Day Count Conventions: Actual/Actual

- The first "actual" refers to the actual number of days in a month.
- The second refers to the actual number of days in a year
- The number of days between June 17, 1992, and October 1, 1992, is 106.
  - 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.

Day Count Conventions: 30/360

- Each month has 30 days and each year 360 days.
- The number of days between June 17, 1992, and October 1, 1992, is 104.
  - 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October.
- In general, the number of days from date (y_1, m_1, d_1) to date (y_2, m_2, d_2) is

$$360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1). \tag{11}$$

Day Count Conventions: 30/360 (continued)

- If d_1 or d_2 is 31, we must change it to 30 *before* applying formula (11).[a]
- Hence:
  - There are 3 days between February 28 and March 1.
  - There are 2 days between February 29 and March 1.
  - There are 29 days between March 1 and March 31.

---

[a]This is the simplest of all the "30/360" variations (called the "30E/360" convention), used mainly in the Eurobond market (Kosowski & Neftci, 2015).

Day Count Conventions: 30/360 (concluded)

- An equivalent formula to (11) on p. 80 without any adjustment is (check it)

$$360 \times (y_2 - y_1) + 30 \times (m_2 - m_1 - 1)$$
$$+ \max(30 - d_1, 0) + \min(d_2, 30).$$

- There are many variations on the "30/360" convention regarding 31, February 28, and February 29.[a]

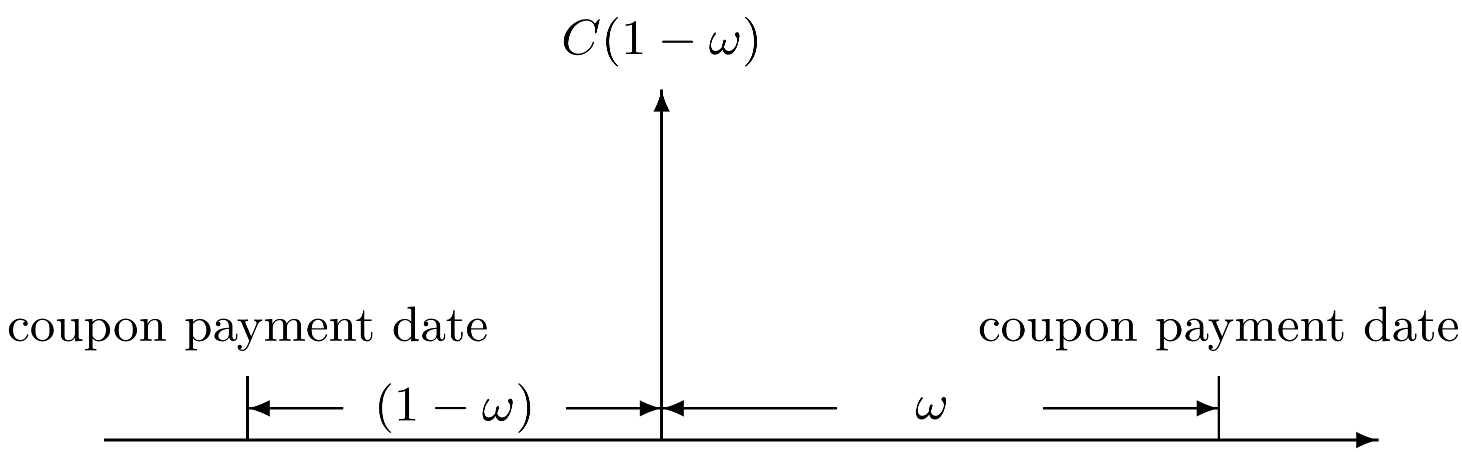
[a] Kosowski & Neftci (2015).

Full Price (Dirty Price, Invoice Price)

- In reality, the settlement date may fall on any day between two coupon payment dates.
- Let

$$\omega \triangleq \frac{\begin{array}{c}\text{number of days between the settlement}\\ \text{and the next coupon payment date}\end{array}}{\text{number of days in the coupon period}}. \tag{12}$$

Full Price (continued)



Full Price (concluded)

- The price is now calculated by

$$\begin{aligned} \text{PV} &= \frac{C}{\left(1+\frac{r}{m}\right)^{\omega}} + \frac{C}{\left(1+\frac{r}{m}\right)^{\omega+1}} \cdots \\ &= \sum_{i=0}^{n-1} \frac{C}{\left(1+\frac{r}{m}\right)^{\omega+i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{\omega+n-1}}. \end{aligned} \tag{13}$$

Accrued Interest

- The quoted price in the U.S./U.K. does not include the accrued interest; it is called the clean price or flat price.
- The buyer pays the invoice price: the quoted price *plus* the accrued interest (AI).
- The accrued interest equals

$$C \times \frac{\begin{array}{c}\text{number of days from the last}\\ \text{coupon payment to the settlement date}\end{array}}{\text{number of days in the coupon period}} = C \times (1 - \omega).$$

Accrued Interest (concluded)

- The yield to maturity is the r satisfying Eq. (13) on p. 85 when PV is the invoice price:

$$\text{clean price} + \text{AI} = \sum_{i=0}^{n-1} \frac{C}{\left(1+\frac{r}{m}\right)^{\omega+i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{\omega+n-1}}.$$

Example ("30/360")

- A bond with a 10% coupon rate and paying interest semiannually, with clean price 111.2891.
- The settlement date is July 1, 1993, and the maturity date is March 1, 1995.
- There are 60 days between July 1, 1993, and the next coupon date, September 1, 1993.
- The accrued interest is $(10/2) \times (1 - \frac{60}{180}) = 3.3333$ per \$100 of par value.

Example ("30/360") (concluded)

- The yield to maturity is 3%.
- This can be verified by Eq. (13) on p. 85 with
  - $\omega = 60/180$,
  - $n = 4$,
  - $m = 2$,
  - $F = 100$,
  - $C = 5$,
  - PV$= 111.2891 + 3.3333$,
  - $r = 0.03$.

Price Behavior (2) Revisited

- Previously, a bond selling at par if the yield to maturity equals the coupon rate.
- But it assumed that the settlement date is on a coupon payment date.
- Suppose the settlement date for a bond selling at par[a] falls between two coupon payment dates.
- Then its yield to maturity is *less* than the coupon rate.[b]
  - The reason: Exponential growth *to* C is replaced by linear growth, hence overpaying the accrued interest.

---

[a] The *quoted price* equals the par value.

[b] See Exercise 3.5.6 of the textbook for proof.

Bond Price Volatility

"Well, Beethoven, what is this?"[a]

— Attributed to Prince Anton Esterházy

[a]Mass in C major.

Price Volatility

- Volatility measures how bond prices respond to interest rate changes.
- It is key to the risk management of interest rate-sensitive securities.

Price Volatility (concluded)

- What is the sensitivity of the percentage price change to changes in interest rates?
- Define price volatility by

$$-\frac{\frac{\partial P}{\partial y}}{P}. \tag{14}$$

Price Volatility of Bonds

- The price volatility of a level-coupon bond is

$$-\frac{(C/y)\,n - \left(C/y^2\right)\left((1+y)^{n+1} - (1+y)\right) - nF}{(C/y)\,((1+y)^{n+1} - (1+y)) + F(1+y)}.$$

  - F is the par value.
  - C is the coupon payment per period.
  - Formula can be simplified a bit with $C = Fc/m$.

- For the above bond,

$$-\frac{\frac{\partial P}{\partial y}}{P} > 0.$$

Macaulay Duration[a]

- The Macaulay duration (MD) is a weighted average of the times to an asset's cash flows.
- The weights are the cash flows' PVs divided by the asset's price.
- Formally,

$$\text{MD} \triangleq \frac{1}{P} \sum_{i=1}^{n} \frac{C_i}{(1+y)^i}\, i.$$

  – What if $C_i = (1+c)^i$ for some constant c and letting $n \to \infty$ and assuming $c > y$?[b]

---

[a]Frederick Macaulay (1882–1970) in 1938.

[b]Contributed by Mr. Chen, Yu-Hsing (B06901048, R11922045) on March 3, 2023.

Macaulay Duration (concluded)

- The Macaulay duration, in periods, is equal to

$$\text{MD} = -(1+y)\,\frac{\partial P}{\partial y}\,\frac{1}{P}. \tag{15}$$

MD of Bonds

- The MD of a level-coupon bond is

$$\mathrm{MD} = \frac{1}{P} \left[\sum_{i=1}^{n} \frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n} \right]. \tag{16}$$

- It can be simplified to

$$\mathrm{MD} = \frac{c(1+y)\,[\,(1+y)^n - 1\,] + ny(y-c)}{cy\,[\,(1+y)^n - 1\,] + y^2},$$

  where c is the period coupon rate.

- The MD of a zero-coupon bond equals n, its term to maturity.

- The MD of a level-coupon bond is less than n.

Remarks

- Formulas (15) on p. 97 and (16) on p. 98 hold only if the coupon C, the par value F, and the maturity n are all independent of the yield y.
  - That is, if the cash flow is independent of yields.
- To see this point, suppose the market yield declines.
- The MD will be lengthened.
- But for securities whose maturity actually decreases as a result, the price volatility[a] may decrease.

[a] As originally defined in formula (14) on p. 94.

How *Not* To Think about MD

- The MD has its origin in measuring the length of time a bond investment is outstanding.
- But it should be seen mainly as measuring *price volatility*.
- Duration of a security can be longer than its maturity or negative!
- Neither makes sense under the maturity interpretation.
- Many, if not most, duration-related terminology can only be comprehended as measuring volatility.

Conversion

- For the MD to be year-based, modify formula (16) on p. 98 to

$$\frac{1}{P}\left[\sum_{i=1}^{n}\frac{i}{k}\,\frac{C}{\left(1+\frac{y}{k}\right)^{i}}+\frac{n}{k}\,\frac{F}{\left(1+\frac{y}{k}\right)^{n}}\right],$$

where y is the *annual* yield and k is the compounding frequency per annum.

- Formula (15) on p. 97 also becomes

$$\text{MD}=-\left(1+\frac{y}{k}\right)\frac{\partial P}{\partial y}\frac{1}{P}.$$

- By definition, MD (in years) $= \dfrac{\text{MD (in periods)}}{k}$.

Modified Duration

- Modified duration is defined as

$$\text{modified duration} \triangleq -\frac{\partial P}{\partial y}\frac{1}{P} = \frac{\text{MD}}{(1+y)}. \tag{17}$$

  – Modified duration equals MD under continuous compounding.

- By the Taylor expansion,

$$\text{percent price change} \approx -\text{modified duration} \times \text{yield change}.$$

Example

- Consider a bond whose modified duration is 11.54 with a yield of 10%.
- If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

$$-11.54 \times 0.001 = -0.01154 = -1.154\%.$$

Modified Duration of a Portfolio

- By calculus, the modified duration of a portfolio equals

$$\sum_i \omega_i D_i.$$

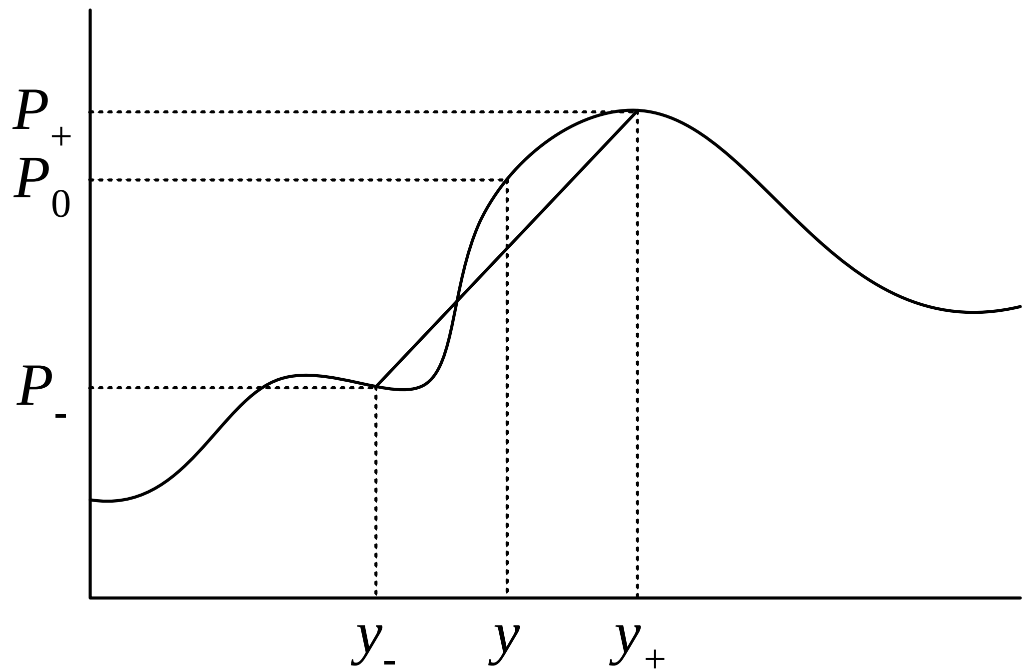
  - D_i is the modified duration of the ith asset.
  - ω_i is the market value of that asset expressed as a percentage of the market value of the portfolio.

Effective Duration

- Yield changes may alter the cash flow or the cash flow may be too complex for simple formulas.
- We need a general numerical formula for volatility.
- The effective duration is defined as

$$\frac{P_{-} - P_{+}}{P_0(y_{+} - y_{-})}.$$

  - P_{-} is the price if the yield is decreased by Δy.
  - P_{+} is the price if the yield is increased by Δy.
  - P_0 is the initial price, y is the initial yield.
  - Δy is small.



P_+
P_0
P_-
y_-
y
y_+

Effective Duration (concluded)

- One can compute the effective duration of just about any financial instrument.
- An alternative is to use

$$\frac{P_0 - P_+}{P_0 \, \Delta y}.$$

  - More economical but theoretically less accurate.

The Practices

- Duration is usually expressed in percentage terms — call it $D_{\%}$ — for quick mental calculation.[a]
- The percentage price change expressed in percentage terms is then approximated by

$$-D_{\%} \times \Delta r$$

when the yield increases instantaneously by $\Delta r\%$.

  - Suppose $D_{\%} = 10$ and $\Delta r = 2$.
  - Price will drop by 20% as $10 \times 2 = 20$.

- $D_{\%}$ in fact equals modified duration (prove it!).

[a] Neftci (2008), "Market professionals do not like to use decimal points."

Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.
- Define dollar duration as

$$\text{modified duration} \times \text{price} = -\frac{\partial P}{\partial y}.$$

- The approximate *dollar* price change is

$$\text{price change} \approx -\text{dollar duration} \times \text{yield change}.$$

- One can hedge a bond portfolio with a dollar duration D by bonds with a dollar duration $-D$.

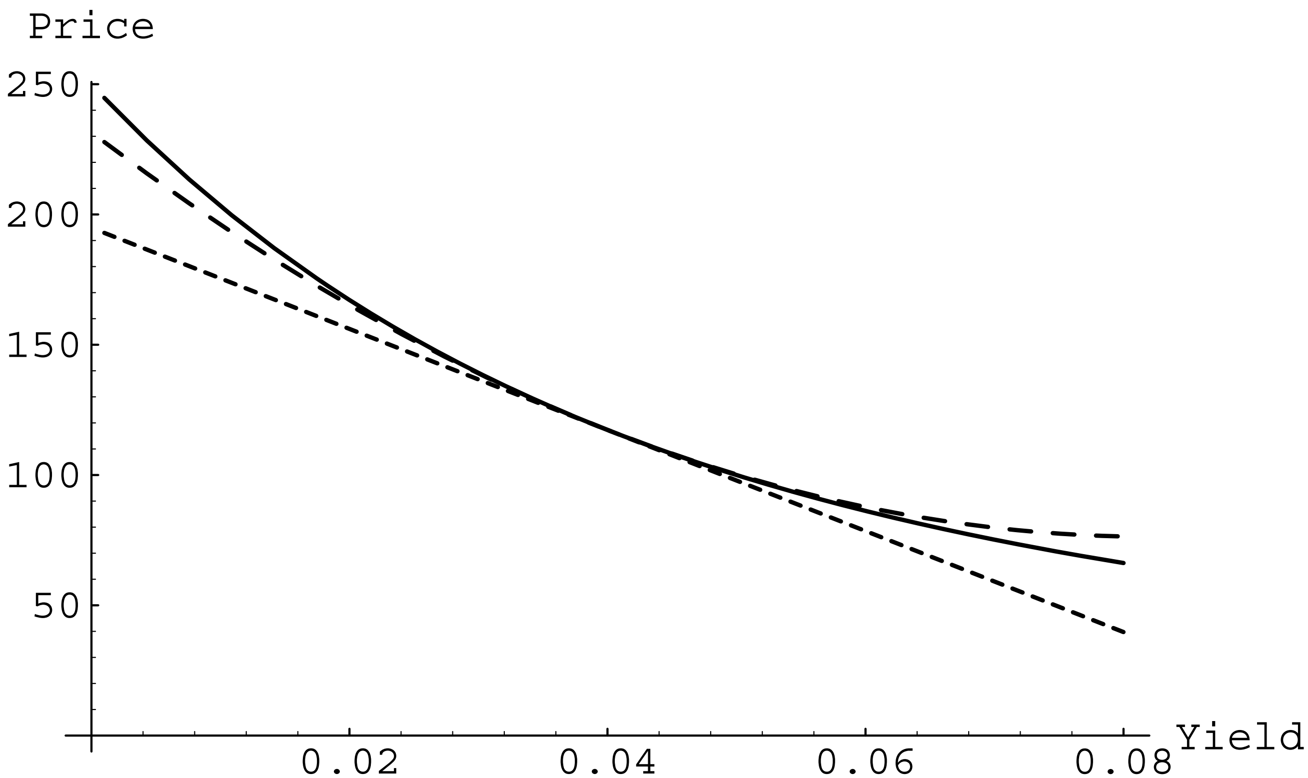
Convexity

- Convexity is defined as

$$\textsf{convexity} \text{ (in periods)} \triangleq \frac{\partial^2 P}{\partial y^2}\,\frac{1}{P}.$$

- The convexity of a level-coupon bond is positive (prove it!).
- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude (see plot next page).
- So between two bonds with the same price and duration, the one with a higher convexity is more valuable.[a]

[a] Do you spot a problem here (Christensen & Sørensen, 1994)?


Price
250
200
150
100
50
0.02
0.04
0.06
0.08
Yield

Convexity (concluded)

- Suppose there are k periods per annum.
- Convexity measured in periods and convexity measured in years are related by

$$\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}.$$

Use of Convexity

- The approximation $\Delta P/P \approx -$ duration $\times$ yield change works for small yield changes.
- For larger yield changes, use

$$
\begin{aligned}
\frac{\Delta P}{P} &\approx \frac{\partial P}{\partial y}\,\frac{1}{P}\,\Delta y + \frac{1}{2}\,\frac{\partial^2 P}{\partial y^2}\,\frac{1}{P}\,(\Delta y)^2 \\
&= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.
\end{aligned}
$$

- Recall the figure on p. 111.

The Practices

- Convexity is usually expressed in percentage terms — call it $C_{\%}$ — for quick mental calculation.
- The percentage price change expressed in percentage terms is approximated by

$$-D_{\%} \times \Delta r + C_{\%} \times (\Delta r)^2/2$$

when the yield increases instantaneously by $\Delta r\%$.

  - Price will drop by 17% if $D_{\%} = 10$, $C_{\%} = 1.5$, and $\Delta r = 2$ because

$$-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.$$

- $C_{\%}$ equals convexity divided by 100 (prove it!).

Effective Convexity

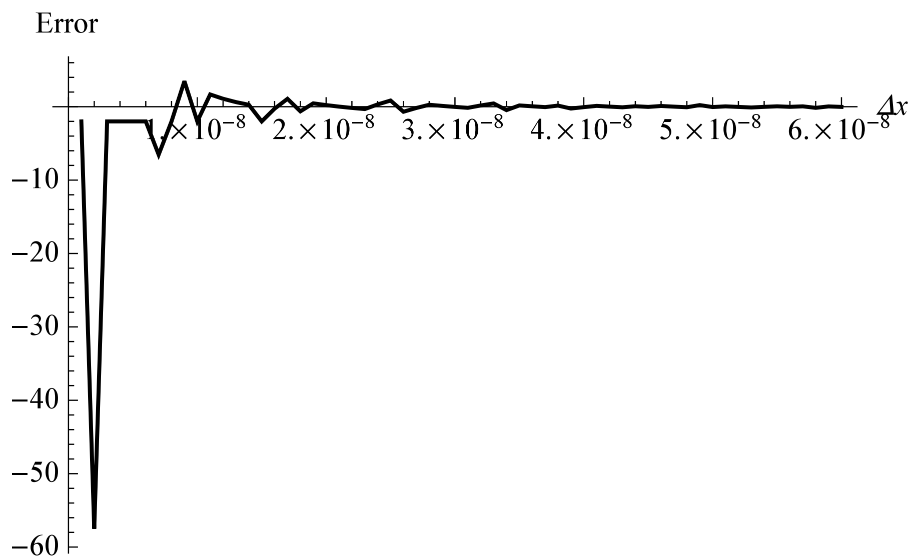
- The effective convexity is defined as

$$\frac{P_+ + P_- - 2P_0}{P_0\,(0.5 \times (y_+ - y_-))^2},$$

  - P_- is the price if the yield is decreased by Δy.
  - P_+ is the price if the yield is increased by Δy.
  - P_0 is the initial price, y is the initial yield.
  - Δy is small.
- Effective convexity is most relevant when a bond's cash flow is interest rate sensitive.
- How to choose the right Δy is a delicate matter.

Approximate $d^2 f(x)^2/dx^2$ at $x = 1$, Where $f(x) = x^2$

- The difference of $[\,(1+\Delta x)^2 + (1-\Delta x)^2 - 2\,]/(\Delta x)^2$ and 2:



- This numerical issue is common in financial engineering but does not have general solutions yet (see pp. 869ff).

Interest Rates and Bond Prices: Which Determines Which?[a]

- If you have one, you have the other.
- So they are just two names given to the same thing: cost of fund.
- Traders most likely work with prices.
- Banks most likely work with interest rates.

[a]Contributed by Mr. Wang, Cheng (R01741064) on March 5, 2014.

Term Structure of Interest Rates

Why is it that the interest of money is lower,
when money is plentiful?
— Samuel Johnson (1709–1784)

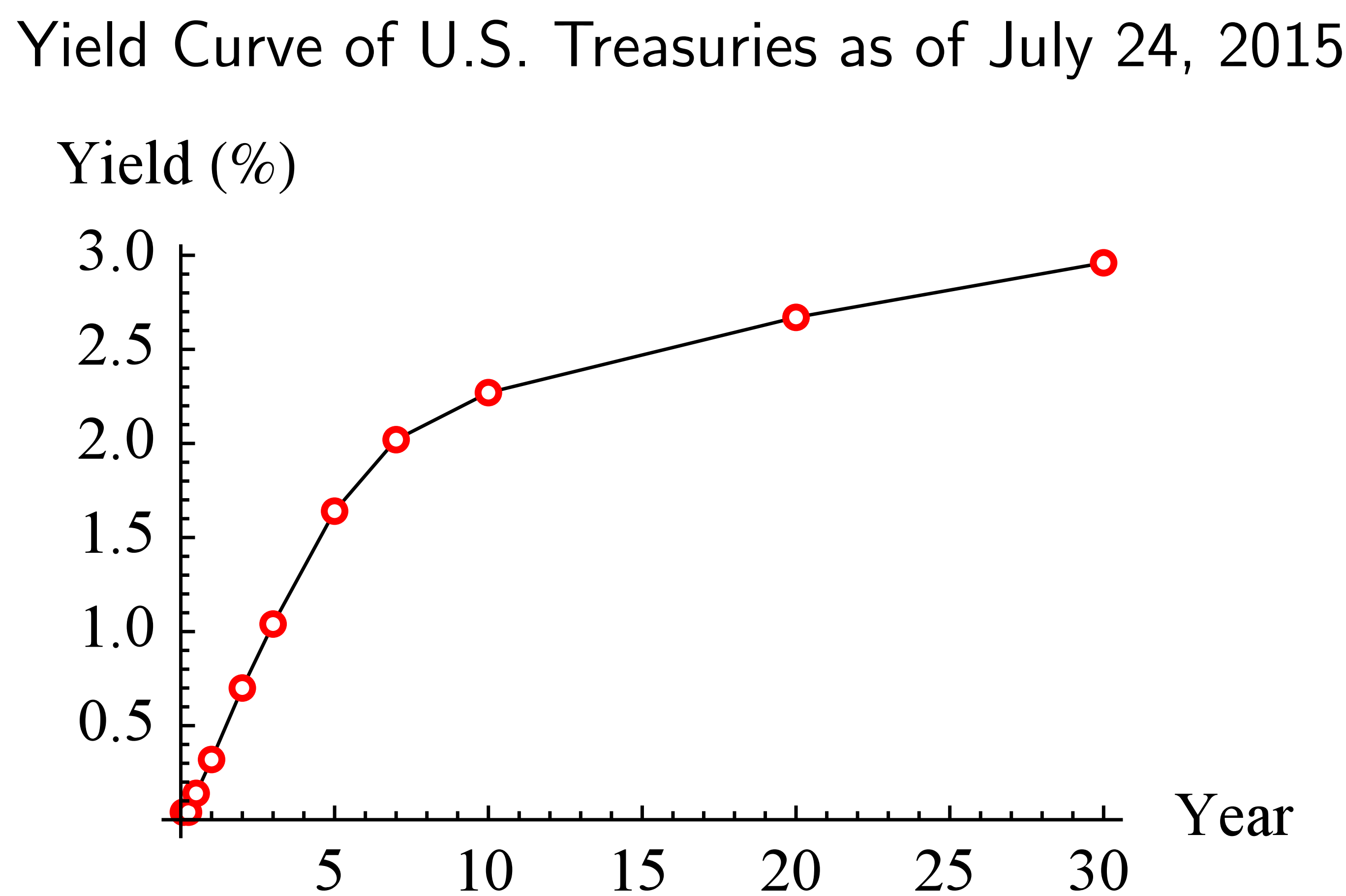
If you have money, don't lend it at interest.
Rather, give [it] to someone
from whom you won't get it back.
— Thomas Gospel 95

Term Structure of Interest Rates

- Concerned with how interest rates change with maturity.
- The set of yields to maturity for bonds form the term structure.
  - The bonds must be of equal quality.
  - They differ solely in their terms to maturity.
- The term structure is fundamental to the valuation of fixed-income securities.

Term Structure of Interest Rates (concluded)

- The term "term structure" often refers exclusively to the yields of zero-coupon bonds.
- A yield curve plots the yields to maturity of coupon bonds against maturity.
- A par yield curve is constructed from bonds trading near par.


Yield Curve of U.S. Treasuries as of July 24, 2015
Yield (%)
3.0
2.5
2.0
1.5
1.0
0.5
Year
5
10
15
20
25
30

Four Typical Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.

Spot Rates

- The i-period spot rate $S(i)$ is the yield to maturity of an i-period zero-coupon bond.
- The PV of one dollar i periods from now is by definition

$$[\,1 + S(i)\,]^{-i}.$$

  - It is the price of an i-period zero-coupon bond.[a]
- The one-period spot rate is called the short rate.
- Spot rate curve:[b] Plot of spot rates against maturity:

$$S(1), S(2), \ldots, S(n).$$

---

[a] Recall Eq. (9) on p. 69.

[b] That is, term structure, per our convention.

Problems with the PV Formula

- In the bond price formula (4) on p. 41,

$$\sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n},$$

every cash flow is discounted at the same yield y.

- Consider two riskless bonds with different yields to maturity because of their different cash flows:

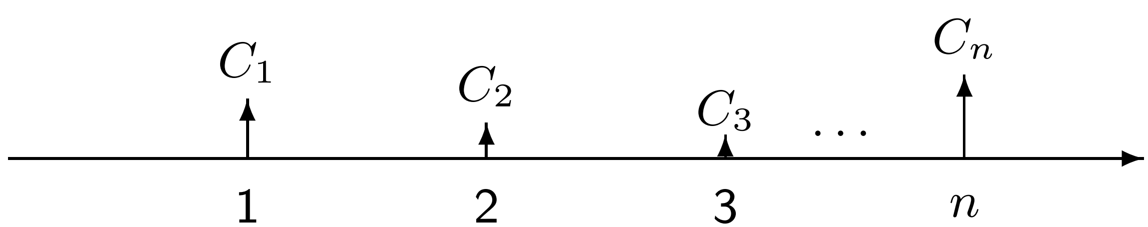
$$\begin{aligned}
\text{PV}_1 &= \sum_{i=1}^{n_1} \frac{C}{(1+y_1)^i} + \frac{F}{(1+y_1)^{n_1}}, \\
\text{PV}_2 &= \sum_{i=1}^{n_2} \frac{C}{(1+y_2)^i} + \frac{F}{(1+y_2)^{n_2}}.
\end{aligned}$$

Problems with the PV Formula (concluded)

- The yield-to-maturity methodology discounts their *contemporaneous* cash flows with *different* rates.
- But shouldn't they be discounted at the *same* rate?

Spot Rate Discount Methodology

- A cash flow $C_1, C_2, \ldots, C_n$ is equivalent to a package of zero-coupon bonds with the ith bond paying C_i dollars at time i.



Spot Rate Discount Methodology (concluded)

- So a level-coupon bond has the price

$$P = \sum_{i=1}^{n} \frac{C}{[\,1 + S(i)\,]^i} + \frac{F}{[\,1 + S(n)\,]^n}. \tag{18}$$

- This pricing method incorporates information from the term structure.
- It discounts each cash flow at the matching spot rate.

Discount Factors

- In general, any riskless security having a cash flow $C_1, C_2, \ldots, C_n$ should have a market price of

$$P = \sum_{i=1}^{n} C_i d(i).$$

  - Above, $d(i) \stackrel{\Delta}{=} [\,1 + S(i)\,]^{-i}$, $i = 1, 2, \ldots, n$, are called the discount factors.
  - $d(i)$ is the PV of one dollar i periods from now.
  - The above formula will be justified on p. 223.

- The discount factors are often interpolated to form a continuous function called the discount function.

Extracting Spot Rates from Yield Curve

- Start with the short rate $S(1)$.
  - Note that short-term Treasuries are zero-coupon bonds.
- Compute $S(2)$ from the two-period coupon bond price P by solving

$$P = \frac{C}{1+S(1)} + \frac{C+100}{[\,1+S(2)\,]^2}.$$

Extracting Spot Rates from Yield Curve (concluded)

- Inductively, we are given the market price P of the n-period coupon bond and

$$S(1), S(2), \ldots, S(n-1).$$

- Then $S(n)$ can be computed from Eq. (18) on p. 128, repeated below,

$$P = \sum_{i=1}^{n} \frac{C}{[\,1 + S(i)\,]^i} + \frac{F}{[\,1 + S(n)\,]^n}.$$

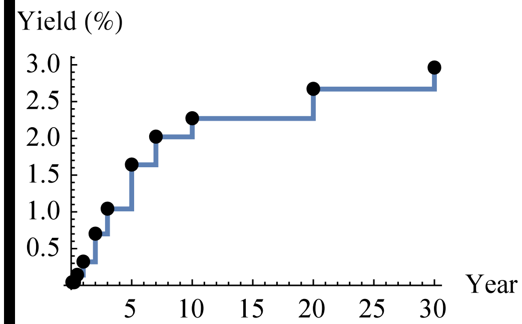
- The running time can be made to be $O(n)$ (see text).
- The procedure is called bootstrapping.

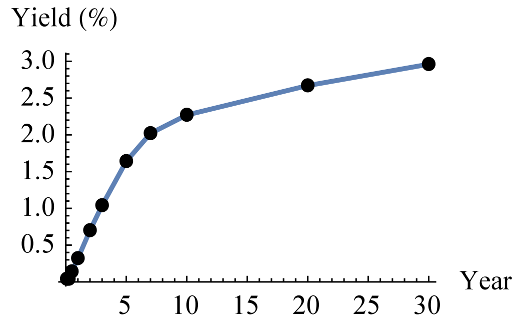
Some Problems

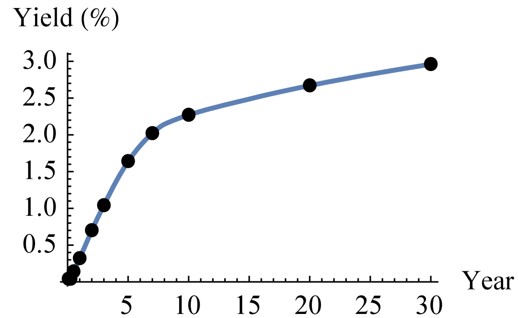
- Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).
- Some maturities might be missing from the data points (the incompleteness problem).
- Treasuries might not be of the same quality.
- Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.[a]

[a] Often without economic justifications.

Which One (from P. 122)?







Yield Spread

- Consider a *risky* bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for P.
- Calculate the IRR of the risky bond.
- Calculate the IRR of a riskless bond with comparable maturity.
- Yield spread is their difference.

Static Spread

- Were the risky bond riskless, it would fetch
$$P^* = \sum_{t=1}^{n} \frac{C_t}{[\,1 + S(t)\,]^t}.$$
- But as risk must be compensated, in reality $P < P^*$.
- The static spread is the amount s by which the spot rate curve has to shift *in parallel* to price the risky bond:
$$P = \sum_{t=1}^{n} \frac{C_t}{[\,1 + s + S(t)\,]^t}.$$
- Unlike the yield spread, the static spread explicitly incorporates information from the term structure.

Of Spot Rate Curve and Yield Curve

- y_i: yield to maturity for the i-period coupon bond.
- $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
- $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
- $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
- $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).
- If the yield curve is flat, the spot rate curve coincides with the yield curve.

Shapes

- The spot rate curve often has the same shape as the yield curve.
  - If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).
- But this is only a trend not a mathematical truth.[a]

[a]See a counterexample in the text.

Forward Rates

- The yield curve contains information regarding future interest rates currently "expected" by the market.
- Invest \$1 for j periods to end up with $[\,1 + S(j)\,]^j$ dollars at time j.
  - The maturity strategy.
- Invest \$1 in bonds for i periods and at time i invest the proceeds in bonds for another $j - i$ periods where $j > i$.
- Will have $[\,1 + S(i)\,]^i [\,1 + S(i,j)\,]^{j-i}$ dollars at time j.
  - $S(i,j)$: $(j - i)$-period spot rate i periods from now.
  - The rollover strategy.

Forward Rates (concluded)

- When $S(i, j)$ equals

$$f(i, j) \triangleq \left[\frac{(1 + S(j))^j}{(1 + S(i))^i} \right]^{1/(j-i)} - 1, \qquad (19)$$

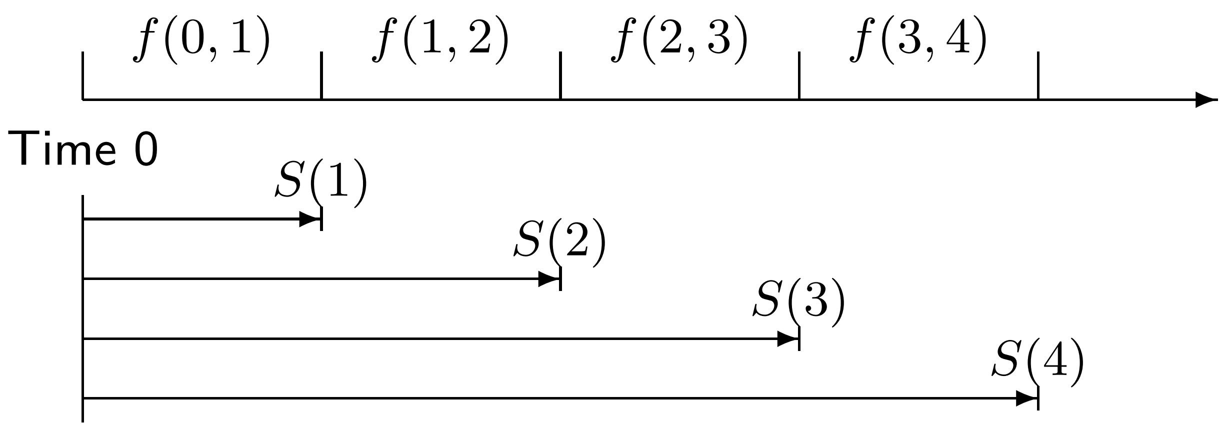
  we will end up with the same $[\,1 + S(j)\,]^j$ dollars.

- As expected,

$$f(0, j) = S(j).$$

- The $f(i, j)$ are the (implied) forward (interest) rates.
  - More precisely, the $(j - i)$-period forward rate i periods from now.

Time Line



Forward Rates and Future Spot Rates

- We did not assume any a priori relation between $f(i, j)$ and future spot rate $S(i, j)$.
  - This is the subject of the term structure theories.
- We merely looked for the future spot rate that, *if realized*, will equate the two investment strategies.
- The $f(i, i+1)$ are the *instantaneous* forward rates or one-period forward rates.

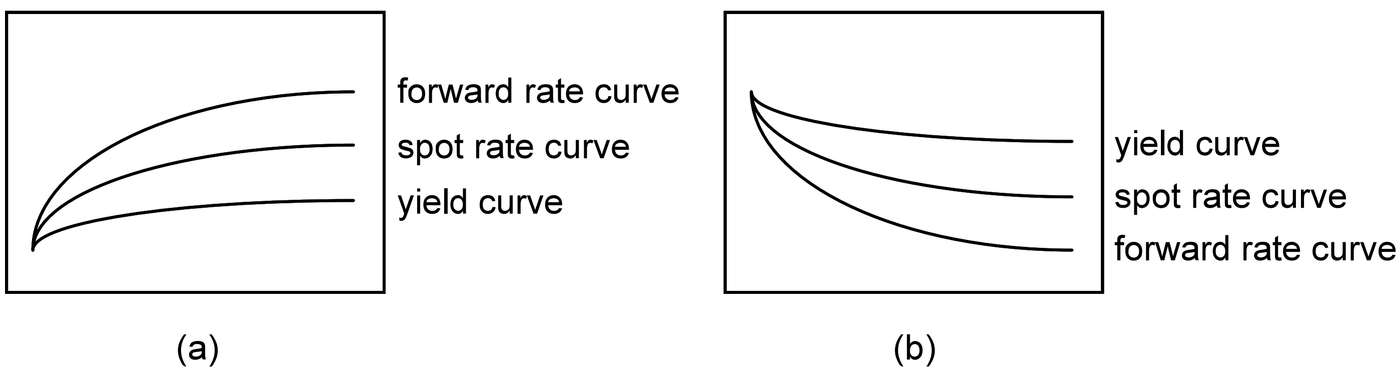
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,

$$f(i, j) > S(j) > \cdots > S(i).$$

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

$$f(i, j) < S(j) < \cdots < S(i).$$


forward rate curve
spot rate curve
yield curve
(a)
yield curve
spot rate curve
forward rate curve
(b)

Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curve

- The FV of \$1 at time n can be derived in two ways.
- Buy n-period zero-coupon bonds and receive

$$[\,1 + S(n)\,]^n.$$

- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is

$$[\,1 + S(1)\,][\,1 + f(1,2)\,]\cdots[\,1 + f(n-1,n)\,].$$

Forward Rates ≡ Spot Rates ≡ Yield Curves (concluded)

- Since they are identical,

$$S(n) = \{[\,1 + S(1)\,][\,1 + f(1,2)\,] \cdots [\,1 + f(n-1,n)\,]\}^{1/n} - 1. \tag{20}$$

- Hence, the forward rates (specifically the one-period forward rates) determine the spot rate curve.
- Other equivalencies can be derived similarly, such as

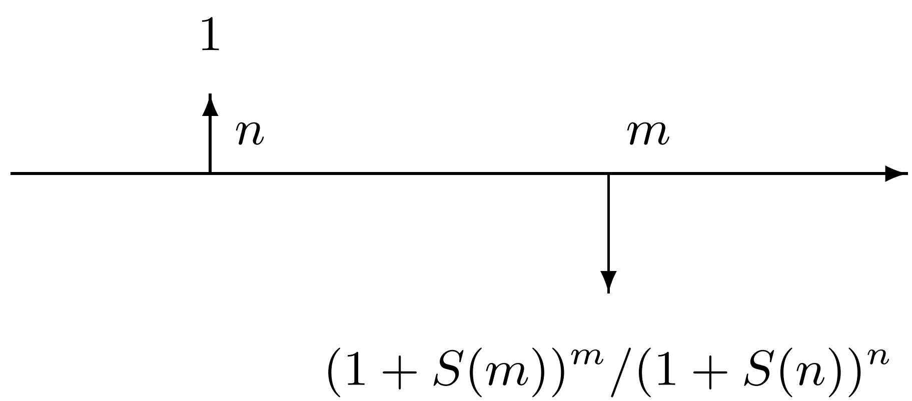
$$f(T, T+1) = \frac{d(T)}{d(T+1)} - 1. \tag{21}$$

Locking in the Forward Rate $f(n, m)$

- Buy one n-period zero-coupon bond for $1/(1 + S(n))^n$ dollars.
- Sell $(1 + S(m))^m/(1 + S(n))^n$ m-period zero-coupon bonds.
- No net initial investment because the cash inflow equals the cash outflow: $1/(1 + S(n))^n$.
- At time n there will be a cash inflow of \$1.
- At time m there will be a cash outflow of $(1 + S(m))^m/(1 + S(n))^n$ dollars.

Locking in the Forward Rate $f(n, m)$ (concluded)

- This implies the interest rate between times n and m equals $f(n, m)$ by formula (19) on p. 139.[a]



[a]Note that $(1 + S(m))^m/(1 + S(n))^n = (1 + f(n, m))^{m-n}$ by that formula.

Forward Loans

- We had generated the cash flow of a type of forward contract called the forward loan.
- Agreed upon today, it enables one to
  - Borrow money at time n in the future, and
  - Repay the loan at time $m > n$ with an interest rate equal to the known forward rate

$$f(n, m).$$

- Can the spot rate curve be arbitrarily drawn?[a]

[a]Contributed by Mr. Dai, Tian-Shyr (B82506025, R86526008, D88526006) in 1998.

Synthetic Bonds

- We had seen that

$$\begin{aligned} & \text{forward loan} \\ = \; & n\text{-period zero} - [\,1 + f(n,m)\,]^{m-n} \times m\text{-period zero}. \end{aligned}$$

- Thus

$$\begin{aligned} & n\text{-period zero} \\ = \; & \text{forward loan} + [\,1 + f(n,m)\,]^{m-n} \times m\text{-period zero}. \end{aligned}$$

- We have created a *synthetic* zero-coupon bond with forward loans and other zero-coupon bonds.
- Useful if the n-period zero is unavailable or illiquid.

Spot and Forward Rates under Continuous Compounding

- The pricing formula:

$$P = \sum_{i=1}^{n} Ce^{-iS(i)} + Fe^{-nS(n)}.$$

- The market discount function:

$$d(n) = e^{-nS(n)}.$$

- The spot rate is an arithmetic average of forward rates,[a]

$$S(n) = \frac{f(0,1) + f(1,2) + \cdots + f(n-1,n)}{n}.$$

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[a] Compare it with formula (20) on p. 145.

Spot and Forward Rates under Continuous Compounding (continued)

- The formula for the forward rate:

$$f(i,j) = \frac{jS(j) - iS(i)}{j - i}. \tag{22}$$

  – Compare the above formula with (19) on p. 139.

- The one-period forward rate:[a]

$$f(j, j+1) = -\ln \frac{d(j+1)}{d(j)}.$$

[a] Compare it with formula (21) on p. 145.

Spot and Forward Rates under Continuous Compounding (concluded)

- Now, the (instantaneous) forward rate curve is:

$$\begin{aligned} f(T) &\stackrel{\Delta}{=} \lim_{\Delta T \to 0} f(T, T + \Delta T) \\ &= S(T) + T \frac{\partial S}{\partial T}. \end{aligned} \tag{23}$$

- So $f(T) > S(T)$ if and only if $\partial S/\partial T > 0$ (i.e., a normal spot rate curve).
- If $S(T) < -T(\partial S/\partial T)$, then $f(T) < 0$.[a]

[a]Consistent with the plot on p. 143. Contributed by Mr. Huang, Hsien-Chun (R03922103) on March 11, 2015.

An Example

- Let the interest rates be continuously compounded.
- Suppose the spot rate curve is[a]

$$S(T) \stackrel{\Delta}{=} 0.08 - 0.05\, e^{-0.18T}.$$

- Then by Eq. (23) on p. 152, the forward rate curve is

$$\begin{aligned} & f(T) \\ = \quad & S(T) + T S'(T) \\ = \quad & 0.08 - 0.05\, e^{-0.18T} + 0.009T\, e^{-0.18T}. \end{aligned}$$

[a]Hull & White (1994).

Unbiased Expectations Theory

- Forward rate equals the average future spot rate,

$$f(a,b) = E[\, S(a,b) \,]. \tag{24}$$

- It does not imply that the forward rate is an accurate predictor for the future spot rate.
- It implies the maturity strategy and the rollover strategy produce the same result at the horizon "on average."

Unbiased Expectations Theory and Spot Rate Curve

- It implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  - $f(j, j+1) > S(j+1)$ if and only if $S(j+1) > S(j)$ from formula (19) on p. 139.
  - So
    $$E[\,S(j, j+1)\,] > S(j+1) > \cdots > S(1)$$
    if and only if $S(j+1) > \cdots > S(1)$.
- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.

A "Bad" Expectations Theory

- The expected returns[a] on all possible riskless bond strategies are equal for *all* holding periods.
- So

$$(1+S(2))^2 = (1+S(1))\, E[\,1+S(1,2)\,] \tag{25}$$

because of the equivalency between buying a two-period bond and rolling over one-period bonds.

- After rearrangement,

$$\frac{1}{E[\,1+S(1,2)\,]} = \frac{1+S(1)}{(1+S(2))^2}.$$

[a] More precisely, the one-plus returns.

A "Bad" Expectations Theory (continued)

- Now consider two one-period strategies.
  - Strategy one buys a two-period bond for $(1 + S(2))^{-2}$ dollars and sells it after one period.
  - The expected return is
$$E[\,(1 + S(1,2))^{-1}\,]/(1 + S(2))^{-2}.$$
  - Strategy two buys a one-period bond with a return of $1 + S(1)$.

A "Bad" Expectations Theory (continued)

- The theory says the returns are equal:

$$\frac{1+S(1)}{(1+S(2))^2} = E\left[\frac{1}{1+S(1,2)}\right].$$

- Combine this with Eq. (25) on p. 156 to obtain

$$E\left[\frac{1}{1+S(1,2)}\right] = \frac{1}{E[\,1+S(1,2)\,]}.$$

A "Bad" Expectations Theory (concluded)

- But this is impossible save for a certain economy.
  - Jensen's inequality states that $E[\,g(X)\,] > g(E[\,X\,])$ for any nondegenerate random variable X and strictly convex function g (i.e., $g''(x) > 0$).
  - Use
$$g(x) \stackrel{\Delta}{=} (1+x)^{-1}$$
    to prove our point.

Local Expectations Theory

- The expected rate of return of any bond over *a single period* equals the prevailing one-period spot rate:

$$\frac{E\left[\,(1+S(1,n))^{-(n-1)}\,\right]}{(1+S(n))^{-n}} = 1+S(1) \ \text{ for } \textit{all } n>1.$$

- This theory is the basis of many interest rate models.