r + 0.06$ for simplicity.