Day Count Conventions: Actual/Actual

- The first “actual” refers to the actual number of days in a month.
- The second refers to the actual number of days in a year.
- The number of days between June 17, 1992, and October 1, 1992, is 106.
  - 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.
Day Count Conventions: 30/360

- Each month has 30 days and each year 360 days.
- The number of days between June 17, 1992, and October 1, 1992, is 104.
  - 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October.
- In general, the number of days from date \((y_1, m_1, d_1)\) to date \((y_2, m_2, d_2)\) is

\[
360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1). \tag{11}
\]
Day Count Conventions: 30/360 (continued)

• If \( d_1 \) or \( d_2 \) is 31, we must change it to 30 before applying formula (11).\(^a\)

• Hence:
  – There are 3 days between February 28 and March 1.
  – There are 2 days between February 29 and March 1.
  – There are 29 days between March 1 and March 31.

\(^a\)This is the simplest of all the “30/360” variations (called the “30E/360” convention), used mainly in the Eurobond market (Kosowski & Neftci, 2015).
Day Count Conventions: 30/360 (concluded)

- An equivalent formula to (11) on p. 80 without any adjustment is (check it)

\[
360 \times (y_2 - y_1) + 30 \times (m_2 - m_1 - 1) \\
+ \max(30 - d_1, 0) + \min(d_2, 30).
\]

- There are many variations on the “30/360” convention regarding 31, February 28, and February 29.\(^a\)

\(^a\)Kosowski & Neftci (2015).
Full Price (Dirty Price, Invoice Price)

- In reality, the settlement date may fall on any day between two coupon payment dates.

- Let

\[ \omega \triangleq \frac{\text{number of days between the settlement and the next coupon payment date}}{\text{number of days in the coupon period}}. \]  

(12)
Full Price (continued)

\[ C(1 - \omega) \]

coupon payment date

(1 − \omega)

\omega

coupon payment date
Full Price (concluded)

- The price is now calculated by

\[
PV = \frac{C}{\left(1 + \frac{r}{m}\right)^\omega} + \frac{C}{\left(1 + \frac{r}{m}\right)^{\omega+1}} \cdots
\]

\[
= \sum_{i=0}^{n-1} \frac{C}{\left(1 + \frac{r}{m}\right)^{\omega+i}} + \frac{F}{\left(1 + \frac{r}{m}\right)^{\omega+n-1}}. \quad (13)
\]
Accrued Interest

- The quoted price in the U.S./U.K. does not include the accrued interest; it is called the clean price or flat price.
- The buyer pays the invoice price: the quoted price plus the accrued interest (AI).
- The accrued interest equals

\[ C \times \frac{\text{number of days from the last coupon payment to the settlement date}}{\text{number of days in the coupon period}} = C \times (1 - \omega). \]
Accrued Interest (concluded)

• The yield to maturity is the $r$ satisfying Eq. (13) on p. 85 when PV is the invoice price:

$$\text{clean price} + \text{AI} = \sum_{i=0}^{n-1} \frac{C}{(1 + \frac{r}{m})^{\omega+i}} + \frac{F}{(1 + \frac{r}{m})^{\omega+n-1}}.$$
Example ("30/360")

- A bond with a 10% coupon rate and paying interest semiannually, with clean price 111.2891.
- The settlement date is July 1, 1993, and the maturity date is March 1, 1995.
- There are 60 days between July 1, 1993, and the next coupon date, September 1, 1993.
- The accrued interest is \((10/2) \times (1 - \frac{60}{180}) = 3.3333\) per $100 of par value.
Example (“30/360”) (concluded)

• The yield to maturity is 3%.

• This can be verified by Eq. (13) on p. 85 with
  \[ \omega = \frac{60}{180}, \]
  \[ n = 4, \]
  \[ m = 2, \]
  \[ F = 100, \]
  \[ C = 5, \]
  \[ PV = 111.2891 + 3.3333, \]
  \[ r = 0.03. \]
Price Behavior (2) Revisited

• Previously, a bond selling at par if the yield to maturity equals the coupon rate.

• But it assumed that the settlement date is on a coupon payment date.

• Suppose the settlement date for a bond selling at par\(^a\) falls between two coupon payment dates.

• Then its yield to maturity is less than the coupon rate.\(^b\)
  – The reason: Exponential growth to \(C\) is replaced by linear growth, hence overpaying the accrued interest.

\(^a\)The quoted price equals the par value.
\(^b\)See Exercise 3.5.6 of the textbook for proof.
Bond Price Volatility
“Well, Beethoven, what is this?”
— Attributed to Prince Anton Esterházy

\textsuperscript{a}Mass in C major.
Price Volatility

- Volatility measures how bond prices respond to interest rate changes.
- It is key to the risk management of interest rate-sensitive securities.
Price Volatility (concluded)

- What is the sensitivity of the percentage price change to changes in interest rates?

- Define price volatility by

\[ - \frac{\partial P}{\partial y} \cdot \frac{P}{P}. \]  

(14)
Price Volatility of Bonds

• The price volatility of a level-coupon bond is

\[- \frac{(C/y)n - (C/y^2)((1+y)^{n+1} - (1+y)) - nF}{(C/y)(((1+y)^{n+1} - (1+y)) + F(1+y))}.\]

– $F$ is the par value.
– $C$ is the coupon payment per period.
– Formula can be simplified a bit with $C = Fc/m$.

• For the above bond,

\[- \frac{\partial P}{\partial y} > 0.\]
Macaulay Duration\textsuperscript{a}

- The Macaulay duration (MD) is a weighted average of the times to an asset’s cash flows.
- The weights are the cash flows’ PVs divided by the asset’s price.
- Formally,

\[
MD \triangleq \frac{1}{P} \sum_{i=1}^{n} \frac{C_i}{(1 + y)^i} i.
\]

- What if \( C_i = (1 + c)^i \) for some constant \( c \) and letting \( n \to \infty \) and assuming \( c > y \)?\textsuperscript{b}

\textsuperscript{a}Frederick Macaulay (1882–1970) in 1938.

\textsuperscript{b}Contributed by Mr. Chen, Yu-Hsing (B06901048, R11922045) on March 3, 2023.
Macaulay Duration (concluded)

- The Macaulay duration, in periods, is equal to

\[
MD = -(1 + y) \frac{\partial P}{\partial y} \frac{1}{P}.
\]  

(15)
**MD of Bonds**

- The MD of a level-coupon bond is

\[
MD = \frac{1}{P} \left[ \sum_{i=1}^{n} \frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n} \right].
\]  

(16)

- It can be simplified to

\[
MD = \frac{c(1+y)[(1+y)^n-1] + ny(y-c)}{cy[(1+y)^n-1]+y^2},
\]

where \( c \) is the period coupon rate.

- The MD of a zero-coupon bond equals \( n \), its term to maturity.

- The MD of a level-coupon bond is less than \( n \).
Remarks

• Formulas (15) on p. 97 and (16) on p. 98 hold only if the coupon $C$, the par value $F$, and the maturity $n$ are all independent of the yield $y$.
  – That is, if the cash flow is independent of yields.

• To see this point, suppose the market yield declines.

• The MD will be lengthened.

• But for securities whose maturity actually decreases as a result, the price volatility\textsuperscript{a} may decrease.

\textsuperscript{a}As originally defined in formula (14) on p. 94.
How Not To Think about MD

- The MD has its origin in measuring the length of time a bond investment is outstanding.

- But it should be seen mainly as measuring *price volatility*.

- Duration of a security can be longer than its maturity or negative!

- Neither makes sense under the maturity interpretation.

- Many, if not most, duration-related terminology can only be comprehended as measuring volatility.
Conversion

• For the MD to be year-based, modify formula (16) on p. 98 to

\[
\frac{1}{P} \left[ \sum_{i=1}^{n} \frac{i}{k} \frac{C}{(1 + \frac{y}{k})^i} + \frac{n}{k} \frac{F}{(1 + \frac{y}{k})^n} \right],
\]

where \(y\) is the annual yield and \(k\) is the compounding frequency per annum.

• Formula (15) on p. 97 also becomes

\[
\text{MD} = - \left(1 + \frac{y}{k}\right) \frac{\partial P}{\partial y} \frac{1}{P}.
\]

• By definition, MD (in years) = \(\frac{\text{MD (in periods)}}{k}\).
Modified Duration

• Modified duration is defined as

\[
\text{modified duration} \triangleq - \frac{\partial P}{\partial y} \frac{1}{P} = \frac{\text{MD}}{(1 + y)}. \tag{17}
\]

– Modified duration equals MD under continuous compounding.

• By the Taylor expansion,

percent price change \approx -\text{modified duration} \times \text{yield change}.
Example

- Consider a bond whose modified duration is 11.54 with a yield of 10%.

- If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

  \[ -11.54 \times 0.001 = -0.01154 = -1.154\% . \]
Modified Duration of a Portfolio

• By calculus, the modified duration of a portfolio equals

\[ \sum_i \omega_i D_i. \]

- \( D_i \) is the modified duration of the \( i \)th asset.
- \( \omega_i \) is the market value of that asset expressed as a percentage of the market value of the portfolio.
Effective Duration

- Yield changes may alter the cash flow or the cash flow may be too complex for simple formulas.
- We need a general numerical formula for volatility.
- The effective duration is defined as
  \[
  \frac{P_- - P_+}{P_0(y_+ - y_-)}.
  \]
  - \(P_-\) is the price if the yield is decreased by \(\Delta y\).
  - \(P_+\) is the price if the yield is increased by \(\Delta y\).
  - \(P_0\) is the initial price, \(y\) is the initial yield.
  - \(\Delta y\) is small.
$P_0$, $P_+$, and $P_-$ are points on the curve, with $y_-$, $y$, and $y_+$ indicating specific values on the x-axis.
Effective Duration (concluded)

• One can compute the effective duration of just about any financial instrument.

• An alternative is to use

$$\frac{P_0 - P_+}{P_0 \Delta y}.$$  

  – More economical but theoretically less accurate.
The Practices

• Duration is usually expressed in percentage terms — call it $D\%$ — for quick mental calculation.\(^a\)

• The percentage price change expressed in percentage terms is then approximated by

$$-D\% \times \Delta r$$

when the yield increases instantaneously by $\Delta r\%$.

– Suppose $D\% = 10$ and $\Delta r = 2$.

– Price will drop by $20\%$ as $10 \times 2 = 20$.

• $D\%$ in fact equals modified duration (prove it!).

\(^a\)Neftci (2008), “Market professionals do not like to use decimal points.”
Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.

- Define dollar duration as

\[ \text{modified duration} \times \text{price} = -\frac{\partial P}{\partial y}. \]

- The approximate dollar price change is

\[ \text{price change} \approx -\text{dollar duration} \times \text{yield change}. \]

- One can hedge a bond portfolio with a dollar duration \( D \) by bonds with a dollar duration \( -D \).
Convexity

• Convexity is defined as

\[
\text{convexity (in periods)} \triangleq \frac{\partial^2 P}{\partial y^2} \frac{1}{P}.
\]

• The convexity of a level-coupon bond is positive (prove it!).

• For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude (see plot next page).

• So between two bonds with the same price and duration, the one with a higher convexity is more valuable.\(^a\)

\(^a\)Do you spot a problem here (Christensen & Sørensen, 1994)?
Convexity (concluded)

- Suppose there are $k$ periods per annum.
- Convexity measured in periods and convexity measured in years are related by

$$\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}.$$
Use of Convexity

• The approximation $\Delta P/P \approx -\text{duration} \times \text{yield change}$ works for small yield changes.

• For larger yield changes, use

$$\frac{\Delta P}{P} \approx \frac{\partial P}{\partial y} \frac{1}{P} \Delta y + \frac{1}{2} \frac{\partial^2 P}{\partial y^2} \frac{1}{P} (\Delta y)^2$$

$$= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.$$ 

• Recall the figure on p. 111.
The Practices

- Convexity is usually expressed in percentage terms — call it $C\%$ — for quick mental calculation.

- The percentage price change expressed in percentage terms is approximated by

\[-D\% \times \Delta r + C\% \times (\Delta r)^2 / 2\]

when the yield increases instantaneously by $\Delta r\%$.

- Price will drop by 17% if $D\% = 10$, $C\% = 1.5$, and $\Delta r = 2$ because

\[-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.\]

- $C\%$ equals convexity divided by 100 (prove it!).
Effective Convexity

• The effective convexity is defined as

\[ \frac{P_+ + P_- - 2P_0}{P_0 \left( 0.5 \times (y_+ - y_-) \right)^2} , \]

- \( P_- \) is the price if the yield is decreased by \( \Delta y \).
- \( P_+ \) is the price if the yield is increased by \( \Delta y \).
- \( P_0 \) is the initial price, \( y \) is the initial yield.
- \( \Delta y \) is small.

• Effective convexity is most relevant when a bond’s cash flow is interest rate sensitive.

• How to choose the right \( \Delta y \) is a delicate matter.
Approximate $d^2 f(x)^2 / dx^2$ at $x = 1$, Where $f(x) = x^2$

- The difference of $[(1 + \Delta x)^2 + (1 - \Delta x)^2 - 2]/(\Delta x)^2$ and 2:

![Graph showing error vs. 4x](image)

- This numerical issue is common in financial engineering but does not have general solutions yet (see pp. 869ff).

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Interest Rates and Bond Prices: Which Determines Which?\textsuperscript{a}

- If you have one, you have the other.
- So they are just two names given to the same thing: cost of fund.
- Traders most likely work with prices.
- Banks most likely work with interest rates.

\textsuperscript{a}Contributed by Mr. Wang, Cheng (R01741064) on March 5, 2014.
Term Structure of Interest Rates
Why is it that the interest of money is lower, when money is plentiful?
— Samuel Johnson (1709–1784)

If you have money, don’t lend it at interest.
Rather, give [it] to someone from whom you won’t get it back.
— Thomas Gospel 95
Term Structure of Interest Rates

• Concerned with how interest rates change with maturity.

• The set of yields to maturity for bonds form the term structure.
  – The bonds must be of equal quality.
  – They differ solely in their terms to maturity.

• The term structure is fundamental to the valuation of fixed-income securities.
Term Structure of Interest Rates (concluded)

- The term “term structure” often refers exclusively to the yields of zero-coupon bonds.

- A yield curve plots the yields to maturity of coupon bonds against maturity.

- A par yield curve is constructed from bonds trading near par.
Yield Curve of U.S. Treasuries as of July 24, 2015
Four Typical Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.
Spot Rates

• The $i$-period spot rate $S(i)$ is the yield to maturity of an $i$-period zero-coupon bond.

• The PV of one dollar $i$ periods from now is by definition
  
  \[ [1 + S(i)]^{-i}. \]

  – It is the price of an $i$-period zero-coupon bond.$^a$

• The one-period spot rate is called the short rate.

• Spot rate curve.$^b$ Plot of spot rates against maturity:
  
  \[ S(1), S(2), \ldots, S(n). \]

---

$^a$Recall Eq. (9) on p. 69.

$^b$That is, term structure, per our convention.
Problems with the PV Formula

• In the bond price formula (4) on p. 41,
\[
\sum_{i=1}^{n} \frac{C}{(1 + y)^i} + \frac{F}{(1 + y)^n},
\]
every cash flow is discounted at the same yield \(y\).

• Consider two riskless bonds with different yields to maturity because of their different cash flows:

\[
PV_1 = \sum_{i=1}^{n_1} \frac{C}{(1 + y_1)^i} + \frac{F}{(1 + y_1)^{n_1}},
\]
\[
PV_2 = \sum_{i=1}^{n_2} \frac{C}{(1 + y_2)^i} + \frac{F}{(1 + y_2)^{n_2}}.
\]
Problems with the PV Formula (concluded)

- The yield-to-maturity methodology discounts their contemporaneous cash flows with different rates.
- But shouldn’t they be discounted at the same rate?
Spot Rate Discount Methodology

• A cash flow $C_1, C_2, \ldots, C_n$ is equivalent to a package of zero-coupon bonds with the $i$th bond paying $C_i$ dollars at time $i$.
Spot Rate Discount Methodology (concluded)

- So a level-coupon bond has the price
\[
P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}.
\]  

- This pricing method incorporates information from the term structure.

- It discounts each cash flow at the matching spot rate.
Discount Factors

• In general, any riskless security having a cash flow $C_1, C_2, \ldots, C_n$ should have a market price of

$$P = \sum_{i=1}^{n} C_i d(i).$$

– Above, $d(i) \triangleq [1 + S(i)]^{-i}$, $i = 1, 2, \ldots, n$, are called the discount factors.

– $d(i)$ is the PV of one dollar $i$ periods from now.

– The above formula will be justified on p. 223.

• The discount factors are often interpolated to form a continuous function called the discount function.
Extracting Spot Rates from Yield Curve

• Start with the short rate $S(1)$.
  – Note that short-term Treasuries are zero-coupon bonds.

• Compute $S(2)$ from the two-period coupon bond price $P$ by solving

$$P = \frac{C}{1 + S(1)} + \frac{C + 100}{[1 + S(2)]^2}.$$
Extracting Spot Rates from Yield Curve (concluded)

• Inductively, we are given the market price $P$ of the $n$-period coupon bond and

$$S(1), S(2), \ldots, S(n - 1).$$

• Then $S(n)$ can be computed from Eq. (18) on p. 128, repeated below,

$$P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}.$$ 

• The running time can be made to be $O(n)$ (see text).

• The procedure is called bootstrapping.
Some Problems

- Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).
- Some maturities might be missing from the data points (the incompleteness problem).
- Treasuries might not be of the same quality.
- Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.\(^a\)

\(^a\)Often without economic justifications.
Which One (from P. 122)?
Yield Spread

- Consider a risky bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for $P$.
- Calculate the IRR of the risky bond.
- Calculate the IRR of a riskless bond with comparable maturity.
- Yield spread is their difference.
Static Spread

- Were the risky bond riskless, it would fetch

\[ P^* = \sum_{t=1}^{n} \frac{C_t}{[1 + S(t)]^t}. \]

- But as risk must be compensated, in reality \( P < P^* \).

- The static spread is the amount \( s \) by which the spot rate curve has to shift in parallel to price the risky bond:

\[ P = \sum_{t=1}^{n} \frac{C_t}{[1 + s + S(t)]^t}. \]

- Unlike the yield spread, the static spread explicitly incorporates information from the term structure.
Of Spot Rate Curve and Yield Curve

• \( y_i \): yield to maturity for the \( i \)-period coupon bond.

• \( S(k) \geq y_k \) if \( y_1 < y_2 < \cdots \) (yield curve is normal).

• \( S(k) \leq y_k \) if \( y_1 > y_2 > \cdots \) (yield curve is inverted).

• \( S(k) \geq y_k \) if \( S(1) < S(2) < \cdots \) (spot rate curve is normal).

• \( S(k) \leq y_k \) if \( S(1) > S(2) > \cdots \) (spot rate curve is inverted).

• If the yield curve is flat, the spot rate curve coincides with the yield curve.
Shapes

- The spot rate curve often has the same shape as the yield curve.
  - If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).
- But this is only a trend not a mathematical truth.\(^a\)

\(^a\)See a counterexample in the text.
Forward Rates

- The yield curve contains information regarding future interest rates currently “expected” by the market.

- Invest $1 for \( j \) periods to end up with \( [1 + S(j)]^j \) dollars at time \( j \).
  - The maturity strategy.

- Invest $1 in bonds for \( i \) periods and at time \( i \) invest the proceeds in bonds for another \( j - i \) periods where \( j > i \).

- Will have \( [1 + S(i)]^i[1 + S(i, j)]^{j-i} \) dollars at time \( j \).
  - \( S(i, j) \): \((j - i)\)-period spot rate \( i \) periods from now.
  - The rollover strategy.
Forward Rates (concluded)

- When $S(i, j)$ equals

$$f(i, j) \triangleq \left[ \frac{(1 + S(j))^j}{(1 + S(i))^i} \right]^{1/(j-i)} - 1,$$  \hspace{1cm} (19)

we will end up with the same $[1 + S(j)]^j$ dollars.

- As expected,

$$f(0, j) = S(j).$$

- The $f(i, j)$ are the (implied) forward (interest) rates.
  - More precisely, the $(j - i)$-period forward rate $i$ periods from now.
Time Line

\[ f(0, 1) \quad f(1, 2) \quad f(2, 3) \quad f(3, 4) \]

Time 0

\( S(1) \quad S(2) \quad S(3) \quad S(4) \)
Forward Rates and Future Spot Rates

• We did not assume any a priori relation between $f(i, j)$ and future spot rate $S(i, j)$.
  – This is the subject of the term structure theories.

• We merely looked for the future spot rate that, if realized, will equate the two investment strategies.

• The $f(i, i + 1)$ are the instantaneous forward rates or one-period forward rates.
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,
  \[ f(i, j) > S(j) > \cdots > S(i) \].

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,
  \[ f(i, j) < S(j) < \cdots < S(i) \].
Forward Rates ≡ Spot Rates ≡ Yield Curve

- The FV of $1 at time $n$ can be derived in two ways.
- Buy $n$-period zero-coupon bonds and receive

$$[1 + S(n)]^n.$$  

- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is

$$[1 + S(1)][1 + f(1, 2)] \cdots [1 + f(n - 1, n)].$$
Forward Rates ≡ Spot Rates ≡ Yield Curves (concluded)

• Since they are identical,

\[ S(n) = \left\{ \left[ 1 + S(1) \right] \left[ 1 + f(1, 2) \right] \right. \]

\[ \cdots \left[ 1 + f(n - 1, n) \right] \}^{1/n} - 1. \]  

(20)

• Hence, the forward rates (specifically the one-period forward rates) determine the spot rate curve.

• Other equivalencies can be derived similarly, such as

\[ f(T, T + 1) = \frac{d(T)}{d(T + 1)} - 1. \]  

(21)
Locking in the Forward Rate $f(n, m)$

- Buy one $n$-period zero-coupon bond for $\frac{1}{(1 + S(n))^n}$ dollars.
- Sell $\frac{(1 + S(m))^m}{(1 + S(n))^n}$ $m$-period zero-coupon bonds.
- No net initial investment because the cash inflow equals the cash outflow: $\frac{1}{(1 + S(n))^n}$.
- At time $n$ there will be a cash inflow of $1$.
- At time $m$ there will be a cash outflow of $\frac{(1 + S(m))^m}{(1 + S(n))^n}$ dollars.
Locking in the Forward Rate $f(n, m)$ (concluded)

- This implies the interest rate between times $n$ and $m$ equals $f(n, m)$ by formula (19) on p. 139.\(^a\)

\[
\frac{(1 + S(m))^m}{(1 + S(n))^n}
\]

\(^a\)Note that \(\frac{(1 + S(m))^m}{(1 + S(n))^n} = (1 + f(n, m))^{m-n}\) by that formula.
Forward Loans

• We had generated the cash flow of a type of forward contract called the forward loan.

• Agreed upon today, it enables one to
  – Borrow money at time $n$ in the future, and
  – Repay the loan at time $m > n$ with an interest rate equal to the known forward rate
    $$f(n, m).$$

• Can the spot rate curve be arbitrarily drawn?\(^a\)

\(^a\)Contributed by Mr. Dai, Tian-Shyr (B82506025, R86526008, D88526006) in 1998.
Synthetic Bonds

• We had seen that

\[
\text{forward loan} = n\text{-period zero} - [1 + f(n, m)]^{m-n} \times m\text{-period zero}.
\]

• Thus

\[
n\text{-period zero} = \text{forward loan} + [1 + f(n, m)]^{m-n} \times m\text{-period zero}.
\]

• We have created a \textit{synthetic} zero-coupon bond with forward loans and other zero-coupon bonds.

• Useful if the \textit{n}-period zero is unavailable or illiquid.
Spot and Forward Rates under Continuous Compounding

- The pricing formula:

\[ P = \sum_{i=1}^{n} C e^{-iS(i)} + F e^{-nS(n)}. \]

- The market discount function:

\[ d(n) = e^{-nS(n)}. \]

- The spot rate is an arithmetic average of forward rates,\(^a\)

\[ S(n) = \frac{f(0, 1) + f(1, 2) + \cdots + f(n - 1, n)}{n}. \]

\(^a\)Compare it with formula (20) on p. 145.
Spot and Forward Rates under Continuous Compounding (continued)

- The formula for the forward rate:

\[ f(i, j) = \frac{jS(j) - iS(i)}{j - i}. \]  \hspace{1cm} (22)

- Compare the above formula with (19) on p. 139.

- The one-period forward rate:

\[ f(j, j + 1) = -\ln \frac{d(j + 1)}{d(j)}. \]

\[ a \text{Compare it with formula (21) on p. 145.} \]
Spot and Forward Rates under Continuous Compounding (concluded)

• Now, the (instantaneous) forward rate curve is:

\[ f(T) \triangleq \lim_{\Delta T \to 0} f(T, T + \Delta T) \]

\[ = S(T) + T \frac{\partial S}{\partial T}. \tag{23} \]

• So \( f(T) > S(T) \) if and only if \( \partial S/\partial T > 0 \) (i.e., a normal spot rate curve).

• If \( S(T) < -T(\partial S/\partial T) \), then \( f(T) < 0 \).\(^a\)

\(^a\)Consistent with the plot on p. 143. Contributed by Mr. Huang, Hsien-Chun (R03922103) on March 11, 2015.
An Example

• Let the interest rates be continuously compounded.

• Suppose the spot rate curve is \( S(T) \triangleq 0.08 - 0.05 e^{-0.18T} \).

• Then by Eq. (23) on p. 152, the forward rate curve is

\[
\begin{align*}
  f(T) &= S(T) + TS'(T) \\
  &= 0.08 - 0.05 e^{-0.18T} + 0.009T e^{-0.18T}.
\end{align*}
\]

\(^{a}\text{Hull & White (1994).}\)
Unbiased Expectations Theory

• Forward rate equals the average future spot rate,

\[ f(a, b) = E[S(a, b)]. \] (24)

• It does not imply that the forward rate is an accurate predictor for the future spot rate.

• It implies the maturity strategy and the rollover strategy produce the same result at the horizon “on average.”
Unbiased Expectations Theory and Spot Rate Curve

- It implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  - \( f(j, j + 1) > S(j + 1) \) if and only if \( S(j + 1) > S(j) \) from formula (19) on p. 139.
  - So
    \[
    E[S(j, j + 1)] > S(j + 1) > \cdots > S(1)
    \]
    if and only if \( S(j + 1) > \cdots > S(1) \).

- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.
A “Bad” Expectations Theory

- The expected returns\(^a\) on all possible riskless bond strategies are equal for all holding periods.

- So

\[
(1 + S(2))^2 = (1 + S(1)) \mathbb{E}[1 + S(1, 2)]
\]

(25)

because of the equivalency between buying a two-period bond and rolling over one-period bonds.

- After rearrangement,

\[
\frac{1}{\mathbb{E}[1 + S(1, 2)]} = \frac{1 + S(1)}{(1 + S(2))^2}.
\]

\(^a\)More precisely, the one-plus returns.
A “Bad” Expectations Theory (continued)

• Now consider two one-period strategies.
  – Strategy one buys a two-period bond for \((1 + S(2))^{-2}\) dollars and sells it after one period.
  – The expected return is 
    
    \[
    E\left[\frac{(1 + S(1, 2))^{-1}}{(1 + S(2))^{-2}}\right].
    \]
  – Strategy two buys a one-period bond with a return of 
    \(1 + S(1)\).
A “Bad” Expectations Theory (continued)

- The theory says the returns are equal:

\[
\frac{1 + S(1)}{(1 + S(2))^2} = E \left[ \frac{1}{1 + S(1, 2)} \right].
\]

- Combine this with Eq. (25) on p. 156 to obtain

\[
E \left[ \frac{1}{1 + S(1, 2)} \right] = \frac{1}{E[1 + S(1, 2)]}.
\]
A “Bad” Expectations Theory (concluded)

• But this is impossible save for a certain economy.
  – Jensen’s inequality states that $E[g(X)] > g(E[X])$
    for any nondegenerate random variable $X$ and
    strictly convex function $g$ (i.e., $g''(x) > 0$).
  – Use
    $$g(x) \triangleq (1 + x)^{-1}$$
    to prove our point.
Local Expectations Theory

- The expected rate of return of any bond over a single period equals the prevailing one-period spot rate:
  \[
  E \left[ \frac{(1 + S(1, n))^{-(n-1)}}{(1 + S(n))^{-n}} \right] = 1 + S(1) \text{ for all } n > 1.
  \]

- This theory is the basis of many interest rate models.