Forward Price: Underlying Pays No Income

Lemma 12 For a forward contract on an underlying asset providing no income,

\[ F = Se^{r\tau}. \]  \hspace{1cm} (60)

• If \( F > Se^{r\tau} \):
  – Borrow \( S \) dollars for \( \tau \) years.
  – Buy the underlying asset.
  – Short the forward contract with delivery price \( F \).
Proof (concluded)

• At maturity:
  – Deliver the asset for $F$.
  – Use $Se^{r\tau}$ to repay the loan, leaving an arbitrage profit of
    $$F - Se^{r\tau} > 0.$$  

• If $F < Se^{r\tau}$, do the opposite.
Example: Zero-Coupon Bonds

- $r$ is the annualized 3-month riskless interest rate.
- $S$ is the spot price of the 6-month zero-coupon bond.
- A new 3-month forward contract on that 6-month zero-coupon bond should command a delivery price of $Se^{r/4}$.
- So if $r = 6\%$ and $S = 970.87$, then the delivery price is $970.87 \times e^{0.06/4} = 985.54$. 
Example: Options

- Suppose $S$ is the spot price of the *European call* that expires at some time later than $T$.

- A $\tau$-year forward contract on that call commands a delivery price of $Se^{r\tau}$.

- So it equals the future value of the Black-Scholes formula on p. 306.
Contract Value: The Underlying Pays No Income

The value of a forward contract is

\[ f = S - X e^{-r\tau}. \]  \hspace{1cm} (61)

- Consider a portfolio consisting of:
  - One long forward contract;
  - Cash amount \( X e^{-r\tau} \);
  - One short position in the underlying asset.
Contract Value: The Underlying Pays No Income (concluded)

- The cash will grow to $X$ at maturity, which can be used to take delivery of the forward contract.
- The delivered asset will then close out the short position.
- Since the value of the portfolio is zero at maturity, its PV must be zero.\(^a\)

- So a forward contract can be replicated by a long position in the underlying and a loan of $Xe^{-r\tau}$ dollars.

\(^a\)Recall p. 222.
Lemma 12 (p. 492) Revisited

- Set $f = 0$ in Eq. (61) on p. 496.
- Then $X = Se^{r\tau}$, the forward price.
Forward Price: Underlying Pays Predictable Income

Lemma 13  For a forward contract on an underlying asset providing a predictable income with a PV of $I$, 

$$F = (S - I) e^{r\tau}.$$  \hfill (62)

- If $F > (S - I) e^{r\tau}$, borrow $S$ dollars for $\tau$ years, buy the underlying asset, and short the forward contract with delivery price $F$.

- Use the income to repay part of the loan.
The Proof (concluded)

• At maturity, the asset is delivered for $F$, and $(S - I) e^{r\tau}$ is used to repay the remaining loan.

• That leaves an arbitrage profit of

$$F - (S - I) e^{r\tau} > 0.$$ 

• If $F < (S - I) e^{r\tau}$, reverse the above.
Example

• Consider a 10-month forward contract on a $50 stock.
• The stock pays a dividend of $1 every 3 months.
• The forward price is

\[
\left( 50 - e^{-r_{3/4}} - e^{-r_{6/2}} - e^{-3\times r_{9/4}} \right) e^{r_{10\times (10/12)}}.
\]

– \( r_i \) is the annualized \( i \)-month interest rate.
Underlying Pays a Continuous Dividend Yield of $q$

- The value of a forward contract at any time prior to $T$ is \(^a\)

\[
f = Se^{-q\tau} - Xe^{-r\tau}.
\]

(63)

- One consequence of Eq. (63) is that the forward price is

\[
F = Se^{(r-q)\tau}.
\]

(64)

\(^a\)See p. 160 of the textbook for proof.
Futures Contracts vs. Forward Contracts

- They are traded on a central exchange.
- A clearinghouse.
  - Credit risk is minimized.
- Futures contracts are standardized instruments.
- Gains and losses are marked to market daily.
  - Adjusted at the end of each trading day based on the settlement price.
Size of a Futures Contract

- The amount of the underlying asset to be delivered under the contract.
  - 5,000 bushels for the corn futures on the Chicago Board of Trade (CBOT).
  - One million U.S. dollars for the Eurodollar futures on the Chicago Mercantile Exchange (CME).\(^a\)

- A position can be closed out (or offset) by entering into a reversing trade to the original one.

- Most futures contracts are closed out in this way rather than have the underlying asset delivered.
  - Forward contracts are meant for delivery.

\(^a\)CME and CBOT merged on July 12, 2007.
Daily Settlements

- Price changes in the futures contract are settled daily.
- Hence the spot price rather than the initial futures price is paid on the delivery date.
- Marking to market nullifies any financial incentive for not making delivery.
  - A farmer enters into a forward contract to sell 100,000 bushels of corn at $2.00 per bushel in November.
  - Suppose the price of corn rises to $2.5 by November.
  - The farmer has incentive to sell his harvest in the spot market for $2.5.
Daily Settlements (concluded)

• (continued)

– With marking to market, the farmer has transferred $0.5 per bushel from his futures account to that of the clearinghouse by November.\(^a\)

– When the farmer makes delivery, he is paid the spot price, $2.5 per bushel.

– The farmer has little incentive to default.

– The net price remains $2.5 − 0.5 = 2 per bushel, the original delivery price.

\(^a\)See p. 507.
Daily Cash Flows

- Let $F_i$ denote the futures price at the end of day $i$.
- The contract’s cash flow on day $i$ is $F_i - F_{i-1}$.
- The net cash flow over the life of the contract is
  \[
  (F_1 - F_0) + (F_2 - F_1) + \cdots + (F_n - F_{n-1})
  = F_n - F_0 = S_T - F_0.
  \]
- A futures contract has the same *accumulated* payoff $S_T - F_0$ as a forward contract.
- The actual payoff may vary because of the reinvestment of daily cash flows and how $S_T - F_0$ is distributed.
Daily Cash Flows (concluded)

\[ F_1 - F_0 \quad F_2 - F_1 \quad F_3 - F_2 \quad \cdots \quad F_n - F_{n-1} \]

0 \quad 1 \quad 2 \quad 3 \quad \cdots \quad n
Delivery and Hedging

- Futures price is the delivery price that makes the futures contract zero-valued.
- Delivery ties the futures price to the spot price.
- On the delivery date, the settlement price of the futures contract is determined by the spot price.
- Hence, when the delivery period is reached, the futures price should be very close to the spot price.\(^a\)
- Changes in futures prices usually track those in spot price, making hedging possible.

\(^a\)But since early 2006, futures for corn, wheat, and soybeans occasionally expired at a price much higher than that day’s spot price (Henriques, 2008).
Forward and Futures Prices

• Surprisingly, futures price equals forward price if interest rates are nonstochastic!\(^a\)

• This result “justifies” treating a futures contract as if it were a forward contract, ignoring its marking-to-market feature.

• The West Texas Intermediate (WTI) futures price was negative on April 20, 2020!\(^b\)

\(^a\)Cox, Ingersoll, & Ross (1981); see p. 164 of the textbook for proof.

\(^b\)April 21 was the last trading day for oil delivery in May to Cushing, Oklahoma.
Remarks

• When interest rates are stochastic, forward and futures prices are no longer theoretically identical.
  – Suppose interest rates are uncertain and futures prices move in the same direction as interest rates.
  – Then futures prices will exceed forward prices.

• For short-term contracts, the differences tend to be small.

• Unless stated otherwise, assume forward and futures prices are identical.
Futures Options

- The underlying of a futures option is a futures contract.
- Upon exercise, the option holder takes a position in the futures contract with a futures price equal to the option’s strike price.
  - A call holder acquires a long futures position.
  - A put holder acquires a short futures position.
- The futures contract is then marked to market.
- And the futures positions of the two parties are at the prevailing futures price (thus zero-valued).
Futures Options (concluded)

• It works as if the *call* holder received a futures contract plus cash equivalent to the prevailing futures price $F_t$ minus the strike price $X$:

\[ F_t - X. \]

– This futures contract has zero value.

• It works as if the *put* holder sold a futures contract for

\[ X - F_t \]

dollars.
Forward Options

- What is delivered is now a forward contract with a delivery price equal to the option’s strike price.
  - Exercising a call forward option results in a long position in a forward contract.
  - Exercising a put forward option results in a short position in a forward contract.

- Exercising a forward option incurs no immediate cash flows as there is no marking to market.
Example

- Consider an American call with strike $100 and an expiration date in September.
- The underlying asset is a forward contract with a delivery date in December.
- Suppose the forward price in July is $110.
- Upon exercise, the call holder receives a forward contract with a delivery price of $100.\(^a\)
- If an offsetting position is then taken in the forward market,\(^b\) a $10 profit \emph{in December} will be assured.
- A call on the futures would realize the $10 profit \emph{in July}.\(^b\)

\(^a\) Recall p. 484.
\(^b\) The counterparty will pay you $110 for the underlying asset.
Some Pricing Relations

• Let $T$ be the delivery date, the current time be 0, and the option\textsuperscript{a} have expiration date $t$ ($t \leq T$).

• Assume a constant, positive interest rate.

• Although forward price equals futures price, a forward option does not have the same value as a futures option.

• The payoffs of calls at time $t$ are, respectively,\textsuperscript{b}

\begin{align*}
\text{futures option} & = \max(F_t - X, 0), & (66) \\
\text{forward option} & = \max(F_t - X, 0) e^{-r(T-t)}. & (67)
\end{align*}

\textsuperscript{a}On the futures or forward contract
\textsuperscript{b}Recall p. 513.
Some Pricing Relations (concluded)

- A European futures option is worth the same as a European option on the underlying asset if $T = t$.
  - Futures price equals spot price at maturity.

- This conclusion is model independent.
Put-Call Parity\textsuperscript{a}

The put-call parity is slightly different from the one in Eq. (31) on p. 229.

\textbf{Theorem 14} (1) For European options on futures contracts,

\[ C = P - (X - F) e^{-rt}. \]

(2) For European options on forward contracts,

\[ C = P - (X - F') e^{-rT}. \]

\textsuperscript{a}See Theorem 12.4.4 of the textbook for proof.
Early Exercise

The early exercise feature is not valuable for *forward* options.\(^a\)

**Theorem 15** *American forward options should not be exercised before expiration as long as the probability of their ending up out of the money is positive.*

Early exercise may be optimal for American *futures* options even if the underlying asset generates no payouts.

**Theorem 16** *American futures options may be exercised optimally before expiration.*

\(^a\)See Theorem 12.4.5 of the textbook for proof.
Black’s Model\textsuperscript{a}

- Formulas for European futures options:

\[ C = Fe^{-rt}N(x) - Xe^{-rt}N(x - \sigma \sqrt{t}), \]  \hspace{1cm} (68)
\[ P = Xe^{-rt}N(-x + \sigma \sqrt{t}) - Fe^{-rt}N(-x), \]  \hspace{1cm} (69)

where \( x \equiv \frac{\ln(F/X) + (\sigma^2/2) t}{\sigma \sqrt{t}} \).

- The above formulas are related to those for options on a stock paying a continuous dividend yield.

- They are Eqs. (44) on p. 337 with \( q \) set to \( r \) and \( S \) replaced by \( F \).

\textsuperscript{a}Black (1976). It is also called the Black-76 model.
Black’s Model (concluded)

- Volatility $\sigma$ is that of the stock price.\(^a\)
  - It is the same as the volatility of the futures price.\(^b\)

- This observation incidentally proves Theorem 16 (p. 519).

- For European forward options, just multiply the above formulas by $e^{-r(T-t)}$.
  - Forward options differ from futures options by a factor of $e^{-r(T-t)}$.\(^c\)

\(^a\)Contributed by Mr. Lu, Yu-Ming (R06723032, D08922008) on April 7, 2021.

\(^b\)Contributed by Mr. Chen, Tung-Li (D09922014) on April 7, 2023.

See also p. 632.

\(^c\)Recall Eqs. (66)–(67) on p. 516.
Binomial Model for Forward and Futures Options

- In a risk-neutral economy, futures price behaves like a stock paying a continuous dividend yield of $r$.
  - Let the futures price at time 0 be $F$.
  - From Lemma 10 (p. 304), the expected value of $S$ at time $\Delta t$ in a risk-neutral economy is
    \[ Se^{r\Delta t}. \]
  - By Eq. (60) on p. 492, the expected futures price at time $\Delta t$ is
    \[ Se^{r\Delta t} e^{r(T-\Delta t)} = Se^{rT} = F. \]
Binomial Model for Forward and Futures Options (continued)

- The above observation continues to hold even if $S$ pays a dividend yield!\(^a\)
  - Let the futures price at time 0 be $F$.
  - From Lemma 10 (p. 304), the expected value of $S$ at time $\Delta t$ in a risk-neutral economy is
    \[ S e^{(r-q)\Delta t}. \]
  - By Eq. (64) on p. 502, the expected futures price at time $\Delta t$ is
    \[ S e^{(r-q)\Delta t} e^{(r-q)(T-\Delta t)} = S e^{(r-q)T} = F. \]

\(^a\)Contributed by Mr. Liu, Yi-Wei (R02723084) on April 16, 2014.
Binomial Model for Forward and Futures Options (concluded)

- Now, under the BOPM, the risk-neutral probability for the futures price is

\[
p_f \triangleq \frac{(1 - d)}{(u - d)}
\]

(70)

by Eq. (45) on p. 339.

- The futures price moves from \( F \) to \( Fu \) with probability \( p_f \) and to \( Fd \) with probability \( 1 - p_f \).

- The original \( u \) and \( d \) are used here.

- The binomial tree algorithm for forward options is identical except that Eq. (67) on p. 516 is the payoff.
Spot and Futures Prices under BOPM

- The futures price is related to the spot price via

\[ F = S e^{rT} \]

if the underlying asset pays no dividends.\(^a\)

- Recall the futures price \( F \) moves to \( Fu \) with probability \( p_f \) per period.

- So the stock price moves from \( S = F e^{-rT} \) to

\[ Fue^{-r(T-\Delta t)} = Sue^{r\Delta t} \]

with probability \( p_f \) per period.

\(^a\)Recall Lemma 12 (p. 492).
Spot and Futures Prices under BOPM (concluded)

- Similarly, the stock price moves from $S = Fe^{-rT}$ to

$$Sde^{r\Delta t}$$

with probability $1 - pf$ per period.

- Note that

$$S(ue^{r\Delta t})(de^{r\Delta t}) = Se^{2r\Delta t} \neq S.$$

- This binomial model for $S$ differs from the CRR tree.

- This model may not be suitable for pricing barrier options (why?).
Negative Probabilities Revisited

- As $0 < p_f < 1$, we have $0 < 1 - p_f < 1$ as well.
- The problem of negative risk-neutral probabilities is solved:
  - Build the tree for the futures price $F$ of the futures contract expiring at the same time as the option.
  - Let the stock pay a continuous dividend yield of $q$.
  - By Eq. (64) on p. 502, recover $S$ from $F$ at each node via
    \[ S = F e^{-(r-q)(T-t)}. \]
Swaps

• Swaps are agreements between two counterparties to exchange cash flows in the future according to some predetermined formula.

• There are two basic types of swaps: interest rate and currency.

• An interest rate swap occurs when two parties exchange interest payments periodically.

• Currency swaps are agreements to deliver one currency against another (our focus here).

• There are theories about why swaps exist.\textsuperscript{a}

\textsuperscript{a}Thanks to a lively discussion on April 16, 2014.
Currency Swaps

- A currency swap involves two parties to exchange cash flows in different currencies.

- Consider the following fixed rates available to party A and party B in U.S. dollars and Japanese yen:

<table>
<thead>
<tr>
<th>Dollars</th>
<th>Yen</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$D_A%$</td>
</tr>
<tr>
<td>B</td>
<td>$D_B%$</td>
</tr>
</tbody>
</table>

- Suppose A wants to take out a fixed-rate loan in yen, and B wants to take out a fixed-rate loan in dollars.
Currency Swaps (continued)

- A straightforward scenario is for A to borrow yen at $Y_A\%$ and B to borrow dollars at $D_B\%$.
- But suppose A is relatively more competitive in the dollar market than the yen market, i.e.,

  \[ Y_B - D_B < Y_A - D_A \quad \text{or} \quad Y_B - Y_A < D_B - D_A. \]

- Consider this alternative arrangement:
  - A borrows dollars.
  - B borrows yen.
  - They enter into a currency swap with a bank (the swap dealer) as the intermediary.
Currency Swaps (concluded)

- The counterparties exchange principal at the beginning and the end of the life of the swap.
- This act transforms A’s loan into a yen loan and B’s yen loan into a dollar loan.
- The total interest rate is originally \((Y_A + D_B)\)%.
- The new arrangement has a smaller total rate of \((D_A + Y_B)\)%.
- So the total gain is \(((D_B - D_A) - (Y_B - Y_A))\)%.
- Transactions will happen only if the gain is distributed so that the cost to each party is less than the original.
Example

- A and B face the following borrowing rates:

<table>
<thead>
<tr>
<th></th>
<th>Dollars</th>
<th>Yen</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>9%</td>
<td>10%</td>
</tr>
<tr>
<td>B</td>
<td>12%</td>
<td>11%</td>
</tr>
</tbody>
</table>

- A wants to borrow yen, and B wants to borrow dollars.
- A can borrow yen directly at 10%.
- B can borrow dollars directly at 12%.
Example (continued)

- The rate differential in dollars (3%) is different from that in yen (1%).

- So a currency swap with a total saving of $3 - 1 = 2\%$ is possible.

- A is relatively more competitive in the dollar market.

- B is relatively more competitive in the yen market.
Example (concluded)

- Next page shows an arrangement which is beneficial to all parties involved.
  - A effectively borrows yen at 9.5% (lower than 10%).
  - B borrows dollars at 11.5% (lower than 12%).
  - The gain is 0.5% for A, 0.5% for B, and, if we treat dollars and yen identically, 1% for the bank.
As a Package of Cash Market Instruments

• Assume no default risk.

• Take B on p. 535 as an example.

• The swap is equivalent to a long position in a yen bond paying 11% annual interest and a short position in a dollar bond paying 11.5% annual interest.

• The pricing formula is \( SP_Y - P_D \).
  - \( P_D \) is the dollar bond’s value in dollars.
  - \( P_Y \) is the yen bond’s value in yen.
  - \( S \) is the $/yen spot exchange rate.
As a Package of Cash Market Instruments (concluded)

- The value of a currency swap depends on:
  - The term structures of interest rates in the currencies involved.
  - The spot exchange rate.
- It has zero value when

$$SP_Y = P_D.$$
Example

• Take a 3-year swap on p. 535 with principal amounts of US$1 million and 100 million yen.

• The payments are made once a year.

• The spot exchange rate is 90 yen/$ and the term structures are flat in both nations—8% in the U.S. and 9% in Japan.

• For B, the value of the swap is (in millions of USD)

\[
\frac{1}{90} \times (11 \times e^{-0.09} + 11 \times e^{-0.09 \times 2} + 111 \times e^{-0.09 \times 3}) \\
- (0.115 \times e^{-0.08} + 0.115 \times e^{-0.08 \times 2} + 1.115 \times e^{-0.08 \times 3}) = 0.074.
\]
As a Package of Forward Contracts

- From Eq. (63) on p. 502, the forward contract maturing \( i \) years from now has a dollar value of

\[
f_i \triangleq (S Y_i) e^{-q_i} - D_i e^{-r_i}.
\]  

- \( Y_i \) is the yen inflow at year \( i \).
- \( S \) is the $/yen spot exchange rate.
- \( q \) is the yen interest rate.
- \( D_i \) is the dollar outflow at year \( i \).
- \( r \) is the dollar interest rate.
As a Package of Forward Contracts (concluded)

- For simplicity, flat term structures were assumed.
- Generalization is straightforward.
Example

• Take the swap in the example on p. 538.

• Every year, B receives 11 million yen and pays 0.115 million dollars.

• In addition, at the end of the third year, B receives 100 million yen and pays 1 million dollars.

• Each of these transactions represents a forward contract.

• \( Y_1 = Y_2 = 11, \ Y_3 = 111, \ S = 1/90, \ D_1 = D_2 = 0.115, \ D_3 = 1.115, \ q = 0.09, \) and \( r = 0.08. \)

• Plug in these numbers to get \( f_1 + f_2 + f_3 = 0.074 \) million dollars as before.