Day Count Conventions: Actual/Actual

- The first “actual” refers to the actual number of days in a month.

- The second refers to the actual number of days in a coupon period.

- The number of days between June 17, 1992, and October 1, 1992, is 106.
  - 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.
Day Count Conventions: 30/360

• Each month has 30 days and each year 360 days.

• The number of days between June 17, 1992, and October 1, 1992, is 104.
  – 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October.

• In general, the number of days from date \((y_1, m_1, d_1)\) to date \((y_2, m_2, d_2)\) is

\[
360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1). \tag{11}
\]
Day Count Conventions: 30/360 (continued)

• If $d_1$ or $d_2$ is 31, we must change it to 30 before applying formula (11).\(^a\)

• Hence:
  – There are 3 days between February 28 and March 1.
  – There are 2 days between February 29 and March 1.
  – There are 29 days between March 1 and March 31.

\(^a\)The simplest of all the “30/360” variations, this is called the “30E/360” convention, used mainly in the Eurobond market (Kosowski & Neftci, 2015).
Day Count Conventions: 30/360 (concluded)

- An equivalent formula to (11) on p. 80 without any adjustment is (check it)

\[
360 \times (y_2 - y_1) + 30 \times (m_2 - m_1 - 1) \\
+ \max(30 - d_1, 0) + \min(d_2, 30).
\]

- There are many variations on the “30/360” convention regarding 31, February 28, and February 29.\(^a\)

\(^a\)Kosowski & Neftci (2015).
Full Price (Dirty Price, Invoice Price)

- In reality, the settlement date may fall on any day between two coupon payment dates.
- Let

\[ \omega \triangleq \frac{\text{number of days between the settlement and the next coupon payment date}}{\text{number of days in the coupon period}}. \]  \hspace{1cm} (12)
Full Price (continued)

\[ C(1 - \omega) \]

coupon payment date

(1 − \omega) \hspace{1cm} \omega

coupon payment date

©2022 Prof. Yuh-Dauh Lyuu, National Taiwan University
Full Price (concluded)

• The price is now calculated by

\[
PV = \frac{C}{(1 + \frac{r}{m})^\omega} + \frac{C}{(1 + \frac{r}{m})^{\omega+1}} \cdots \\
= \sum_{i=0}^{n-1} \frac{C}{(1 + \frac{r}{m})^{\omega+i}} + \frac{F}{(1 + \frac{r}{m})^{\omega+n-1}}.
\]  

(13)
Accrued Interest

- The quoted price in the U.S./U.K. does not include the accrued interest; it is called the clean price or flat price.
- The buyer pays the invoice price: the quoted price plus the accrued interest (AI).
- The accrued interest equals

\[
C \times \frac{\text{number of days from the last coupon payment to the settlement date}}{\text{number of days in the coupon period}} = C \times (1 - \omega).
\]
Accrued Interest (concluded)

• The yield to maturity is the $r$ satisfying Eq. (13) on p. 85 when PV is the invoice price:

\[
clean\text{ price} + AI = \sum_{i=0}^{n-1} \frac{C}{(1 + \frac{r}{m})^{\omega+i}} + \frac{F}{(1 + \frac{r}{m})^{\omega+n-1}}.
\]
Example ("30/360")

- A bond with a 10% coupon rate and paying interest semiannually, with clean price 111.2891.
- The settlement date is July 1, 1993, and the maturity date is March 1, 1995.
- There are 60 days between July 1, 1993, and the next coupon date, September 1, 1993.
- The accrued interest is \((10/2) \times (1 - \frac{60}{180}) = 3.3333\) per $100 of par value.
Example ("30/360") (concluded)

- The yield to maturity is 3%.

- This can be verified by Eq. (13) on p. 85 with
  - \( \omega = \frac{60}{180} \),
  - \( n = 4 \),
  - \( m = 2 \),
  - \( F = 100 \),
  - \( C = 5 \),
  - \( PV = 111.2891 + 3.3333 \),
  - \( r = 0.03 \).
Price Behavior (2) Revisited

- Before: A bond selling at par if the yield to maturity equals the coupon rate.

- But it assumed that the settlement date is on a coupon payment date.

- Now suppose the settlement date for a bond selling at par\(^a\) falls between two coupon payment dates.

- Then its yield to maturity is less than the coupon rate.\(^b\)
  - The short reason: Exponential growth to \(C\) is replaced by linear growth, hence overpaying.

\(^a\)The quoted price equals the par value.
\(^b\)See Exercise 3.5.6 of the textbook for proof.
Bond Price Volatility
“Well, Beethoven, what is this?”
— Attributed to Prince Anton Esterházy
Price Volatility

- Volatility measures how bond prices respond to interest rate changes.
- It is key to the risk management of interest rate-sensitive securities.
Price Volatility (concluded)

• What is the sensitivity of the percentage price change to changes in interest rates?

• Define price volatility by

\[-\frac{\partial P}{\partial y} \cdot \frac{1}{P}.\]  \hspace{1cm} (14)
Price Volatility of Bonds

• The price volatility of a level-coupon bond is

\[
- \frac{(C/y)n - (C/y^2)((1 + y)^{n+1} - (1 + y)) - nF}{(C/y)((1 + y)^{n+1} - (1 + y)) + F(1 + y)}.
\]

- \( F \) is the par value.
- \( C \) is the coupon payment per period.
- Formula can be simplified a bit with \( C = Fc/m \).

• For the above bond,

\[
- \frac{\partial P}{\partial y} > 0.
\]
Macaulay Duration\textsuperscript{a}

• The Macaulay duration (MD) is a weighted average of the times to an asset’s cash flows.

• The weights are the cash flows’ PVs divided by the asset’s price.

• Formally,

\[
MD \equiv \frac{1}{P} \sum_{i=1}^{n} \frac{C_i}{(1+y)^i} i.
\]

• The Macaulay duration, in periods, is equal to

\[
MD = -(1+y) \frac{\partial P}{\partial y} \frac{1}{P}. \tag{15}
\]

\textsuperscript{a}Macaulay (1938).
MD of Bonds

- The MD of a level-coupon bond is

\[ MD = \frac{1}{P} \left[ \sum_{i=1}^{n} \frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n} \right]. \] (16)

- It can be simplified to

\[ MD = \frac{c(1+y) \left[ (1+y)^n - 1 \right] + ny(y-c)}{cy \left[ (1+y)^n - 1 \right] + y^2}, \]

where \( c \) is the period coupon rate.

- The MD of a zero-coupon bond equals \( n \), its term to maturity.

- The MD of a level-coupon bond is less than \( n \).
Remarks

• Formulas (15) on p. 96 and (16) on p. 97 hold only if the coupon \( C \), the par value \( F \), and the maturity \( n \) are all independent of the yield \( y \).
  – That is, if the cash flow is independent of yields.

• To see this point, suppose the market yield declines.

• The MD will be lengthened.

• But for securities whose maturity actually decreases as a result, the price volatility\(^a\) may decrease.

\(^a\)As originally defined in formula (14) on p. 94.
How Not To Think about MD

- The MD has its origin in measuring the length of time a bond investment is outstanding.
- But it should be seen mainly as measuring *price volatility*.
- Duration of a security can be longer than its maturity or negative!
- Neither makes sense under the maturity interpretation.
- Many, if not most, duration-related terminology can only be comprehended as measuring volatility.
Conversion

• For the MD to be year-based, modify formula (16) on p. 97 to

\[
\frac{1}{P} \left[ \sum_{i=1}^{n} \frac{i}{k} \frac{C}{(1 + \frac{y}{k})^i} + \frac{n}{k} \frac{F}{(1 + \frac{y}{k})^n} \right],
\]

where \( y \) is the annual yield and \( k \) is the compounding frequency per annum.

• Formula (15) on p. 96 also becomes

\[
\text{MD} = - \left( 1 + \frac{y}{k} \right) \frac{\partial P}{\partial y} \frac{1}{P}.
\]

• By definition, \( \text{MD} \) (in years) = \( \frac{\text{MD (in periods)}}{k} \).
Modified Duration

- Modified duration is defined as

\[
\text{modified duration} \triangleq -\frac{\partial P}{\partial y} \frac{1}{P} = \frac{\text{MD}}{(1 + y)}. \tag{17}
\]

- Modified duration equals MD under continuous compounding.

- By the Taylor expansion,

percent price change \( \approx -\) modified duration \( \times \) yield change.
Example

- Consider a bond whose modified duration is 11.54 with a yield of 10%.

- If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be
  \[-11.54 \times 0.001 = -0.01154 = -1.154\%\].
Modified Duration of a Portfolio

- By calculus, the modified duration of a portfolio equals

\[ \sum_i \omega_i D_i. \]

- \( D_i \) is the modified duration of the \( i \)th asset.
- \( \omega_i \) is the market value of that asset expressed as a percentage of the market value of the portfolio.
Effective Duration

- Yield changes may alter the cash flow or the cash flow may be too complex for simple formulas.
- We need a general numerical formula for volatility.
- The effective duration is defined as

\[ \frac{P_- - P_+}{P_0(y_+ - y_-)}. \]

- \( P_- \) is the price if the yield is decreased by \( \Delta y \).
- \( P_+ \) is the price if the yield is increased by \( \Delta y \).
- \( P_0 \) is the initial price, \( y \) is the initial yield.
- \( \Delta y \) is small.
Effective Duration (concluded)

• One can compute the effective duration of just about any financial instrument.

• An alternative is to use

\[
\frac{P_0 - P_+}{P_0 \Delta y}.
\]

– More economical but theoretically less accurate.
The Practices

• Duration is usually expressed in percentage terms — call it $D\%$ — for quick mental calculation.\textsuperscript{a}

• The percentage price change expressed in percentage terms is then approximated by

$$-D\% \times \Delta r$$

when the yield increases instantaneously by $\Delta r\%$.

- Price will drop by 20% if $D\% = 10$ and $\Delta r = 2$ because $10 \times 2 = 20$.

• $D\%$ in fact equals modified duration (prove it!).

\textsuperscript{a}Neftci (2008), “Market professionals do not like to use decimal points.”
Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.

- Define dollar duration as

\[ \text{modified duration} \times \text{price} = - \frac{\partial P}{\partial y}. \]

- The approximate dollar price change is

\[ \text{price change} \approx -\text{dollar duration} \times \text{yield change}. \]

- One can hedge a bond portfolio with a dollar duration \( D \) by bonds with a dollar duration \( -D \).
Convexity

• Convexity is defined as

\[ \text{convexity (in periods)} \triangleq \frac{\partial^2 P}{\partial y^2} \frac{1}{P} \]

• The convexity of a level-coupon bond is positive (prove it!).

• For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude (see plot next page).

• So between two bonds with the same price and duration, the one with a higher convexity is more valuable.\(^a\)

\(^a\)Do you spot a problem here (Christensen & Sørensen, 1994)?
Convexity (concluded)

• Suppose there are $k$ periods per annum.

• Convexity measured in periods and convexity measured in years are related by

$$\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}.$$
Use of Convexity

- The approximation $\Delta P/P \approx - \text{duration} \times \text{yield change}$ works for small yield changes.

- For larger yield changes, use

$$\frac{\Delta P}{P} \approx \frac{\partial P}{\partial y} \frac{1}{P} \Delta y + \frac{1}{2} \frac{\partial^2 P}{\partial y^2} \frac{1}{P} (\Delta y)^2$$

$$= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.$$ 

- Recall the figure on p. 110.
The Practices

• Convexity is usually expressed in percentage terms — call it $C\%$ — for quick mental calculation.

• The percentage price change expressed in percentage terms is approximated by

$$-D\% \times \Delta r + C\% \times (\Delta r)^2 / 2$$

when the yield increases instantaneously by $\Delta r\%$.

– Price will drop by 17% if $D\% = 10$, $C\% = 1.5$, and $\Delta r = 2$ because

$$-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.$$  

• $C\%$ equals convexity divided by 100 (prove it!).
Effective Convexity

• The effective convexity is defined as

\[
\frac{P_+ + P_- - 2P_0}{P_0 \left(0.5 \times (y_+ - y_-)\right)^2},
\]

- \(P_-\) is the price if the yield is decreased by \(\Delta y\).
- \(P_+\) is the price if the yield is increased by \(\Delta y\).
- \(P_0\) is the initial price, \(y\) is the initial yield.
- \(\Delta y\) is small.

• Effective convexity is most relevant when a bond’s cash flow is interest rate sensitive.

• How to choose the right \(\Delta y\) is a delicate matter.
Approximate $d^2 f(x)^2 / dx^2$ at $x = 1$, Where $f(x) = x^2$

- The difference of $[(1 + \Delta x)^2 + (1 - \Delta x)^2 - 2]/(\Delta x)^2$ and 2:

- This numerical issue is common in financial engineering but does not admit general solutions yet (see pp. 875ff).
Interest Rates and Bond Prices: Which Determines Which?\(^a\)

- If you have one, you have the other.
- So they are just two names given to the same thing: cost of fund.
- Traders most likely work with prices.
- Banks most likely work with interest rates.

\(^a\)Contributed by Mr. Wang, Cheng (R01741064) on March 5, 2014.
Term Structure of Interest Rates
Why is it that the interest of money is lower, when money is plentiful?
— Samuel Johnson (1709–1784)

If you have money, don’t lend it at interest.
Rather, give [it] to someone from whom you won’t get it back.
— Thomas Gospel 95
Term Structure of Interest Rates

• Concerned with how interest rates change with maturity.

• The set of yields to maturity for bonds form the term structure.
  – The bonds must be of equal quality.
  – They differ solely in their terms to maturity.

• The term structure is fundamental to the valuation of fixed-income securities.
Term Structure of Interest Rates (concluded)

- The term “term structure” often refers exclusively to the yields of zero-coupon bonds.

- A yield curve plots the yields to maturity of coupon bonds against maturity.

- A par yield curve is constructed from bonds trading near par.
Yield Curve of U.S. Treasuries as of July 24, 2015

Yield (%)
Four Typical Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.
Spot Rates

- The $i$-period spot rate $S(i)$ is the yield to maturity of an $i$-period zero-coupon bond.

- The PV of one dollar $i$ periods from now is by definition

$$[1 + S(i)]^{-i}.$$ 

  - It is the price of an $i$-period zero-coupon bond.$^a$

- The one-period spot rate is called the short rate.

- Spot rate curve.$^b$ Plot of spot rates against maturity:

$$S(1), S(2), \ldots, S(n).$$

---

$^a$Recall Eq. (9) on p. 69.

$^b$That is, term structure.
Problems with the PV Formula

• In the bond price formula (4) on p. 41,

\[
\sum_{i=1}^{n} \frac{C}{(1 + y)^i} + \frac{F}{(1 + y)^n},
\]

every cash flow is discounted at the same yield \( y \).

• Consider two riskless bonds with different yields to maturity because of their different cash flows:

\[
PV_1 = \sum_{i=1}^{n_1} \frac{C}{(1 + y_1)^i} + \frac{F}{(1 + y_1)^{n_1}},
\]

\[
PV_2 = \sum_{i=1}^{n_2} \frac{C}{(1 + y_2)^i} + \frac{F}{(1 + y_2)^{n_2}}.
\]
Problems with the PV Formula (concluded)

- The yield-to-maturity methodology discounts their contemporaneous cash flows with different rates.
- But shouldn’t they be discounted at the same rate?
Spot Rate Discount Methodology

- A cash flow $C_1, C_2, \ldots, C_n$ is equivalent to a package of zero-coupon bonds with the $i$th bond paying $C_i$ dollars at time $i$. 

\[ \begin{align*} &C_1 \quad C_2 \quad C_3 \quad \cdots \quad C_n \\ &1 \quad 2 \quad 3 \quad \cdots \quad n \end{align*} \]
Spot Rate Discount Methodology (concluded)

• So a level-coupon bond has the price

\[
P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}.
\]  

(18)

• This pricing method incorporates information from the term structure.

• It discounts each cash flow at the matching spot rate.
Discount Factors

- In general, any riskless security having a cash flow $C_1, C_2, \ldots, C_n$ should have a market price of

$$P = \sum_{i=1}^{n} C_i d(i).$$

- Above, $d(i) \equiv [1 + S(i)]^{-i}$, $i = 1, 2, \ldots, n$, are called the discount factors.
- $d(i)$ is the PV of one dollar $i$ periods from now.
- The above formula will be justified on p. 222.

- The discount factors are often interpolated to form a continuous function called the discount function.
Extracting Spot Rates from Yield Curve

• Start with the short rate $S(1)$.
  – Note that short-term Treasuries are zero-coupon bonds.

• Compute $S(2)$ from the two-period coupon bond price $P$ by solving

$$P = \frac{C}{1 + S(1)} + \frac{C + 100}{[1 + S(2)]^2}.$$
Extracting Spot Rates from Yield Curve (concluded)

• Inductively, we are given the market price $P$ of the $n$-period coupon bond and

$$S(1), S(2), \ldots, S(n-1).$$

• Then $S(n)$ can be computed from Eq. (18) on p. 127, repeated below,

$$P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}.$$ 

• The running time can be made to be $O(n)$ (see text).

• The procedure is called bootstrapping.
Some Problems

- Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).
- Some maturities might be missing from the data points (the incompleteness problem).
- Treasuries might not be of the same quality.
- Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.\(^a\)

\(^a\)Often without economic justifications.
Yield Spread

• Consider a risky bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for $P$.

• Calculate the IRR of the risky bond.

• Calculate the IRR of a riskless bond with comparable maturity.

• Yield spread is their difference.
Static Spread

- Were the risky bond riskless, it would fetch

\[ P^* = \sum_{t=1}^{n} \frac{C_t}{1 + S(t)}^t. \]

- But as risk must be compensated, in reality \( P < P^* \).

- The static spread is the amount \( s \) by which the spot rate curve has to shift \textit{in parallel} to price the risky bond:

\[ P = \sum_{t=1}^{n} \frac{C_t}{1 + s + S(t)}^t. \]

- Unlike the yield spread, the static spread explicitly incorporates information from the term structure.
Of Spot Rate Curve and Yield Curve

• $y_k$: yield to maturity for the $k$-period coupon bond.

• $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).

• $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).

• $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).

• $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).

• If the yield curve is flat, the spot rate curve coincides with the yield curve.
Shapes

• The spot rate curve often has the same shape as the yield curve.
  – If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).

• But this is only a trend not a mathematical truth.\textsuperscript{a}

\textsuperscript{a}See a counterexample in the text.
Forward Rates

- The yield curve contains information regarding future interest rates currently “expected” by the market.

- Invest $1 for $j$ periods to end up with $[1 + S(j)]^j$ dollars at time $j$.
  - The maturity strategy.

- Invest $1$ in bonds for $i$ periods and at time $i$ invest the proceeds in bonds for another $j - i$ periods where $j > i$.

- Will have $[1 + S(i)]^i[1 + S(i, j)]^{j-i}$ dollars at time $j$.
  - $S(i, j)$: $(j - i)$-period spot rate $i$ periods from now.
  - The rollover strategy.
Forward Rates (concluded)

- When $S(i, j)$ equals

$$f(i, j) \triangleq \left[ \frac{(1 + S(j))^j}{(1 + S(i))^i} \right]^{1/(j-i)} - 1,$$

we will end up with $[1 + S(j)]^j$ dollars again.

- As expected,

$$f(0, j) = S(j).$$

- The $f(i, j)$ are the (implied) forward (interest) rates.
  - More precisely, the $(j - i)$-period forward rate $i$ periods from now.
Time Line

\[ f(0, 1) \quad f(1, 2) \quad f(2, 3) \quad f(3, 4) \]

Time 0

\[ S(1) \quad S(2) \quad S(3) \quad S(4) \]
Forward Rates and Future Spot Rates

- We did not assume any a priori relation between $f(i, j)$ and future spot rate $S(i, j)$.
  - This is the subject of the term structure theories.
- We merely looked for the future spot rate that, if realized, will equate the two investment strategies.
- The $f(i, i + 1)$ are the instantaneous forward rates or one-period forward rates.
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,
  \[ f(i, j) > S(j) > \cdots > S(i). \]

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,
  \[ f(i, j) < S(j) < \cdots < S(i). \]
Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curve

- The FV of $1$ at time $n$ can be derived in two ways.
- Buy $n$-period zero-coupon bonds and receive 
  \[
  [1 + S(n)]^n.
  \]
- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is 
  \[
  [1 + S(1)][1 + f(1, 2)] \cdots [1 + f(n - 1, n)].
  \]