The PV Formula (p. 41) Justified

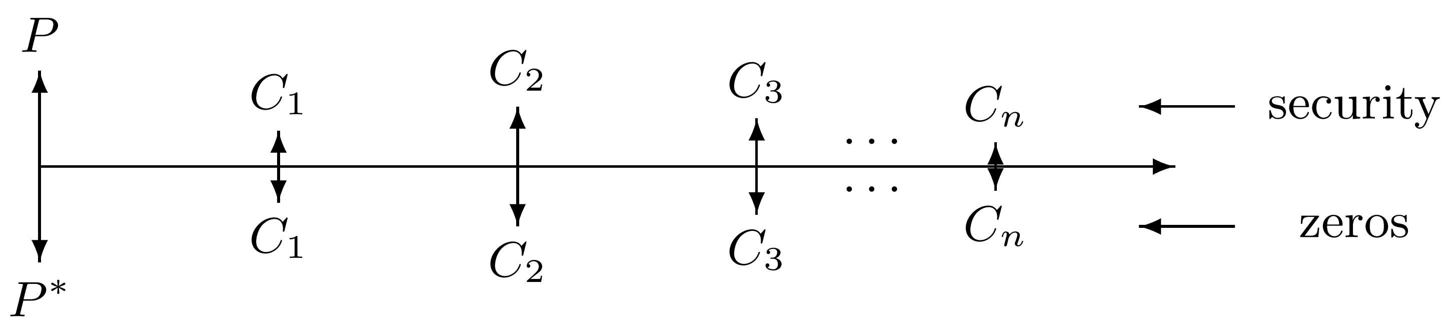
Theorem 1 *For a certain cash flow* $C_1, C_2, \ldots, C_n$,

$$P = \sum_{i=1}^{n} C_i d(i).$$

- Suppose the price $P^* < P$.
- Short[a] the n zeros that match the security's n cash flows.
- The proceeds are P dollars.

---

[a] A key assumption.


P
P^*
C_1
C_1
C_2
C_2
C_3
C_3
$\cdots$
C_n
C_n
security
zeros

The Proof (concluded)

- Then use P^* of the proceeds to buy the security.
- The cash inflows of the security will offset exactly the obligations of the zeros.
- A riskless profit of $P - P^*$ dollars has been realized now.
- If $P^* > P$, just reverse the trades.

One More Example

Theorem 2 *A put or a call must have a nonnegative value.*

- Suppose otherwise and the option has a negative price.
- Buy the option for a positive cash flow now.
- It will end up with a nonnegative amount at expiration.
- So an arbitrage profit is realized now.

Relative Option Prices

- These relations hold regardless of the model for stock prices.
- Assume, among other things, that there are no transactions costs[a] or margin requirements, borrowing and lending are available at the riskless interest rate, interest rates are *nonnegative*, and there are no arbitrage opportunities.

---

[a]Schwab cut the fees of online trades of stocks and ETFs to zero on October 7, 2019.

Relative Option Prices (concluded)

- Let the current time be time zero.
- PV(x) stands for the PV of x dollars at expiration.
- Hence
$$\text{PV}(x) = xd(\tau),$$
where τ is the time to expiration.

Put-Call Parity[a]

$$C = P + S - \mathrm{PV}(X). \tag{31}$$

- Consider the portfolio of:
  - One short European call;
  - One long European put;
  - One share of stock;
  - A loan of $\mathrm{PV}(X)$.
- All options are assumed to carry the same strike price X and time to expiration, τ.
- The initial cash flow is therefore

$$C - P - S + \mathrm{PV}(X).$$

[a]Castelli (1877).

The Proof (continued)

- At expiration, if the stock price $S_\tau \leq X$, the put will be worth $X - S_\tau$ and the call will expire worthless.
- The loan is now X.
- The net future cash flow is zero:

$$0 + (X - S_\tau) + S_\tau - X = 0.$$

- On the other hand, if $S_\tau > X$, the call will be worth $S_\tau - X$ and the put will expire worthless.
- The net future cash flow is again zero:

$$-(S_\tau - X) + 0 + S_\tau - X = 0.$$

The Proof (concluded)

- The net future cash flow is zero in either case.
- The no-arbitrage principle[a] implies that the initial investment to set up the portfolio must be nil as well.

---

[a] Recall p. 222.

Consequences of Put-Call Parity

- There is only one kind of European option.
  - The other can be replicated from it in combination with stock and riskless lending or borrowing.
  - Combinations such as this create synthetic securities.[a]
- $S = C - P + \text{PV}(X)$: A stock is equivalent to a portfolio containing a long call, a short put, and lending $\text{PV}(X)$.
- $C - P = S - \text{PV}(X)$: A long call and a short put amount to a long position in stock and borrowing the PV of the strike price (buying stock on margin).

---

[a]Like the synthetic bonds on p. 149.

Intrinsic Value

Lemma 3 *An American call or a European call on a non-dividend-paying stock is never worth less than its intrinsic value.*

- An American call cannot be worth less than its intrinsic value.[a]
- For European options, the put-call parity implies

$$C = (S - X) + (X - \mathrm{PV}(X)) + P \geq S - X.$$

- Recall $C \geq 0$ (p. 226).
- It follows that $C \geq \max(S - X, 0)$, the intrinsic value.

[a]See Lemma 8.3.1 of the textbook.

Intrinsic Value (concluded)

A European *put* on a non-dividend-paying stock may be worth less than its intrinsic value $X - S$.

Lemma 4 *For European puts,* $P \geq \max(\mathrm{PV}(X) - S, 0)$.

- Prove it with the put-call parity.[a]
- Can explain the right figure on p. 196 why $P < X - S$ when S is small.

[a]See Lemma 8.3.2 of the textbook.

Early Exercise of American Calls

European calls and American calls are identical when the underlying stock pays no dividends!

Theorem 5 (Merton, 1973) *An American call on a non-dividend-paying stock should not be exercised before expiration.*

- By Exercise 8.3.2 of the text, $C \geq \max(S - \text{PV}(X), 0)$.
- If the call is exercised, the value is $S - X$.
- But
$$\max(S - \text{PV}(X), 0) \geq S - X.$$

Remarks

- The above theorem does *not* mean American calls should be kept until maturity.
- What it does imply is that when early exercise is being considered, a *better* alternative is to sell it.
- Early exercise may become optimal for American calls on a dividend-paying stock, however.
  - Options are assumed to be unprotected.
  - Stock price declines as the stock goes ex-dividend.

Early Exercise of American Calls: Dividend Case

Surprisingly, an American call should be exercised only at a few dates.[a]

Theorem 6 (Merton, 1973) *An American call will only be exercised at expiration or just before an ex-dividend date.*

In contrast, it might be optimal to exercise an American put even if the underlying stock does not pay dividends.

[a]See Theorem 8.4.2 of the textbook.

A General Result[a]

Theorem 7 (Cox & Rubinstein, 1985) *Any piecewise linear payoff function can be replicated using a portfolio of calls and puts.*

Corollary 8 *Any sufficiently well-behaved payoff function can be approximated by a portfolio of calls and puts.*

Theorem 9 (Bakshi & Madan, 2000) *Any payoff function with bounded expectation can be replicated by a continuum of out-of-the-money European calls and puts.*

---

[a]See Exercise 8.3.6 of the textbook.

Option Pricing Models

Black insisted that anything one could do
with a mouse could be done better
with macro redefinitions
of particular keys on the keyboard.
— Emanuel Derman (2004),
My Life as a Quant

So we would bring in smart folks.
They didn't know anything about finance.[a]
James Simons (2015, May 13, 33:27)

[a] `https://www.youtube.com/watch?v=QNznD9hMEh0`

The Setting

- The no-arbitrage principle is insufficient to pin down the exact option value.
- Need a model of probabilistic behavior of stock prices.
- An obstacle is that it seems a risk-adjusted interest rate is needed to discount the option's expected payoff.[a]
- Breakthrough came in 1973 when Black (1938–1995) and Scholes with help from Merton published their celebrated option pricing model.[b]
  - Known as the Black-Scholes option pricing model.

---

[a]Like Eq. (30) on p. 183.

[b]The results were obtained as early as June 1969. Merton and Scholes were winners of the 1997 Nobel Prize in Economic Sciences.

Fischer Black (1938–1995)

Myron Scholes (1941–)

Robert C. Merton (1944–)

Terms and Approach

- C: call value.
- P: put value.
- X: strike price
- S: stock price
- $\hat{r} > 0$: the continuously compounded riskless rate per period.
- $R \stackrel{\Delta}{=} e^{\hat{r}}$: gross return.
- Start from the discrete-time binomial model.

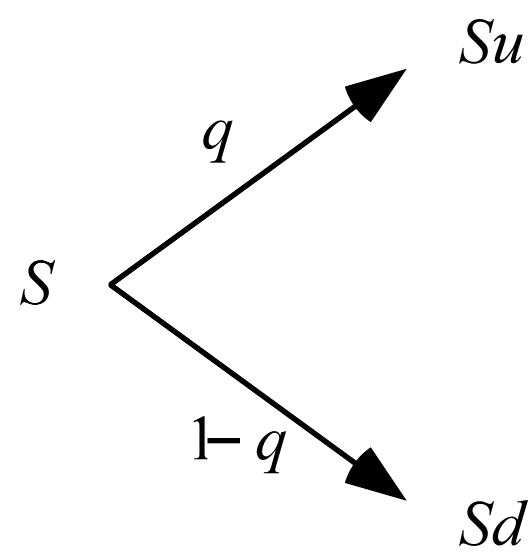
Binomial Option Pricing Model (BOPM)

- Time is discrete and measured in periods.
- If the current stock price is S, it can go to Su with probability q and Sd with probability $1 - q$, where $0 < q < 1$ and $d < u$.
  - In fact, $d \leq R \leq u$ must hold to rule out arbitrage.[a]
- Six pieces of information will suffice to determine the option value based on arbitrage considerations:

  S, u, d, X, $\hat{r}$, and the number of periods to expiration.

---

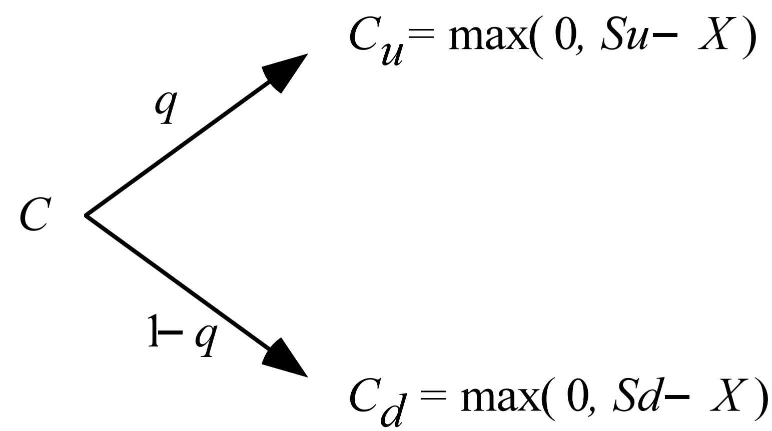
[a]See Exercise 9.2.1 of the textbook. The sufficient condition is $d < R < u$ (Björk, 2009), which we shall assume.


Su
q
S
1 − q
Sd

Call on a Non-Dividend-Paying Stock: Single Period

- The expiration date is only one period from now.
- C_u is the call price at time 1 if the stock price moves to Su.
- C_d is the call price at time 1 if the stock price moves to Sd.
- Clearly,

$$\begin{aligned} C_u &= \max(0, Su - X), \\ C_d &= \max(0, Sd - X). \end{aligned}$$


$C_u = \max(0, Su - X)$
q
C
$1 - q$
$C_d = \max(0, Sd - X)$

Call on a Non-Dividend-Paying Stock: Single Period (continued)

- Set up a portfolio of h shares of stock and B dollars in riskless bonds.
  - This costs $hS + B$.
  - We call h the hedge ratio or delta.
- The value of this portfolio at time one is

$$\begin{aligned} &hSu + RB, &\quad \text{up move,} \\ &hSd + RB, &\quad \text{down move.} \end{aligned}$$

Call on a Non-Dividend-Paying Stock: Single Period (continued)

- Choose h and B such that the portfolio *replicates* the payoff of the call,

$$\begin{aligned} hSu + RB &= C_u, \\ hSd + RB &= C_d. \end{aligned}$$

Call on a Non-Dividend-Paying Stock: Single Period (concluded)

- Solve the above equations to obtain

$$h = \frac{C_u - C_d}{Su - Sd} \geq 0, \tag{32}$$

$$B = \frac{uC_d - dC_u}{(u - d)\, R}. \tag{33}$$

- By the no-arbitrage principle, the European call should cost the same as the equivalent portfolio,[a]

$$C = hS + B.$$

- As $uC_d - dC_u < 0$, the equivalent portfolio is a *levered* long position in stocks.

[a] Or the replicating portfolio, as it replicates the option.

American Call Pricing in One Period

- Have to consider immediate exercise.
- $C = \max(hS + B, S - X)$.
  - When $hS + B \geq S - X$, the call should not be exercised immediately.
  - When $hS + B < S - X$, the option should be exercised immediately.
- For non-dividend-paying stocks, early exercise is not optimal by Theorem 5 (p. 235).
- So

$$C = hS + B.$$

Put Pricing in One Period

- Puts can be similarly priced.
- The delta for the put is $(P_u - P_d)/(Su - Sd) \leq 0$, where

$$\begin{aligned} P_u &= \max(0, X - Su), \\ P_d &= \max(0, X - Sd). \end{aligned}$$

- Let $B = \frac{uP_d - dP_u}{(u-d)\,R}$.
- The European put is worth $hS + B$.
- The American put is worth $\max(hS + B, X - S)$.
  - Early exercise is possible with American puts.

Risk

- Surprisingly, the option value is independent of q.[a]
- Hence it is independent of the expected value of the stock,
$$qSu + (1 - q)\, Sd.$$
- The option value depends on the sizes of price changes, u and d, which the investors must agree upon.
- Then the set of possible stock prices is the same whatever q is.

[a] More precisely, not directly dependent on q. Thanks to a lively class discussion on March 16, 2011.

Pseudo Probability

- After substitution and rearrangement,

$$hS + B = \frac{\left(\frac{R-d}{u-d}\right) C_u + \left(\frac{u-R}{u-d}\right) C_d}{R}.$$

- Rewrite it as

$$hS + B = \frac{pC_u + (1-p)\, C_d}{R},$$

where

$$p \triangleq \frac{R-d}{u-d}. \tag{34}$$

Pseudo Probability (concluded)

- As $0 < p < 1$, it may be interpreted as probability.
- Alternatively,

$$\left(\frac{R-d}{u-d}\right) C_u + \left(\frac{u-R}{u-d}\right) C_d$$

interpolates the value at SR through points (Su, C_u) and (Sd, C_d).

Risk-Neutral Probability

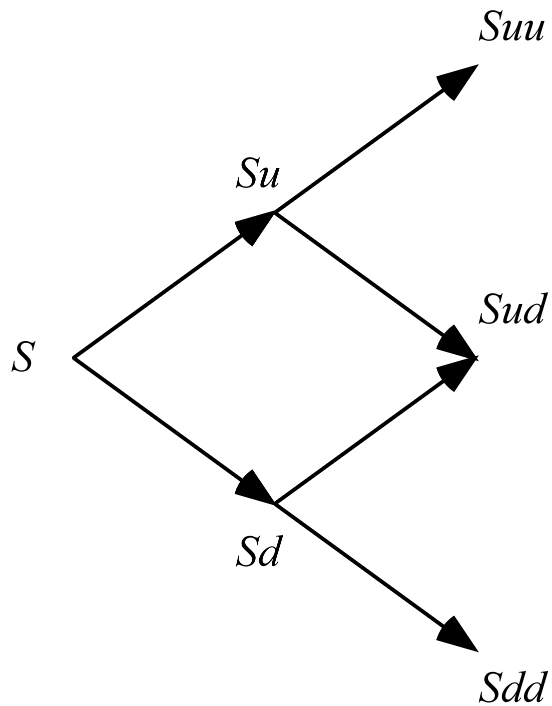
- The expected rate of return for the stock is equal to the riskless rate $\hat{r}$ under p as
$$pSu + (1 - p)\, Sd = RS. \tag{35}$$
- The expected rates of return of all securities must be the riskless rate when investors are risk-neutral.
- For this reason, p is called the risk-neutral probability.
- The value of an option is the expectation of its discounted future payoff in a risk-neutral economy.
- So the rate used for discounting the FV is the riskless rate[a] *in a risk-neutral economy.*

[a] Recall the question on p. 241.

Option on a Non-Dividend-Paying Stock: Multi-Period

- Consider a call with two periods remaining before expiration.
- Under the binomial model, the stock can take on 3 possible prices at time two: Suu, Sud, and Sdd.
  - There are 4 paths.
  - But the tree *combines* or *recombines*; hence there are only 3 terminal prices.
- At any node, the next two stock prices only depend on the current price, not the prices of earlier times.[a]

[a]It is Markovian.


Suu
Su
Sud
S
Sd
Sdd

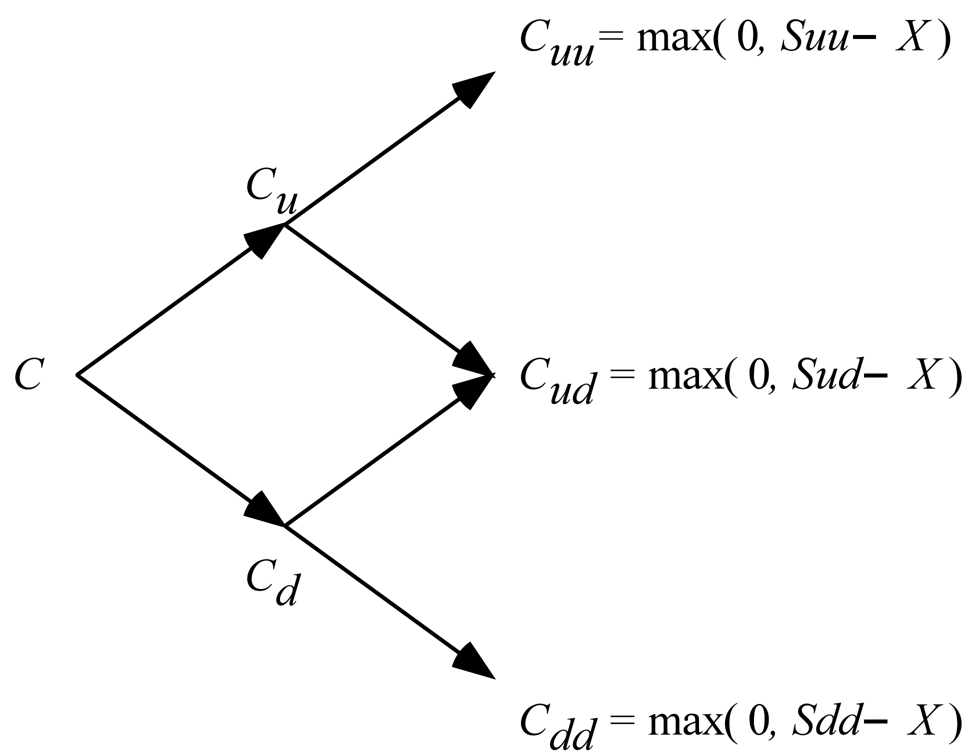
Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- Let C_{uu} be the call's value at time two if the stock price is Suu.
- Thus,

$$C_{uu} = \max(0, Suu - X).$$

- C_{ud} and C_{dd} can be calculated analogously,

$$\begin{aligned} C_{ud} &= \max(0, Sud - X), \\ C_{dd} &= \max(0, Sdd - X). \end{aligned}$$


$C_{uu} = \max(0, Suu - X)$
C_u
C
$C_{ud} = \max(0, Sud - X)$
C_d
$C_{dd} = \max(0, Sdd - X)$

Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- The call values at time 1 can be obtained by applying the same logic:

$$
\begin{aligned}
C_u &= \frac{pC_{uu} + (1-p)\, C_{ud}}{R}, \qquad (36) \\
C_d &= \frac{pC_{ud} + (1-p)\, C_{dd}}{R}.
\end{aligned}
$$

- Deltas can be derived from Eq. (32) on p. 252.
- For example, the delta at C_u is

$$
\frac{C_{uu} - C_{ud}}{Suu - Sud}.
$$

Option on a Non-Dividend-Paying Stock: Multi-Period (concluded)

- We now reach the current period.
- Compute
$$\frac{pC_u + (1-p)\,C_d}{R}$$
as the option price.
- The values of delta h and B can be derived from Eqs. (32)–(33) on p. 252.

Early Exercise

- Since the call will not be exercised at time 1 even if it is American, $C_u \geq Su - X$ and $C_d \geq Sd - X$.
- Therefore,

$$\begin{aligned} hS + B &= \frac{pC_u + (1-p)\, C_d}{R} \geq \frac{[\, pu + (1-p)\, d\,]\, S - X}{R} \\ &= S - \frac{X}{R} > S - X. \end{aligned}$$

  – The call again will not be exercised at present.[a]

- So

$$C = hS + B = \frac{pC_u + (1-p)\, C_d}{R}.$$

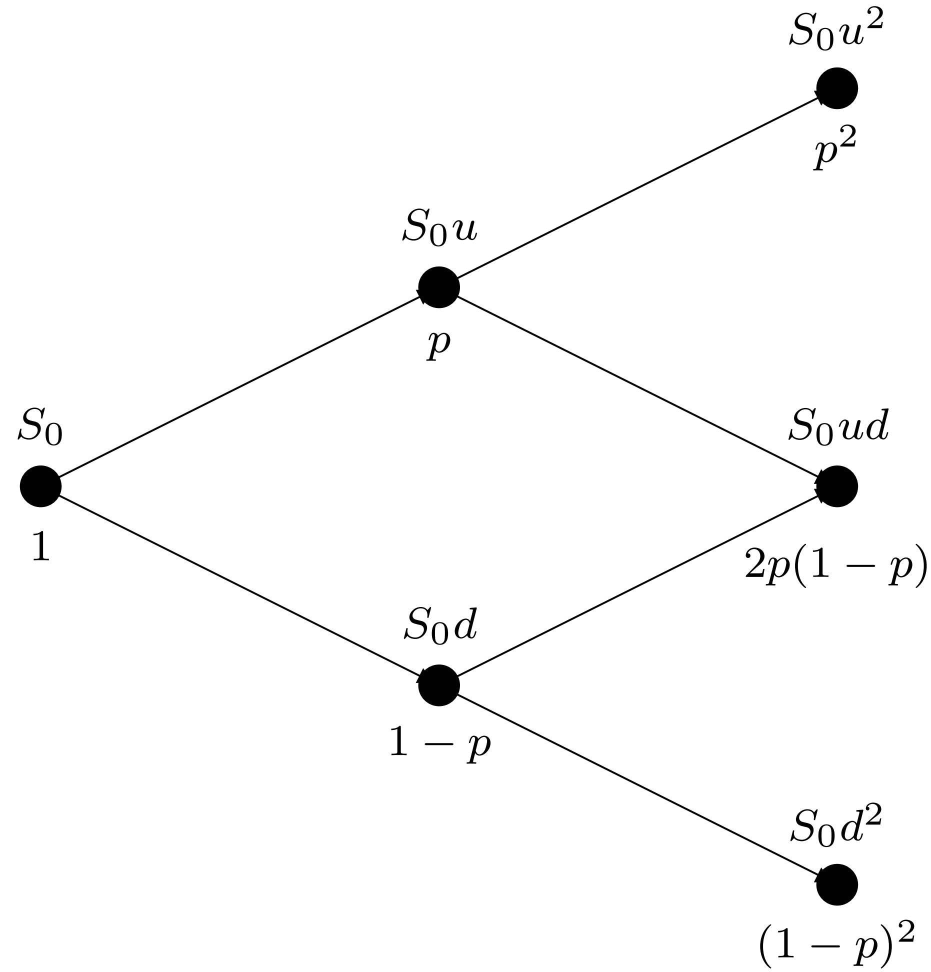
[a]Consistent with Theorem 5 (p. 235).

Backward Induction[a]

- The above expression calculates C from the two successor nodes C_u and C_d and none beyond.
- The same computation happened at C_u and C_d, too, as demonstrated in Eq. (36) on p. 263.
- This recursive procedure is called backward induction.
- C equals

$$
\begin{aligned}
& [\,p^2 C_{uu} + 2p(1-p)\, C_{ud} + (1-p)^2 C_{dd}](1/R^2) \\
= \; & [\,p^2 \max\big(0, Su^2 - X\big) + 2p(1-p) \max\,(0, Sud - X) \\
& +(1-p)^2 \max\big(0, Sd^2 - X\big)\,]/R^2.
\end{aligned}
$$

[a]Ernst Zermelo (1871–1953).


S_0u^2
p^2
S_0u
p
S_0
1
S_0ud
$2p(1-p)$
S_0d
$1-p$
S_0d^2
$(1-p)^2$

Backward Induction (continued)

- In the n-period case,

$$C = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, Su^j d^{n-j} - X\right)}{R^n}.$$

  - The value of a call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.

- Similarly,

$$P = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, X - Su^j d^{n-j}\right)}{R^n}.$$

Backward Induction (concluded)

- Note that
$$p_j \stackrel{\Delta}{=} \frac{\binom{n}{j} p^j (1-p)^{n-j}}{R^n}$$
is the state price[a] for the state $Su^j d^{n-j}$, $j = 0, 1, \ldots, n$.

- In general,
$$\text{option price} = \sum_j (p_j \times \text{payoff at state } j).$$

---

[a]Recall p. 213. One can obtain the *undiscounted* state price $\binom{n}{j} p^j (1-p)^{n-j}$—the risk-neutral probability—for the state $Su^j d^{n-j}$ with $(X_M - X_L)^{-1}$ units of the butterfly spread where $X_L = Su^{j-1} d^{n-j+1}$, $X_M = Su^j d^{n-j}$, and $X_H = Su^{j-1+1} d^{n-j-1}$. See Bahra (1997).

Risk-Neutral Pricing Methodology

- Every derivative can be priced as if the economy were risk-neutral.
- For a European-style derivative with the terminal payoff function $\mathcal{D}$, its value is

$$e^{-\hat{r}n} E^{\pi}[\,\mathcal{D}\,]. \tag{37}$$

  - E^{π} means the expectation is taken under the risk-neutral probability.
- The "equivalence" between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) fundamental theorem of asset pricing.[a]

[a] Dybvig & Ross (1987).

Self-Financing

- Delta changes over time.
- The maintenance of an equivalent portfolio is dynamic.
- But it does *not* depend on predicting future stock prices.
- The portfolio's value at the end of the current period is precisely the amount needed to set up the next portfolio.
- The trading strategy is *self-financing* because there is neither injection nor withdrawal of funds throughout.[a]
  - Changes in value are due entirely to capital gains.

---

[a]Except at the beginning, of course, when the option premium is paid before the replication starts.

Binomial Distribution

- Denote the binomial distribution with parameters n and p by

$$b(j;n,p) \stackrel{\Delta}{=} \binom{n}{j} p^j (1-p)^{n-j} = \frac{n!}{j!\,(n-j)!}\, p^j (1-p)^{n-j}.$$

  – $n! = 1 \times 2 \times \cdots \times n$.

  – Convention: $0! = 1$.

- Suppose you flip a coin n times with p being the probability of getting heads.

- Then $b(j;n,p)$ is the probability of getting j heads.

The Binomial Option Pricing Formula

- The stock prices at time n are

$$Su^n, Su^{n-1}d, \ldots, Sd^n.$$

- Let a be the minimum number of upward price moves for the call to finish in the money.
- So a is the smallest nonnegative integer j such that

$$Su^j d^{n-j} \geq X,$$

or, equivalently,

$$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil.$$

The Binomial Option Pricing Formula (concluded)

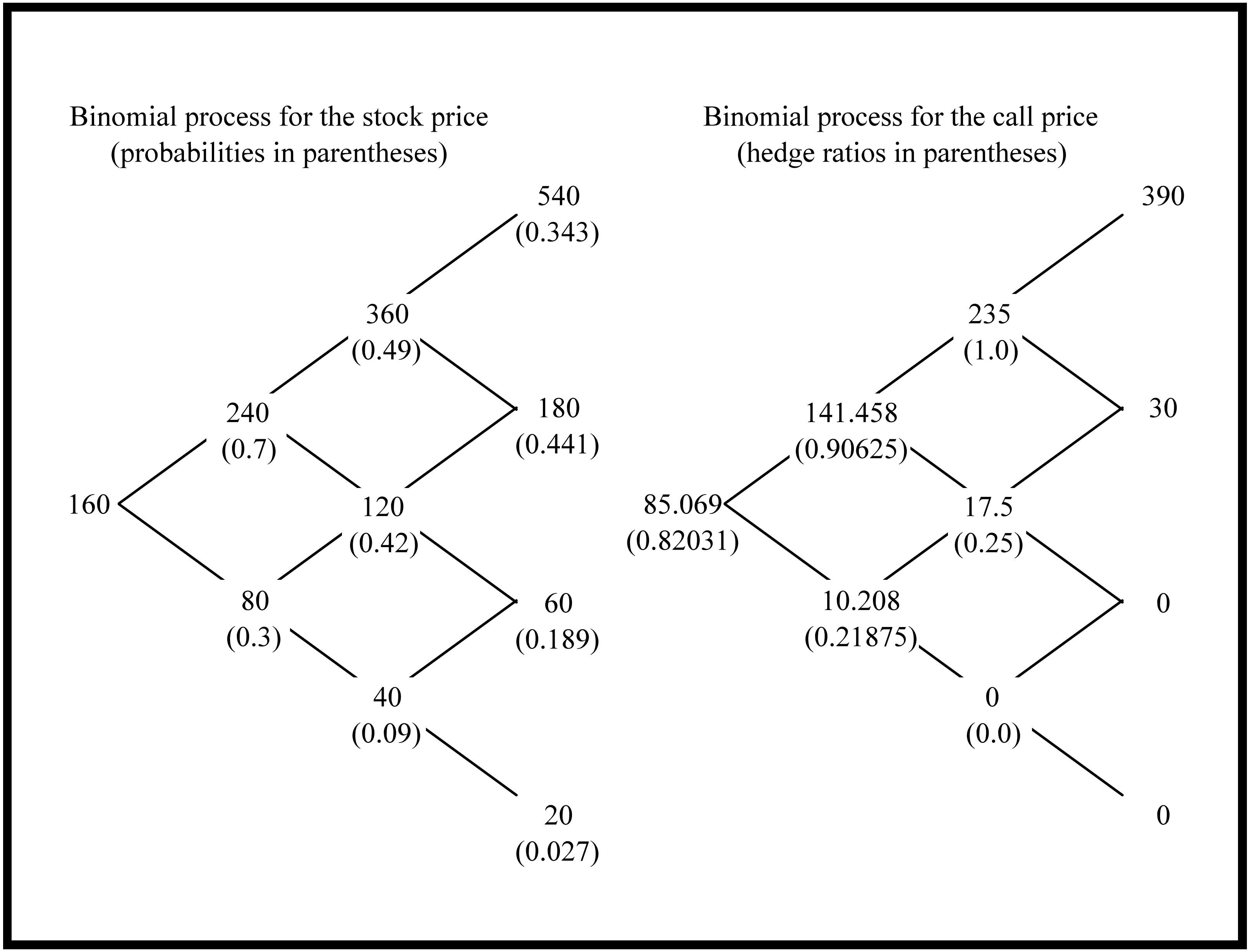
- Hence,

$$
\begin{aligned}
& C \\
= \; & \frac{\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \left(S u^j d^{n-j} - X \right)}{R^n} && (38) \\
= \; & S \sum_{j=a}^{n} \binom{n}{j} \frac{(pu)^j [\,(1-p)\, d\,]^{n-j}}{R^n} \\
& - \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \\
= \; & S \sum_{j=a}^{n} b\,(j; n, pu/R) - X e^{-\hat{r} n} \sum_{j=a}^{n} b(j; n, p). && (39)
\end{aligned}
$$

Numerical Examples

- A non-dividend-paying stock is selling for \$160.
- $u = 1.5$ and $d = 0.5$.
- $r = 18.232\%$ per period ($R = e^{0.18232} = 1.2$).
  - Hence $p = (R - d)/(u - d) = 0.7$.
- Consider a European call on this stock with $X = 150$ and $n = 3$.
- The call value is \$85.069 by backward induction.
- Or, the PV of the expected payoff at expiration:

$$\frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.$$


Binomial process for the stock price
(probabilities in parentheses)
540
(0.343)
360
(0.49)
240
(0.7)
180
(0.441)
160
120
(0.42)
80
(0.3)
60
(0.189)
40
(0.09)
20
(0.027)
Binomial process for the call price
(hedge ratios in parentheses)
390
235
(1.0)
141.458
(0.90625)
30
85.069
(0.82031)
17.5
(0.25)
10.208
(0.21875)
0
0
(0.0)
0

Numerical Examples (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for \$90 instead.
- Sell the call for \$90.
- Invest \$85.069 in the *replicating* portfolio with 0.82031 shares of stock as required by the delta.
- Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.
- The fund that remains,

$$90 - 85.069 = 4.931 \text{ dollars},$$

is the arbitrage profit, as we will see.

Numerical Examples (continued)

Time 1:

- Suppose the stock price moves to \$240.
- The new delta is 0.90625.
- Buy

$$0.90625 - 0.82031 = 0.08594$$

  more shares at the cost of $0.08594 \times 240 = 20.6256$ dollars financed by borrowing.
- Debt now totals $20.6256 + 46.1806 \times 1.2 = 76.04232$ dollars.

Numerical Examples (continued)

- The trading strategy is self-financing because the portfolio has a value of

$$0.90625 \times 240 - 76.04232 = 141.45768.$$

- It matches the corresponding call value by backward induction![a]

[a] See p. 276.

Numerical Examples (continued)

Time 2:

- Suppose the stock price plunges to \$120.
- The new delta is 0.25.
- Sell $0.90625 - 0.25 = 0.65625$ shares.
- This generates an income of $0.65625 \times 120 = 78.75$ dollars.
- Use this income to reduce the debt to

$$76.04232 \times 1.2 - 78.75 = 12.5$$

dollars.

Numerical Examples (continued)

Time 3 (the case of rising price):

- The stock price moves to $180.
- The call we wrote finishes in the money.
- Close out the call's short position by buying back the call or buying a share of stock for delivery.
- This results in a loss of $180 - 150 = 30$ dollars.
- Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.
- It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.

Numerical Examples (concluded)

Time 3 (the case of declining price):

- The stock price moves to \$60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of

$$0.25 \times 60 = 15$$

  dollars.
- Use it to repay the debt of $12.5 \times 1.2 = 15$ dollars.

Applications besides Exploiting Arbitrage Opportunities[a]

- Replicate an option using stocks and bonds.
  - Set up a portfolio to replicate the call with \$85.069.
- Hedge the options we issued.
  - Use \$85.069 to set up a portfolio to replicate the call to counterbalance its values exactly.[b]
- $\cdots$
- Without hedge, one may end up forking out \$390 in the worst case (see p. 276)![c]

---

[a] Thanks to a lively class discussion on March 16, 2011.

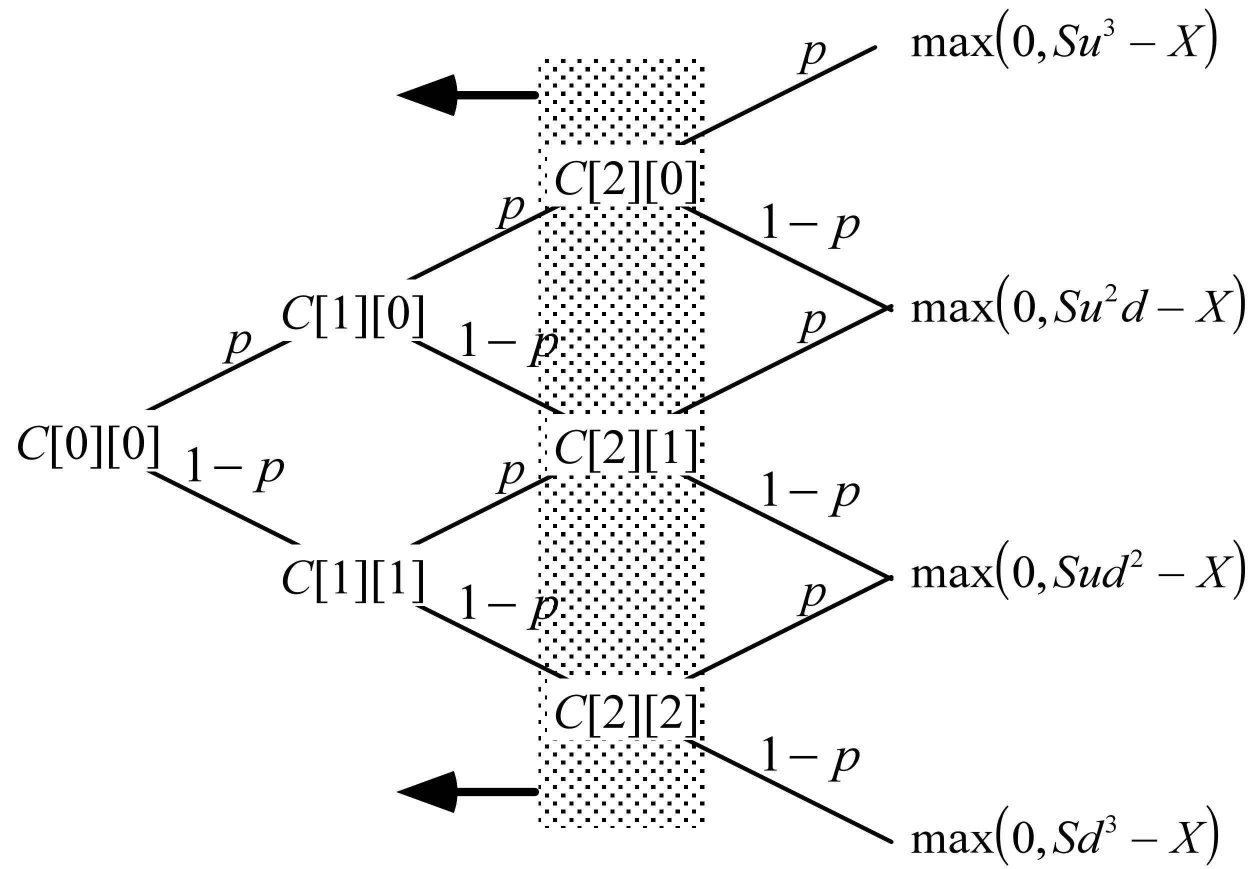
[b] Hedging and replication are mirror images.

[c] Thanks to a lively class discussion on March 16, 2016.

Binomial Tree Algorithms for European Options

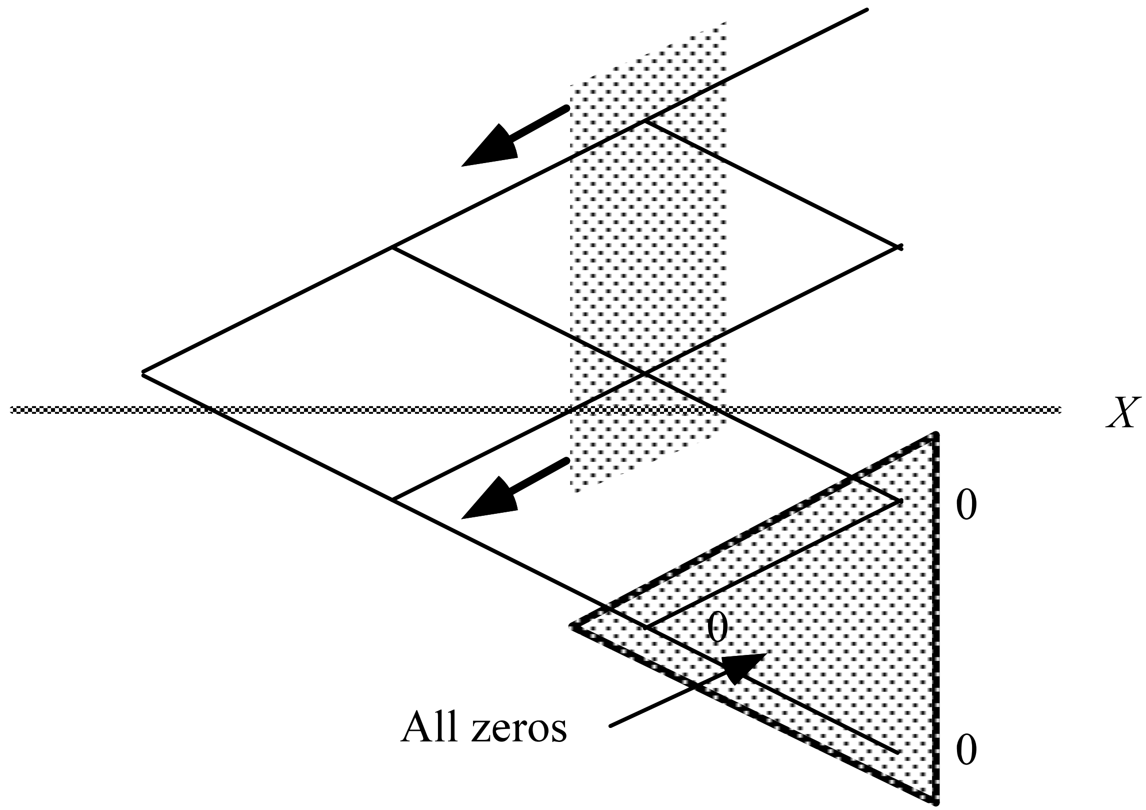
- The BOPM implies the binomial tree algorithm that applies backward induction.
- The total running time is $O(n^2)$ because there are $\sim n^2/2$ nodes.
- The memory requirement is $O(n^2)$.
  - Can be easily reduced to $O(n)$ by reusing space.[a]
- To find the hedge ratio, apply formula (32) on p. 252.
- To price European puts, simply replace the payoff.

[a]But watch out for the proper updating of array entries.



max(0, Su³ − X)
p
C[2][0]
1 − p
p
max(0, Su²d − X)
C[1][0]
1 − p
p
p
C[0][0]
1 − p
C[2][1]
p
1 − p
max(0, Sud² − X)
C[1][1]
1 − p
p
C[2][2]
1 − p
max(0, Sd³ − X)

Further Time Improvement for Calls



Optimal Algorithm

- We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.
- Note that

$$b(j; n, p) = \frac{p(n-j+1)}{(1-p)\,j}\, b(j-1; n, p).$$

Optimal Algorithm (continued)

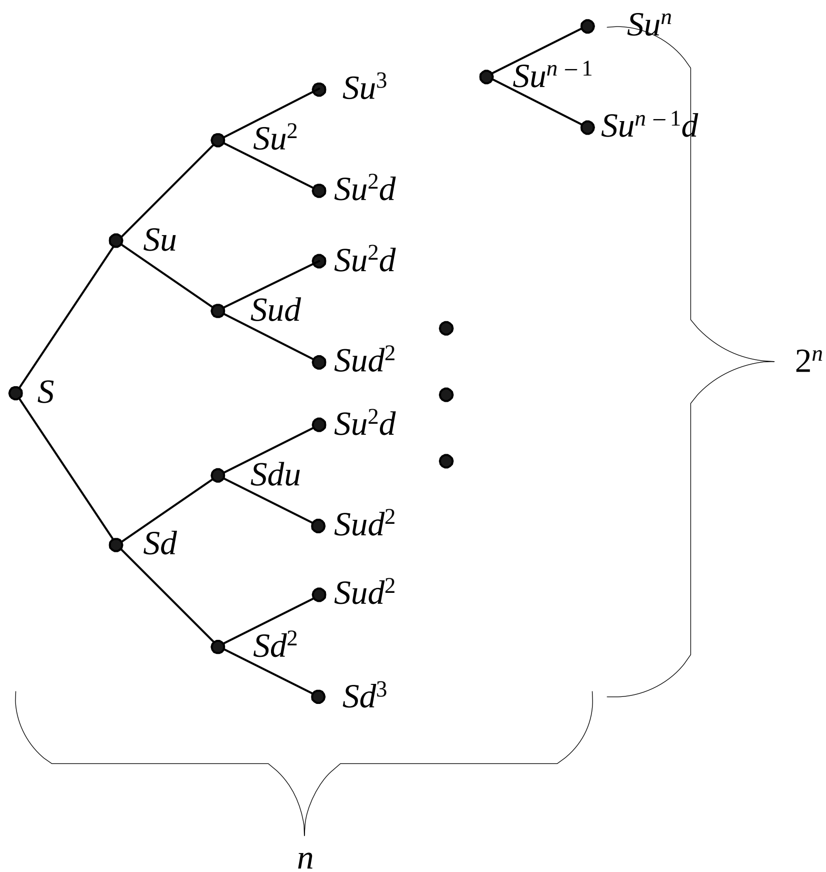
- The following program computes $b(j; n, p)$ in $b[\,j\,]$:
- It runs in $O(n)$ steps.

$b[\,a\,] := \binom{n}{a}\, p^a (1-p)^{n-a};$
for $j = a+1, a+2, \ldots, n$ **do**
    $b[\,j\,] := b[\,j-1\,] \times p \times (n-j+1)/((1-p) \times j);$
end for

Optimal Algorithm (concluded)

- With the $b(j; n, p)$ available, the risk-neutral valuation formula (38) on p. 274 is trivial to compute.
- But we only need a single variable to store the $b(j; n, p)$s as they are being sequentially computed.
- This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.
- This forward-induction approach *cannot* be applied to American options because of early exercise.
- So binomial tree algorithms for American options usually run in $O(n^2)$ time.

The Bushy Tree



Toward the Black-Scholes Formula

- The binomial model seems to suffer from two unrealistic assumptions.
  - The stock price takes on only two values in a period.
  - Trading occurs at discrete points in time.
- As n increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.[a]
- Need to calibrate the BOPM's parameters u, d, and R to make it converge to the continuous-time model.
- We now skim through the proof.

[a]Continuous-time trading may create arbitrage opportunities in practice (Budish, Cramton, & Shim, 2015)!

Toward the Black-Scholes Formula (continued)

- Let τ denote the time to expiration of the option measured in years.
- Let r be the continuously compounded annual rate.
- With n periods during the option's life, each period represents a time interval of τ/n.
- Need to adjust the period-based u, d, and interest rate $\hat{r}$ to match the empirical results as $n \to \infty$.

Toward the Black-Scholes Formula (continued)

- First, $\hat{r} = r\tau/n$.
  - Each period is $\Delta t \triangleq \tau/n$ years long.
  - The period gross return $R = e^{\hat{r}}$.
- Let
$$\widehat{\mu} \triangleq \frac{1}{n} E\left[\ln \frac{S_\tau}{S}\right]$$
denote the expected value of the continuously compounded rate of return per period of the BOPM.
- Let
$$\widehat{\sigma}^2 \triangleq \frac{1}{n} \mathrm{Var}\left[\ln \frac{S_\tau}{S}\right]$$
denote the variance of that return.

Toward the Black-Scholes Formula (continued)

- Under the BOPM, it is not hard to show that[a]

$$\begin{aligned} \widehat{\mu} &= q \ln(u/d) + \ln d, \\ \widehat{\sigma}^2 &= q(1-q) \ln^2(u/d). \end{aligned}$$

- Assume the stock's *true* continuously compounded rate of return over τ years has mean $\mu\tau$ and variance $\sigma^2\tau$.
- Call σ the stock's (annualized) volatility.

[a]It follows the Bernoulli distribution.

Toward the Black-Scholes Formula (continued)

- The BOPM converges to the distribution only if

$$n\widehat{\mu} = n[\,q\ln(u/d) + \ln d\,] \to \mu\tau, \tag{40}$$

$$n\widehat{\sigma}^2 = nq(1-q)\ln^2(u/d) \to \sigma^2\tau. \tag{41}$$

- We need one more condition to have a solution for u, d, q.

Toward the Black-Scholes Formula (continued)

- Impose

$$ud = 1.$$

  - It makes nodes at the same horizontal level of the tree have identical price (review p. 286).
  - Other choices are possible (see text).

- Exact solutions for u, d, q are feasible if Eqs. (40)–(41) are replaced by equations: 3 equations for 3 variables.[a]

[a]Chance (2008).

Toward the Black-Scholes Formula (continued)

- The above requirements can be satisfied by

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad q = \frac{1}{2} + \frac{1}{2}\,\frac{\mu}{\sigma}\sqrt{\Delta t}\,. \tag{42}$$

- With Eqs. (42), it can be checked that

$$\begin{aligned} n\widehat{\mu} &= \mu\tau, \\ n\widehat{\sigma}^2 &= \left[1 - \left(\frac{\mu}{\sigma}\right)^2 \Delta t\right] \sigma^2\tau \to \sigma^2\tau. \end{aligned}$$

- With the above choice, even if σ is not calibrated correctly, the mean is still matched![a]

---

[a]Recall Eq. (35) on p. 258. So u and d are related to volatility exclusively in the CRR model. Both are independent of r and μ.

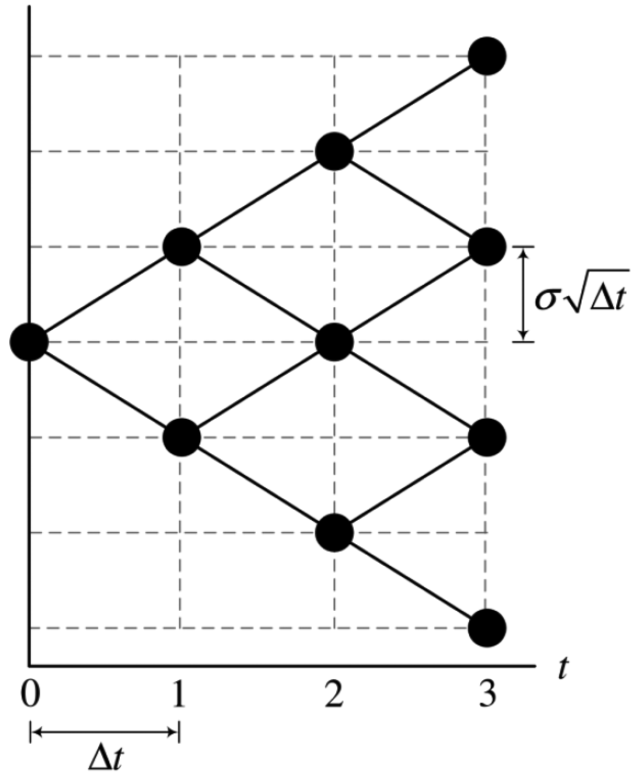
Toward the Black-Scholes Formula (continued)

- The choices (42) result in the CRR binomial model.[a]
  - Black (1992), "This method is probably used more than the original formula in practical situations."
  - OptionMetrics's (2015) IvyDB uses the CRR model.[b]
- The CRR model is best seen in logarithmic price:

$$\ln S \to \begin{cases} \ln S + \sigma\sqrt{\Delta t}, & \text{up move,} \\ \ln S - \sigma\sqrt{\Delta t}, & \text{down move.} \end{cases}$$

[a]Cox, Ross, & Rubinstein (1979).

[b]See http://www.ckgsb.com/uploads/report/file/201611/02/1478069847635278.pd


$\sigma\sqrt{\Delta t}$
t
0
1
2
3
Δt

Toward the Black-Scholes Formula (continued)

- The no-arbitrage inequalities $d < R < u$ may not hold under Eqs. (42) on p. 297 or Eq. (34) on p. 256.
  - If this happens, the probabilities lie outside $[0, 1]$.[a]
- The problem disappears when n satisfies $e^{\sigma\sqrt{\Delta t}} > e^{r\Delta t}$, i.e., when

$$n > \frac{r^2}{\sigma^2}\,\tau. \tag{43}$$

  - So it goes away if n is large enough.
  - Other solutions can be found in the textbook[b] or will be presented later.

[a] Many papers and programs forget to check this condition!

[b] See Exercise 9.3.1 of the textbook.

Toward the Black-Scholes Formula (continued)

- The central limit theorem says $\ln(S_\tau/S)$ converges to $N(\mu\tau, \sigma^2\tau)$.[a]
- So $\ln S_\tau$ approaches $N(\mu\tau + \ln S, \sigma^2\tau)$.
- Conclusion: S_τ has a lognormal distribution in the limit.

[a] The normal distribution with mean $\mu\tau$ and variance $\sigma^2\tau$. As our probabilities depend on n, this argument is heuristic.

Toward the Black-Scholes Formula (continued)

Lemma 10 *The continuously compounded rate of return* $\ln(S_\tau/S)$ *approaches the normal distribution with mean* $(r-\sigma^2/2)\,\tau$ *and variance* $\sigma^2\tau$ *in a risk-neutral economy.*

- Let q equal the risk-neutral probability

$$p \stackrel{\Delta}{=} (e^{r\tau/n} - d)/(u - d).$$

- Let $n \to \infty$.
- Then $\mu = r - \sigma^2/2$.[a]

---

[a] See Lemma 9.3.3 of the textbook. Now, $p = \frac{1}{2} + \frac{\mu}{2\sigma}\,\Delta t^{0.5} + \frac{\sigma^4+4\sigma^2\mu+6\mu^2}{24\sigma}\,\Delta t^{1.5} + O\left(\Delta t^{2.5}\right)$, consistent with Eq. (42) on p. 297.

Toward the Black-Scholes Formula (continued)

- The expected stock price at expiration in a risk-neutral economy is[a]

$$Se^{r\tau}.$$

- The stock's expected annual rate of return is thus the riskless rate r.
  - By rate of return we mean $(1/\tau) \ln E[\, S_\tau / S \,]$ (arithmetic average rate of return) not $(1/\tau) E[\, \ln(S_\tau / S) \,]$ (geometric average rate of return).
  - The latter would give $r - \sigma^2/2$ by Lemma 10.

---

[a]By Lemma 10 (p. 302) and Eq. (29) on p. 181.

Toward the Black-Scholes Formula (continued)[a]

Theorem 11 (The Black-Scholes Formula, 1973)

$$\begin{aligned} C &= SN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\ P &= Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - SN(-x), \end{aligned}$$

where

$$x \triangleq \frac{\ln(S/X) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

---

[a]On a United flight from San Francisco to Tokyo on March 7, 2010, a real-estate manager mentioned this formula to me!

Toward the Black-Scholes Formula (concluded)

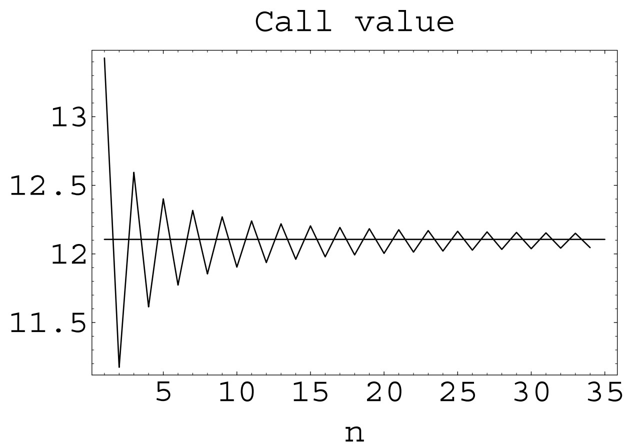
- See Eq. (39) on p. 274 for the meaning of x.
- See Exercise 13.2.12 of the textbook for an interpretation of the probability associated with $N(x)$ and $N(-x)$.

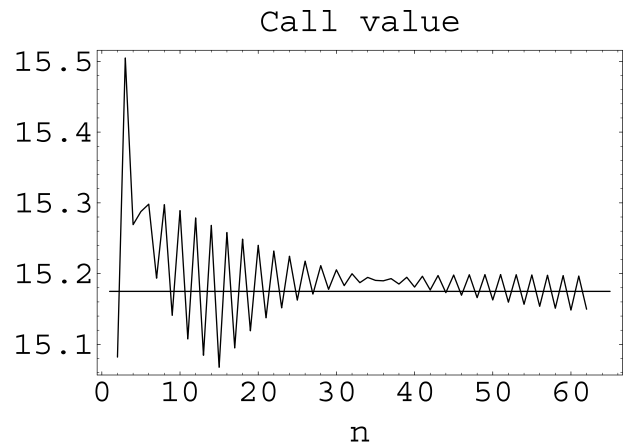
BOPM and Black-Scholes Model

- The Black-Scholes formula needs 5 parameters: S, X, σ, τ, and r.
- Binomial tree algorithms take 6 inputs: S, X, u, d, $\hat{r}$, and n.
- The connections are

$$
\begin{aligned}
u &= e^{\sigma\sqrt{\tau/n}}, \\
d &= e^{-\sigma\sqrt{\tau/n}}, \\
\hat{r} &= r\tau/n.
\end{aligned}
$$

  – This holds for the CRR model as well.





- $S = 100$, $X = 100$ (left), and $X = 95$ (right).

BOPM and Black-Scholes Model (concluded)

- The binomial tree algorithms converge reasonably fast.
- The error is $O(1/n)$.[a]
- Oscillations are inherent, however.
- Oscillations can be dealt with by judicious choices of u and d.[b]

---

[a] F. Diener & M. Diener (2004); L. Chang & Palmer (2007).
[b] See Exercise 9.3.8 of the textbook.

Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market's opinion of the volatility.[a]
  - Solve for σ given the option price, S, X, τ, and r with numerical methods.
  - How about American options?

---

[a]Implied volatility is hard to compute when τ is small (why?).

Implied Volatility (concluded)

- Implied volatility is

  the wrong number to put in the wrong formula to get the right price of plain-vanilla options.[a]

- Think of it as an alternative to quoting option prices.
- Implied volatility is often preferred to historical (statistical) volatility in practice.
  - Using the historical volatility is like driving a car with your eyes on the rearview mirror?[b]
- Volatility is meaningful only if seen through a model![c]

---

[a]Rebonato (2004).
[b]E.g., 1:16:23 of https://www.youtube.com/watch?v=8TJQhQ2GZ0Y
[c]Alexander (2001).

Problems; the Smile[a]

- Options written on the same underlying asset usually do not produce the same implied volatility.
- A typical pattern is a "smile" in relation to the strike price.
  - The implied volatility is lowest for at-the-money options.
  - It becomes higher the further the option is in- or out-of-the-money.
- This is common for foreign exchange options.

---

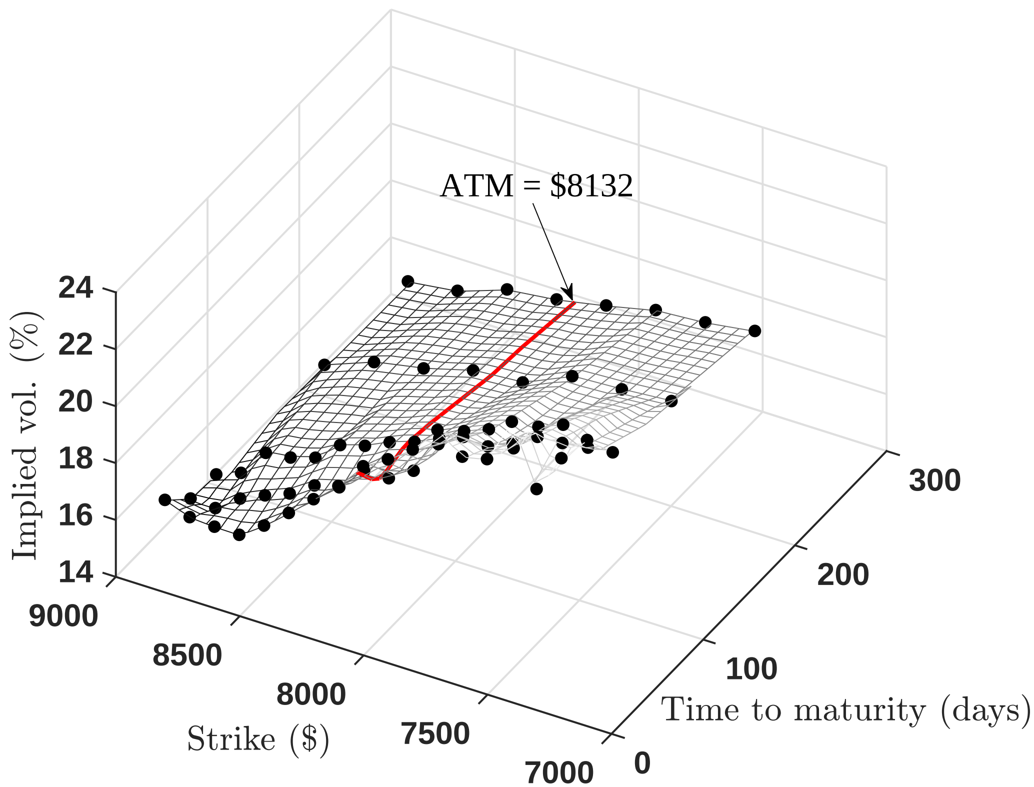
[a] Alexander (2001).

Problems; the Smile (concluded)

- Other patterns have also been observed.
- For stock options, low-strike options tend to have higher implied volatilities.
- One explanation is the high demand for insurance provided by out-of-the-money puts.
- Another reason is volatility rises when stock falls,[a] making in-the-money calls more likely to become in the money again.

[a]This is called the leverage effect (Black, 1992).

TXO Calls (September 25, 2015)[a]



[a]The underlying Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) closed at 8132. Plot supplied by Mr. Lok, U Hou (D99922028) on December 6, 2017.