Terms and Approach

- $C$: call value.
- $P$: put value.
- $X$: strike price
- $S$: stock price
- $\hat{r} > 0$: the continuously compounded riskless rate per period.
- $R \triangleq e^{\hat{r}}$: gross return.
- Start from the discrete-time binomial model.
Binomial Option Pricing Model (BOPM)

- Time is discrete and measured in periods.
- If the current stock price is $S$, it can go to $Su$ with probability $q$ and $Sd$ with probability $1 - q$, where $0 < q < 1$ and $d < u$.
  - In fact, $d \leq R \leq u$ must hold to rule out arbitrage.\(^a\)
- Six pieces of information will suffice to determine the option value based on arbitrage considerations:
  
  $S$, $u$, $d$, $X$, $\hat{r}$, and the number of periods to expiration.

\(^a\)See Exercise 9.2.1 of the textbook. The sufficient condition is $d < R < u$ (Björk, 2009), which we shall assume.
Call on a Non-Dividend-Paying Stock: Single Period

• The expiration date is only one period from now.
• \( C_u \) is the call price at time 1 if the stock price moves to \( S_u \).
• \( C_d \) is the call price at time 1 if the stock price moves to \( S_d \).
• Clearly,

\[
C_u = \max(0, S_u - X), \\
C_d = \max(0, S_d - X).
\]
\[ C_u = \max(0, Su - X) \]
\[ C_d = \max(0, Sd - X) \]
Call on a Non-Dividend-Paying Stock: Single Period (continued)

- Set up a portfolio of $h$ shares of stock and $B$ dollars in riskless bonds.
  - This costs $hS + B$.
  - We call $h$ the hedge ratio or delta.

- The value of this portfolio at time one is
  
  $hSu + RB$, up move,
  
  $hSd + RB$, down move.
Call on a Non-Dividend-Paying Stock: Single Period (continued)

- Choose $h$ and $B$ such that the portfolio replicates the payoff of the call,

$$hS_u + RB = C_u,$$
$$hS_d + RB = C_d.$$
Call on a Non-Dividend-Paying Stock: Single Period (concluded)

- Solve the above equations to obtain

\[ h = \frac{C_u - C_d}{S_u - S_d} \geq 0, \quad (32) \]
\[ B = \frac{uC_d - dC_u}{(u - d) R}. \quad (33) \]

- By the no-arbitrage principle, the European call should cost the same as the equivalent portfolio,\(^a\)

\[ C = hS + B. \]

- As \( uC_d - dC_u < 0 \), the equivalent portfolio is a levered long position in stocks.

\(^a\)Or the replicating portfolio, as it replicates the option.
American Call Pricing in One Period

- Have to consider immediate exercise.
- \( C = \max(hS + B, S - X) \).
  - When \( hS + B \geq S - X \), the call should not be exercised immediately.
  - When \( hS + B < S - X \), the option should be exercised immediately.
- For non-dividend-paying stocks, early exercise is not optimal by Theorem 5 (p. 235).
- So
  \[
  C = hS + B.
  \]
Put Pricing in One Period

- Puts can be similarly priced.

- The delta for the put is \((P_u - P_d)/(S_u - S_d) \leq 0\), where
  \[
  P_u = \max(0, X - S_u), \quad P_d = \max(0, X - S_d).
  \]

- Let \(B = \frac{uP_d - dP_u}{(u-d)R}\).

- The European put is worth \(hS + B\).

- The American put is worth \(\max(hS + B, X - S)\).
  - Early exercise can be optimal with American puts.
Risk

• Surprisingly, the option value is independent of $q$.\(^a\)

• Hence it is independent of the expected value of the stock,

$$qS_u + (1 - q) S_d.$$

• The option value depends on the sizes of price changes, $u$ and $d$, which the investors must agree upon.

• Then the set of possible stock prices is the same whatever $q$ is.

\(^a\)More precisely, not directly dependent on $q$. Thanks to a lively class discussion on March 16, 2011.
Pseudo Probability

• After substitution and rearrangement,

$$hS + B = \frac{\left(\frac{R-d}{u-d}\right) C_u + \left(\frac{u-R}{u-d}\right) C_d}{R}.$$ 

• Rewrite it as

$$hS + B = \frac{pC_u + (1-p) C_d}{R},$$

where

$$p \triangleq \frac{R-d}{u-d}. \quad (34)$$
Pseudo Probability (concluded)

• As $0 < p < 1$, it may be interpreted as probability.

• Alternatively,

$\left( \frac{R-d}{u-d} \right) C_u + \left( \frac{u-R}{u-d} \right) C_d$

interpolates the value at $SR$ through points $(Su, C_u)$ and $(Sd, C_d)$.
Risk-Neutral Probability

• The expected rate of return for the stock is equal to the riskless rate $\hat{r}$ under $p$ as

$$pSu + (1 - p) Sd = RS.$$ (35)

• The expected rates of return of all securities must be the riskless rate when investors are risk-neutral.

• For this reason, $p$ is called the risk-neutral probability.

• The value of an option is the expectation of its discounted future payoff in a risk-neutral economy.

• So the rate used for discounting the FV is the riskless rate$^a$ in a risk-neutral economy.

$^a$Recall the question on p. 241.
Option on a Non-Dividend-Paying Stock: Multi-Period

• Consider a call with two periods remaining before expiration.

• Under the binomial model, the stock can take on 3 possible prices at time two: $S_{uu}$, $S_{ud}$, and $S_{dd}$.
  – There are 4 paths.
  – But the tree combines or recombines; hence there are only 3 terminal prices.

• At any node, the next two stock prices only depend on the current price, not the prices of earlier times.$^a$

$^a$It is Markovian.
Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- Let $C_{uu}$ be the call’s value at time two if the stock price is $S_{uu}$.
- Thus,
  \[ C_{uu} = \max(0, S_{uu} - X). \]
- $C_{ud}$ and $C_{dd}$ can be calculated analogously,
  \[ C_{ud} = \max(0, S_{ud} - X), \]
  \[ C_{dd} = \max(0, S_{dd} - X). \]
\[ C_{uu} = \max(0, Suu - X) \]

\[ C_{ud} = \max(0, Sud - X) \]

\[ C_{dd} = \max(0, Sdd - X) \]
Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- The call values at time 1 can be obtained by applying the same logic:

\[
C_u = \frac{pC_{uu} + (1-p)C_{ud}}{R}, \\
C_d = \frac{pC_{ud} + (1-p)C_{dd}}{R}.
\]  

- Deltas can be derived from Eq. (32) on p. 252.

- For example, the delta at \( C_u \) is

\[
\frac{C_{uu} - C_{ud}}{S_{uu} - S_{ud}}.
\]
Option on a Non-Dividend-Paying Stock: Multi-Period (concluded)

- We now reach the current period.
- Compute

\[
\frac{p C_u + (1 - p) C_d}{R}
\]

as the option price.
- Again, the values of delta \( h \) and \( B \) can be derived from Eqs. (32)–(33) on p. 252.
Early Exercise

- Since the call will not be exercised at time 1 even if it is American, \( C_u \geq S_u - X \) and \( C_d \geq S_d - X \).

- Therefore,

\[
hS + B = \frac{pC_u + (1 - p)C_d}{R} \geq \frac{[pu + (1 - p)d]S - X}{R} \]

\[
= S - \frac{X}{R} > S - X.
\]

- The call again will not be exercised at present.\(^a\)

- So

\[
C = hS + B = \frac{pC_u + (1 - p)C_d}{R}.
\]

\(^a\)Consistent with Theorem 5 (p. 235).
Backward Induction\textsuperscript{a}

- The above expression calculates $C$ from the two successor nodes $C_u$ and $C_d$ and none beyond.
- The same computation happened at $C_u$ and $C_d$, too, as demonstrated in Eq. (36) on p. 263.
- This recursive procedure is called backward induction.
- $C$ equals

$$
[p^2 C_{uu} + 2p(1-p)C_{ud} + (1-p)^2 C_{dd}](1/R^2) \\
= [p^2 \max(0, S_u^2 - X) + 2p(1-p)\max(0, S_{ud} - X) \\
+ (1-p)^2 \max(0, S_d^2 - X)]/R^2.
$$

\textsuperscript{a}Ernst Zermelo (1871–1953).
Backward Induction (continued)

- In the $n$-period case,

$$C = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \times \max(0, Su^j d^{n-j} - X)}{R^n}.$$ 

- The value of a call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.

- Similarly,

$$P = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \times \max(0, X - Su^j d^{n-j})}{R^n}.$$
Backward Induction (concluded)

• Note that

\[ p_j \overset{\Delta}{=} \frac{{n \choose j} p^j (1 - p)^{n-j}}{R^n} \]

is the state price\(^a\) for the state \( Su^j d^{n-j} \), \( j = 0, 1, \ldots, n \).

• In general,

\[ \text{option price} = \sum_j (p_j \times \text{payoff at state } j). \]

\(^a\)Recall p. 213. One can obtain the undiscounted state price \( {n \choose j} p^j (1 - p)^{n-j} \)—the risk-neutral probability—for the state \( Su^j d^{n-j} \) with \( (X_M - X_L)^{-1} \) units of the butterfly spread where \( X_L = Su^j d^{n-j+1} \), \( X_M = Su^j d^{n-j} \), and \( X_H = Su^{j-1} d^{n-j-1} \) (Bahra, 1997).
Risk-Neutral Pricing Methodology

• Every derivative can be priced as if the economy were risk-neutral.

• For a European-style derivative with the terminal payoff function $D$, its value is

\[ e^{-r_n} E^\pi [D]. \tag{37} \]

– $E^\pi$ means the expectation is taken under the risk-neutral probability.

• The “equivalence” between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) fundamental theorem of asset pricing.\(^a\)

\(^a\)Dybvig & Ross (1987).
Philip H. Dybvig\textsuperscript{a} (1955–)

\textsuperscript{a}Co-winner of the 2022 Nobel Prize in Economic Sciences.
Stephen Ross (1944–2017)
Self-Financing

- Delta changes over time.
- The maintenance of an equivalent portfolio is dynamic.
- But it does not depend on predicting future stock prices.
- The portfolio’s value at the end of the current period is precisely the amount needed to set up the next portfolio.
- The trading strategy is self-financing because there is neither injection nor withdrawal of funds throughout.\(^a\)
  - Changes in value are due entirely to capital gains.

\(^a\)Except at the beginning, of course, when the option premium is paid before the replication starts.
Binomial Distribution

- Denote the binomial distribution with parameters $n$ and $p$ by
  \[ b(j; n, p) \triangleq \binom{n}{j} p^j (1 - p)^{n-j} = \frac{n!}{j! (n - j)!} p^j (1 - p)^{n-j}. \]

  - $n! = 1 \times 2 \times \cdots \times n$.
  - Convention: $0! = 1$.

- Suppose you flip a coin $n$ times with $p$ being the probability of getting heads.

- Then $b(j; n, p)$ is the probability of getting $j$ heads.
The Binomial Option Pricing Formula

- The stock prices at time $n$ are
  \[ Su^n, Su^{n-1}d, \ldots, Sd^n. \]

- Let $a$ be the minimum number of upward price moves for the call to finish in the money.

- So $a$ is the smallest nonnegative integer $j$ such that
  \[ Su^j d^{n-j} \geq X, \]
  or, equivalently,
  \[ a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil. \]
The Binomial Option Pricing Formula (concluded)

• Hence,

\[
C = \frac{\sum_{j=a}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \left( S w^j d^{n-j} - X \right)}{R^n}
\]

\[
= S \sum_{j=a}^{n} \binom{n}{j} (p u)^j \left( (1 - p) d \right)^{n-j} \frac{1}{R^n} 
\]

\[
- \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1 - p)^{n-j}
\]

\[
= S \sum_{j=a}^{n} b(j; n, pu/R) - X e^{-\hat{r} n} \sum_{j=a}^{n} b(j; n, p). \quad (39)
\]
Numerical Examples

• A non-dividend-paying stock is selling for $160.

• $u = 1.5$ and $d = 0.5$.

• $r = 18.232\%$ per period ($R = e^{0.18232} = 1.2$).
  
  - Hence $p = (R - d)/(u - d) = 0.7$.

• Consider a European call on this stock with $X = 150$ and $n = 3$.

• The call value is $85.069$ by backward induction.

• Or, the PV of the expected payoff at expiration:

$$
\frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.
$$
Numerical Examples (continued)

• Mispricing leads to arbitrage profits.

• Suppose the option is selling for $90 instead.

• Sell the call for $90.

• Invest $85.069 in the replicating portfolio with 0.82031 shares of stock as required by the delta.

• Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.

• The fund that remains,

$$90 - 85.069 = 4.931$$ dollars,

is the arbitrage profit, as we will see.
Numerical Examples (continued)

Time 1:

• Suppose the stock price moves to $240.
• The new delta is 0.90625.
• Buy

  \[0.90625 - 0.82031 = 0.08594\]

  more shares at the cost of \(0.08594 \times 240 = 20.6256\) dollars financed by borrowing.

• Debt now totals \(20.6256 + 46.1806 \times 1.2 = 76.04232\) dollars.
Numerical Examples (continued)

• The trading strategy is self-financing because the portfolio has a value of

\[ 0.90625 \times 240 - 76.04232 = 141.45768. \]

• It matches the corresponding call value by backward induction!\(^a\)

\(^a\)See p. 278.
Numerical Examples (continued)

Time 2:

• Suppose the stock price plunges to $120.

• The new delta is 0.25.

• Sell \( 0.90625 - 0.25 = 0.65625 \) shares.

• This generates an income of \( 0.65625 \times 120 = 78.75 \) dollars.

• Use this income to reduce the debt to

\[
76.04232 \times 1.2 - 78.75 = 12.5
\]

dollars.
Numerical Examples (continued)

Time 3 (the case of rising price):

- The stock price moves to $180.
- The call we wrote finishes in the money.
- Close out the call’s short position by buying back the call or buying a share of stock for delivery.
- This results in a loss of $180 - $150 = 30 dollars.
- Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.
- It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.
Numerical Examples (concluded)

Time 3 (the case of declining price):

- The stock price moves to $60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of

\[ 0.25 \times 60 = 15 \]

dollars.
- Use it to repay the debt of \( 12.5 \times 1.2 = 15 \) dollars.
Applications besides Exploiting Arbitrage Opportunities\textsuperscript{a}

- Replicate an option using stocks and bonds.
  - Set up a portfolio to replicate the call with $85.069.

- Hedge the options we issued.
  - Use $85.069 to set up a portfolio to replicate the call to counterbalance its values exactly.\textsuperscript{b}

- ...\textsuperscript{c}

- Without hedge, one may end up forking out $390 in the worst case (see p. 278)!

\textsuperscript{a}Thanks to a lively class discussion on March 16, 2011.
\textsuperscript{b}Hedging and replication are mirror images.
\textsuperscript{c}Thanks to a lively class discussion on March 16, 2016.
Binomial Tree Algorithms for European Options

- The BOPM implies the binomial tree algorithm that applies backward induction.

- The total running time is $O(n^2)$ because there are $\sim n^2/2$ nodes.

- The memory requirement is $O(n^2)$.
  - Can be easily reduced to $O(n)$ by reusing space.\(^a\)

- To find the hedge ratio, apply formula (32) on p. 252.

- To price European puts, simply replace the payoff.

\(^a\)But watch out for the proper updating of array entries.
Further Time Improvement for Calls
Optimal Algorithm

- We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.

- Note that

$$b(j; n, p) = \frac{p(n - j + 1)}{(1 - p) j} b(j - 1; n, p).$$
Optimal Algorithm (continued)

- The following program computes $b(j; n, p)$ in $b[j]$:
- It runs in $O(n)$ steps.

1: $b[a] := \binom{n}{a} p^a (1 - p)^{n-a};$
2: for $j = a + 1, a + 2, \ldots , n$ do
3: $b[j] := b[j - 1] \times p \times (n - j + 1)/((1 - p) \times j);$ 
4: end for
Optimal Algorithm (concluded)

• With the $b(j; n, p)$ available, the risk-neutral valuation formula (38) on p. 276 is trivial to compute.

• But we only need a single variable to store the $b(j; n, p)$s as they are being sequentially computed.

• This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.

• This forward-induction approach cannot be applied to American options because of early exercise.

• So binomial tree algorithms for American options usually run in $O(n^2)$ time.
The Bushy Tree
Toward the Black-Scholes Formula

• The binomial model seems to suffer from two unrealistic assumptions.
  – The stock price takes on only two values in a period.
  – Trading occurs at discrete points in time.

• As $n$ increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.\(^a\)

• Need to calibrate the BOPM’s parameters $u$, $d$, and $R$ to make it converge to the continuous-time model.

• We now skim through the proof.

\(^a\)Continuous-time trading may create arbitrage opportunities in practice (Budish, Cramton, & Shim, 2015)!
Toward the Black-Scholes Formula (continued)

- Let $\tau$ denote the time to expiration of the option measured in years.
- Let $r$ be the continuously compounded annual rate.
- With $n$ periods during the option’s life, each period represents a time interval of $\tau/n$.
- Need to adjust the period-based $u$, $d$, and interest rate $\hat{r}$ to match the empirical results as $n \to \infty$. 
Toward the Black-Scholes Formula (continued)

• First, $\hat{r} = r\tau/n$.
  
  – Each period is $\Delta t \equiv \tau/n$ years long.
  
  – The period gross return $R = e^{\hat{r}}$.

• Let

\[
\hat{\mu} \triangleq \frac{1}{n} E \left[ \ln \frac{S_\tau}{S} \right]
\]

denote the expected value of the continuously compounded rate of return per period of the BOPM.

• Let

\[
\hat{\sigma}^2 \triangleq \frac{1}{n} \text{Var} \left[ \ln \frac{S_\tau}{S} \right]
\]

denote the variance of that return.
Toward the Black-Scholes Formula (continued)

- Under the BOPM, it is not hard to show that\(^a\)

\[
\hat{\mu} = q \ln(u/d) + \ln d, \\
\hat{\sigma}^2 = q(1 - q) \ln^2(u/d).
\]

- Assume the stock’s \textit{true} continuously compounded rate of return over \(\tau\) years has mean \(\mu \tau\) and variance \(\sigma^2 \tau\).

- Call \(\sigma\) the stock’s (annualized) volatility.

\(^a\)It follows the Bernoulli distribution.
Toward the Black-Scholes Formula (continued)

• The BOPM converges to the distribution only if

\[ n \hat{\mu} = n[q \ln(u/d) + \ln d] \to \mu \tau, \quad (40) \]
\[ n \hat{\sigma}^2 = nq(1 - q) \ln^2(u/d) \to \sigma^2 \tau. \quad (41) \]

• We need one more condition to have a solution for \( u, d, q \).
Toward the Black-Scholes Formula (continued)

- Impose \( ud = 1 \).
  - It makes nodes at the same horizontal level of the tree have identical price (review p. 288).
  - Other choices are possible (see text).

- Exact solutions for \( u, d, q \) are feasible if Eqs. (40)–(41) are replaced by equations: 3 equations for 3 variables.\(^a\)

\(^a\)Chance (2008).
Toward the Black-Scholes Formula (continued)

• The above requirements can be satisfied by

\[ u = e^{\sigma \sqrt{\Delta t}}, \quad d = e^{-\sigma \sqrt{\Delta t}}, \quad q = \frac{1}{2} + \frac{1}{2} \frac{\mu}{\sigma} \sqrt{\Delta t}. \] (42)

• With Eqs. (42), it can be checked that

\[ \hat{n} \mu = \mu \tau, \]
\[ \hat{n} \sigma^2 = \left[ 1 - \left( \frac{\mu}{\sigma} \right)^2 \Delta t \right] \sigma^2 \tau \to \sigma^2 \tau. \]

• With the above choice, even if \( \sigma \) is not calibrated correctly, the mean is still matched!\(^a\)

\(^a\)Recall Eq. (35) on p. 258. So \( u \) and \( d \) are related to volatility exclusively in the CRR model. Both are independent of \( r \) and \( \mu \).
Toward the Black-Scholes Formula (continued)

• The choices (42) result in the CRR binomial model.\textsuperscript{a}
  
  – Black (1992), “This method is probably used more than the original formula in practical situations.”
  
  – OptionMetrics’s (2015) IvyDB uses the CRR model.\textsuperscript{b}

• The CRR model is best seen in logarithmic price:

\[
\ln S \rightarrow \begin{cases} 
\ln S + \sigma \sqrt{\Delta t}, & \text{up move,} \\
\ln S - \sigma \sqrt{\Delta t}, & \text{down move.}
\end{cases}
\]

\textsuperscript{a}Cox, Ross, & Rubinstein (1979).

\textsuperscript{b}See \url{http://www.ckgsb.com/uploads/report/file/201611/02/1478069847635278.pdf}
Toward the Black-Scholes Formula (continued)

- The no-arbitrage inequalities $d < R < u$ may not hold under Eqs. (42) on p. 299 or Eq. (34) on p. 256.
  - If this happens, the probabilities lie outside $[0, 1]$.

- The problem disappears when $n$ satisfies $e^{\sigma \sqrt{\Delta t}} > e^{r \Delta t}$, i.e., when

  \[ n > \frac{r^2}{\sigma^2} \tau. \]  

  - So it goes away if $n$ is large enough.
  - Other solutions can be found in the textbook or will be presented later.

\[ a \text{Many papers and programs forget to check this condition!} \]
\[ b \text{See Exercise 9.3.1 of the textbook.} \]
Toward the Black-Scholes Formula (continued)

- The central limit theorem says \( \ln(S_\tau/S) \) converges to \( N(\mu\tau, \sigma^2\tau) \).\(^a\)

- So \( \ln S_\tau \) approaches \( N(\mu\tau + \ln S, \sigma^2\tau) \).

- Conclusion: \( S_\tau \) has a lognormal distribution in the limit.

\(^a\)The normal distribution with mean \( \mu\tau \) and variance \( \sigma^2\tau \). As our probabilities depend on \( n \), this argument is heuristic.
Toward the Black-Scholes Formula (continued)

Lemma 10 The continuously compounded rate of return \( \ln(S_{\tau}/S) \) approaches the normal distribution with mean \( (r - \sigma^2/2)\tau \) and variance \( \sigma^2\tau \) in a risk-neutral economy.

- Let \( q \) equal the risk-neutral probability
  \[
  p \triangleq (e^{r\tau/n} - d)/(u - d).
  \]

- Let \( n \to \infty \).

- Then \( \mu = r - \sigma^2/2 \).\(^a\)

\(^a\)See Lemma 9.3.3 of the textbook. Now, \( p = \frac{1}{2} + \frac{\mu}{2\sigma} \Delta t^{0.5} + \frac{\sigma^4 + 4\sigma^2\mu + 6\mu^2}{24\sigma} \Delta t^{1.5} + O(\Delta t^{2.5}) \), consistent with Eq. (42) on p. 299.
Toward the Black-Scholes Formula (continued)

• The expected stock price at expiration in a risk-neutral economy is\(^a\)
  \[ S e^{r\tau}. \]

• The stock’s expected annual rate of return is thus the riskless rate \( r \).
  
  – By rate of return we mean \( (1/\tau) \ln E[S_\tau/S] \) (arithmetic average rate of return) not \( (1/\tau)E[\ln(S_\tau/S)] \) (geometric average rate of return).
  
  – The latter would give \( r - \sigma^2/2 \) by Lemma 10.

\(^a\)By Lemma 10 (p. 304) and Eq. (29) on p. 181.
Theorem 11 (The Black-Scholes Formula, 1973)

\[
\begin{align*}
C &= SN(x) - X e^{-r \tau} N(x - \sigma \sqrt{\tau}), \\
P &= X e^{-r \tau} N(-x + \sigma \sqrt{\tau}) - SN(-x),
\end{align*}
\]

where

\[
x \equiv \frac{\ln(S/X) + (r + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}.
\]

\(^a\)On a United flight from San Francisco to Tokyo on March 7, 2010, a real-estate manager mentioned this formula to me!
Toward the Black-Scholes Formula (concluded)

- See Eq. (39) on p. 276 for the meaning of $x$.

- See Exercise 13.2.12 of the textbook for an interpretation of the probability associated with $N(x)$ and $N(-x)$.
BOPM and Black-Scholes Model

- The Black-Scholes formula needs 5 parameters: $S$, $X$, $\sigma$, $\tau$, and $r$.

- Binomial tree algorithms take 6 inputs: $S$, $X$, $u$, $d$, $\hat{r}$, and $n$.

- The connections are

$$u = e^{\sigma \sqrt{\tau/n}},$$  
$$d = e^{-\sigma \sqrt{\tau/n}},$$  
$$\hat{r} = r\tau/n.$$  

- This holds for the CRR model as well.
- \( S = 100, \ X = 100 \) (left), and \( X = 95 \) (right).
BOPM and Black-Scholes Model (concluded)

- The binomial tree algorithms converge reasonably fast.
- The error is $O(1/n)$.
- Oscillations are inherent, however.
- Oscillations can be dealt with by judicious choices of $u$ and $d$.

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\(^a\)F. Diener & M. Diener (2004); L. Chang & Palmer (2007).
\(^b\)See Exercise 9.3.8 of the textbook.
Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market’s opinion of the volatility.\(^a\)
  - Solve for \(\sigma\) given the option price, \(S\), \(X\), \(\tau\), and \(r\) with numerical methods.
  - How about American options?

\(^a\)Implied volatility is hard to compute when \(\tau\) is small (why?).
Implied Volatility (concluded)

- Implied volatility is the wrong number to put in the wrong formula to get the right price of plain-vanilla options.\(^a\)

- Think of it as an alternative to quoting option prices.

- Implied volatility is often preferred to historical (statistical) volatility in practice.
  - Using the historical volatility is like driving a car with your eyes on the rearview mirror?\(^b\)

- Volatility is meaningful only if seen through a model!\(^c\)

\(^a\) Rebonato (2004).
\(^b\) E.g., 1:16:23 of https://www.youtube.com/watch?v=8TJQhQ2GZ0Y
\(^c\) Alexander (2001).
Problems; the Smile$^a$

- Options written on the same underlying asset usually do not produce the same implied volatility.

- A typical pattern is a “smile” in relation to the strike price.
  - The implied volatility is lowest for at-the-money options.
  - It becomes higher the further the option is in- or out-of-the-money.

- This is common for foreign exchange options.

\footnote{Alexander (2001).}
Problems; the Smile (concluded)

- Other patterns have also been observed.
- For stock options, low-strike options tend to have higher implied volatilities.
- One explanation is the high demand for insurance provided by out-of-the-money puts.
- Another reason is volatility rises when stock falls, making in-the-money calls more likely to become in the money again.

\(^a\)This is called the leverage effect (Black, 1992).
The underlying Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) closed at 8132. Plot supplied by Mr. Lok, U Hou (D99922028) on December 6, 2017.

\[ \text{ATM} = \$8132 \]
Tackling the Smile

• To address this issue, volatilities are often combined to produce a composite implied volatility.

• This practice is not sound theoretically.

• The existence of different implied volatilities for options on the same underlying asset shows the Black-Scholes model cannot be literally true.
Binomial Tree Algorithms for American Puts

- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs
  \[
  \max(0, X - S u^j d^{m-j})
  \]
  and applies backward induction.
- At each intermediate node, compare the payoff if exercised and the *continuation value*.
- Keep the larger one.
Bermudan Options

- Some American options can be exercised only at discrete time points instead of continuously.
- They are called Bermudan options.
- Their pricing algorithm is identical to that for American options.
- But early exercise is considered for only those nodes when early exercise is permitted.
Time-Dependent Volatility

• Suppose the (instantaneous) volatility can change over time but otherwise predictable: $\sigma(t)$ instead of $\sigma$.

• In the limit, the variance of $\ln(S_\tau/S)$ is

$$\int_0^\tau \sigma^2(t) \, dt$$

rather than $\sigma^2 \tau$.

• The annualized volatility to be used in the Black-Scholes formula should now be

$$\sqrt{\int_0^\tau \frac{\sigma^2(t) \, dt}{\tau}}.$$  

---

$^a$Merton (1973).
Time-Dependent Instantaneous Volatility (concluded)

- For the binomial model, $u$ and $d$ depend on time:

$$u = e^{\sigma(t)\sqrt{\tau/n}},$$
$$d = e^{-\sigma(t)\sqrt{\tau/n}}.$$  

- But how to make the binomial tree combine?\(^a\)

\(^a\)Amin (1991); C. I. Chen (R98922127) (2011).
Volatility (1990–2016)\textsuperscript{a}

\textsuperscript{a}Supplied by Mr. Lok, U Hou (D99922028) on July 17, 2017.
Time-Dependent Short Rates

• Suppose the short rate (i.e., the one-period spot rate or forward rate) changes over time but predictable.

• The annual riskless rate $r$ in the Black-Scholes formula should be the spot rate with a time to maturity equal to $\tau$.

• In other words,

$$r = \frac{\sum_{i=0}^{n-1} r_i}{\tau},$$

where $r_i$ is the continuously compounded short rate measured in periods for period $i$.

• Will the binomial tree fail to combine?  

\[\text{^aThat is, one-period forward rate.}\]
Trading Days and Calendar Days

- Interest accrues based on the calendar day.

- But $\sigma$ is usually calculated based on trading days only.
  - Stock price seems to have lower volatilities when the exchange is closed.$^a$

- How to harmonize these two different times into the Black-Scholes formula and binomial tree algorithms?$^b$

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$^a$Fama (1965); K. French (1980); K. French & Roll (1986).

$^b$Recall p. 163 about dating issues.
Trading Days and Calendar Days (continued)

- Think of $\sigma$ as measuring the annualized volatility of stock price \textit{one year from now}.
- Suppose a year has $m$ (say 253) trading days.
- We can replace $\sigma$ in the Black-Scholes formula with $\sqrt{\frac{365}{m} \times \frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}}$.

\footnote{D. French (1984).}
Trading Days and Calendar Days (concluded)

- This works only for European options.
- How about binomial tree algorithms?\textsuperscript{a}

\textsuperscript{a}Contributed by Mr. Lu, Zheng-Liang (D00922011) in 2015.
Options on a Stock That Pays Dividends

- Early exercise must be considered.

- Proportional dividend payout model is tractable (see text).
  - The dividend amount is a constant proportion of the prevailing stock price.

- In general, the corporate dividend policy is a complex issue.
Known Dividends

• Constant dividends introduce complications.
• Use $D$ to denote the amount of the dividend.
• Suppose an ex-dividend date falls in the first period.
• At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$.
• Follow the stock price one more period.
• The number of possible stock prices is not three but four: $(Su - D)u$, $(Su - D)d$, $(Sd - D)u$, $(Sd - D)d$.
  – The binomial tree no longer combines.
$(Su - D)u$

$(Su - D)\downarrow$

$Su - D$

$S$

$(Su - D)d$

$(Sd - D)u$

$Sd - D$

$(Sd - D)d$
An Ad-Hoc Approximation

- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends.a
- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.
- The riskless component at any time is the PV of future dividends during the life of the option.
  - Then, $\sigma$ is the volatility of the process followed by the risky component.
- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

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aRoll (1977); Heath & Jarrow (1988).
An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all future dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.
The Ad-Hoc Approximation vs. P. 328 (Step 1)

\[(S - D/R)u^2\]

\[(S - D/R)u\]

\[(S - D/R)ud\]

\[(S - D/R)d\]

\[(S - D/R)d^2\]
The Ad-Hoc Approximation vs. P. 328 (Step 2)

\[(S - D/R) + D/R = S\]

\[(S - D/R)u\]

\[(S - D/R)u^2\]

\[(S - D/R)d\]

\[(S - D/R)d^2\]
The Ad-Hoc Approximation vs. P. 328\textsuperscript{a}

- The trees are different.

- The stock prices at maturity are also different.
  - \((Su - D)u, (Su - D)d, (Sd - D)u, (Sd - D)d\) (p. 328).
  - \((S - D/R)u^2, (S - D/R)ud, (S - D/R)d^2\) (ad hoc).

- Note that, as \(d < R < u\),
  \[
  (Su - D)u > (S - D/R)u^2, \\
  (Sd - D)d < (S - D/R)d^2,
  \]

\textsuperscript{a}Contributed by Mr. Yang, Jui-Chung (D97723002) on March 18, 2009.
The Ad-Hoc Approximation vs. P. 328 (concluded)

- So the ad hoc approximation has a smaller dynamic range.

- This explains why in practice the volatility is usually increased when using the ad hoc approximation.
A General Approach\textsuperscript{a}

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases (see pp. 790ff).
- Other approaches include adjusting $\sigma$ and approximating the known dividend with a dividend yield.\textsuperscript{b}

\textsuperscript{a}Dai (B82506025, R86526008, D8852600) & Lyuu (2004). Also Arealy & Rodrigues (2013).

\textsuperscript{b}Geske & Shastri (1985). It works well for American options but not European ones (Dai, 2009).
Continuous Dividend Yields

• Dividends are paid continuously.
  – Approximates a broad-based stock market portfolio.

• The payment of a continuous dividend yield at rate $q$ reduces the growth rate of the stock price by $q$.
  – A stock that grows from $S$ to $S_\tau$ with a continuous dividend yield of $q$ would have grown from $S$ to $S_\tau e^{q\tau}$ without the dividends.

• A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays no dividends.\(^a\)

\(^a\)In pricing European options, only the distribution of $S_\tau$ matters.
Continuous Dividend Yields (continued)

- So the Black-Scholes formulas hold with $S$ replaced by $Se^{-q\tau}$:

$$C = Se^{-q\tau}N(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}), \quad (44)$$
$$P = Xe^{-r\tau}N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau}N(-x), \quad (44')$$

where

$$x \Delta \frac{\ln(S/X) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

- Formulas (44) and (44') remain valid as long as the dividend yield is predictable.

\footnote{Merton (1973).}
Continuous Dividend Yields (continued)

- To run binomial tree algorithms, replace $u$ with $ue^{-q\Delta t}$ and $d$ with $de^{-q\Delta t}$, where $\Delta t \overset{\Delta}{=} \tau/n$.
  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.

- Other than the changes, binomial tree algorithms stay the same.
  - In particular, $p$ should use the original $u$ and $d$!\(^a\)

\(^a\)Contributed by Ms. Wang, Chuan-Ju (F95922018) on May 2, 2007.
Continuous Dividend Yields (concluded)

- Alternatively, pick the risk-neutral probability as
  \[
  \frac{e^{(r-q)\Delta t} - d}{u - d},
  \tag{45}
  \]
  where \( \Delta t \equiv \tau/n \).

  - The reason: The stock price grows at an expected rate of \( r - q \) in a risk-neutral economy.

- The \( u \) and \( d \) remain unchanged.

- Except the change in Eq. (45), binomial tree algorithms stay the same as if there were no dividends.