Introduction to Term Structure Modeling

The fox often ran to the hole
by which they had come in,
to find out if his body was still thin enough
to slip through it.
— *Grimm's Fairy Tales*

And the worst thing you can have
is models and spreadsheets.
— Warren Buffet (2008, May 3)

Renaissance is 100% model driven.[a]
James Simons (2015, May 13, 37:09)

[a] https://www.youtube.com/watch?v=QNznD9hMEh0

Outline

- Use the binomial interest rate tree to model stochastic term structure.
  - Illustrates the basic ideas underlying future models.
  - Applications are generic in that pricing and hedging methodologies can be easily adapted to other models.
- Although the idea is similar to the earlier one used in option pricing, the current task is more complicated.
  - The evolution of an entire term structure, not just a single stock price, is to be modeled.
  - Interest rates of various maturities cannot evolve arbitrarily, or arbitrage profits may occur.

Goals

- A stochastic interest rate model performs two tasks.
  - Provides a stochastic process that defines future term structures without arbitrage profits.
  - "Consistent" with the observed term structures.

History

- The methodology was founded by Merton (1970).
- Modern interest rate modeling is often traced to 1977 when Vasicek and Cox, Ingersoll, and Ross developed simultaneously their influential models.
- Early models have fitting problems because they may not price today's benchmark bonds correctly.
- An alternative approach pioneered by Ho and Lee (1986) makes fitting the market yield curve mandatory.
- Models based on such a paradigm are called arbitrage-free or no-arbitrage models.[a]

[a] Somewhat misleadingly.

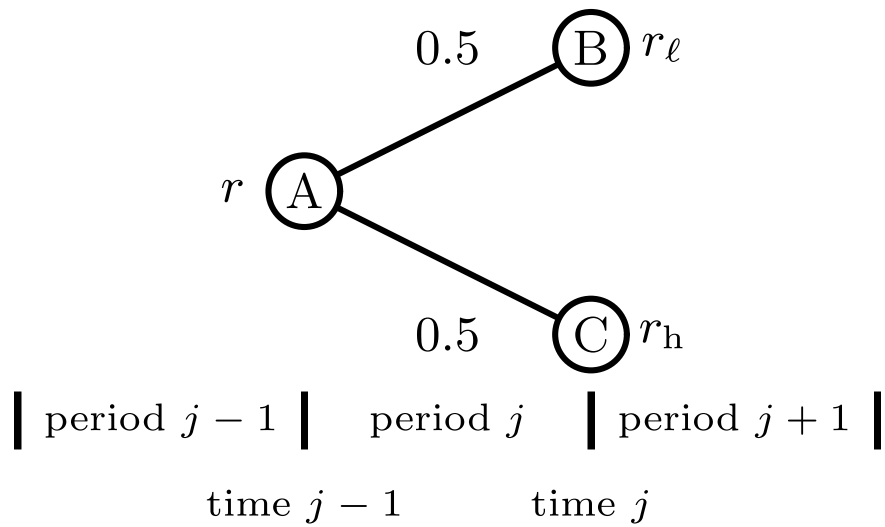
Binomial Interest Rate Tree

- Goal is to construct a no-arbitrage interest rate tree consistent with the yields — and sometimes yield volatilities — of zero-coupon bonds of all maturities.
  - This procedure is called calibration.[a]
- Pick a binomial tree model in which the logarithm of the future short rate obeys the binomial distribution.
  - Like the CRR tree for pricing options.
- The limiting distribution of the short rate at any future time is hence lognormal.

[a]Derman (2004), "complexity without calibration is pointless."

Binomial Interest Rate Tree (continued)

- A binomial tree of future short rates is constructed.
- Every short rate is followed by two short rates in the following period.
- In the figure on p. 1026, node A coincides with the start of period j during which the short rate r is in effect.
- At the conclusion of period j, a new short rate goes into effect for period $j + 1$.


0.5
B
r_ℓ
r
A
0.5
C
r_{h}
period $j-1$
period j
period $j+1$
time $j-1$
time j

Binomial Interest Rate Tree (continued)

- This may take one of two possible values:
  - r_ℓ: the "low" short-rate outcome at node B.
  - r_{h}: the "high" short-rate outcome at node C.
- Each branch has a 50% chance of occurring in a risk-neutral economy.
- We require that the paths combine as the binomial process unfolds.
- Tuckman (2002) attributes this model to Salomon Brothers.

Binomial Interest Rate Tree (continued)

- The short rate r can go to r_{h} and r_{ℓ} with equal risk-neutral probability $1/2$ in a period of length Δt.
- Hence the volatility of $\ln r$ after Δt time is[a]

$$\sigma = \frac{1}{2} \frac{1}{\sqrt{\Delta t}} \ln \left(\frac{r_{\mathrm{h}}}{r_{\ell}} \right). \tag{137}$$

- Above, σ is annualized,[b] whereas r_{ℓ} and r_{h} are period based.

[a] See Exercise 23.2.3 in text.

[b] You may remove the $1/\sqrt{\Delta t}$ term to return it to being period based.

Binomial Interest Rate Tree (continued)

- Note that
$$\frac{r_{\mathrm{h}}}{r_{\ell}} = e^{2\sigma\sqrt{\Delta t}}.$$
- Thus greater volatility, hence uncertainty, leads to larger r_{h}/r_{ℓ} and wider ranges of possible short rates.
- The ratio r_{h}/r_{ℓ} may depend on time if the volatility is a function of time.
- Note that r_{h}/r_{ℓ} has nothing to do with the current short rate r if σ is independent of r.

Binomial Interest Rate Tree (continued)

- In general there are j possible rates for *period* j,[a]

$$r_j,\, r_j v_j,\, r_j v_j^2,\, \ldots,\, r_j v_j^{j-1},$$

  where

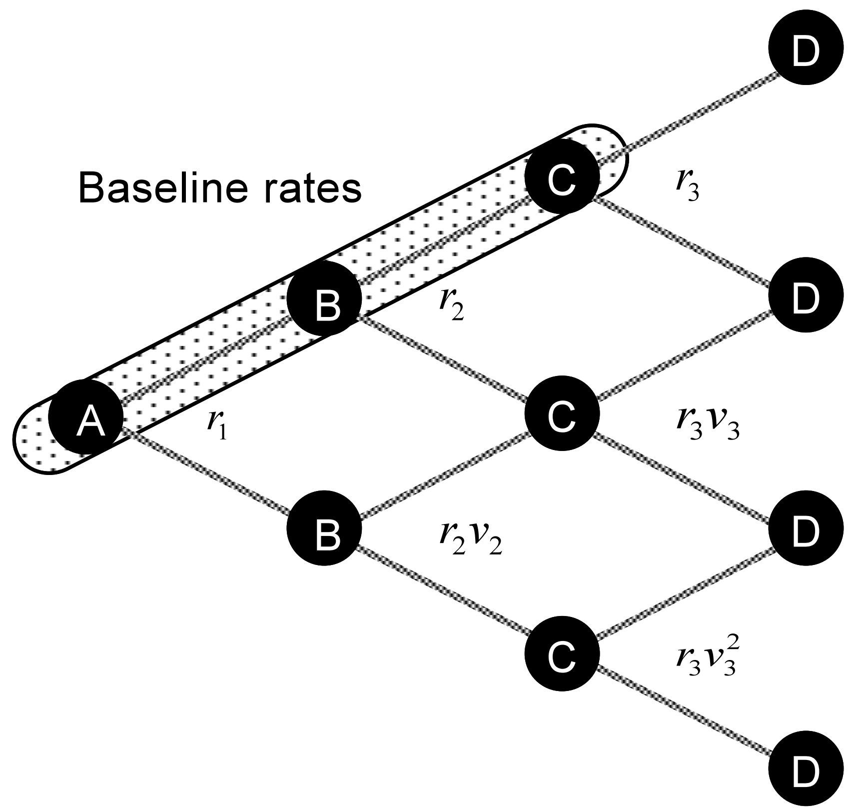
$$v_j \stackrel{\Delta}{=} e^{2\sigma_j \sqrt{\Delta t}} = 1 + O\left(\sqrt{\Delta t}\right) \tag{138}$$

  is the multiplicative ratio for the rates in period j (see figure on next page).

- We shall call r_j the baseline rates.

- The subscript j in σ_j means to emphasize that the short rate volatility may be time dependent.

[a] Not $j + 1$.


Baseline rates
A
B
B
C
C
C
D
D
D
D
r_1
r_2
$r_2 v_2$
r_3
$r_3 v_3$
$r_3 v_3^2$

Binomial Interest Rate Tree (concluded)

- In the limit, the short rate follows

$$r(t) = \mu(t)\, e^{\sigma(t)\, W(t)}. \tag{139}$$

  - The (percent) short rate volatility $\sigma(t)$ is a deterministic function of time.

- The expected value of $r(t)$ equals $\mu(t)\, e^{\sigma(t)^2 (t/2)}$.

- Hence a *declining* short rate volatility is needed to preclude the short rate from assuming implausibly high values.

- This is how the binomial interest rate tree achieves mean reversion to some long-term mean.

Memory Issues

- Path independency: The term structure at any node is independent of the path taken to reach it.
- So only the baseline rates r_i and the multiplicative ratios v_i need to be stored in computer memory.
- This takes up only $O(n)$ space.[a]
- Storing the whole tree would take up $O(n^2)$ space.
  - Daily interest rate movements for 30 years require roughly $(30 \times 365)^2/2 \approx 6 \times 10^7$ double-precision floating-point numbers (half a gigabyte!).

[a]Throughout, n denotes the depth of the tree.

Set Things in Motion

- The abstract process is now in place.
- We need the yields to maturities of the riskless bonds that make up the benchmark yield curve and their volatilities.
- In the U.S., for example, the on-the-run yield curve obtained by the most recently issued Treasury securities may be used as the benchmark curve.

Set Things in Motion (concluded)

- The term structure of (yield) volatilities[a] can be estimated from:
  - Historical data (historical volatility).
  - Or interest rate option prices such as cap prices (implied volatility).
- The binomial tree should be found that is consistent with both term structures.
- Here we focus on the term structure of interest rates.

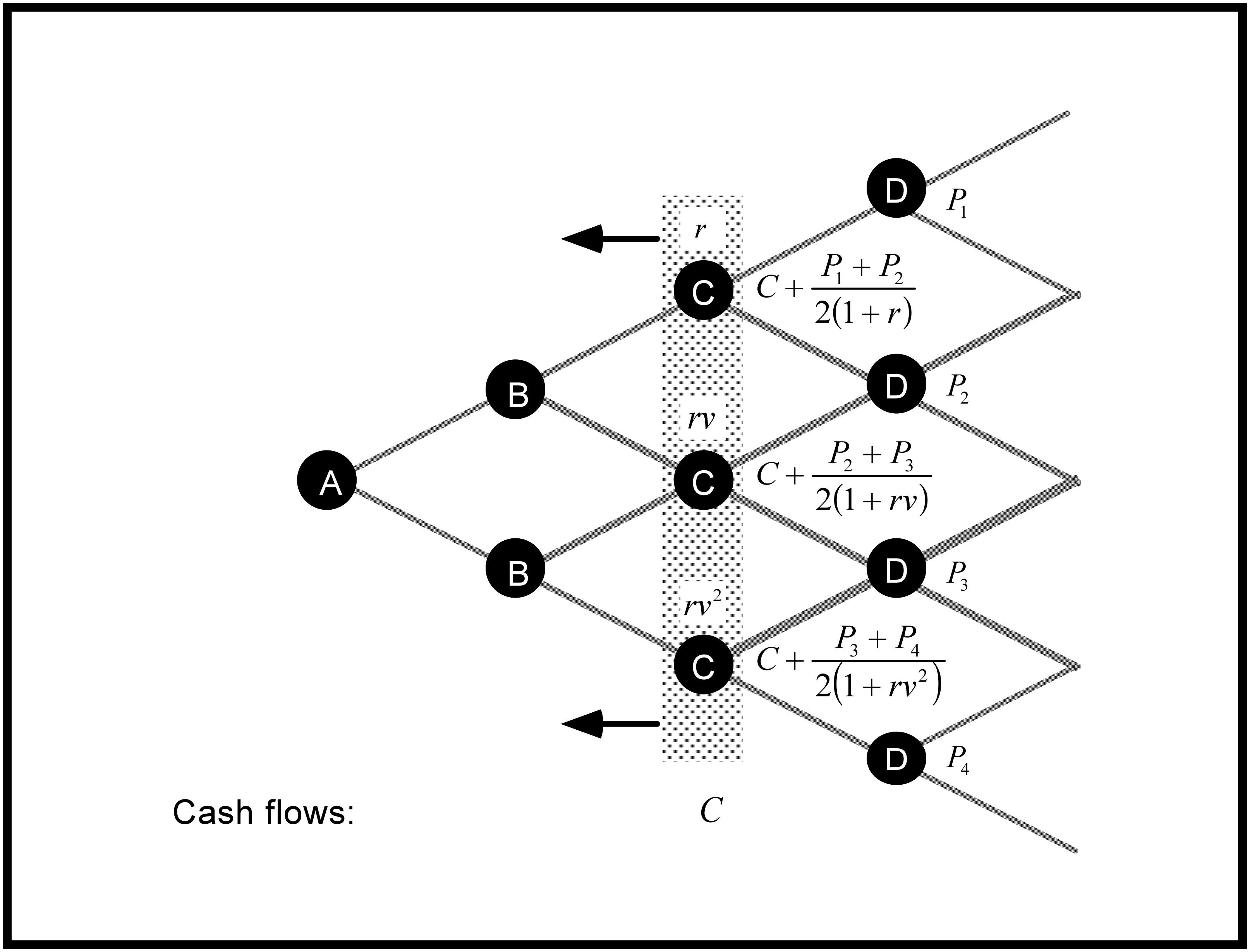
[a]Or simply the volatility (term) structure.

Model Term Structures

- The model price is computed by backward induction.
- Refer back to the figure on p. 1026.
- Given that the values at nodes B and C are P_{B} and P_{C}, respectively, the value at node A is then

$$\frac{P_{\mathrm{B}} + P_{\mathrm{C}}}{2(1+r)} + \text{cash flow at node A}.$$

- We compute the values column by column (see next page).
- This takes $O(n^2)$ time and $O(n)$ space.


r
rv
rv^2
A
B
B
C
C
C
D
D
D
D
$C+\frac{P_1+P_2}{2(1+r)}$
$C+\frac{P_2+P_3}{2(1+rv)}$
$C+\frac{P_3+P_4}{2(1+rv^2)}$
P_1
P_2
P_3
P_4
Cash flows:
C

Term Structure Dynamics

- An n-period zero-coupon bond's price can be computed by assigning \$1 to every node at time n and then applying backward induction.
- Repeat this step for $n = 1, 2, \ldots$ to obtain the market discount function implied by the tree.
- The tree therefore determines a term structure.
- It also contains a term structure dynamics.
  - Every node in the tree induces a binomial interest rate tree and a term structure.

Sample Term Structure

- We shall construct interest rate trees consistent with the sample term structure in the table below.
  - This is calibration (the reverse of pricing).
- Assume the short rate volatility is such that

$$v \stackrel{\Delta}{=} \frac{r_{\rm h}}{r_\ell} = 1.5,$$

independent of time.

Period	1	2	3
Spot rate (%)	4	4.2	4.3
One-period forward rate (%)	4	4.4	4.5
Discount factor	0.96154	0.92101	0.88135

An Approximate Calibration Scheme

- Start with the implied one-period forward rates.
- Equate the expected short rate with the forward rate.[a]
- For the first period, the forward rate is today's one-period spot rate.
- In general, let f_j denote the forward rate in period j.
- This forward rate can be derived from the market discount function via[b]

$$f_j = \frac{d(j)}{d(j+1)} - 1.$$

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[a]See Exercise 5.6.6 in text for the motivation.

[b]See Exercise 5.6.3 in text.

An Approximate Calibration Scheme (continued)

- As the ith short rate $r_j v_j^{i-1}$, $1 \leq i \leq j$, occurs with probability $2^{-(j-1)} \binom{j-1}{i-1}$, we set up

$$\sum_{i=1}^{j} 2^{-(j-1)} \binom{j-1}{i-1} r_j v_j^{i-1} = f_j.$$

- Thus

$$r_j = \left(\frac{2}{1+v_j} \right)^{j-1} f_j. \tag{140}$$

- This binomial interest rate tree is trivial to set up (implicitly), in $O(n)$ time.

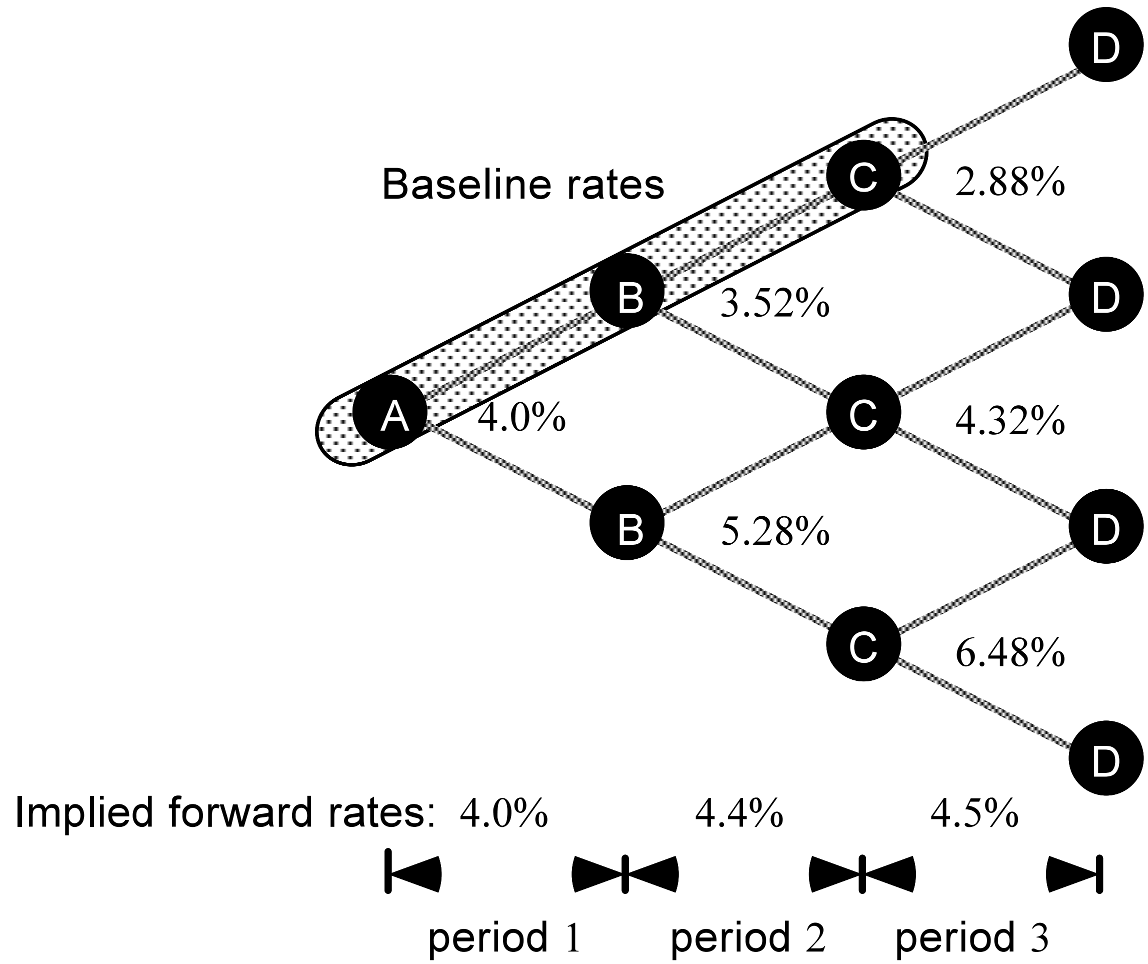
An Approximate Calibration Scheme (continued)

- The ensuing tree for the sample term structure appears in figure on the next page.
- For example, the price of the zero-coupon bond paying \$1 at the end of the third period is

$$\frac{1}{4} \times \frac{1}{1.04} \times \left(\frac{1}{1.0352} \times \left(\frac{1}{1.0288} + \frac{1}{1.0432} \right) + \frac{1}{1.0528} \times \left(\frac{1}{1.0432} + \frac{1}{1.0648} \right) \right)$$

  or 0.88155, which exceeds discount factor 0.88135.

- The tree is *not* calibrated.


Baseline rates
A
B
B
C
C
C
D
D
D
D
4.0%
3.52%
5.28%
2.88%
4.32%
6.48%
Implied forward rates: 4.0%
4.4%
4.5%
period 1
period 2
period 3

An Approximate Calibration Scheme (concluded)

- This bias is inherent: The tree *overprices* the bonds.[a]
- Suppose we replace the baseline rates r_j by $r_j v_j$.
- Then the resulting tree *underprices* the bonds.[b]
- The true baseline rates are thus bounded between r_j and $r_j v_j$.

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[a]See Exercise 23.2.4 in text.

[b]Lyuu & C. Wang (`F95922018`) (2009, 2011).

Issues in Calibration

- The model prices generated by the binomial interest rate tree should match the observed market prices.
- Perhaps the most crucial aspect of model building.
- Treat the backward induction for the model price of the m-period zero-coupon bond as computing some function $f(r_m)$ of the unknown baseline rate r_m for period m.
- A root-finding method is applied to solve $f(r_m) = P$ for r_m given the zero's price P and $r_1, r_2, \ldots, r_{m-1}$.
- This procedure is carried out for $m = 1, 2, \ldots, n$.
- It runs in $O(n^3)$ time.

Binomial Interest Rate Tree Calibration

- Calibration can be accomplished in $O(n^2)$ time by the use of forward induction.[a]
- The scheme records how much \$1 at a node contributes to the model price.
- This number is called the state price.[b]
  - It is the price of a state contingent claim that pays \$1 at that particular node (state) and 0 elsewhere.
- The column of state prices will be established by moving *forward* from time 0 to time n.

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[a] Jamshidian (1991).

[b] Recall p. 213. Alternative names are the Arrow-Debreu price and Green's function.

Binomial Interest Rate Tree Calibration (continued)

- Suppose we are at *time* j and there are $j+1$ nodes.
  - $P_1, P_2, \ldots, P_j$ are the known state prices at the *earlier* time $j-1$.
  - The unknown baseline rate for *period* j is $r \stackrel{\Delta}{=} r_j$.
  - The known multiplicative ratio is $v \stackrel{\Delta}{=} v_j$.
  - The rates for period j are thus $r, rv, \ldots, rv^{j-1}$.[a]
- By definition, $\sum_{i=1}^{j} P_i$ is the price of the $(j-1)$-period zero-coupon bond.
- We want to find r based on $P_1, P_2, \ldots, P_j$ and the price of the j-period zero-coupon bond.

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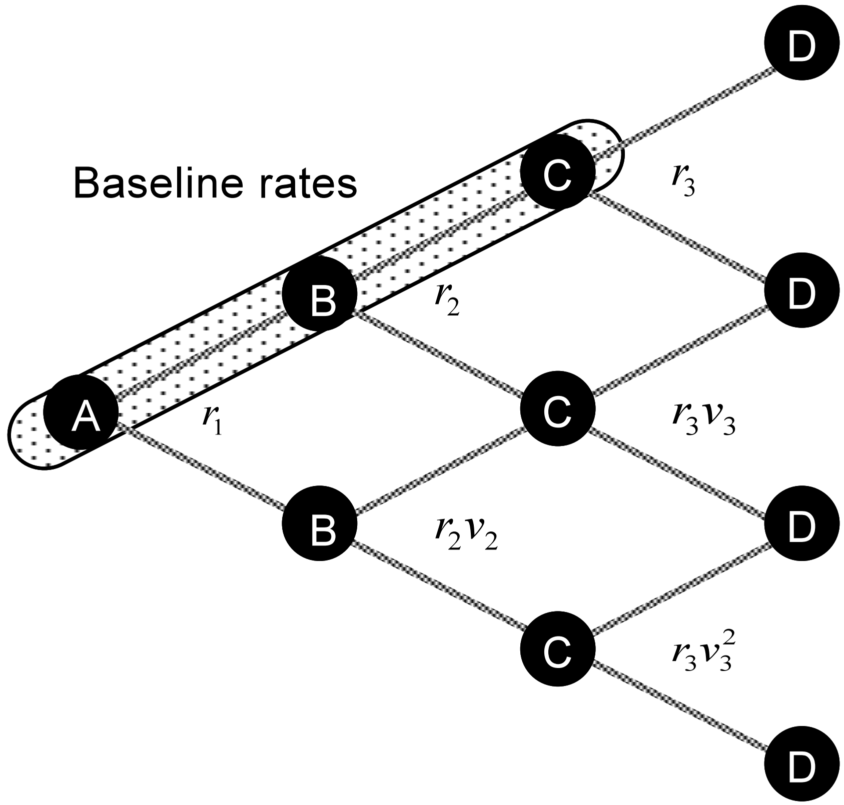
[a] Recall p. 1031, repeated on next page with $j = 3$.

Binomial Interest Rate Tree Calibration (continued)



Binomial Interest Rate Tree Calibration (continued)

- One dollar at time j has a known market value of $1/[\,1+S(j)\,]^j$, where $S(j)$ is the j-period spot rate.
- Alternatively, this dollar has a present value of

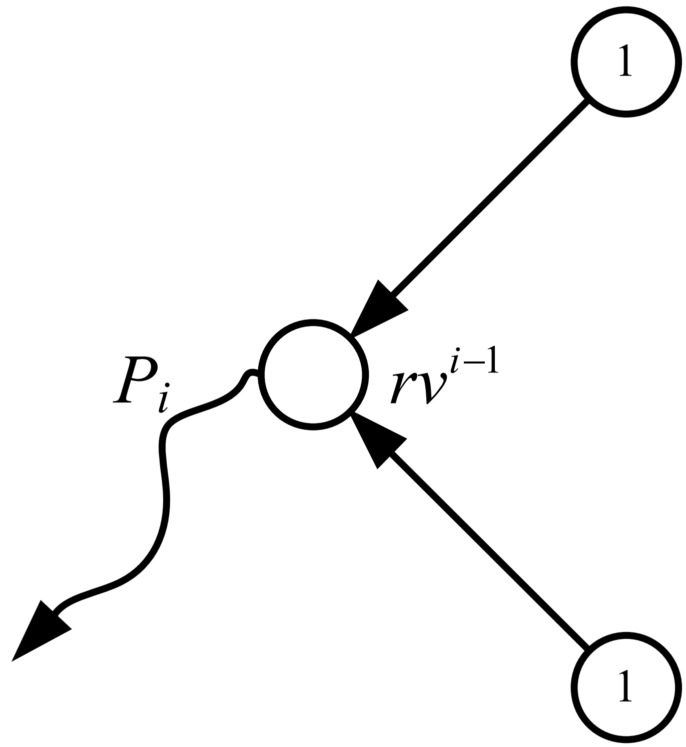
$$g(r) \triangleq \frac{P_1}{(1+r)} + \frac{P_2}{(1+rv)} + \frac{P_3}{(1+rv^2)} + \cdots + \frac{P_j}{(1+rv^{j-1})}$$

(see the next plot).

- So we solve

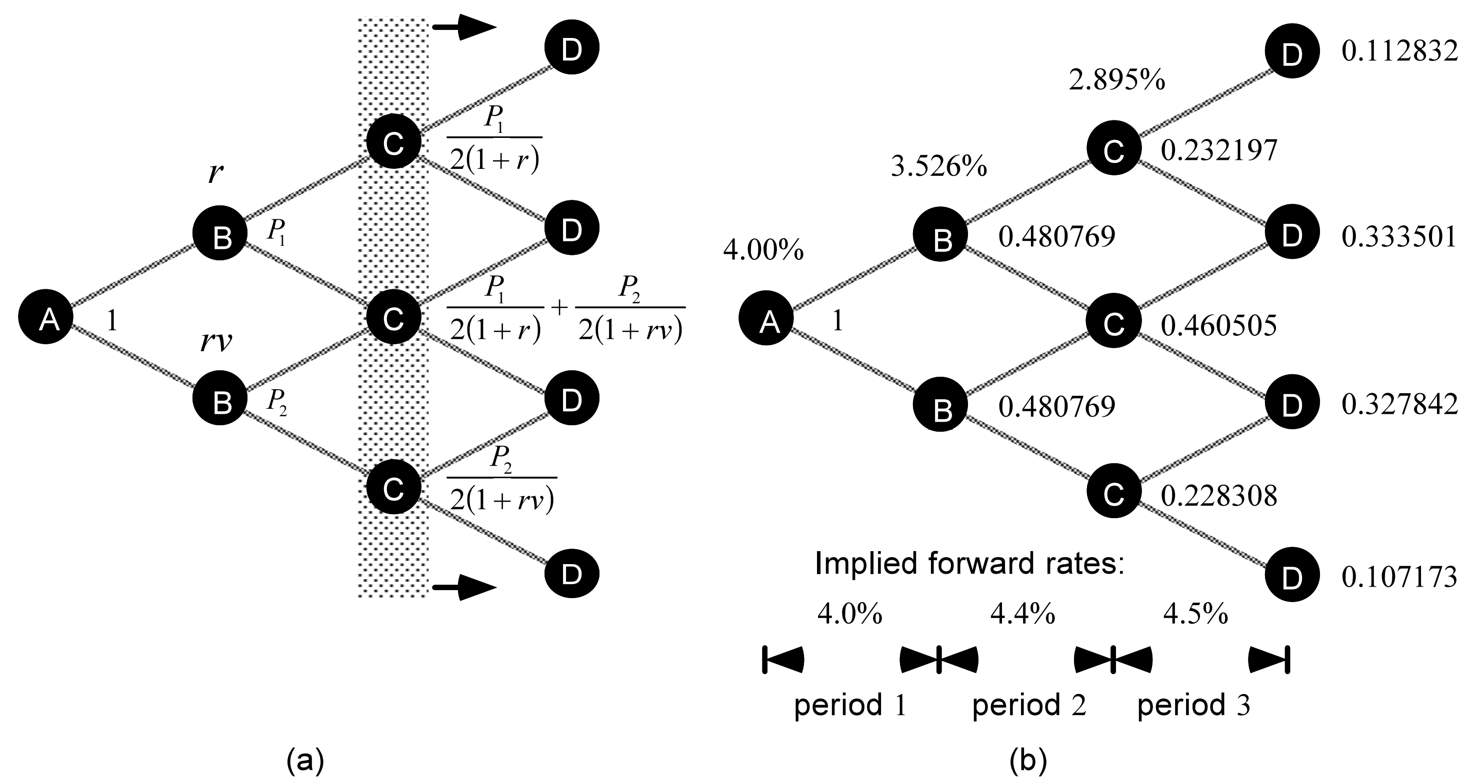
$$g(r) = \frac{1}{[\,1+S(j)\,]^j} \tag{141}$$

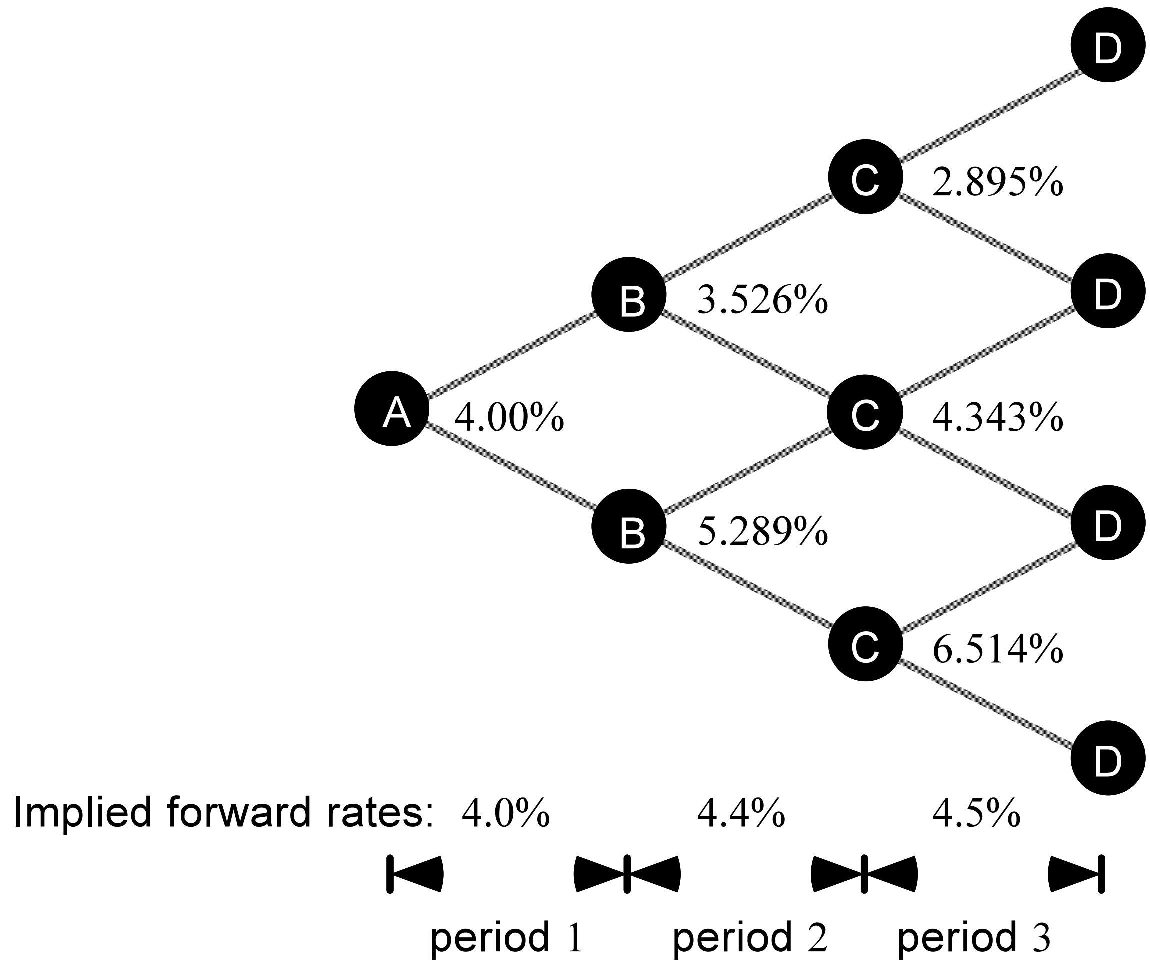
for r.


1
P_i
rv^{i-1}
1

Binomial Interest Rate Tree Calibration (continued)

- Given a decreasing market discount function, a unique positive real-number solution for r is guaranteed.
- The state prices at time j can now be calculated (see panel (a) of the next page with $j = 2$).
- We call a tree with these state prices a binomial state price tree (see panel (b) of the next page).
- The calibrated tree is depicted on p. 1053.


A
B
C
D
1
r
rv
P_1
P_2
$\frac{P_1}{2(1+r)}$
$\frac{P_1}{2(1+r)} + \frac{P_2}{2(1+rv)}$
$\frac{P_2}{2(1+rv)}$
(a)
4.00%
3.526%
2.895%
0.480769
0.480769
0.232197
0.460505
0.228308
0.112832
0.333501
0.327842
0.107173
Implied forward rates:
4.0%
4.4%
4.5%
period 1
period 2
period 3
(b)


D
C
2.895%
B
3.526%
D
A
4.00%
C
4.343%
B
5.289%
D
C
6.514%
D
Implied forward rates: 4.0%
4.4%
4.5%
period 1
period 2
period 3

Binomial Interest Rate Tree Calibration (concluded)

- Use the Newton-Raphson method to solve for the r in Eq. (141) on p. 1049 as $g'(r)$ is easy to evaluate.
- The monotonicity and the convexity of $g(r)$ facilitates root finding.
- The total running time is $O(n^2)$ as each root-finding routine consumes $O(j)$ time.
- With a good initial guess,[a] the Newton-Raphson method converges in only a few steps.[b]

[a] Such as $r_j = (\frac{2}{1+v_j})^{j-1} f_j$ on p. 1041.
[b] Lyuu (1999).

A Numerical Example

- One dollar at the end of the second period should have a present value of 0.92101 by the sample term structure.
- The baseline rate for the second period, r_2, satisfies
$$\frac{0.480769}{1+r_2} + \frac{0.480769}{1+1.5\times r_2} = 0.92101.$$
- The result is $r_2 = 3.526\%$.
- This is used to derive the next column of state prices shown in panel (b) on p. 1052 as 0.232197, 0.460505, and 0.228308.
- Their sum matches the market discount factor 0.92101.

A Numerical Example (concluded)

- The baseline rate for the third period, r_3, satisfies

$$\frac{0.232197}{1+r_3} + \frac{0.460505}{1+1.5 \times r_3} + \frac{0.228308}{1+(1.5)^2 \times r_3} = 0.88135.$$

- The result is $r_3 = 2.895\%$.
- Now, redo the calculation on p. 1042 using the new rates:

$$\frac{1}{4} \times \frac{1}{1.04} \times \left[\frac{1}{1.03526} \times \left(\frac{1}{1.02895} + \frac{1}{1.04343} \right) + \frac{1}{1.05289} \times \left(\frac{1}{1.04343} + \frac{1}{1.06514} \right) \right],$$

which equals 0.88135, an exact match.

- The tree on p. 1053 prices without bias the benchmark securities.

Spread of Nonbenchmark Bonds

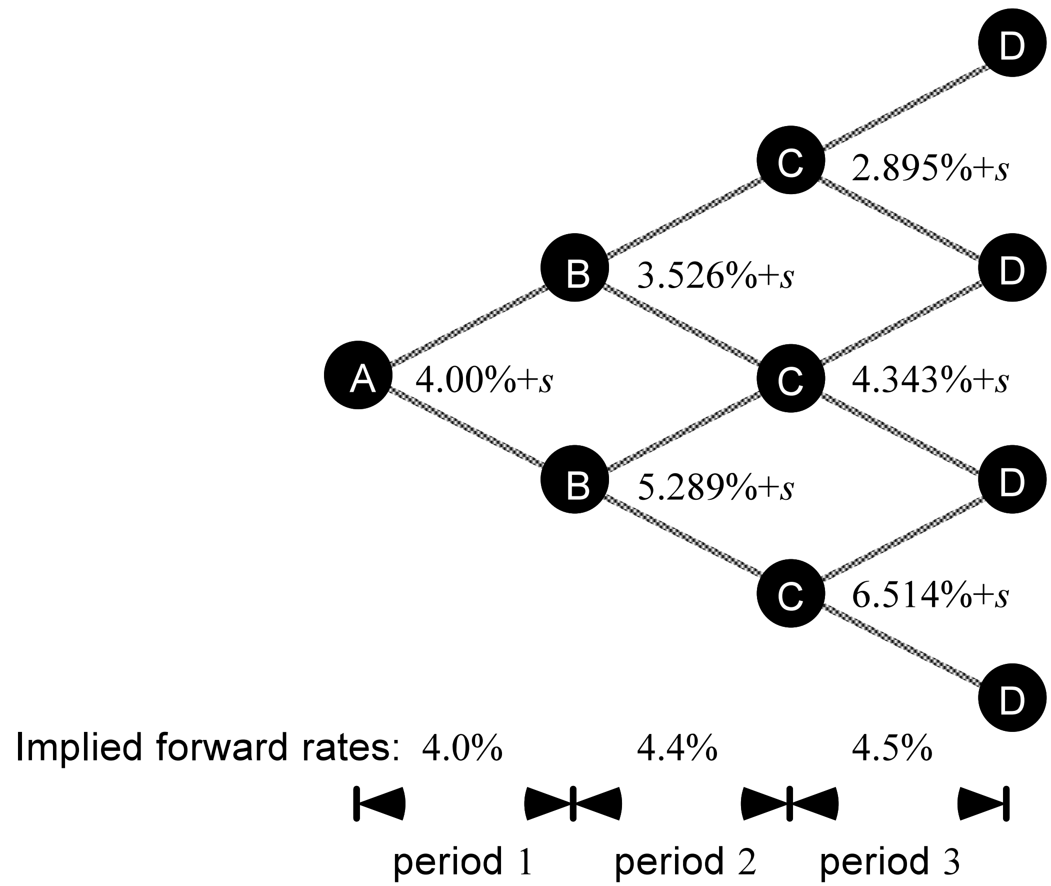
- Model prices by the calibrated tree seldom match the market prices of nonbenchmark bonds.
- The incremental return over the benchmark bonds is called spread.
- If we add the spread uniformly over the short rates in the tree, the model price will equal the market price.
- We will apply the spread concept to option-free bonds next.

Spread of Nonbenchmark Bonds (continued)

- We illustrate the idea with an example.
- Start with the tree on p. 1059.
- Consider a security with cash flow C_i at time i for $i = 1, 2, 3$.
- Its model price is $p(s)$, which is equal to

$$\frac{1}{1.04+s} \times \left[C_1 + \frac{1}{2} \times \frac{1}{1.03526+s} \times \left(C_2 + \frac{1}{2} \left(\frac{C_3}{1.02895+s} + \frac{C_3}{1.04343+s} \right) \right) + \frac{1}{2} \times \frac{1}{1.05289+s} \times \left(C_2 + \frac{1}{2} \left(\frac{C_3}{1.04343+s} + \frac{C_3}{1.06514+s} \right) \right) \right].$$

- Given a market price of P, the spread is the s that solves $P = p(s)$.


D
C
2.895%+s
B
3.526%+s
D
A
4.00%+s
C
4.343%+s
B
5.289%+s
D
C
6.514%+s
D
Implied forward rates: 4.0%
4.4%
4.5%
period 1
period 2
period 3

Spread of Nonbenchmark Bonds (continued)

- The model price $p(s)$ is a monotonically decreasing, convex function of s.
- Employ any root-finding method to solve

$$p(s) - P = 0$$

  for s.
- But a quick look at the equation for $p(s)$ reveals that evaluating $p'(s)$ directly is infeasible.
- Fortunately, the tree can be used to evaluate both $p(s)$ and $p'(s)$ during backward induction.

Spread of Nonbenchmark Bonds (continued)

- Consider an arbitrary node A in the tree associated with the short rate r.
- While computing the model price $p(s)$, a price $p_{\mathrm{A}}(s)$ is computed at A.
- Prices computed at A's two successor nodes B and C are discounted by $r + s$ to obtain $p_{\mathrm{A}}(s)$ as follows,

$$p_{\mathrm{A}}(s) = c + \frac{p_{\mathrm{B}}(s) + p_{\mathrm{C}}(s)}{2(1 + r + s)},$$

where c denotes the cash flow at A.

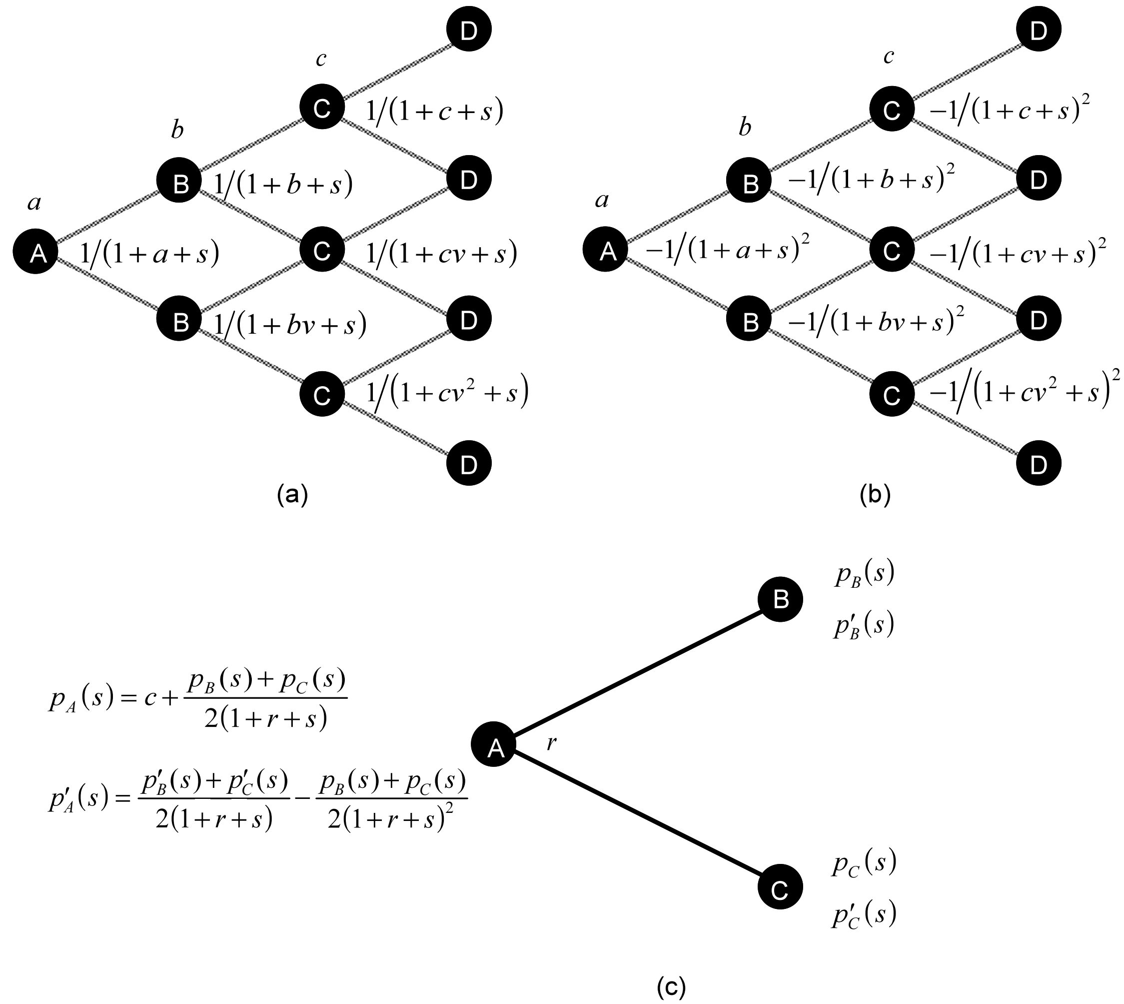
Spread of Nonbenchmark Bonds (continued)

- To compute $p'_{\mathrm{A}}(s)$ as well, node A calculates

$$p'_{\mathrm{A}}(s) = \frac{p'_{\mathrm{B}}(s) + p'_{\mathrm{C}}(s)}{2(1 + r + s)} - \frac{p_{\mathrm{B}}(s) + p_{\mathrm{C}}(s)}{2(1 + r + s)^2}. \tag{142}$$

- This is easy if $p'_{\mathrm{B}}(s)$ and $p'_{\mathrm{C}}(s)$ are also computed at nodes B and C.
- When A is a terminal node, simply use the payoff function for $p_{\mathrm{A}}(s)$.[a]

[a] Contributed by Mr. Chou, Ming-Hsin (R02723073) on May 28, 2014.



a
b
c
A
B
B
C
C
C
D
D
D
D
$1/(1+a+s)$
$1/(1+b+s)$
$1/(1+bv+s)$
$1/(1+c+s)$
$1/(1+cv+s)$
$1/(1+cv^2+s)$
(a)
$-1/(1+a+s)^2$
$-1/(1+b+s)^2$
$-1/(1+bv+s)^2$
$-1/(1+c+s)^2$
$-1/(1+cv+s)^2$
$-1/(1+cv^2+s)^2$
(b)
$p_A(s) = c + \frac{p_B(s)+p_C(s)}{2(1+r+s)}$
$p'_A(s) = \frac{p'_B(s)+p'_C(s)}{2(1+r+s)} - \frac{p_B(s)+p_C(s)}{2(1+r+s)^2}$
r
$p_B(s)$
$p'_B(s)$
$p_C(s)$
$p'_C(s)$
(c)

Spread of Nonbenchmark Bonds (continued)

- Apply the above procedure inductively to yield $p(s)$ and $p'(s)$ at the root (p. 1063).
- This is called the differential tree method.[a]
  - Similar ideas can be found in automatic differentiation[b] (AD) and backpropagation[c] in artificial neural networks.
- The total running time is $O(n^2)$.
- The memory requirement is $O(n)$.

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[a] Lyuu (1999).
[b] Rall (1981).
[c] Werbos (1974); Rumelhart, Hinton, & Williams (1986).

Spread of Nonbenchmark Bonds (continued)

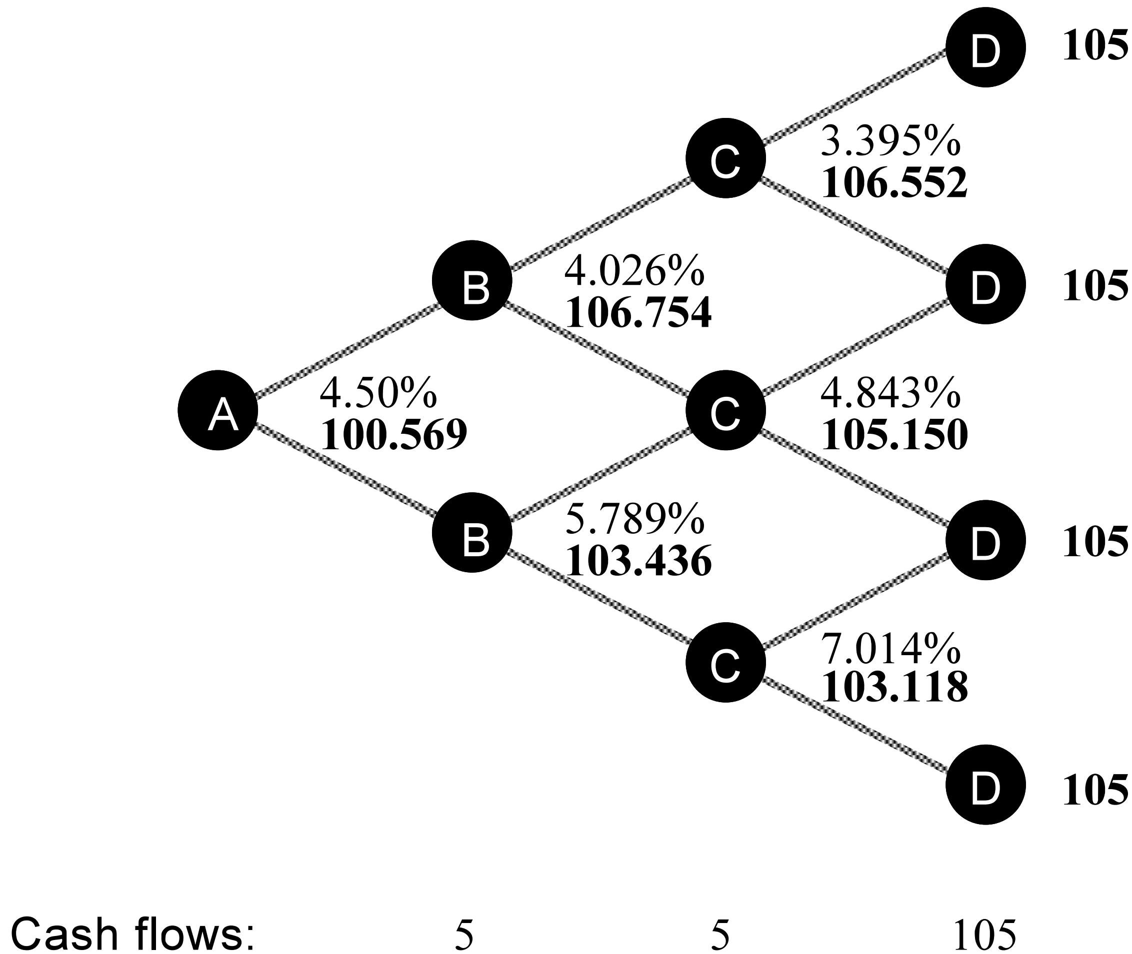
Number of partitions n	Running time (s)	Number of iterations	Number of partitions	Running time (s)	Number of iterations
500	7.850	5	10500	3503.410	5
1500	71.650	5	11500	4169.570	5
2500	198.770	5	12500	4912.680	5
3500	387.460	5	13500	5714.440	5
4500	641.400	5	14500	6589.360	5
5500	951.800	5	15500	7548.760	5
6500	1327.900	5	16500	8502.950	5
7500	1761.110	5	17500	9523.900	5
8500	2269.750	5	18500	10617.370	5
9500	2834.170	5	....	....	....

75MHz Sun SPARCstation 20.

Spread of Nonbenchmark Bonds (concluded)

- Consider a three-year, 5% bond with a market price of 100.569.
- Assume the bond pays annual interest.
- The spread is 50 basis points over the tree.[a]
- Note that the idea of spread does not assume parallel shifts in the term structure.
- It also differs from the yield spread (p. 134) and static spread (p. 135) of the nonbenchmark bond over an otherwise identical benchmark bond.

[a]See plot on the next page.


A
4.50%
100.569
B
4.026%
106.754
B
5.789%
103.436
C
3.395%
106.552
C
4.843%
105.150
C
7.014%
103.118
D
105
D
105
D
105
D
105
Cash flows:
5
5
105

More Applications of the Differential Tree: Calculating Implied Volatility (in seconds)[a]

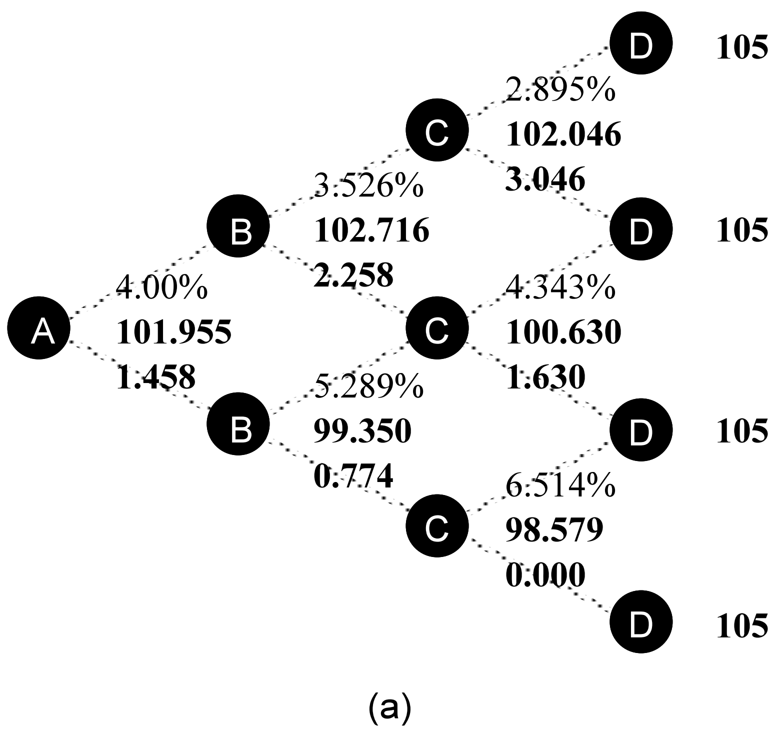
American call			American put		
Number of partitions	Running time	Number of iterations	Number of partitions	Running time	Number of iterations
100	0.008210	2	100	0.013845	3
200	0.033310	2	200	0.036335	3
300	0.072940	2	300	0.120455	3
400	0.129180	2	400	0.214100	3
500	0.201850	2	500	0.333950	3
600	0.290480	2	600	0.323260	2
700	0.394090	2	700	0.435720	2
800	0.522040	2	800	0.569605	2

Intel 166MHz Pentium, running on Microsoft Windows 95.

[a]Lyuu (1999).

Fixed-Income Options

- Consider a 2-year 99 European call on the 3-year, 5% Treasury.
- Assume the Treasury pays annual interest.
- On p. 1070 the 3-year Treasury's price *minus* the \$5 interest at year 2 are \$102.046, \$100.630, and \$98.579.
  - The accrued interest is *not* included as it belongs to the bond seller.
- Now compare the strike price against the bond prices.
- The call is in the money in the first two scenarios out of the money in the third.



A
4.00%
101.955
1.458
B
3.526%
102.716
2.258
B
5.289%
99.350
0.774
C
2.895%
102.046
3.046
C
4.343%
100.630
1.630
C
6.514%
98.579
0.000
D
105
(a)
A
4.00%
101.955
0.096
B
3.526%
102.716
0.000
B
5.289%
99.350
0.200
C
2.895%
102.046
0.000
C
4.343%
100.630
0.000
C
6.514%
98.579
0.421
D
105
(b)

Fixed-Income Options (continued)

- The option value is calculated to be \$1.458 on p. 1070(a).
- European interest rate puts can be valued similarly.
- Consider a two-year 99 European put on the same security.
- At expiration, the put is in the money only when the Treasury is worth \$98.579.
- The option value is computed to be \$0.096 on p. 1070(b).

Fixed-Income Options (concluded)

- The present value of the strike price is $\text{PV}(X) = 99 \times 0.92101 = 91.18$.
- The Treasury is worth $B = 101.955$.
- The present value of the interest payments during the life of the options is[a]

$$\text{PV}(I) = 5 \times 0.96154 + 5 \times 0.92101 = 9.41275.$$

- The call and the put are worth $C = 1.458$ and $P = 0.096$, respectively.
- The put-call parity is preserved:

$$C = P + B - \text{PV}(I) - \text{PV}(X).$$

[a] There is no coupon today.

Delta or Hedge Ratio

- How much does the option price change in response to changes in the *price* of the underlying bond?
- This relation is called delta (or hedge ratio), defined as

$$\frac{O_{\mathrm{h}} - O_{\ell}}{P_{\mathrm{h}} - P_{\ell}}.$$

- In the above P_{h} and P_{ℓ} denote the bond prices if the short rate moves up and down, respectively.
- Similarly, O_{h} and O_{ℓ} denote the option values if the short rate moves up and down, respectively.

Delta or Hedge Ratio (concluded)

- Delta measures the sensitivity of the option value to changes in the underlying bond price.
- So it shows how to hedge one with the other.
- Take the call and put on p. 1070 as examples.
- Their deltas are

$$\begin{aligned}\frac{0.774 - 2.258}{99.350 - 102.716} &= 0.441,\\ \frac{0.200 - 0.000}{99.350 - 102.716} &= -0.059,\end{aligned}$$

respectively.

Volatility Term Structures

- The binomial interest rate tree can be used to calculate the yield volatility of zero-coupon bonds.
- Consider an n-period zero-coupon bond.
- First find its yield to maturity y_{h} (y_ℓ, respectively) at the end of the initial period if the short rate rises (declines, respectively).
- The yield volatility for our model is defined as

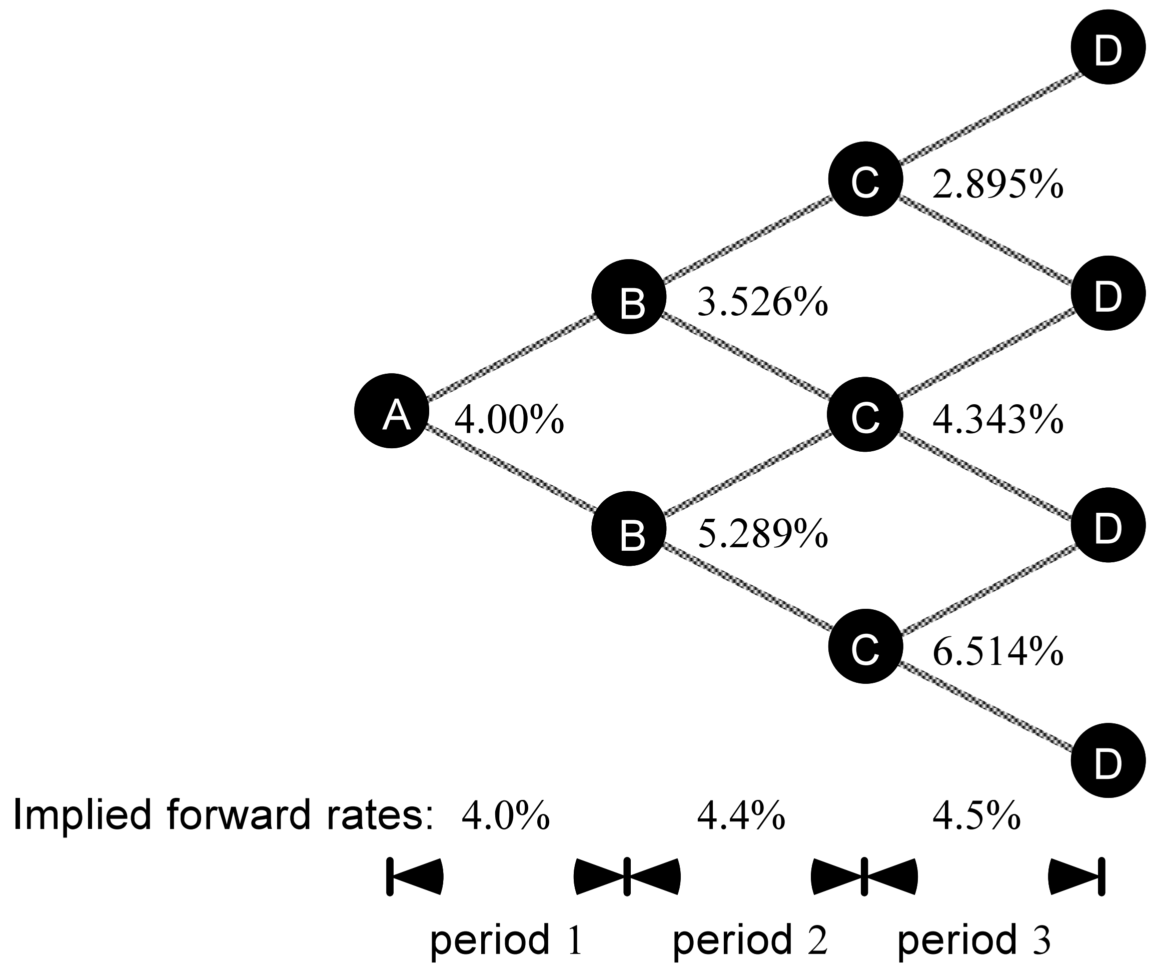
$$\frac{1}{2} \ln \left(\frac{y_{\mathrm{h}}}{y_\ell} \right). \tag{143}$$

Volatility Term Structures (continued)

- For example, take the tree on p. 1053 (repeated on next page).
- The two-year zero's yield at the end of the first period is 5.289% if the rate rises and 3.526% if the rate declines.
- Its yield volatility is therefore

$$\frac{1}{2} \ln \left(\frac{0.05289}{0.03526} \right) = 20.273\%.$$

Volatility Term Structures (continued)



Volatility Term Structures (continued)

- Consider the three-year zero-coupon bond.
- If the short rate rises, the price of the zero one year from now will be

$$\frac{1}{2} \times \frac{1}{1.05289} \times \left(\frac{1}{1.04343} + \frac{1}{1.06514} \right) = 0.90096.$$

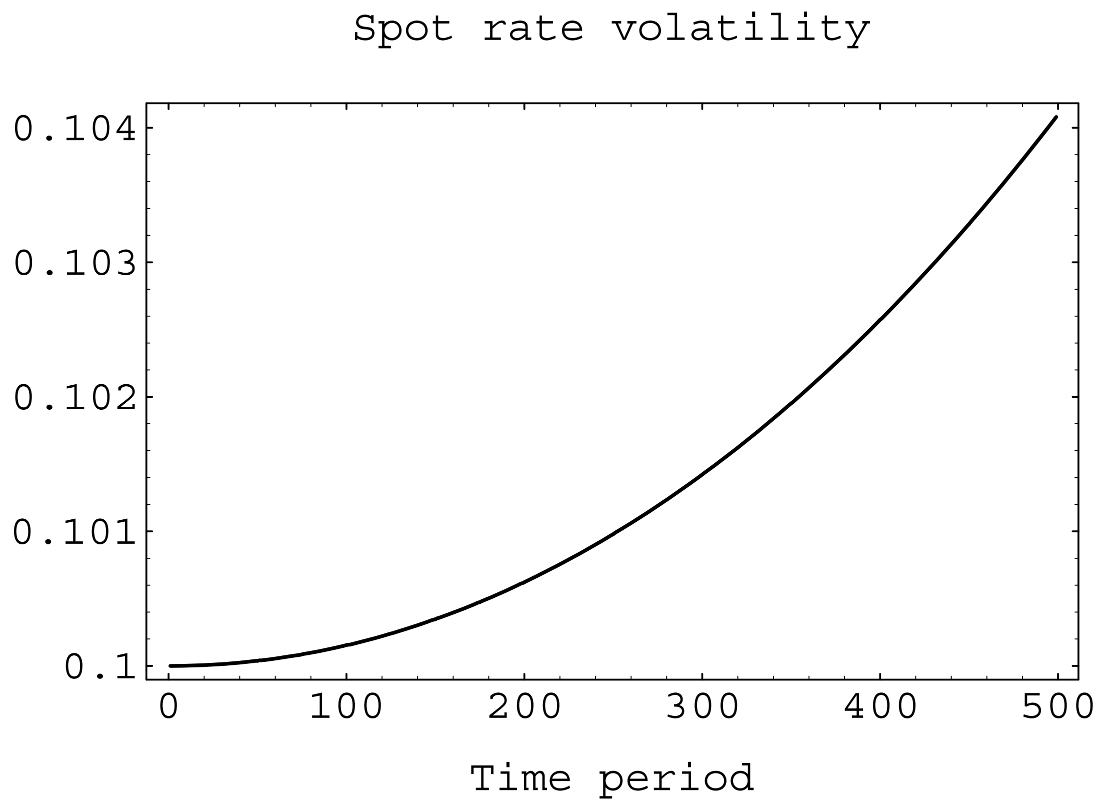
- Thus its yield is $\sqrt{\frac{1}{0.90096}} - 1 = 0.053531$.
- If the short rate declines, the price of the zero one year from now will be

$$\frac{1}{2} \times \frac{1}{1.03526} \times \left(\frac{1}{1.02895} + \frac{1}{1.04343} \right) = 0.93225.$$

Volatility Term Structures (continued)

- Thus its yield is $\sqrt{\frac{1}{0.93225}} - 1 = 0.0357$.
- The yield volatility is hence
$$\frac{1}{2} \ln\left(\frac{0.053531}{0.0357}\right) = 20.256\%,$$
slightly less than the one-year yield volatility.
- This is consistent with the reality that longer-term bonds typically have lower yield volatilities than shorter-term bonds.[a]
- The procedure can be repeated for longer-term zeros to obtain their yield volatilities.

[a]The relation is reversed for *price* volatilities (duration).



(Short rate volatility given a flat %10 volatility structure.)

Volatility Term Structures (concluded)

- We started with v_i and then derived the volatility term structure.
- In practice, the steps are reversed.
- The volatility term structure is supplied by the user along with the term structure.
- The v_i—hence the short rate volatilities via Eq. (138) on p. 1030—and the r_i are then simultaneously determined.
- The result is the Black-Derman-Toy (1990) model of Goldman Sachs.

Foundations of Term Structure Modeling

[Meriwether] scoring especially high marks
in mathematics — an indispensable subject
for a bond trader.

— Roger Lowenstein,
When Genius Failed (2000)

[The] fixed-income traders I knew
seemed smarter than the equity trader $[\cdots]$
there's no competitive edge to
being smart in the equities business[.]
— Emanuel Derman,
My Life as a Quant (2004)

Bond market terminology was designed less
to convey meaning than to bewilder outsiders.
— Michael Lewis, *The Big Short* (2011)

Terminology

- A period denotes a unit of elapsed time.
  - Viewed at time t, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.
- Bonds will be assumed to have a par value of one — unless stated otherwise.
- The time unit for continuous-time models will usually be measured by the year.

Standard Notations

The following notation will be used throughout.

t**:** a point in time.

$r(t)$**:** the one-period riskless rate prevailing at time t for repayment one period later.[a]

$P(t, T)$**:** the present value at time t of one dollar at time T.

[a] Alternatively, the instantaneous spot rate, or short rate, at time t.

Standard Notations (continued)

$r(t, T)$**:** the $(T - t)$-period interest rate prevailing at time t stated on a per-period basis and compounded once per period.[a]

$F(t, T, M)$**:** the forward price at time t of a forward contract that delivers at time T a zero-coupon bond maturing at time $M \geq T$.

[a] In other words, the $(T - t)$-period spot rate at time t.

Standard Notations (concluded)

$f(t, T, L)$**:** the L-period forward rate at time T implied at time t stated on a per-period basis and compounded once per period.

$f(t, T)$**:** the one-period or instantaneous forward rate at time T as seen at time t stated on a per period basis and compounded once per period.

- It is $f(t, T, 1)$ in the discrete-time model and $f(t, T, dt)$ in the continuous-time model.
- Note that $f(t, t)$ equals the short rate $r(t)$.

Fundamental Relations

- The price of a zero-coupon bond equals

$$P(t,T) = \begin{cases} (1+r(t,T))^{-(T-t)}, & \text{in discrete time,} \\ e^{-r(t,T)(T-t)}, & \text{in continuous time.} \end{cases} \tag{144}$$

- $r(t,T)$ as a function of T defines the spot rate curve at time t.
- By definition,

$$f(t,t) = \begin{cases} r(t,t+1), & \text{in discrete time,} \\ r(t,t), & \text{in continuous time.} \end{cases}$$

Fundamental Relations (continued)

- Forward prices and zero-coupon bond prices are related:

$$F(t, T, M) = \frac{P(t, M)}{P(t, T)}, \quad T \leq M. \tag{145}$$

  - The forward price equals the future value at time T of the underlying asset.[a]

- The above identity holds for discrete-time and continuous-time models.

---

[a] See Exercise 24.2.1 of the textbook for proof.

Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by

$$f(t, T, L) = \left(\frac{1}{F(t, T, T + L)} \right)^{1/L} - 1 = \left(\frac{P(t, T)}{P(t, T + L)} \right)^{1/L} - 1 \tag{146}$$

  in discrete time.

- The analog under simple compounding is

$$f(t, T, L) = \frac{1}{L} \left(\frac{P(t, T)}{P(t, T + L)} - 1 \right).$$

Fundamental Relations (continued)

- In continuous time,

$$\begin{aligned} f(t,T,L) &= -\frac{\ln F(t,T,T+L)}{L} \\ &= \frac{\ln(P(t,T)/P(t,T+L))}{L} \qquad (147) \end{aligned}$$

  by Eq. (145) on p. 1090.

- Furthermore,

$$\begin{aligned} f(t,T,\Delta t) &= \frac{\ln(P(t,T)/P(t,T+\Delta t))}{\Delta t} \to -\frac{\partial \ln P(t,T)}{\partial T} \\ &= -\frac{\partial P(t,T)/\partial T}{P(t,T)}. \end{aligned}$$

Fundamental Relations (continued)

- So

$$f(t,T) \triangleq -\frac{\partial \ln P(t,T)}{\partial T} = -\frac{\partial P(t,T)/\partial T}{P(t,T)}, \quad t \leq T. \tag{148}$$

- Because the above identity is equivalent to

$$P(t,T) = e^{-\int_t^T f(t,s)\,ds}, \tag{149}$$

the spot rate curve is

$$r(t,T) = \frac{\int_t^T f(t,s)\,ds}{T-t}.$$

Fundamental Relations (concluded)

- The discrete analog to Eq. (149) is

$$P(t,T) = \frac{1}{(1+r(t))(1+f(t,t+1))\cdots(1+f(t,T-1))}.$$

- The short rate and the market discount function are related by

$$r(t) = -\left.\frac{\partial P(t,T)}{\partial T}\right|_{T=t}.$$

Risk-Neutral Pricing

- Assume the local expectations theory.
- The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  - For all $t+1 < T$,

$$\frac{E_t[\,P(t+1,T)\,]}{P(t,T)} = 1 + r(t). \qquad (150)$$

  - Relation (150) in fact follows from the risk-neutral valuation principle.[a]

[a] Recall Theorem 17 on p. 567.

Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability π.
- Equation (150) on p. 1095 can also be expressed as

$$E_t[\,P(t+1,T)\,] = F(t,t+1,T).$$

  - Verify that with, e.g., Eq. (145) on p. 1090.
- Hence the forward price for the next period is an unbiased estimator of the expected bond price.[a]
  - But the forward rate is *not* an unbiased estimator of the expected future short rate.[b]

[a] Under the local expectations theory.
[b] Recall p. 1044.

Risk-Neutral Pricing (continued)

- Rewrite Eq. (150) on p. 1095 as

$$\frac{E_t^{\pi}[\,P(t+1,T)\,]}{1+r(t)} = P(t,T). \tag{151}$$

  - It says the current market discount function equals the expected market discount function one period from now discounted by the short rate.

Risk-Neutral Pricing (concluded)

- Apply the above equality iteratively to obtain

$$
\begin{aligned}
& P(t,T) \\
= \;& E_t^{\pi}\left[\frac{P(t+1,T)}{1+r(t)}\right] \\
= \;& E_t^{\pi}\left[\frac{E_{t+1}^{\pi}[\,P(t+2,T)\,]}{(1+r(t))(1+r(t+1))}\right] = \cdots \\
= \;& E_t^{\pi}\left[\frac{1}{(1+r(t))(1+r(t+1))\cdots(1+r(T-1))}\right].
\end{aligned}
$$

Continuous-Time Risk-Neutral Pricing

- In continuous time, the local expectations theory implies

$$P(t,T) = E_t\left[e^{-\int_t^T r(s)\,ds} \right], \quad t < T. \tag{152}$$

- Note that $e^{\int_t^T r(s)\,ds}$ is the bank account process, which denotes the rolled-over money market account.