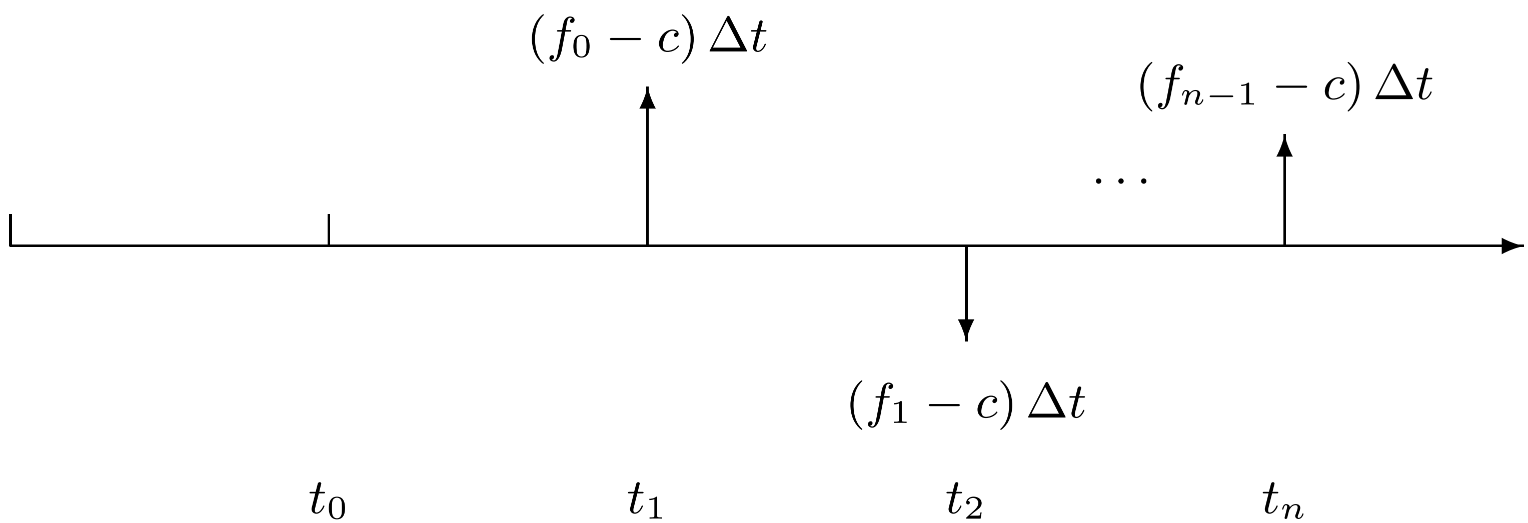
## Interest Rate Swaps

- Consider an interest rate swap made at time $t$ (now) with payments to be exchanged at times $t_1, t_2, \ldots, t_n$.
- For simplicity, assume $t_{i+1} - t_i$ is a fixed constant $\Delta t$ for all $i$, and the notional principal is one dollar.
- The fixed rate is $c$ per annum.
- The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.
- The payoff at time $t_{i+1}$ for the *fixed-rate payer* is $(f_i - c)\,\Delta t$.

# Interest Rate Swaps (continued)



## Interest Rate Swaps (continued)

- Simple rates are adopted here.
- Hence $f_i$ satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$

- If $t < t_0$, we have a forward interest rate swap.
- The ordinary swap corresponds to $t = t_0$.

## Interest Rate Swaps (continued)

- The value of the swap at time $t$ is thus

$$
\begin{aligned}
& \sum_{i=1}^{n} E_t^{\pi} \left[ e^{-\int_t^{t_i} r(s)\,ds} (f_{i-1} - c)\,\Delta t \right] \\
= \; & \sum_{i=1}^{n} E_t^{\pi} \left[ e^{-\int_t^{t_i} r(s)\,ds} \left( \frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right) \right] \\
= \; & \sum_{i=1}^{n} E_t^{\pi} \left[ e^{-\int_t^{t_i} r(s)\,ds} \left( e^{\int_{t_{i-1}}^{t_i} r(s)\,ds} - (1 + c\Delta t) \right) \right] \\
= \; & \sum_{i=1}^{n} [\, P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i) \,] \\
= \; & P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).
\end{aligned}
$$

## Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds, statically.
- In fact, it can be priced by simple PV calculations.

## Swap Rate

- The swap rate, which gives the swap zero value, equals

$$S_n(t) \triangleq \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} P(t, t_i)\, \Delta t}. \tag{153}$$

- The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.
- For an ordinary swap, $P(t, t_0) = 1$.
- The swap rate is called a forward swap rate if $t_0 > t$.

## The Term Structure Equation[a]

- Let us start with the zero-coupon bonds and the money market account.
- Let the zero-coupon bond price $P(r, t, T)$ follow

$$\frac{dP}{P} = \mu_p \, dt + \sigma_p \, dW.$$

- At time $t$, short one unit of a bond maturing at time $s_1$ and buy $\alpha$ units of a bond maturing at time $s_2$.

[a]Vasicek (1977). Vasicek co-founded KMV, which was sold to Moody's for USD$210 million in 2002.

## The Term Structure Equation (continued)

- The net wealth change follows

$$
\begin{aligned}
& -dP(r,t,s_1) + \alpha\, dP(r,t,s_2) \\
= \;& (-P(r,t,s_1)\,\mu_p(r,t,s_1) + \alpha P(r,t,s_2)\,\mu_p(r,t,s_2))\,dt \\
& + (-P(r,t,s_1)\,\sigma_p(r,t,s_1) + \alpha P(r,t,s_2)\,\sigma_p(r,t,s_2))\,dW.
\end{aligned}
$$

- Pick

$$\alpha \triangleq \frac{P(r,t,s_1)\,\sigma_p(r,t,s_1)}{P(r,t,s_2)\,\sigma_p(r,t,s_2)}.$$

## The Term Structure Equation (continued)

- Then the net wealth has no volatility and must earn the riskless return:

$$\frac{-P(r,t,s_1)\,\mu_p(r,t,s_1)+\alpha P(r,t,s_2)\,\mu_p(r,t,s_2)}{-P(r,t,s_1)+\alpha P(r,t,s_2)}=r.$$

- Simplify the above to obtain

$$\frac{\sigma_p(r,t,s_1)\,\mu_p(r,t,s_2)-\sigma_p(r,t,s_2)\,\mu_p(r,t,s_1)}{\sigma_p(r,t,s_1)-\sigma_p(r,t,s_2)}=r.$$

- This becomes

$$\frac{\mu_p(r,t,s_2)-r}{\sigma_p(r,t,s_2)}=\frac{\mu_p(r,t,s_1)-r}{\sigma_p(r,t,s_1)}$$

after rearrangement.

## The Term Structure Equation (continued)

- Since the above equality holds for any $s_1$ and $s_2$,

$$\frac{\mu_p(r,t,s) - r}{\sigma_p(r,t,s)} \triangleq \lambda(r,t) \tag{154}$$

  for some $\lambda$ *independent* of the bond maturity $s$.

- As $\mu_p = r + \lambda\sigma_p$, all assets are expected to appreciate at a rate equal to the sum of the short rate and a constant times the asset's volatility.

- The term $\lambda(r,t)$ is called the market price of risk.

- The market price of risk must be the same for all bonds to preclude arbitrage opportunities.

## The Term Structure Equation (continued)

- Assume a Markovian short rate model,

$$dr = \mu(r,t)\,dt + \sigma(r,t)\,dW.$$

- Then the bond price process is also Markovian.
- By Eq. (14.15) on p. 202 of the textbook,

$$\mu_p = \left[-\frac{\partial P}{\partial T} + \mu(r,t)\,\frac{\partial P}{\partial r} + \frac{\sigma(r,t)^2}{2}\,\frac{\partial^2 P}{\partial r^2}\right]/P, \tag{155}$$

$$\sigma_p = \sigma(r,t)\,\frac{\partial P}{\partial r}/P, \tag{155'}$$

subject to $P(\,\cdot\,,T,T)=1$.

# The Term Structure Equation (concluded)

- Substitute $\mu_p$ and $\sigma_p$ into Eq. (154) on p. 1109 to obtain

$$-\frac{\partial P}{\partial T} + [\,\mu(r,t) - \lambda(r,t)\,\sigma(r,t)\,]\,\frac{\partial P}{\partial r} + \frac{1}{2}\,\sigma(r,t)^2\,\frac{\partial^2 P}{\partial r^2} = rP. \tag{156}$$

- This is called the term structure equation.
- It applies to all interest rate derivatives: The differences are the terminal and boundary conditions.
- Once $P$ is available, the spot rate curve emerges via

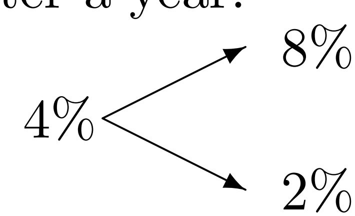
$$r(t,T) = -\frac{\ln P(t,T)}{T-t}.$$

## Numerical Examples

- Assume this spot rate curve:

| Year | 1 | 2 |
|---|---|---|
| Spot rate | 4% | 5% |

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:



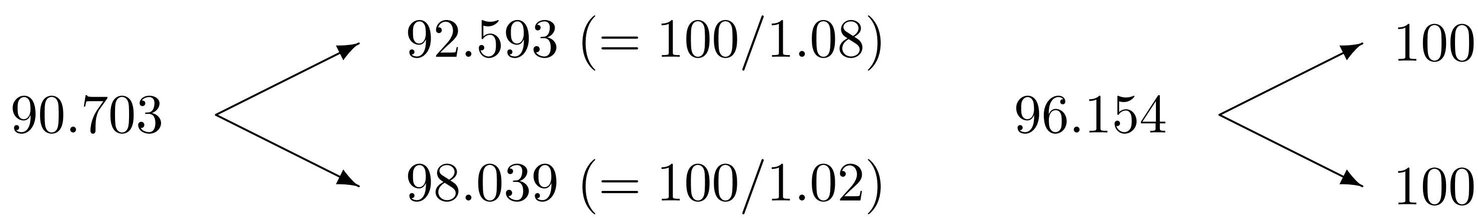
## Numerical Examples (continued)

- *No* real-world probabilities are given.
- The prices of one- and two-year zero-coupon bonds are, respectively,

$$\begin{aligned} 100/1.04 &= 96.154, \\ 100/(1.05)^2 &= 90.703. \end{aligned}$$

- They follow the binomial processes on p. 1114.

# Numerical Examples (continued)




The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.

## Numerical Examples (continued)

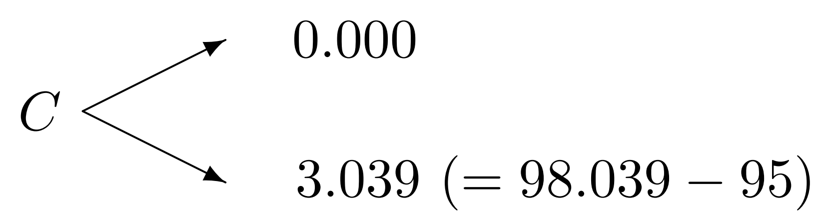
- The pricing of derivatives can be simplified by assuming investors are risk-neutral.
- Suppose all securities have the same expected one-period rate of return, the riskless rate.
- Then
$$(1-p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\%,$$
where $p$ denotes the risk-neutral probability of a down move in rates.

## Numerical Examples (concluded)

- Solving the equation leads to $p = 0.319$.
- Interest rate contingent claims can be priced under this probability.

## Numerical Examples: Fixed-Income Options

- A one-year European call on the two-year zero with a \$95 strike price has the payoffs,




- To solve for the option value $C$, we replicate the call by a portfolio of $x$ one-year and $y$ two-year zeros.

# Numerical Examples: Fixed-Income Options (continued)

- This leads to the simultaneous equations,

$$x \times 100 + y \times 92.593 = 0.000,$$
$$x \times 100 + y \times 98.039 = 3.039.$$

- They give $x = -0.5167$ and $y = 0.5580$.
- Consequently,

$$C = x \times 96.154 + y \times 90.703 \approx 0.93$$

to prevent arbitrage.

## Numerical Examples: Fixed-Income Options (continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.

## Numerical Examples: Fixed-Income Options (concluded)

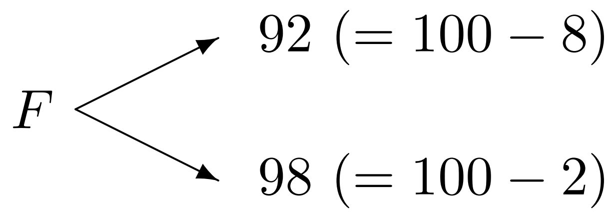
- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth

$$C = \frac{(1-p) \times 0 + p \times 3.039}{1.04} \approx 0.93,$$

  the same as before.
- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.

# Numerical Examples: Futures and Forward Prices

- A one-year futures contract on the one-year rate has a payoff of $100 - r$, where $r$ is the one-year rate at maturity:




- As the futures price $F$ is the expected future payoff,[a]

$$F = (1 - p) \times 92 + p \times 98 = 93.914.$$

[a] See Exercise 13.2.11 of the textbook or p. 568.

## Numerical Examples: Futures and Forward Prices (concluded)

- The forward price for a one-year forward contract on a one-year zero-coupon bond is[a]

$$90.703/96.154 = 94.331\%.$$

- The forward price exceeds the futures price.[b]

---

[a]By Eq. (145) on p. 1090.
[b]Unlike the nonstochastic case on p. 510.

# *Equilibrium Term Structure Models*

The nature of modern trade
is to give to those who have much
and take from those who have little.
— Walter Bagehot (1867),
*The English Constitution*

8. What's your problem? Any moron
can understand bond pricing models.
— *Top Ten Lies Finance Professors
Tell Their Students*

## Introduction

- We now survey equilibrium models.
- Recall that the spot rates satisfy

$$r(t,T) = -\frac{\ln P(t,T)}{T-t}$$

by Eq. (144) on p. 1089.

- Hence the discount function $P(t,T)$ suffices to establish the spot rate curve.
- All models to follow are short rate models.
- Unless stated otherwise, the processes are risk-neutral.

# The Vasicek Model[a]

- The short rate follows

$$dr = \beta(\mu - r)\, dt + \sigma\, dW.$$

- The short rate is pulled to the long-term mean level $\mu$ at rate $\beta$.
- Superimposed on this "pull" is a normally distributed stochastic term $\sigma\, dW$.
- Since the process is an Ornstein-Uhlenbeck process,

$$E[\, r(T) \,|\, r(t) = r \,] = \mu + (r - \mu)\, e^{-\beta(T-t)}$$

from Eq. (89) on p. 635.

[a]Vasicek (1977).

# The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

$$P(t,T) = A(t,T)\, e^{-B(t,T)\, r(t)}, \tag{157}$$

where

$$A(t,T) = \begin{cases} \exp\left[ \frac{(B(t,T)-T+t)(\beta^2\mu-\sigma^2/2)}{\beta^2} - \frac{\sigma^2 B(t,T)^2}{4\beta} \right], & \text{if } \beta \neq 0, \\ \exp\left[ \frac{\sigma^2 (T-t)^3}{6} \right], & \text{if } \beta = 0, \end{cases}$$

and

$$B(t,T) = \begin{cases} \frac{1-e^{-\beta(T-t)}}{\beta}, & \text{if } \beta \neq 0, \\ T-t, & \text{if } \beta = 0. \end{cases}$$

## The Vasicek Model (continued)

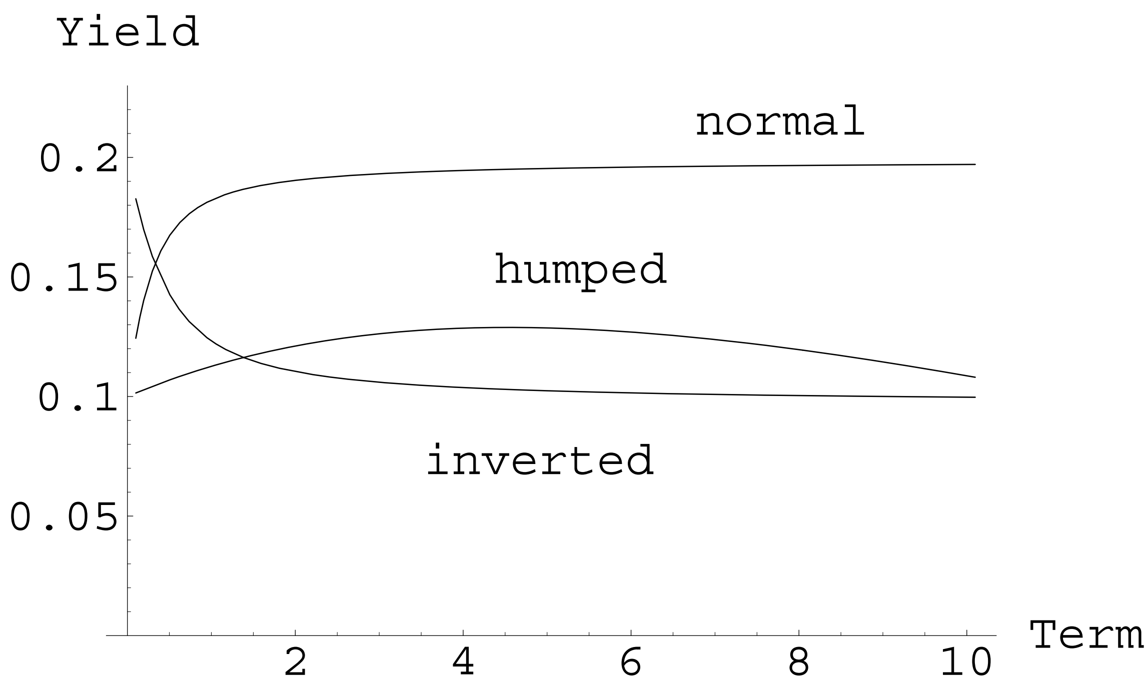
- If $\beta = 0$, then $P$ goes to infinity as $T \to \infty$.
- Sensibly, $P$ goes to zero as $T \to \infty$ if $\beta \neq 0$.
- But even if $\beta \neq 0$, $P$ may exceed one for a finite $T$.
- The long rate $r(t, \infty)$ is the constant

$$\mu - \frac{\sigma^2}{2\beta^2},$$

independent of the current short rate.

# The Vasicek Model (concluded)

- The spot rate volatility structure is the curve
$$\sigma \frac{\partial r(t,T)}{\partial r} = \frac{\sigma B(t,T)}{T-t}.$$
- As it depends only on $T-t$ not on $t$ by itself, the same curve is maintained for any future time $t$.
- When $\beta > 0$, the curve tends to decline with maturity.
  - The long rate's volatility is zero unless $\beta = 0$.
- The speed of mean reversion, $\beta$, controls the shape of the curve.
- Higher $\beta$ leads to greater attenuation of volatility with maturity.


Yield
normal
humped
inverted
0.2
0.15
0.1
0.05
2
4
6
8
10
Term

# The Vasicek Model: Options on Zeros[a]

- Consider a European call with strike price $X$ expiring at time $T$ on a zero-coupon bond with par value \$1 and maturing at time $s > T$.
- Its price is given by

$$P(t, s)\, N(x) - X P(t, T)\, N(x - \sigma_v).$$

[a]Jamshidian (1989).

# The Vasicek Model: Options on Zeros (concluded)

- Above

$$\begin{aligned} x &\stackrel{\Delta}{=} \frac{1}{\sigma_v} \ln\left(\frac{P(t,s)}{P(t,T)\,X}\right) + \frac{\sigma_v}{2}, \\ \sigma_v &\equiv v(t,T)\,B(T,s), \\ v(t,T)^2 &\stackrel{\Delta}{=} \begin{cases} \frac{\sigma^2\left[1-e^{-2\beta(T-t)}\right]}{2\beta}, & \text{if } \beta \neq 0 \\ \sigma^2(T-t), & \text{if } \beta = 0 \end{cases}. \end{aligned}$$

- By the put-call parity, the price of a European put is

$$XP(t,T)\,N(-x+\sigma_v) - P(t,s)\,N(-x).$$

# Binomial Vasicek[a]

- Consider a binomial model for the short rate in the time interval $[0, T]$ divided into $n$ identical pieces.
- Let $\Delta t \triangleq T/n$ and[b]

$$p(r) \triangleq \frac{1}{2} + \frac{\beta(\mu - r)\sqrt{\Delta t}}{2\sigma}.$$

- The following binomial model converges to the Vasicek model,[c]

$$r(k+1) = r(k) + \sigma\sqrt{\Delta t}\,\xi(k), \quad 0 \leq k < n.$$

[a]Nelson & Ramaswamy (1990).

[b]The same form as Eq. (42) on p. 299 for the BOPM.

[c]Same as the CRR tree except that the probabilities vary here.

## Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with

$$\text{Prob}[\,\xi(k) = 1\,] = \begin{cases} p(r(k)), & \text{if } 0 \leq p(r(k)) \leq 1 \\ 0, & \text{if } p(r(k)) < 0, \\ 1, & \text{if } 1 < p(r(k)). \end{cases}$$

- Observe that the probability of an up move, $p$, is a decreasing function of the interest rate $r$.
- This is consistent with mean reversion.

## Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its *constant* volatility, $\sigma$.

## The Cox-Ingersoll-Ross Model[a]

- It is the following square-root short rate model:

$$dr = \beta(\mu - r)\,dt + \sigma\sqrt{r}\,dW. \tag{158}$$

- The diffusion differs from the Vasicek model by a multiplicative factor $\sqrt{r}$.
- The parameter $\beta$ determines the speed of adjustment.
- If $r(0) > 0$, then the short rate can reach zero *only if*

$$2\beta\mu < \sigma^2.$$

  – This is called the Feller (1951) condition.
- See text for the bond pricing formula.

[a]Cox, Ingersoll, & Ross (1985).

# Binomial CIR

- We want to approximate the short rate process in the time interval $[0, T]$.
- Divide it into $n$ periods of duration $\Delta t \triangleq T/n$.
- Assume $\mu, \beta \geq 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will *not* combine.

# Binomial CIR (continued)

- Instead, consider the transformed process[a]

$$x(r) \stackrel{\Delta}{=} 2\sqrt{r}/\sigma.$$

- By Ito's lemma (p. 610),

$$dx = m(x)\, dt + dW,$$

where

$$m(x) \stackrel{\Delta}{=} 2\beta\mu/(\sigma^2 x) - (\beta x/2) - 1/(2x).$$

- This new process has a *constant* volatility.
- Thus its binomial tree combines.

[a]See pp. 1148ff for justification.

## Binomial CIR (continued)

- Construct the combining tree for $r$ as follows.
- First, construct a tree for $x$.
- Then transform each node of the tree into one for $r$ via the inverse transformation (see next page)

$$r = f(x) \triangleq \frac{x^2 \sigma^2}{4}.$$

- But when $x \approx 0$ (so $r \approx 0$), the moments may not be matched well.[a]

[a] Nawalkha & Beliaeva (2007).

$$
\begin{array}{ccccccccccc}
 & & & & x+2\sqrt{\Delta t} & & & & & & f(x+2\sqrt{\Delta t})\\
 & & & \nearrow & & & & & & \nearrow & \\
 & & x+\sqrt{\Delta t} & & & & & & f(x+\sqrt{\Delta t}) & & \\
 & \nearrow & & \searrow & & & & \nearrow & & \searrow & \\
x & & & & x & & f(x) & & & & f(x)\\
 & \searrow & & \nearrow & & & & \searrow & & \nearrow & \\
 & & x-\sqrt{\Delta t} & & & & & & f(x-\sqrt{\Delta t}) & & \\
 & & & \searrow & & & & & & \searrow & \\
 & & & & x-2\sqrt{\Delta t} & & & & & & f(x-2\sqrt{\Delta t})
\end{array}
$$

## Binomial CIR (continued)

- The probability of an up move at each node $r$ is

$$p(r) \triangleq \frac{\beta(\mu - r)\,\Delta t + r - r^-}{r^+ - r^-}.$$

  - $r^+ \triangleq f(x + \sqrt{\Delta t})$ denotes the result of an up move from $r$.
  - $r^- \triangleq f(x - \sqrt{\Delta t})$ the result of a down move.
- Finally, set the probability $p(r)$ to one as $r$ goes to zero to make the probability stay between zero and one.

# Binomial CIR (concluded)

- It can be shown that

$$p(r) = \left(\beta\mu - \frac{\sigma^2}{4}\right)\sqrt{\frac{\Delta t}{r}} - B\sqrt{r\Delta t} + C,$$

for some $B \geq 0$ and $C > 0$.[a]

- If $\beta\mu - (\sigma^2/4) \geq 0$, the up-move probability $p(r)$ decreases if and only if short rate $r$ increases.
- Even if $\beta\mu - (\sigma^2/4) < 0$, $p(r)$ *tends* to decrease as $r$ increases and decrease as $r$ declines.
- This phenomenon agrees with mean reversion.

[a]Thanks to a lively class discussion on May 28, 2014.

## Numerical Examples

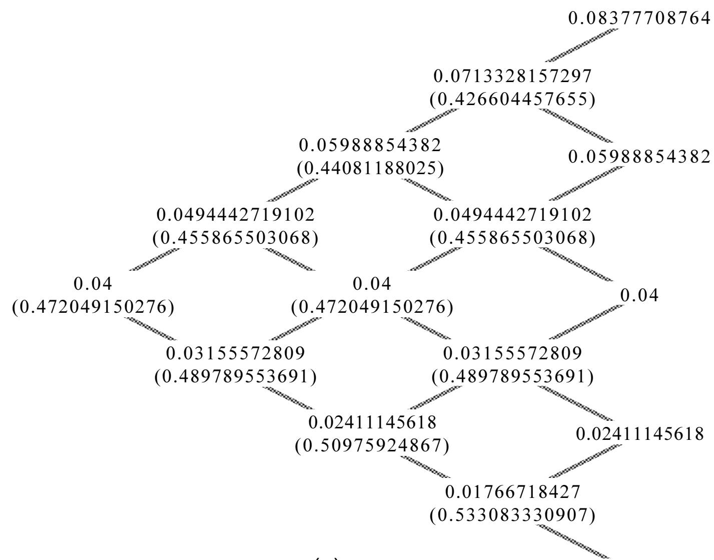
- Consider the process,

$$0.2\,(0.04 - r)\,dt + 0.1\sqrt{r}\,dW,$$

  for the time interval $[\,0, 1\,]$ given the initial rate $r(0) = 0.04$.

- We shall use $\Delta t = 0.2$ (year) for the binomial approximation.

- See p. 1144(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.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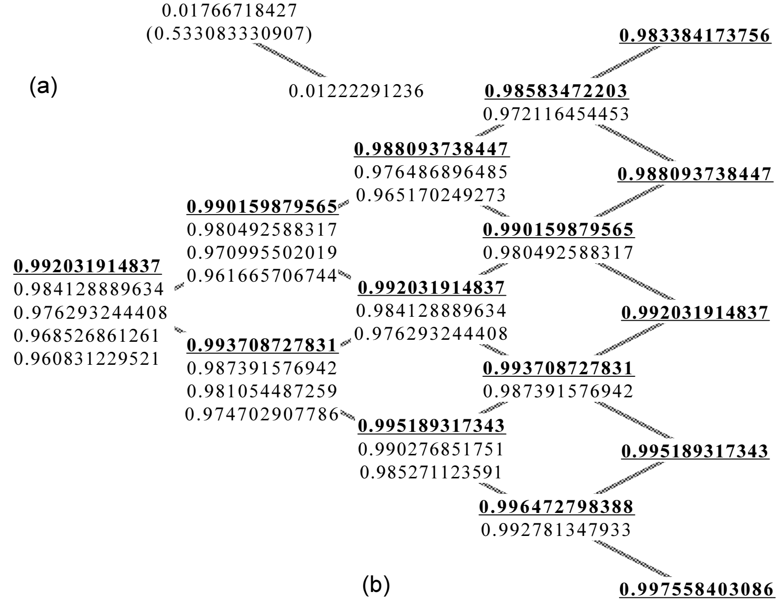
(a)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0.996472798388
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0.997558403086

(b)

# Numerical Examples (concluded)

- Consider the node which is the result of an up move from the root.
- Since the root has $x = 2\sqrt{r(0)}/\sigma = 4$, this particular node's $x$ value equals $4 + \sqrt{\Delta t} = 4.4472135955$.
- Use the inverse transformation to obtain the short rate

$$\frac{x^2 \times (0.1)^2}{4} \approx 0.0494442719102.$$

- Once the short rates are in place, computing the probabilities is easy.
- Convergence is quite good.[a]

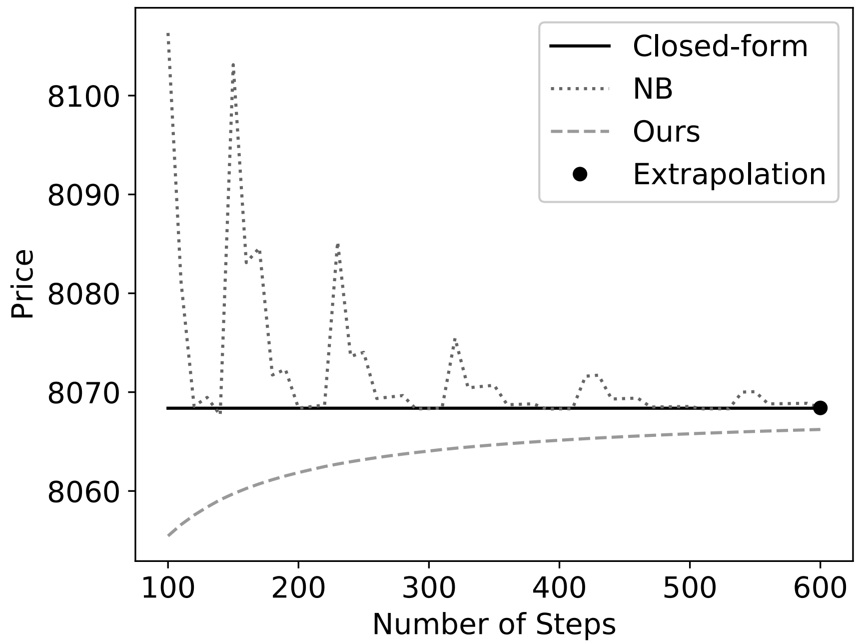
[a] See p. 369 of the textbook.

# Trinomial CIR

- The binomial CIR tree does not have the degree of freedom to match the mean and variance exactly.
- It actually fails to match them at very low $x$.
- A trinomial tree for the CIR model with $O(n^{1.5})$ nodes that matches the mean and variance exactly is available.[a]

[a] Z. Lu (D00922011) & Lyuu (2018); H. Huang (R03922103) (2019).

# A Comparison[a]




$r(0) = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\beta = 1.2$, $T = 5$, principal is 10,000.

[a]Plot from H. Huang (R03922103) (2019).

# A General Method for Constructing Binomial Models[a]

- We are given a continuous-time process,

$$dy = \alpha(y,t)\,dt + \sigma(y,t)\,dW.$$

- Need to make sure the binomial model's drift and diffusion converge to the above process.
- Set the probability of an up move to

$$\frac{\alpha(y,t)\,\Delta t + y - y_{\rm d}}{y_{\rm u} - y_{\rm d}}.$$

- Here $y_{\rm u} \stackrel{\Delta}{=} y + \sigma(y,t)\sqrt{\Delta t}$ and $y_{\rm d} \stackrel{\Delta}{=} y - \sigma(y,t)\sqrt{\Delta t}$ represent the two rates that follow the current rate $y$.

[a]Nelson & Ramaswamy (1990).

## A General Method (continued)

- The displacements are identical, at $\sigma(y, t)\sqrt{\Delta t}$.
- But the binomial tree may not combine as

$$\begin{aligned} & \sigma(y, t)\sqrt{\Delta t} - \sigma(y_{\mathrm{u}}, t + \Delta t)\sqrt{\Delta t} \\ \neq \quad & -\sigma(y, t)\sqrt{\Delta t} + \sigma(y_{\mathrm{d}}, t + \Delta t)\sqrt{\Delta t} \end{aligned}$$

  in general.
- When $\sigma(y, t)$ is a constant independent of $y$, equality holds and the tree combines.

## A General Method (continued)

- To achieve this, define the transformation

$$x(y,t) \stackrel{\Delta}{=} \int^{y} \sigma(z,t)^{-1}\, dz.$$

- Then $x$ follows

$$dx = m(y,t)\, dt + dW$$

for some $m(y,t)$.[a]

- The diffusion term is now a constant, and the binomial tree for $x$ combines.

[a] See Exercise 25.2.13 of the textbook.

## A General Method (concluded)

- The transformation is unique.[a]
- The probability of an up move remains

$$\frac{\alpha(y(x,t),t)\,\Delta t + y(x,t) - y_{\mathrm{d}}(x,t)}{y_{\mathrm{u}}(x,t) - y_{\mathrm{d}}(x,t)},$$

  where $y(x,t)$ is the inverse transformation of $x(y,t)$ from $x$ back to $y$.
- Note that

$$\begin{aligned} y_{\mathrm{u}}(x,t) &\stackrel{\Delta}{=} y(x+\sqrt{\Delta t}, t+\Delta t), \\ y_{\mathrm{d}}(x,t) &\stackrel{\Delta}{=} y(x-\sqrt{\Delta t}, t+\Delta t). \end{aligned}$$

[a] H. Chiu (R98723059) (2012).

# Examples

- The transformation is

$$\int^r (\sigma\sqrt{z})^{-1}\, dz = \frac{2\sqrt{r}}{\sigma}$$

  for the CIR model.

- The transformation is

$$\int^S (\sigma z)^{-1}\, dz = \frac{\ln S}{\sigma}$$

  for the Black-Scholes model $dS = \mu S\, dt + \sigma S\, dW$.

- The familiar BOPM and CRR discretize $\ln S$ not $S$.

## On One-Factor Short Rate Models

- By using only the short rate, they ignore other rates on the yield curve.
- Such models also restrict the volatility to be a function of interest rate *levels* only.
- The prices of all bonds move in the same direction at the same time (their magnitudes may differ).
- The returns on all bonds thus become highly correlated.
- In reality, there seems to be a certain amount of independence between short- and long-term rates.

## On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.
- Derivatives whose values depend on the correlation structure will be mispriced.
- The calibrated models may not generate term structures as concave as the data suggest.
- The term structure empirically changes in slope and curvature as well as makes parallel moves.
- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.

## On One-Factor Short Rate Models (concluded)

- Multifactor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.
- But they are much harder to think about and work with.
- They also take much more computer time—the curse of dimensionality.
- These practical concerns limit the use of multifactor models to two- or three-factor ones.[a]

[a] Kamakura (2019) has a 10-factor HJM model for the U.S. Treasuries (see `http://www.kamakuraco.com/KamakuraReleasesNewStochasticVolatilityModel.aspx`).

## Options on Coupon Bonds[a]

- Assume the market discount function $P$ is a monotonically decreasing function of the short rate $r$.
  - Such as a one-factor short rate model.
- The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.
- Consider a European call expiring at time $T$ on a bond with par value \$1.
- Let $X$ denote the strike price.

[a]Jamshidian (1989).

## Options on Coupon Bonds (continued)

- The bond has cash flows $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$, where $t_i > T$ for all $i$.
- The payoff for the option is

$$\max\left\{\left[\sum_{i=1}^{n} c_i P(r(T), T, t_i)\right] - X, 0\right\}.$$

- At time $T$, there is a unique value $r^*$ for $r(T)$ that renders the coupon bond's price equal the strike price $X$.

## Options on Coupon Bonds (continued)

- This $r^*$ can be obtained by solving

$$X = \sum_{i=1}^{n} c_i P(r, T, t_i)$$

numerically for $r$.

- Let

$$X_i \stackrel{\Delta}{=} P(r^*, T, t_i),$$

the value at time $T$ of a zero-coupon bond with par value \$1 and maturing at time $t_i$ if $r(T) = r^*$.

- Note that $P(r, T, t_i) \geq X_i$ if and only if $r \leq r^*$.

# Options on Coupon Bonds (concluded)

- As $X = \sum_i c_i X_i$, the option's payoff equals

$$\max\left\{ \left[ \sum_{i=1}^{n} c_i P(r(T), T, t_i) \right] - \left[ \sum_{i=1}^{n} c_i X_i \right], 0 \right\}$$

$$= \sum_{i=1}^{n} c_i \times \max(P(r(T), T, t_i) - X_i, 0).$$

- Thus the call is a package of $n$ options on the underlying zero-coupon bond.

- Why can't we do the same thing for Asian options?[a]

[a]Contributed by Mr. Yang, Jui-Chung (D97723002) on May 20, 2009.

# *No-Arbitrage Term Structure Models*

How much of the structure of our theories
really tells us about things in nature,
and how much do we contribute ourselves?
— Arthur Eddington (1882–1944)

How can I apply this model
if I don't understand it?
— Edward I. Altman (2019)

## Motivations

- Recall the difficulties facing equilibrium models mentioned earlier.
  - They usually require the estimation of the market price of risk.[a]
  - They cannot fit the market term structure.
  - But consistency with the market is often mandatory in practice.

---

[a] Recall p. 1109.

## No-Arbitrage Models

- No-arbitrage models utilize the full information of the term structure.
- They accept the observed term structure as consistent with an unobserved and unspecified equilibrium.
- From there, arbitrage-free movements of interest rates or bond prices over time are modeled.
- By definition, the market price of risk must be reflected in the current term structure; hence the resulting interest rate process is risk-neutral.

## No-Arbitrage Models (concluded)

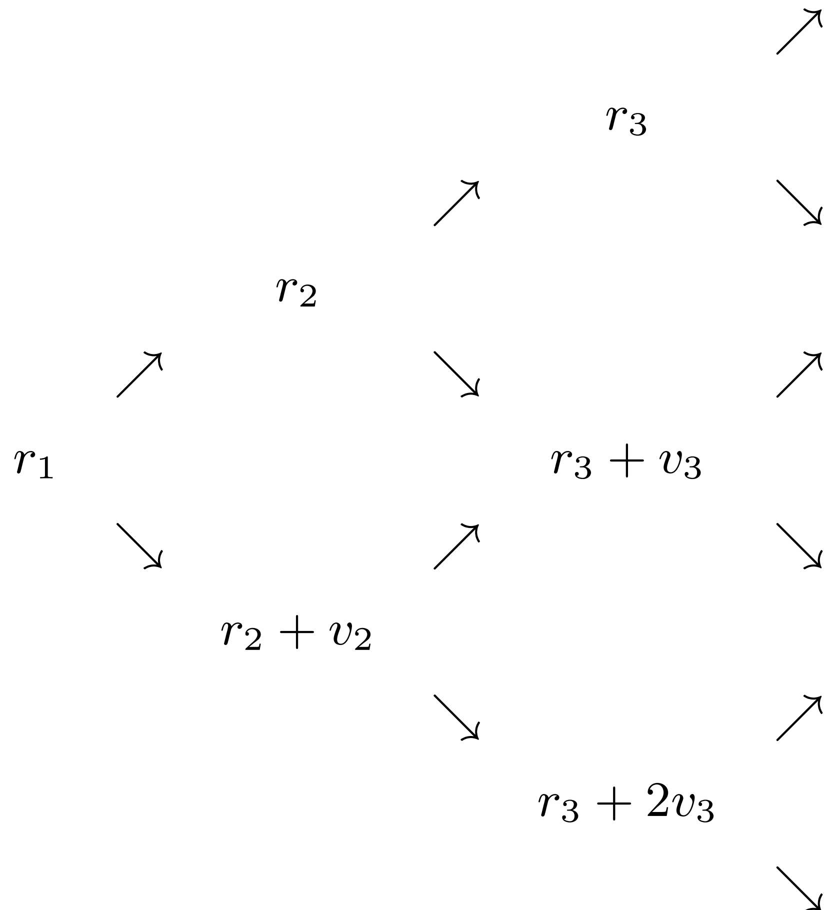
- No-arbitrage models can specify the dynamics of zero-coupon bond prices, forward rates, or the short rate.
- Bond price and forward rate models are usually non-Markovian (path dependent).
- In contrast, short rate models are generally constructed to be explicitly Markovian (path independent).
- Markovian models are easier to handle computationally.

## The Ho-Lee Model[a]

- The short rates at any given time are evenly spaced.
- Let $p$ denote the risk-neutral probability that the short rate makes an up move.
- We shall adopt continuous compounding.

---

[a] T. Ho & S. B. Lee (1986).


$r_3$
$r_2$
$r_1$
$r_3 + v_3$
$r_2 + v_2$
$r_3 + 2v_3$

## The Ho-Lee Model (continued)

- The Ho-Lee model starts with zero-coupon bond prices $P(t, t+1), P(t, t+2), \ldots$ at time $t$ identified with the root of the tree.
- Let the discount factors in the next period be

$$\begin{array}{ll} P_{\rm d}(t+1, t+2), P_{\rm d}(t+1, t+3), \ldots, & \text{if short rate moves down,} \\ P_{\rm u}(t+1, t+2), P_{\rm u}(t+1, t+3), \ldots, & \text{if short rate moves up.} \end{array}$$

- By backward induction, it is not hard to see that for $n \geq 2$,[a]

$$P_{\rm u}(t+1, t+n) = P_{\rm d}(t+1, t+n)\, e^{-(v_2 + \cdots + v_n)}. \tag{159}$$

[a] See p. 376 of the textbook.

# The Ho-Lee Model (continued)

- It is also not hard to check that the $n$-period zero-coupon bond has yields

$$\begin{aligned} y_{\mathrm{d}}(n) &\stackrel{\Delta}{=} -\frac{\ln P_{\mathrm{d}}(t+1,t+n)}{n-1} \\ y_{\mathrm{u}}(n) &\stackrel{\Delta}{=} -\frac{\ln P_{\mathrm{u}}(t+1,t+n)}{n-1} = y_{\mathrm{d}}(n) + \frac{v_2 + \cdots + v_n}{n-1} \end{aligned}$$

- The volatility of the yield to maturity for this bond is therefore

$$\begin{aligned} \kappa_n &\stackrel{\Delta}{=} \sqrt{p y_{\mathrm{u}}(n)^2 + (1-p)\, y_{\mathrm{d}}(n)^2 - [\, p y_{\mathrm{u}}(n) + (1-p)\, y_{\mathrm{d}}(n)\,]^2} \\ &= \sqrt{p(1-p)}\, (y_{\mathrm{u}}(n) - y_{\mathrm{d}}(n)) \\ &= \sqrt{p(1-p)}\, \frac{v_2 + \cdots + v_n}{n-1}. \end{aligned}$$

## The Ho-Lee Model (concluded)

- In particular, the short rate volatility is determined by taking $n = 2$:

$$\sigma = \sqrt{p(1-p)}\, v_2. \tag{160}$$

- The volatility of the short rate therefore equals

$$\sqrt{p(1-p)}\,(r_{\mathrm{u}} - r_{\mathrm{d}}),$$

where $r_{\mathrm{u}}$ and $r_{\mathrm{d}}$ are the two successor rates.[a]

[a]Contrast this with the lognormal model (137) of the binomial interest rate tree on p. 1028.

# The Ho-Lee Model: Volatility Term Structure

- The volatility term structure is composed of

$$\kappa_2, \kappa_3, \ldots .$$

  - The volatility structure is supplied by the market.
  - For the Ho-Lee model, it is independent of

$$r_2, r_3, \ldots .$$

- It is easy to compute the $v_i$s from the volatility structure, and vice versa.[a]
- The $r_i$s can be computed by forward induction.

[a] Review p. 1168.

## The Ho-Lee Model: Bond Price Process

- In a risk-neutral economy, the initial discount factors satisfy[a]

$$P(t, t+n) = [\, p P_{\mathrm{u}}(t+1, t+n) + (1-p)\, P_{\mathrm{d}}(t+1, t+n) \,]\, P(t, t+1).$$

- Combine the above with Eq. (159) on p. 1167 and assume $p = 1/2$ to obtain[b]

$$P_{\mathrm{d}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2 \times \exp[\, v_2 + \cdots + v_n \,]}{1 + \exp[\, v_2 + \cdots + v_n \,]},$$

$$P_{\mathrm{u}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1 + \exp[\, v_2 + \cdots + v_n \,]}.$$

---

[a] Recall Eq. (151) on p. 1097.

[b] In the limit, only the volatility matters; the first formula is similar to multiple logistic regression.

# The Ho-Lee Model: Bond Price Process (concluded)

- The bond price tree combines.[a]
- Suppose all $v_i$ equal some constant $v$ and $\delta \stackrel{\Delta}{=} e^v > 0$.
- Then

$$P_{\mathrm{d}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2\delta^{n-1}}{1+\delta^{n-1}},$$

$$P_{\mathrm{u}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1+\delta^{n-1}}.$$

- Short rate volatility $\sigma = v/2$ by Eq. (160) on p. 1169.
- Price derivatives by taking expectations under the risk-neutral probability.

[a]See Exercise 26.2.3 of the textbook.

## Calibration

- The Ho-Lee model can be calibrated in $O(n^2)$ time using state prices.
- But it can actually be calibrated in $O(n)$ time.[a]
  - Derive the $v_i$'s in linear time.
  - Derive the $r_i$'s in linear time.

[a] See Programming Assignment 26.2.6 of the textbook.

## The Ho-Lee Model: Yields and Their Covariances

- The one-period rate of return of an $n$-period zero-coupon bond is[a]

$$r(t, t+n) \triangleq \ln\left(\frac{P(t+1, t+n)}{P(t, t+n)}\right).$$

- Its two possible value are

$$\ln \frac{P_{\mathrm{d}}(t+1, t+n)}{P(t, t+n)} \quad \text{and} \quad \ln \frac{P_{\mathrm{u}}(t+1, t+n)}{P(t, t+n)}.$$

- Thus the variance of return is[b]

$$\mathrm{Var}[\, r(t, t+n)\,] = p(1-p)\,[\,(n-1)\,v\,]^2 = (n-1)^2\sigma^2.$$

[a] So $r(t, t+n)$ does not mean the $n$-period spot rate at time $t$ here.

[b] Recall that $\sigma$ is the short rate volatility by Eq. (160) on p. 1169.

## The Ho-Lee Model: Yields and Their Covariances (concluded)

- The covariance between $r(t, t+n)$ and $r(t, t+m)$ is[a]

$$(n-1)(m-1)\,\sigma^2.$$

- As a result, the correlation between any two one-period rates of return is one.
- Strong correlation between rates is inherent in all one-factor Markovian models.

[a]See Exercise 26.2.7 of the textbook.

# The Ho-Lee Model: Short Rate Process

- The continuous-time limit of the Ho-Lee model is[a]

$$dr = \theta(t)\,dt + \sigma\,dW. \tag{161}$$

- This is Vasicek's model with the mean-reverting drift replaced by a deterministic, time-dependent drift.
- A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying,

$$dr = \theta(t)\,dt + \sigma(t)\,dW.$$

- This corresponds to the discrete-time model in which $v_i$ are not all identical.

---

[a]See Exercise 26.2.10 of the textbook.

## The Ho-Lee Model: Some Problems

- Future (nominal) interest rates may be negative.
- The short rate volatility is independent of the rate level.
- It has all the problems associated with a one-factor model.[a]

---

[a]Recall pp. 1153ff. See T. Ho & S. B. Lee (2004) for a multifactor Ho-Lee model.

## Problems with No-Arbitrage Models in General

- Interest rate movements should reflect shifts in the model's state variables (factors) not its parameters.
- Model *parameters*, such as the drift $\theta(t)$ in the continuous-time Ho-Lee model, should be stable over time.
- But in practice, no-arbitrage models capture yield curve shifts through the recalibration of parameters.
  – A new model is thus born every day.

## Problems with No-Arbitrage Models in General (concluded)

- This in effect says the model estimated at some time does not describe the term structure of interest rates and their volatilities at other times.
- Consequently, a model's intertemporal behavior is suspect, and using it for hedging and risk management may be unreliable.

## The Black-Derman-Toy Model[a]

- This model is extensively used by practitioners.
- The BDT short rate process is the lognormal binomial interest rate process described on pp. 1024ff.[b]
- The volatility structure[c] is given by the market.
- From it, the short rate volatilities (thus $v_i$) are determined together with the baseline rates $r_i$.

---

[a] Black, Derman, & Toy (BDT) (1990), but essentially finished in 1986 according to Mehrling (2005).

[b] Repeated on next page.

[c] Recall Eq. (143) on p. 1075.

$$
\begin{array}{ccccccc}
 & & & & & & r_4 \\
 & & & & & \nearrow & \\
 & & & & r_3 & & \\
 & & & \nearrow & & \searrow & \\
 & & r_2 & & & & r_4 v_4 \\
 & \nearrow & & \searrow & & \nearrow & \\
r_1 & & & & r_3 v_3 & & \\
 & \searrow & & \nearrow & & \searrow & \\
 & & r_2 v_2 & & & & r_4 v_4^2 \\
 & & & \searrow & & \nearrow & \\
 & & & & r_3 v_3^2 & & \\
 & & & & & \searrow & \\
 & & & & & & r_4 v_4^3
\end{array}
$$

## The Black-Derman-Toy Model (concluded)

- Our earlier binomial interest rate tree, in contrast, assumes $v_i$ are given a priori.
- Lognormal models preclude negative short rates.

## The BDT Model: Volatility Structure

- The volatility structure defines the yield volatilities of zero-coupon bonds of various maturities.
- Let the yield volatility of the $i$-period zero-coupon bond be denoted by $\kappa_i$.
- $P_{\mathrm{u}}$ is the price of the $i$-period zero-coupon bond one period from now if the short rate makes an up move.
- $P_{\mathrm{d}}$ is the price of the $i$-period zero-coupon bond one period from now if the short rate makes a down move.

## The BDT Model: Volatility Structure (concluded)

- Corresponding to these two prices are the following yields to maturity,

$$y_{\rm u} \stackrel{\Delta}{=} P_{\rm u}^{-1/(i-1)} - 1,$$
$$y_{\rm d} \stackrel{\Delta}{=} P_{\rm d}^{-1/(i-1)} - 1.$$

- The yield volatility is defined as[a]

$$\kappa_i \stackrel{\Delta}{=} \frac{\ln(y_{\rm u}/y_{\rm d})}{2}.$$

---

[a]Ryecall Eq. (143) on p. 1075.

# The BDT Model: Calibration

- The inputs to the BDT model are riskless zero-coupon bond yields and their volatilities.
- For economy of expression, all numbers are period based.
- Suppose inductively that we have calculated

$$(r_1, v_1), (r_2, v_2), \ldots, (r_{i-1}, v_{i-1}).$$

  - They define the binomial tree up to time $i-2$ (thus period $i-1$).[a]
  - Thus the spot rates up to time $i-1$ have been matched.

[a] Recall that $(r_{i-1}, v_{i-1})$ defines $i-1$ short rates at time $i-2$, which are applicable to period $i-1$.

## The BDT Model: Calibration (continued)

- We now proceed to calculate $r_i$ and $v_i$ to extend the tree to cover period $i$.
- Assume the price of the $i$-period zero can move to $P_{\mathrm{u}}$ or $P_{\mathrm{d}}$ one period from now.
- Let $y$ denote the current $i$-period spot rate, which is known.
- In a risk-neutral economy,

$$\frac{P_{\mathrm{u}} + P_{\mathrm{d}}}{2(1 + r_i)} = \frac{1}{(1 + y)^i}. \tag{162}$$

- Obviously, $P_{\mathrm{u}}$ and $P_{\mathrm{d}}$ are functions of the unknown $r_i$ and $v_i$.

# The BDT Model: Calibration (continued)

- Viewed from now, the future $(i-1)$-period spot rate at time 1 is uncertain.
- Recall that $y_{\mathrm{u}}$ and $y_{\mathrm{d}}$ represent the spot rates at the up node and the down node, respectively.[a]
- With $\kappa_i^2$ denoting their variance, we have

$$\kappa_i = \frac{1}{2} \ln \left( \frac{{P_{\mathrm{u}}}^{-1/(i-1)} - 1}{{P_{\mathrm{d}}}^{-1/(i-1)} - 1} \right). \tag{163}$$

---

[a]Recall p. 1184.

## The BDT Model: Calibration (continued)

- Solving Eqs. (162)–(163) for $r_i$ and $v_i$ with backward induction takes $O(i^2)$ time.
  - That leads to a cubic-time algorithm.
- We next employ forward induction to derive a quadratic-time calibration algorithm.[a]
- Forward induction figures out, by moving *forward* in time, how much \$1 at a node contributes to the price.[b]
- This number is called the state price and is the price of the claim that pays \$1 at that node and zero elsewhere.

---

[a] W. J. Chen (R84526007) & Lyuu (1997); Lyuu (1999).

[b] Review p. 1052(a).

# The BDT Model: Calibration (continued)

- Let the unknown baseline rate for period $i$ be $r_i = r$.
- Let the unknown multiplicative ratio be $v_i = v$.
- Let the state prices at time $i-1$ be

$$P_1, P_2, \ldots, P_i.$$

- The rates *from* them are

$$r, rv, \ldots, rv^{i-1}$$

  for period $i$, respectively.
- One dollar at time $i$ has a present value of

$$f(r,v) \stackrel{\Delta}{=} \frac{P_1}{1+r} + \frac{P_2}{1+rv} + \frac{P_3}{1+rv^2} + \cdots + \frac{P_i}{1+rv^{i-1}}.$$

# The BDT Model: Calibration (continued)

- By Eq. (163) on p. 1187, the yield volatility is

$$g(r,v) \triangleq \frac{1}{2} \ln \left[ \frac{\left( \frac{P_{\mathrm{u},1}}{1+rv} + \frac{P_{\mathrm{u},2}}{1+rv^2} + \cdots + \frac{P_{\mathrm{u},i-1}}{1+rv^{i-1}} \right)^{-1/(i-1)} - 1}{\left( \frac{P_{\mathrm{d},1}}{1+r} + \frac{P_{\mathrm{d},2}}{1+rv} + \cdots + \frac{P_{\mathrm{d},i-1}}{1+rv^{i-2}} \right)^{-1/(i-1)} - 1} \right].$$

- Above, $P_{\mathrm{u},1}, P_{\mathrm{u},2}, \ldots$ denote the state prices at time $i-1$ *of* the subtree rooted at the up node.[a]

- And $P_{\mathrm{d},1}, P_{\mathrm{d},2}, \ldots$ denote the state prices at time $i-1$ *of* the subtree rooted at the down node.[b]

---

[a]Like $r_2 v_2$ on p. 1181.

[b]Like $r_2$ on p. 1181.

## The BDT Model: Calibration (concluded)

- Note that every node maintains *three* state prices: $P_*, P_{\mathrm{u},*}, P_{\mathrm{d},*}$.
- Now solve

$$
\begin{aligned}
f(r, v) &= \frac{1}{(1+y)^i}, \\
g(r, v) &= \kappa_i,
\end{aligned}
$$

for $r = r_i$ and $v = v_i$.

- Finally, calculate the state prices at time $i$.
- This $O(n^2)$-time algorithm appears on p. 382 of the textbook.

# Calibrating the BDT Model with the Differential Tree (in seconds)[a]

| Number of years | Running time | Number of years | Running time | Number of years | Running time |
|---|---|---|---|---|---|
| 3000 | 398.880 | 39000 | 8562.640 | 75000 | 26182.080 |
| 6000 | 1697.680 | 42000 | 9579.780 | 78000 | 28138.140 |
| 9000 | 2539.040 | 45000 | 10785.850 | 81000 | 30230.260 |
| 12000 | 2803.890 | 48000 | 11905.290 | 84000 | 32317.050 |
| 15000 | 3149.330 | 51000 | 13199.470 | 87000 | 34487.320 |
| 18000 | 3549.100 | 54000 | 14411.790 | 90000 | 36795.430 |
| 21000 | 3990.050 | 57000 | 15932.370 | 120000 | 63767.690 |
| 24000 | 4470.320 | 60000 | 17360.670 | 150000 | 98339.710 |
| 27000 | 5211.830 | 63000 | 19037.910 | 180000 | 140484.180 |
| 30000 | 5944.330 | 66000 | 20751.100 | 210000 | 190557.420 |
| 33000 | 6639.480 | 69000 | 22435.050 | 240000 | 249138.210 |
| 36000 | 7611.630 | 72000 | 24292.740 | 270000 | 313480.390 |

75MHz Sun SPARCstation 20, one period per year.

[a]Lyuu (1999).

## The BDT Model: Continuous-Time Limit

- The continuous-time limit of the BDT model is[a]

$$d \ln r = \left[ \theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln r \right] dt + \sigma(t)\, dW.$$

- The short rate volatility $\sigma(t)$ should be a *declining* function of time for the model to display mean reversion.
  - That makes $\sigma'(t) < 0$.
- In particular, constant $\sigma(t)$ will not attain mean reversion.

[a] Jamshidian (1991).

## The Black-Karasinski Model[a]

- The BK model stipulates that the short rate follows

$$d \ln r = \kappa(t)(\theta(t) - \ln r)\, dt + \sigma(t)\, dW.$$

- This explicitly mean-reverting model depends on time through $\kappa(\cdot)$, $\theta(\cdot)$, and $\sigma(\cdot)$.
- The BK model hence has one more degree of freedom than the BDT model.
- The speed of mean reversion $\kappa(t)$ and the short rate volatility $\sigma(t)$ are independent.

[a] Black & Karasinski (1991).

## The Black-Karasinski Model: Discrete Time

- The discrete-time version of the BK model has the same representation as the BDT model.
- To maintain a combining binomial tree, however, requires some manipulations.
- The next plot illustrates the ideas in which

$$
\begin{aligned}
t_2 &\stackrel{\Delta}{=} t_1 + \Delta t_1, \\
t_3 &\stackrel{\Delta}{=} t_2 + \Delta t_2.
\end{aligned}
$$

$$
\begin{array}{ccccc}
 & & & \nearrow & \\
 & & \ln r_{\mathrm{d}}(t_2) & & \\
 & \nearrow & & \searrow & \\
\ln r(t_1) & & & & \ln r_{\mathrm{du}}(t_3) = \ln r_{\mathrm{ud}}(t_3) \\
 & \searrow & & \nearrow & \\
 & & \ln r_{\mathrm{u}}(t_2) & & \\
 & & & \searrow &
\end{array}
$$

## The Black-Karasinski Model: Discrete Time (continued)

- Note that

$$\begin{aligned}\ln r_{\mathrm{d}}(t_2) &= \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1))\,\Delta t_1 - \sigma(t_1)\sqrt{\Delta t_1}\,,\\ \ln r_{\mathrm{u}}(t_2) &= \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1))\,\Delta t_1 + \sigma(t_1)\sqrt{\Delta t_1}\,.\end{aligned}$$

- To make sure an up move followed by a down move coincides with a down move followed by an up move,

$$\begin{aligned}&\ln r_{\mathrm{d}}(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_{\mathrm{d}}(t_2))\,\Delta t_2 + \sigma(t_2)\sqrt{\Delta t_2}\,,\\ =\ &\ln r_{\mathrm{u}}(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_{\mathrm{u}}(t_2))\,\Delta t_2 - \sigma(t_2)\sqrt{\Delta t_2}\,.\end{aligned}$$

# The Black-Karasinski Model: Discrete Time (continued)

- They imply

$$\kappa(t_2) = \frac{1 - (\sigma(t_2)/\sigma(t_1))\sqrt{\Delta t_2/\Delta t_1}}{\Delta t_2}. \tag{164}$$

- So from $\Delta t_1$, we can calculate the $\Delta t_2$ that satisfies the combining condition and then iterate.
  - $t_0 \to \Delta t_1 \to t_1 \to \Delta t_2 \to t_2 \to \Delta t_3 \to \cdots \to T$ (roughly).[a]

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[a] As $\kappa(t), \theta(t), \sigma(t)$ are independent of $r$, the $\Delta t_i$ will not depend on $r$ either.

# The Black-Karasinski Model: Discrete Time (concluded)

- Unequal durations $\Delta t_i$ are often necessary to ensure a combining tree.[a]

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[a] Amin (1991); C. I. Chen (`R98922127`) (2011); Lok (`D99922028`) & Lyuu (2016, 2017).

## Problems with Lognormal Models in General

- Lognormal models such as BDT and BK share the problem that $E^{\pi}[\, M(t)\,] = \infty$ for any finite $t$ if they model the continuously compounded rate.[a]
- So periodically compounded rates should be modeled.[b]
- Another issue is computational.
- Lognormal models usually do not admit of analytical solutions to even basic fixed-income securities.
- As a result, to price short-dated derivatives on long-term bonds, the tree has to be built over the life of the underlying asset instead of the life of the derivative.

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[a] Hogan & Weintraub (1993).

[b] Sandmann & Sondermann (1993).

# Problems with Lognormal Models in General (concluded)

- This problem can be somewhat mitigated by adopting variable-duration time steps.[a]
  - Use a fine time step up to the maturity of the short-dated derivative.
  - Use a coarse time step beyond the maturity.
- A down side of this procedure is that it has to be tailor-made for each derivative.
- Finally, empirically, interest rates do not follow the lognormal distribution.

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[a]Hull & White (1993).