Forward Price: Underlying Pays No Income

Lemma 12 *For a forward contract on an underlying asset providing no income,*

$$F = Se^{r\tau}. \tag{60}$$

- If $F > Se^{r\tau}$:
  - Borrow S dollars for τ years.
  - Buy the underlying asset.
  - Short the forward contract with delivery price F.

Proof (concluded)

- At maturity:
  - Deliver the asset for F.
  - Use $Se^{r\tau}$ to repay the loan, leaving an arbitrage profit of
$$F - Se^{r\tau} > 0.$$
- If $F < Se^{r\tau}$, do the opposite.

Example: Zero-Coupon Bonds

- r is the annualized 3-month riskless interest rate.
- S is the spot price of the 6-month zero-coupon bond.
- A new 3-month forward contract on that 6-month zero-coupon bond should command a delivery price of $Se^{r/4}$.
- So if $r = 6\%$ and $S = 970.87$, then the delivery price is

$$970.87 \times e^{0.06/4} = 985.54.$$

Example: Options

- Suppose S is the spot price of the *European call* that expires at some time later than T.
- A τ-year forward contract on that call commands a delivery price of $Se^{r\tau}$.
- So it equals the future value of the Black-Scholes formula on p. 306.

Contract Value: The Underlying Pays No Income

The value of a forward contract is

$$f = S - Xe^{-r\tau}. \tag{61}$$

- Consider a portfolio consisting of:
  - One long forward contract;
  - Cash amount $Xe^{-r\tau}$;
  - One short position in the underlying asset.

Contract Value: The Underlying Pays No Income (concluded)

- The cash will grow to X at maturity, which can be used to take delivery of the forward contract.
- The delivered asset will then close out the short position.
- Since the value of the portfolio is zero at maturity, its PV must be zero.[a]
- So a forward contract can be replicated by a long position in the underlying and a loan of $Xe^{-r\tau}$ dollars.

---

[a]Recall p. 222.

Lemma 12 (p. 492) Revisited

- Set $f = 0$ in Eq. (61) on p. 496.
- Then $X = Se^{r\tau}$, the forward price.

Forward Price: Underlying Pays Predictable Income

Lemma 13 *For a forward contract on an underlying asset providing a predictable income with a PV of I,*

$$F = (S - I)\, e^{r\tau}. \tag{62}$$

- If $F > (S - I)\, e^{r\tau}$, borrow S dollars for τ years, buy the underlying asset, and short the forward contract with delivery price F.
- Use the income to repay part of the loan.

The Proof (concluded)

- At maturity, the asset is delivered for F, and $(S - I)\, e^{r\tau}$ is used to repay the *remaining* loan.
- That leaves an arbitrage profit of

$$F - (S - I)\, e^{r\tau} > 0.$$

- If $F < (S - I)\, e^{r\tau}$, reverse the above.

Example

- Consider a 10-month forward contract on a \$50 stock.
- The stock pays a dividend of \$1 every 3 months.
- The forward price is

$$\left(50 - e^{-r_3/4} - e^{-r_6/2} - e^{-3\times r_9/4}\right) e^{r_{10}\times(10/12)}.$$

  – r_i is the annualized i-month interest rate.

Underlying Pays a Continuous Dividend Yield of q

- The value of a forward contract at any time prior to T is[a]

$$f = Se^{-q\tau} - Xe^{-r\tau}. \tag{63}$$

- One consequence of Eq. (63) is that the forward price is

$$F = Se^{(r-q)\,\tau}. \tag{64}$$

[a]See p. 160 of the textbook for proof.

Futures Contracts vs. Forward Contracts

- They are traded on a central exchange.
- A clearinghouse.
  - Credit risk is minimized.
- Futures contracts are standardized instruments.
- Gains and losses are marked to market daily.
  - Adjusted at the end of each trading day based on the settlement price.

Size of a Futures Contract

- The amount of the underlying asset to be delivered under the contract.
  - 5,000 bushels for the corn futures on the Chicago Board of Trade (CBOT).
  - One million U.S. dollars for the Eurodollar futures on the Chicago Mercantile Exchange (CME).[a]
- A position can be closed out (or offset) by entering into a reversing trade to the original one.
- Most futures contracts are closed out in this way rather than have the underlying asset delivered.
  - Forward contracts are meant for delivery.

[a]CME and CBOT merged on July 12, 2007.

Daily Settlements

- Price changes in the futures contract are settled daily.
- Hence the spot price rather than the initial futures price is paid on the delivery date.
- Marking to market nullifies any financial incentive for not making delivery.
  - A farmer enters into a forward contract to sell 100,000 bushels of corn at \$2.00 per bushel in November.
  - Suppose the price of corn rises to \$2.5 by November.
  - The farmer has incentive to sell his harvest in the spot market for \$2.5.

Daily Settlements (concluded)

- (continued)
  - With marking to market, the farmer has transferred \$0.5 per bushel from his futures account to that of the clearinghouse by November.[a]
  - When the farmer makes delivery, he is paid the spot price, \$2.5 per bushel.
  - The farmer has little incentive to default.
  - The *net* price remains $\$2.5 - 0.5 = 2$ per bushel, the original delivery price.

---

[a]See p. 507.

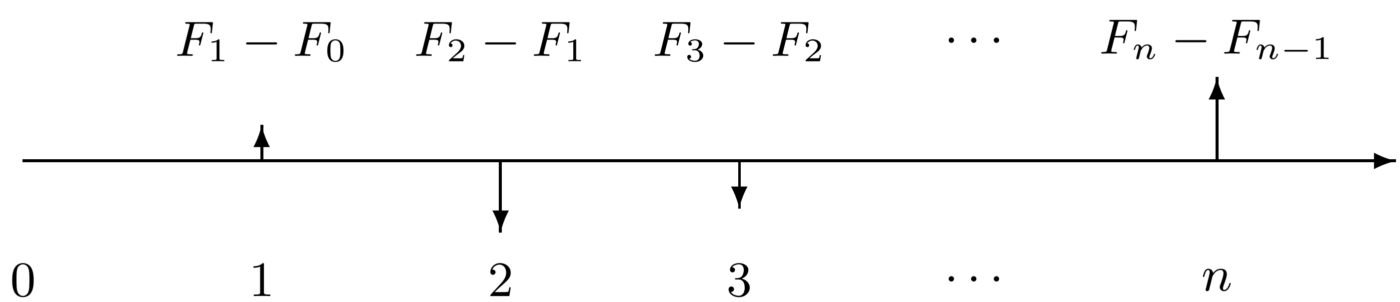
Daily Cash Flows

- Let F_i denote the futures price at the end of day i.
- The contract's cash flow on day i is $F_i - F_{i-1}$.
- The net cash flow over the life of the contract is

$$\begin{aligned}(F_1 - F_0) + (F_2 - F_1) + \cdots + (F_n - F_{n-1}) \\ = F_n - F_0 = S_T - F_0.\end{aligned}$$

- A futures contract has the same *accumulated* payoff $S_T - F_0$ as a forward contract.
- The actual payoff may vary because of the reinvestment of daily cash flows and how $S_T - F_0$ is distributed.

Daily Cash Flows (concluded)



Delivery and Hedging

- Futures price is the delivery price that makes the futures contract zero-valued.
- Delivery ties the futures price to the spot price.
- On the delivery date, the settlement price of the futures contract is determined by the spot price.
- Hence, when the delivery period is reached, the futures price should be very close to the spot price.[a]
- Changes in futures prices usually track those in spot price, making hedging possible.

---

[a]But since early 2006, futures for corn, wheat, and soybeans occasionally expired at a price much higher than that day's spot price (Henriques, 2008).

Forward and Futures Prices

- Surprisingly, futures price equals forward price if interest rates are nonstochastic![a]
- This result "justifies" treating a futures contract as if it were a forward contract, ignoring its marking-to-market feature.
- The West Texas Intermediate (WTI) futures price was negative on April 20, 2020![b]

---

[a] Cox, Ingersoll, & Ross (1981); see p. 164 of the textbook for proof.

[b] April 21 was the last trading day for oil delivery in May to Cushing, Oklahoma.

Remarks

- When interest rates are stochastic, forward and futures prices are no longer theoretically identical.
  - Suppose interest rates are uncertain and futures prices move in the same direction as interest rates.
  - Then futures prices will exceed forward prices.
- For short-term contracts, the differences tend to be small.
- Unless stated otherwise, assume forward and futures prices are identical.

Futures Options

- The underlying of a futures option is a futures contract.
- Upon exercise, the option holder takes a position in the futures contract with a futures price equal to the option's strike price.
  - A call holder acquires a *long* futures position.
  - A put holder acquires a *short* futures position.
- The futures contract is then marked to market.
- And the futures positions of the two parties are at the prevailing futures price (thus zero-valued).

Futures Options (concluded)

- It works as if the *call* holder received a futures contract plus cash equivalent to the prevailing futures price F_t minus the strike price X:

$$F_t - X.$$

  - This futures contract has zero value.

- It works as if the *put* holder sold a futures contract for

$$X - F_t$$

dollars.

Forward Options

- What is delivered is now a forward contract with a delivery price equal to the option's strike price.
  - Exercising a call forward option results in a *long* position in a forward contract.
  - Exercising a put forward option results in a *short* position in a forward contract.
- Exercising a forward option incurs no immediate cash flows as there is no marking to market.

Example

- Consider an American call with strike \$100 and an expiration date in September.
- The underlying asset is a *forward* contract with a delivery date in December.
- Suppose the forward price in July is \$110.
- Upon exercise, the call holder receives a forward contract with a delivery price of \$100.[a]
- If an offsetting position is then taken in the forward market,[b] a \$10 profit *in December* will be assured.
- A call on the futures would realize the \$10 profit *in July*.

[a] Recall p. 484.

[b] The counterparty will pay you \$110 for the underlying asset.

Some Pricing Relations

- Let T be the delivery date, the current time be 0, and the option[a] have expiration date t ($t \leq T$).
- Assume a constant, positive interest rate.
- Although forward price equals futures price, a forward option does *not* have the same value as a futures option.
- The payoffs of calls at time t are, respectively,[b]

$$\text{futures option} = \max(F_t - X, 0), \quad (66)$$

$$\text{forward option} = \max(F_t - X, 0)\, e^{-r(T-t)}. \quad (67)$$

---

[a] On the futures or forward contract

[b] Recall p. 513.

Some Pricing Relations (concluded)

- A European futures option is worth the same as a European option on the underlying asset if $T = t$.
  - Futures price equals spot price at maturity.
- This conclusion is model independent.

Put-Call Parity[a]

The put-call parity is slightly different from the one in Eq. (31) on p. 229.

Theorem 14 *(1) For European options on futures contracts,*

$$C = P - (X - F)\, e^{-rt}.$$

(2) For European options on forward contracts,

$$C = P - (X - F)\, e^{-rT}.$$

[a] See Theorem 12.4.4 of the textbook for proof.

Early Exercise

The early exercise feature is not valuable for *forward* options.[a]

Theorem 15 *American forward options should not be exercised before expiration as long as the probability of their ending up out of the money is positive.*

Early exercise may be optimal for American *futures* options even if the underlying asset generates no payouts.

Theorem 16 *American futures options may be exercised optimally before expiration.*

[a]See Theorem 12.4.5 of the textbook for proof.

Black's Model[a]

- Formulas for European futures options:

$$C = Fe^{-rt} N(x) - Xe^{-rt} N(x - \sigma\sqrt{t}\,), \qquad (68)$$

$$P = Xe^{-rt} N(-x + \sigma\sqrt{t}\,) - Fe^{-rt} N(-x), \qquad (69)$$

  where $x \stackrel{\Delta}{=} \frac{\ln(F/X) + (\sigma^2/2)\, t}{\sigma\sqrt{t}}$.

- The above formulas are related to those for options on a stock paying a continuous dividend yield.

- They are Eqs. (44) on p. 337 with q set to r and S replaced by F.

[a]Black (1976). It is also called the Black-76 model.

Black's Model (concluded)

- Volatility σ is that of the stock price.[a]
  - It is the same as the volatlity of the futures price.[b]
- This observation incidentally proves Theorem 16 (p. 519).
- For European forward options, just multiply the above formulas by $e^{-r(T-t)}$.
  - Forward options differ from futures options by a factor of $e^{-r(T-t)}$.[c]

---

[a] Contributed by Mr. Lu, Yu-Ming (`R06723032`, `D08922008`) on April 7, 2021.

[b] Contributed by Mr. Chen, Tung-Li (`D09922014`) on April 7, 2023. See also p. 632.

[c] Recall Eqs. (66)–(67) on p. 516.

Binomial Model for Forward and Futures Options

- In a risk-neutral economy, futures price behaves *like* a stock paying a continuous dividend yield of r.
  - Let the futures price at time 0 be F.
  - From Lemma 10 (p. 304), the expected value of S at time Δt in a risk-neutral economy is

$$Se^{r\Delta t}.$$

  - By Eq. (60) on p. 492, the expected futures price at time Δt is

$$Se^{r\Delta t}e^{r(T-\Delta t)} = Se^{rT} = F.$$

Binomial Model for Forward and Futures Options (continued)

- The above observation continues to hold even if S pays a dividend yield![a]
  - Let the futures price at time 0 be F.
  - From Lemma 10 (p. 304), the expected value of S at time Δt in a risk-neutral economy is
$$Se^{(r-q)\,\Delta t}.$$
  - By Eq. (64) on p. 502, the expected futures price at time Δt is
$$Se^{(r-q)\,\Delta t}e^{(r-q)(T-\Delta t)} = Se^{(r-q)\,T} = F.$$

[a] Contributed by Mr. Liu, Yi-Wei (R02723084) on April 16, 2014.

Binomial Model for Forward and Futures Options (concluded)

- Now, under the BOPM, the risk-neutral probability for the futures price is

$$p_{\mathrm{f}} \stackrel{\Delta}{=} (1-d)/(u-d) \tag{70}$$

  by Eq. (45) on p. 339.

  - The futures price moves from F to Fu with probability p_{f} and to Fd with probability $1-p_{\mathrm{f}}$.
  - The *original* u and d are used here.

- The binomial tree algorithm for *forward* options is identical except that Eq. (67) on p. 516 is the payoff.

Spot and Futures Prices under BOPM

- The futures price is related to the spot price via

$$F = Se^{rT}$$

  if the underlying asset pays no dividends.[a]

- Recall the futures price F moves to Fu with probability p_{f} per period.

- So the stock price moves from $S = Fe^{-rT}$ to

$$Fue^{-r(T-\Delta t)} = Sue^{r\Delta t}$$

  with probability p_{f} per period.

---

[a] Recall Lemma 12 (p. 492).

Spot and Futures Prices under BOPM (concluded)

- Similarly, the stock price moves from $S = Fe^{-rT}$ to

$$Sde^{r\Delta t}$$

  with probability $1 - p_{\mathrm{f}}$ per period.

- Note that

$$S(ue^{r\Delta t})(de^{r\Delta t}) = Se^{2r\Delta t} \neq S.$$

- This binomial model for S differs from the CRR tree.
- This model may not be suitable for pricing barrier options (why?).

Negative Probabilities Revisited

- As $0 < p_{\mathrm{f}} < 1$, we have $0 < 1 - p_{\mathrm{f}} < 1$ as well.
- The problem of negative risk-neutral probabilities is solved:
  - Build the tree for the futures price F of the futures contract expiring at the same time as the option.
  - Let the stock pay a continuous dividend yield of q.
  - By Eq. (64) on p. 502, recover S from F at each node via
$$S = F e^{-(r-q)(T-t)}.$$

Swaps

- Swaps are agreements between two counterparties to exchange cash flows in the future according to some predetermined formula.
- There are two basic types of swaps: interest rate and currency.
- An interest rate swap occurs when two parties exchange interest payments periodically.
- Currency swaps are agreements to deliver one currency against another (our focus here).
- There are theories about why swaps exist.[a]

[a]Thanks to a lively discussion on April 16, 2014.

Currency Swaps

- A currency swap involves two parties to exchange cash flows in different currencies.
- Consider the following fixed rates available to party A and party B in U.S. dollars and Japanese yen:

	Dollars	Yen
A	$D_{\rm A}$%	$\boxed{Y_{\rm A}\%}$
B	$\boxed{D_{\rm B}\%}$	$Y_{\rm B}$%

- Suppose A wants to take out a fixed-rate loan in yen, and B wants to take out a fixed-rate loan in dollars.

Currency Swaps (continued)

- A straightforward scenario is for A to borrow yen at $Y_{\mathrm{A}}\%$ and B to borrow dollars at $D_{\mathrm{B}}\%$.
- But suppose A is *relatively* more competitive in the dollar market than the yen market, i.e.,

$$Y_{\mathrm{B}} - D_{\mathrm{B}} < Y_{\mathrm{A}} - D_{\mathrm{A}} \quad \text{or} \quad Y_{\mathrm{B}} - Y_{\mathrm{A}} < D_{\mathrm{B}} - D_{\mathrm{A}}.$$

- Consider this alternative arrangement:
  - A borrows dollars.
  - B borrows yen.
  - They enter into a currency swap with a bank (the swap dealer) as the intermediary.

Currency Swaps (concluded)

- The counterparties exchange principal at the beginning and the end of the life of the swap.
- This act transforms A's loan into a yen loan and B's yen loan into a dollar loan.
- The total interest rate is originally $(Y_{\mathrm{A}} + D_{\mathrm{B}})\%$.
- The new arrangement has a smaller total rate of $(D_{\mathrm{A}} + Y_{\mathrm{B}})\%$.
- So the total gain is $((D_{\mathrm{B}} - D_{\mathrm{A}}) - (Y_{\mathrm{B}} - Y_{\mathrm{A}}))\%$.
- Transactions will happen only if the gain is distributed so that the cost to each party is less than the original.

Example

- A and B face the following borrowing rates:

	Dollars	Yen
A	9%	10%
B	12%	11%

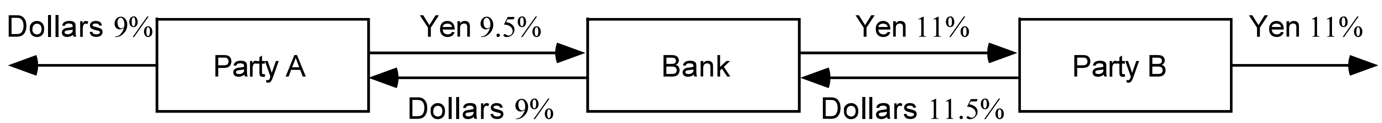
- A wants to borrow yen, and B wants to borrow dollars.
- A can borrow yen directly at 10%.
- B can borrow dollars directly at 12%.

Example (continued)

- The rate differential in dollars (3%) is different from that in yen (1%).
- So a currency swap with a total saving of $3 - 1 = 2\%$ is possible.
- A is relatively more competitive in the dollar market.
- B is relatively more competitive in the yen market.

Example (concluded)

- Next page shows an arrangement which is beneficial to all parties involved.
  - A effectively borrows yen at 9.5% (lower than 10%).
  - B borrows dollars at 11.5% (lower than 12%).
  - The gain is 0.5% for A, 0.5% for B, and, if we treat dollars and yen identically, 1% for the bank.


Dollars 9%
Party A
Yen 9.5%
Dollars 9%
Bank
Yen 11%
Dollars 11.5%
Party B
Yen 11%

As a Package of Cash Market Instruments

- Assume no default risk.
- Take B on p. 535 as an example.
- The swap is equivalent to a long position in a yen bond paying 11% annual interest and a short position in a dollar bond paying 11.5% annual interest.
- The pricing formula is $SP_{\mathrm{Y}} - P_{\mathrm{D}}$.
  - P_{D} is the dollar bond's value in dollars.
  - P_{Y} is the yen bond's value in yen.
  - S is the $/yen spot exchange rate.

As a Package of Cash Market Instruments (concluded)

- The value of a currency swap depends on:
  - The term structures of interest rates in the currencies involved.
  - The spot exchange rate.
- It has zero value when

$$SP_{\mathrm{Y}} = P_{\mathrm{D}}.$$

Example

- Take a 3-year swap on p. 535 with principal amounts of US$1 million and 100 million yen.
- The payments are made once a year.
- The spot exchange rate is 90 yen/$ and the term structures are flat in both nations—8% in the U.S. and 9% in Japan.
- For B, the value of the swap is (in millions of USD)

$$\frac{1}{90} \times \left(11 \times e^{-0.09} + 11 \times e^{-0.09\times 2} + 111 \times e^{-0.09\times 3}\right)$$
$$-\left(0.115 \times e^{-0.08} + 0.115 \times e^{-0.08\times 2} + 1.115 \times e^{-0.08\times 3}\right) \quad = \quad 0.074.$$

As a Package of Forward Contracts

- From Eq. (63) on p. 502, the forward contract maturing i years from now has a *dollar* value of

$$f_i \triangleq (SY_i)\, e^{-qi} - D_i e^{-ri}. \tag{71}$$

  - Y_i is the yen inflow at year i.
  - S is the \$/yen spot exchange rate.
  - q is the yen interest rate.
  - D_i is the dollar outflow at year i.
  - r is the dollar interest rate.

As a Package of Forward Contracts (concluded)

- For simplicity, flat term structures were assumed.
- Generalization is straightforward.

Example

- Take the swap in the example on p. 538.
- Every year, B receives 11 million yen and pays 0.115 million dollars.
- In addition, at the end of the third year, B receives 100 million yen and pays 1 million dollars.
- Each of these transactions represents a forward contract.
- $Y_1 = Y_2 = 11$, $Y_3 = 111$, $S = 1/90$, $D_1 = D_2 = 0.115$, $D_3 = 1.115$, $q = 0.09$, and $r = 0.08$.
- Plug in these numbers to get $f_1 + f_2 + f_3 = 0.074$ million dollars as before.