The Extended Vasicek Model[a]

- Hull and White proposed models that extend the Vasicek model and the CIR model.
- They are called the extended Vasicek model and the extended CIR model.
- The extended Vasicek model adds time dependence to the original Vasicek model,

$$dr = (\theta(t) - a(t)\, r)\, dt + \sigma(t)\, dW.$$

- Like the Ho-Lee model, this is a normal model.
- The inclusion of $\theta(t)$ allows for an exact fit to the current spot rate curve.

[a] Hull & White (1990).

The Extended Vasicek Model (concluded)

- Function $\sigma(t)$ defines the short rate volatility, and $a(t)$ determines the shape of the volatility structure.
- Many European-style securities can be evaluated analytically.
- Efficient numerical procedures can be developed for American-style securities.

The Hull-White Model

- The Hull-White model is the following special case,

$$dr = (\theta(t) - ar)\, dt + \sigma\, dW. \tag{166}$$

- When the current term structure is matched,[a]

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

  – Recall that $f(0,t)$ defines the forward rate curve.

[a] Hull & White (1993).

The Extended CIR Model

- In the extended CIR model the short rate follows

$$dr = (\theta(t) - a(t)\, r)\, dt + \sigma(t)\sqrt{r}\, dW.$$

- The functions $\theta(t)$, $a(t)$, and $\sigma(t)$ are implied from market observables.
- With constant parameters, there exist analytical solutions to a small set of interest rate-sensitive securities.

The Hull-White Model: Calibration[a]

- We describe a trinomial forward induction scheme to calibrate the Hull-White model given a and σ.
- As with the Ho-Lee model, the set of achievable short rates is evenly spaced.
- Let r_0 be the annualized, continuously compounded short rate at time zero.
- Every short rate on the tree takes on a value

$$r_0 + j\Delta r$$

for some integer j.

[a]Hull & White (1993).

The Hull-White Model: Calibration (continued)

- Time increments on the tree are also equally spaced at Δt apart.
- Hence nodes are located at times $i\Delta t$ for $i = 0, 1, 2, \ldots$.
- We shall refer to the node on the tree with

$$\begin{aligned} t_i &\stackrel{\Delta}{=} i\Delta t, \\ r_j &\stackrel{\Delta}{=} r_0 + j\Delta r, \end{aligned} \tag{167}$$

  as the (i, j) node.
- The short rate at node (i, j), which equals r_j, is effective for the time period $[\,t_i, t_{i+1})$.

The Hull-White Model: Calibration (continued)

- Use

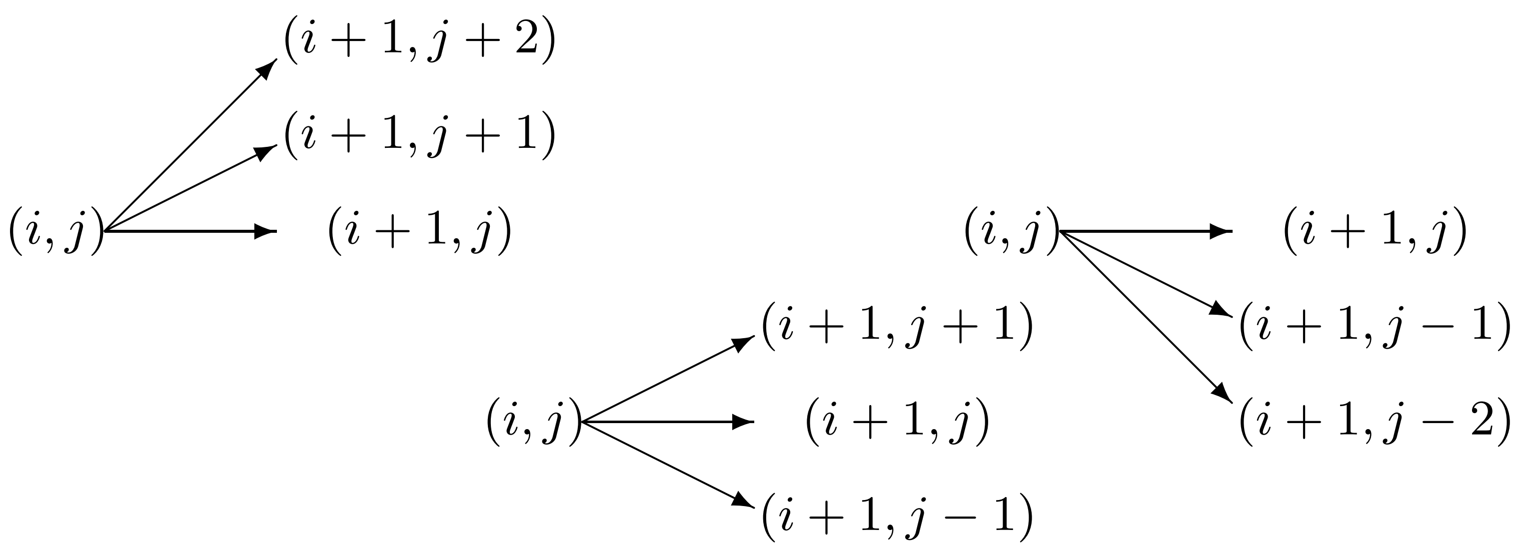
$$\mu_{i,j} \stackrel{\Delta}{=} \theta(t_i) - ar_j \tag{168}$$

to denote the drift rate[a] of the short rate as seen from node (i, j).

- The three distinct possibilities for node (i, j) with three branches incident from it are displayed on p. 1215.
- The middle branch may be an increase of Δr, no change, or a decrease of Δr.

[a]Or, the annualized expected change.

The Hull-White Model: Calibration (continued)



The Hull-White Model: Calibration (continued)

- The upper and the lower branches bracket the middle branch.
- Define

$$
\begin{aligned}
p_1(i,j) &\stackrel{\Delta}{=} \text{the probability of following the upper branch from node } (i,j),\\
p_2(i,j) &\stackrel{\Delta}{=} \text{the probability of following the middle branch from node } (i,j),\\
p_3(i,j) &\stackrel{\Delta}{=} \text{the probability of following the lower branch from node } (i,j).
\end{aligned}
$$

- The root of the tree is set to the current short rate r_0.
- Inductively, the drift $\mu_{i,j}$ at node (i,j) is a function of (the still unknown) $\theta(t_i)$.

The Hull-White Model: Calibration (continued)

- Once $\theta(t_i)$ is available, $\mu_{i,j}$ can be derived via Eq. (168) on p. 1214.
- This in turn determines the branching scheme at every node (i, j) for each j, as we will see shortly.
- The value of $\theta(t_i)$ must thus be made consistent with the spot rate $r(0, t_{i+2})$.[a]

[a]Not $r(0, t_{i+1})$!

The Hull-White Model: Calibration (continued)

- The branches emanating from node (i, j) with their probabilities[a] must be chosen to be consistent with $\mu_{i,j}$ and σ.
- This is done by selecting the middle node to be as closest to the current short rate r_j plus the drift $\mu_{i,j} \Delta t$.[b]

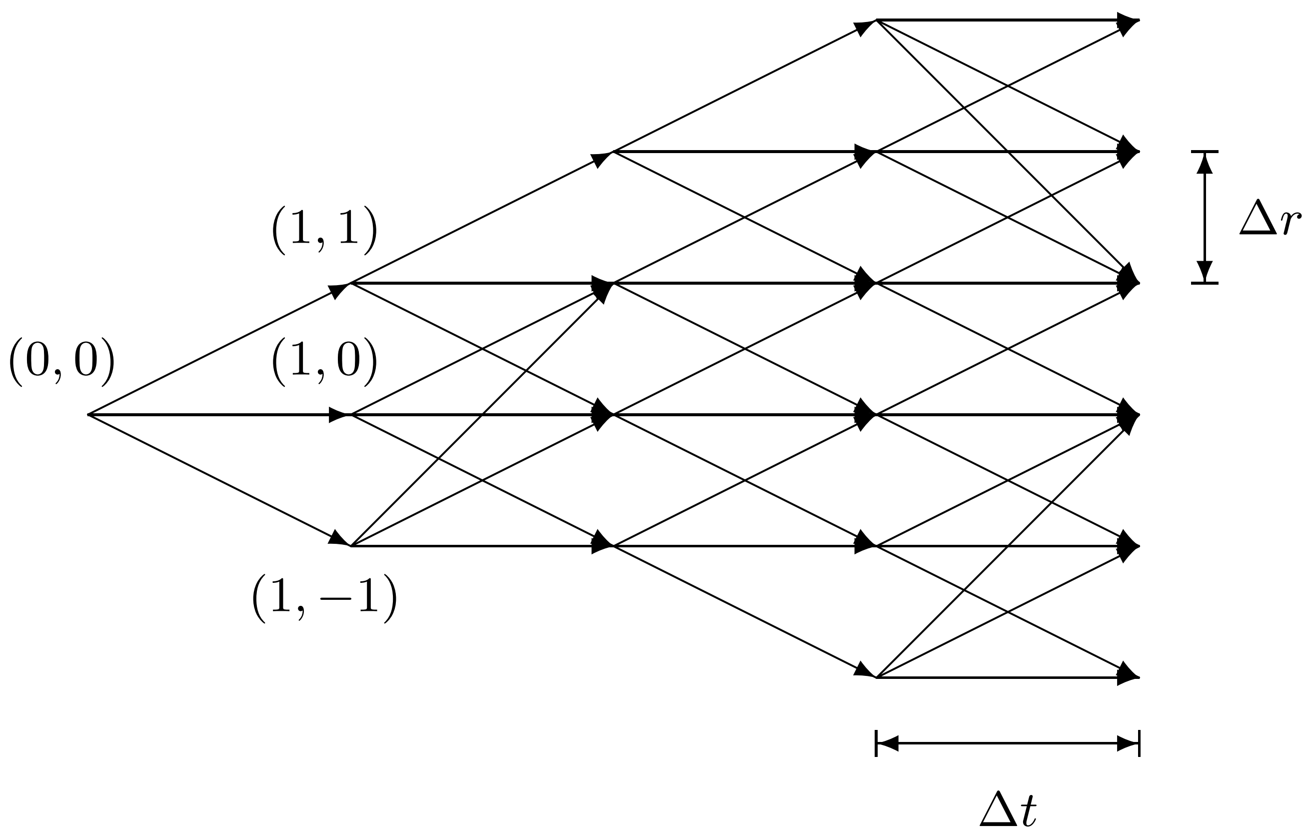
---

[a] That is, $p_1(i, j)$, $p_2(i, j)$, and $p_3(i, j)$.

[b] A precursor of Lyuu and C. Wu's (R90723065) (2003, 2005) mean-tracking idea, which in turn is the precursor of the binomial-trinomial tree of Dai (B82506025, R86526008, D8852600) & Lyuu (2006, 2008, 2010).

The Hull-White Model: Calibration (continued)

- Let k be the number among $\{ j-1, j, j+1 \}$ that makes the short rate reached by the middle branch, r_k, closest to
$$r_j + \mu_{i,j} \Delta t.$$
  - But note that $\mu_{i,j}$ is still *not* computed yet.
- Then the three nodes following node (i, j) are nodes
$$(i+1, k+1), (i+1, k), (i+1, k-1).$$
- See p. 1220 for a possible geometry.
- The resulting tree combines.


(0, 0)
(1, 1)
(1, 0)
(1, −1)
Δr
Δt

The Hull-White Model: Calibration (continued)

- The probabilities for moving along these branches are functions of $\mu_{i,j}$, σ, j, and k:

$$p_1(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} + \frac{\eta}{2\Delta r}, \tag{169}$$

$$p_2(i,j) = 1 - \frac{\sigma^2 \Delta t + \eta^2}{(\Delta r)^2}, \tag{169$'$}$$

$$p_3(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} - \frac{\eta}{2\Delta r}, \tag{169$''$}$$

where

$$\eta \stackrel{\Delta}{=} \mu_{i,j} \Delta t + (j-k)\, \Delta r.$$

The Hull-White Model: Calibration (continued)

- As trinomial tree algorithms are but explicit methods in disguise,[a] certain relations must hold for Δr and Δt to guarantee stability.
- It can be shown that their values must satisfy

$$\frac{\sigma\sqrt{3\Delta t}}{2} \leq \Delta r \leq 2\sigma\sqrt{\Delta t}$$

for the probabilities to lie between zero and one.

  – For example, Δr can be set to $\sigma\sqrt{3\Delta t}$.[b]

- Now it only remains to determine $\theta(t_i)$.

---

[a]Recall p. 850.

[b]Hull & White (1988).

The Hull-White Model: Calibration (continued)

- At this point at time t_i,

$$r(0, t_1), r(0, t_2), \ldots, r(0, t_{i+1})$$

  have already been matched.

- Let $Q(i, j)$ be the state price at node (i, j).
- By construction, the state prices $Q(i, j)$ for all j are known by now.
- We begin with state price $Q(0, 0) = 1$.

The Hull-White Model: Calibration (continued)

- Let $\hat{r}(i)$ refer to the short rate value at time t_i.
- The value at time zero of a zero-coupon bond maturing at time t_{i+2} is then

$$
\begin{aligned}
& e^{-r(0,t_{i+2})(i+2)\,\Delta t} \\
= & \sum_j Q(i,j)\, e^{-r_j \Delta t}\, E^{\pi}\left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right]. \qquad (170)
\end{aligned}
$$

- The right-hand side represents the value of \$1 at time t_{i+2} as seen at node (i,j) at time[a] t_i before being discounted by $Q(i,j)$.

---

[a]Thus $\hat{r}(i+1)$ is stochastic.

The Hull-White Model: Calibration (continued)

- The expectation in Eq. (170) can be approximated by[a]

$$E^{\pi}\left[\, e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right]$$
$$\approx \quad e^{-r_j \Delta t}\left(1 - \mu_{i,j}(\Delta t)^2 + \frac{\sigma^2(\Delta t)^3}{2}\right). \quad (171)$$

  – This solves the chicken-egg problem!

- Substitute Eq. (171) into Eq. (170) and replace $\mu_{i,j}$ with $\theta(t_i) - a r_j$ to obtain

$$\theta(t_i) \approx \frac{\sum_j Q(i,j)\, e^{-2r_j \Delta t}\left(1 + a r_j (\Delta t)^2 + \sigma^2 (\Delta t)^3/2\right) - e^{-r(0,t_{i+2})(i+2)\,\Delta t}}{(\Delta t)^2 \sum_j Q(i,j)\, e^{-2 r_j \Delta t}}.$$

[a]See Exercise 26.4.2 of the textbook.

The Hull-White Model: Calibration (continued)

- For the Hull-White model, the expectation in Eq. (171) is actually known analytically by Eq. (29) on p. 180:

$$
\begin{aligned}
& E^{\pi}\left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right] \\
= \quad & e^{-r_j \Delta t + (-\theta(t_i) + a r_j + \sigma^2 \Delta t/2)(\Delta t)^2}.
\end{aligned}
$$

- Therefore, alternatively,

$$
\theta(t_i) = \frac{r(0, t_{i+2})(i+2)}{\Delta t} + \frac{\sigma^2 \Delta t}{2} + \frac{\ln \sum_j Q(i,j)\, e^{-2 r_j \Delta t + a r_j (\Delta t)^2}}{(\Delta t)^2}.
$$

- With $\theta(t_i)$ in hand, we can compute $\mu_{i,j}$.[a]

---

[a]See Eq. (168) on p. 1214.

The Hull-White Model: Calibration (concluded)

- With $\mu_{i,j}$ available, we compute the probabilities.[a]
- Finally the state prices at time t_{i+1}:

$$Q(i+1, j) = \sum_{(i, j^*) \text{ is connected to } (i+1, j) \text{ with probability } p_{j^*}} p_{j^*} e^{-r_{j^*} \Delta t} Q(i, j^*).$$

- There are at most 5 choices for j^* (why?).
- The total running time is $O(n^2)$.
- The space requirement is $O(n)$ (why?).

---

[a] See Eqs. (169) on p. 1221.

Comments on the Hull-White Model

- One can try different values of a and σ for each option.
- Or have an a value common to all options but use a different σ value for each option.
- Either approach can match all the option prices exactly.
- But suppose the demand is for a single set of parameters to apply to *all* option prices.
- Then the Hull-White model can be calibrated to all the observed option prices by choosing a and σ that minimize the mean-squared pricing error.[a]

[a] Hull & White (1995).

The Hull-White Model: Calibration with Irregular Trinomial Trees

- The previous calibration algorithm is quite general.
- For example, it can be modified to apply to cases where the diffusion term has the form σr^b.
- But it has at least two shortcomings.
- First, the resulting trinomial tree is irregular (p. 1220).
  - So it is harder to program (for nonprogrammers).
- The second shortcoming is a consequence of the tree's irregular shape.

The Hull-White Model: Calibration with Irregular Trinomial Trees (concluded)

- Recall that the algorithm figured out $\theta(t_i)$ that matches the spot rate $r(0, t_{i+2})$ in order to determine the branching schemes for the nodes at time t_i.
- But without those branches, the tree was not specified, and backward induction on the tree was not possible.
- To avoid this chicken-egg dilemma, the algorithm turned to the continuous-time model to evaluate Eq. (170) on p. 1224 that helps derive $\theta(t_i)$.
- The resulting $\theta(t_i)$ hence might not yield a tree that matches the spot rates exactly.

The Hull-White Model: Calibration with Regular Trinomial Trees[a]

- The next, simpler algorithm exploits the fact that the Hull-White model has a constant diffusion term σ.
- The resulting trinomial tree will be regular.
- All the $\theta(t_i)$ terms can be chosen by backward induction to match the spot rates exactly.
- The tree is constructed in two phases.

[a]Hull & White (1994).

The Hull-White Model: Calibration with Regular Trinomial Trees (continued)

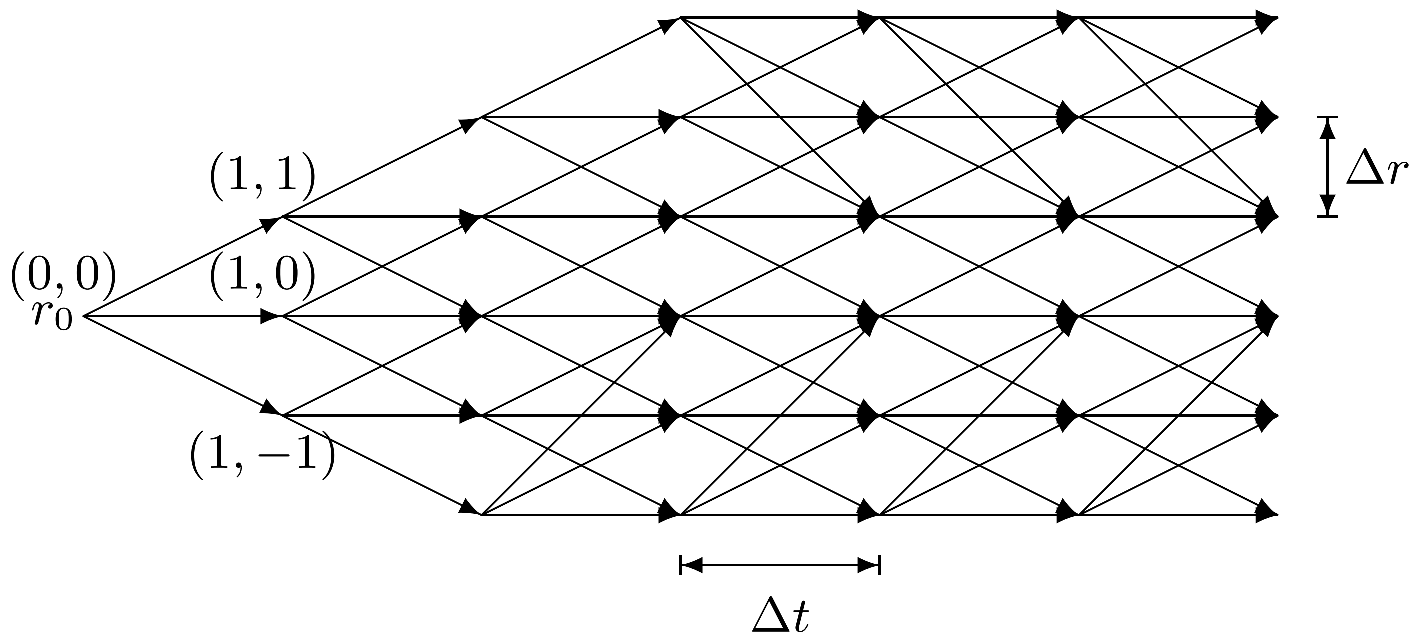
- In the first phase, a tree is built for the $\theta(t) = 0$ case, which is an Ornstein-Uhlenbeck process:

$$dr = -ar\,dt + \sigma\,dW, \quad r(0) = 0.$$

  – The tree is dagger-shaped (see p. 1233).

  – The number of nodes above the r_0-line is $j_{\max}$, and that below the line is $j_{\min}$.

  – They will be picked so that the probabilities (169) on p. 1221 are positive for all nodes.



The short rate at node $(0, 0)$ equals $r_0 = 0$; here $j_{\max} = 3$ and $j_{\min} = 2$.

The Hull-White Model: Calibration with Regular Trinomial Trees (concluded)

- The tree's branches and probabilities are now in place.
- Phase two fits the term structure.
  - Backward induction is applied to calculate the β_i to *add* to the short rates on the tree at time t_i so that the spot rate $r(0, t_{i+1})$ is matched.[a]

---

[a]Contrast this with the previous algorithm, where it was $r(0, t_{i+2})$ that was matched!

The Hull-White Model: Calibration

- Assume that $a > 0$.
- Set $\Delta r = \sigma\sqrt{3\Delta t}$.[a]
- Node (i, j) is a top node if $j = j_{\max}$ and a bottom node if $j = -j_{\min}$.
- Because the root has a short rate of $r_0 = 0$, phase one sets $r_j = j\Delta r$.[b]
- Hence the probabilities in Eqs. (169) on p. 1221 use

$$\eta \stackrel{\Delta}{=} -aj\Delta r\Delta t + (j-k)\,\Delta r.$$

- Recall that k tracks the middle branch.

[a] Recall p. 1222.

[b] Compare it with formula (167) on p. 1213.

The Hull-White Model: Calibration (continued)

- The probabilities become

$$
\begin{aligned}
& p_1(i, j) \\
= \; & \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 - aj\Delta t + (j-k)}{2}, \qquad (172) \\
& p_2(i, j) \\
= \; & \frac{2}{3} - \left[a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 \right], \qquad (173) \\
& p_3(i, j) \\
= \; & \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 + aj\Delta t - (j-k)}{2}. \qquad (174)
\end{aligned}
$$

- p_1: up move; p_2: flat move; p_3: down move.

The Hull-White Model: Calibration (continued)

- The dagger shape dictates this:
  - Let $k = j - 1$ if node (i, j) is a top node.
  - Let $k = j + 1$ if node (i, j) is a bottom node.
  - Let $k = j$ for the rest of the nodes.
- Note that the probabilities are identical for nodes (i, j) with the *same* j.
- Note also the symmetry,

$$p_1(i, j) = p_3(i, -j).$$

The Hull-White Model: Calibration (continued)

- The inequalities

$$\frac{3-\sqrt{6}}{3} < ja\Delta t < \sqrt{\frac{2}{3}} \tag{175}$$

ensure that all the branching probabilities are positive in the upper half of the tree, that is, $j > 0$ (verify this).

- Similarly, the inequalities

$$-\sqrt{\frac{2}{3}} < ja\Delta t < -\frac{3-\sqrt{6}}{3}$$

ensure that the probabilities are positive in the lower half of the tree, that is, $j < 0$.

The Hull-White Model: Calibration (continued)

- To further make the tree symmetric across the r_0-line, we let $j_{\min} = j_{\max}$.
- As
$$\frac{3-\sqrt{6}}{3} \approx 0.184,$$
a good choice is
$$j_{\max} = \left\lceil \frac{0.184}{a\Delta t} \right\rceil = O(n).$$

The Hull-White Model: Calibration (continued)

- Phase two computes the β_is to fit the spot rates.
- We begin with state price $Q(0, 0) = 1$.
- Inductively, suppose that spot rates

$$r(0, t_1), r(0, t_2), \ldots, r(0, t_i)$$

  have already been matched.
- By construction, the state prices $Q(i, j)$ for all j are known by now.

The Hull-White Model: Calibration (continued)

- The value of a zero-coupon bond maturing at time t_{i+1} equals

$$e^{-r(0,t_{i+1})(i+1)\,\Delta t} = \sum_j Q(i,j)\, e^{-(\beta_i + r_j)\Delta t}$$

  by risk-neutral valuation.

- Hence

$$\beta_i = \frac{r(0,t_{i+1})(i+1)\,\Delta t + \ln \sum_j Q(i,j)\, e^{-r_j\,\Delta t}}{\Delta t}. \tag{176}$$

The Hull-White Model: Calibration (concluded)

- The short rate at node (i, j) now equals $\beta_i + r_j$.
- The state prices at time t_{i+1},

$$Q(i+1, j)$$

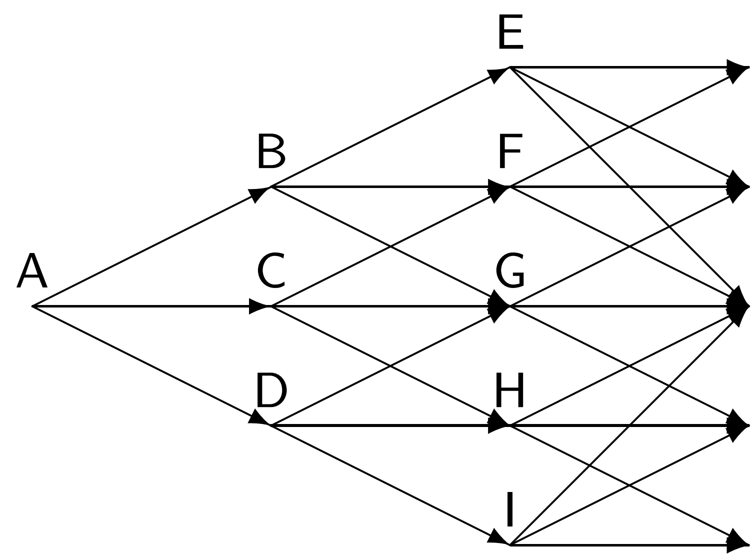
for $-\min(i+1, j_{\max}) \leq j \leq \min(i+1, j_{\max})$, can now be calculated as before.[a]

- The total running time is $O(n j_{\max})$.
- The space requirement is $O(n)$.

[a] Recall p. 1227.

A Numerical Example

- Assume $a = 0.1$, $\sigma = 0.01$, and $\Delta t = 1$ (year).
- Immediately, $\Delta r = 1.73205\%$ and $j_{\max} = 2$.
- The plot on p. 1244 illustrates the 3-period trinomial tree after phase one.
- For example, the branching probabilities for node E are calculated by Eqs. (172)–(174) on p. 1236 with $j = 2$ and $k = 1$.



Node	A, C, G	B, F	E	D, H	I
r (%)	0.00000	1.73205	3.46410	-1.73205	-3.46410
p_1	0.16667	0.12167	0.88667	0.22167	0.08667
p_2	0.66667	0.65667	0.02667	0.65667	0.02667
p_3	0.16667	0.22167	0.08667	0.12167	0.88667

A Numerical Example (continued)

- Suppose that phase two is to fit the spot rate curve
$$0.08 - 0.05 \times e^{-0.18 \times t}.$$
- The annualized continuously compounded spot rates are
$$r(0,1) = 3.82365\%, r(0,2) = 4.51162\%, r(0,3) = 5.08626\%.$$
- Start with state price $Q(0,0) = 1$ at node A.

A Numerical Example (continued)

- Now, by Eq. (176) on p. 1241,

$$\beta_0 = r(0,1) + \ln Q(0,0)\, e^{-r_0} = r(0,1) = 3.82365\%.$$

- Hence the short rate at node A equals

$$\beta_0 + r_0 = 3.82365\%.$$

- The state prices at year one are calculated as

$$\begin{aligned} Q(1,1) &= p_1(0,0)\, e^{-(\beta_0 + r_0)} = 0.160414, \\ Q(1,0) &= p_2(0,0)\, e^{-(\beta_0 + r_0)} = 0.641657, \\ Q(1,-1) &= p_3(0,0)\, e^{-(\beta_0 + r_0)} = 0.160414. \end{aligned}$$

A Numerical Example (continued)

- The 2-year rate spot rate $r(0,2)$ is matched by picking

$$\beta_1 = r(0,2) \times 2 + \ln\left[Q(1,1)\, e^{-\Delta r} + Q(1,0) + Q(1,-1)\, e^{\Delta r} \right] = 5.20459\%.$$

- Hence the short rates at nodes B, C, and D equal

$$\beta_1 + r_j,$$

  where $j = 1, 0, -1$, respectively.

- They are found to be 6.93664%, 5.20459%, and 3.47254%.

A Numerical Example (continued)

- The state prices at year two are calculated as

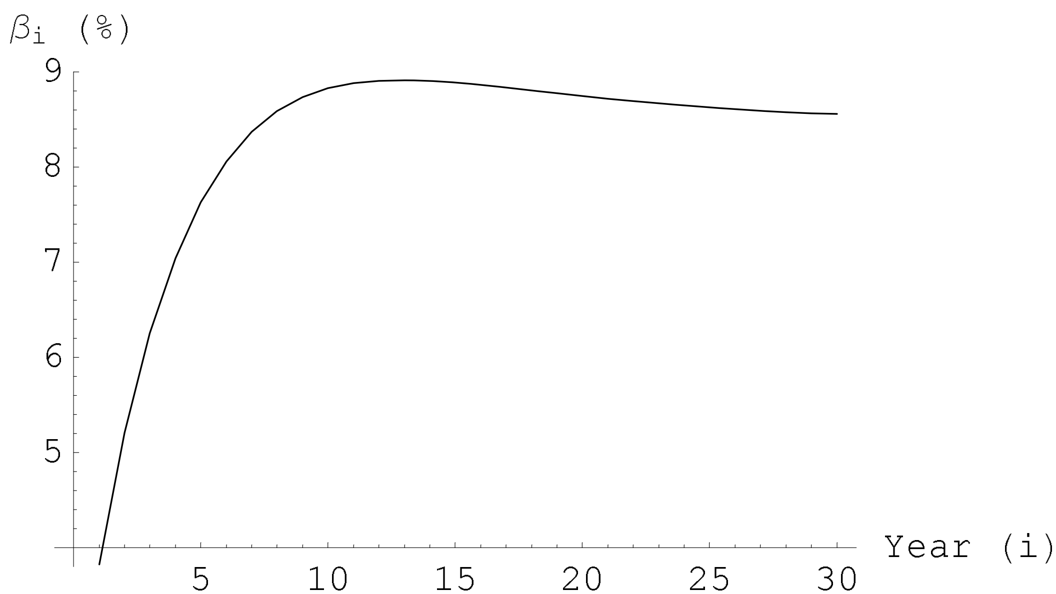
$$
\begin{aligned}
Q(2,2) &= p_1(1,1)\, e^{-(\beta_1+r_1)} Q(1,1) = 0.018209,\\
Q(2,1) &= p_2(1,1)\, e^{-(\beta_1+r_1)} Q(1,1) + p_1(1,0)\, e^{-(\beta_1+r_0)} Q(1,0)\\
&= 0.199799,\\
Q(2,0) &= p_3(1,1)\, e^{-(\beta_1+r_1)} Q(1,1) + p_2(1,0)\, e^{-(\beta_1+r_0)} Q(1,0)\\
&\quad + p_1(1,-1)\, e^{-(\beta_1+r_{-1})} Q(1,-1) = 0.473597,\\
Q(2,-1) &= p_3(1,0)\, e^{-(\beta_1+r_0)} Q(1,0) + p_2(1,-1)\, e^{-(\beta_1+r_{-1})} Q(1,-1)\\
&= 0.203263,\\
Q(2,-2) &= p_3(1,-1)\, e^{-(\beta_1+r_{-1})} Q(1,-1) = 0.018851.
\end{aligned}
$$

A Numerical Example (concluded)

- The 3-year rate spot rate $r(0,3)$ is matched by picking

$$\begin{aligned}\beta_2 \;&=\; r(0,3)\times 3+\ln\Big[\,Q(2,2)\,e^{-2\times\Delta r}+Q(2,1)\,e^{-\Delta r}+Q(2,0)\\ &\quad +Q(2,-1)\,e^{\Delta r}+Q(2,-2)\,e^{2\times\Delta r}\Big]=6.25359\%.\end{aligned}$$

- Hence the short rates at nodes E, F, G, H, and I equal $\beta_2 + r_j$, where $j = 2, 1, 0, -1, -2$, respectively.
- They are found to be 9.71769%, 7.98564%, 6.25359%, 4.52154%, and 2.78949%.
- The figure on p. 1250 plots β_i for $i = 0, 1, \ldots, 29$.


β_i (%)
9
8
7
6
5
Year (i)
5
10
15
20
25
30

The (Whole) Yield Curve Approach

- We have seen several Markovian short rate models.
- The Markovian approach is computationally efficient.
- But it is difficult to model the behavior of yields and bond prices of different maturities.
- The alternative yield curve approach regards the whole term structure as the state of a process and directly specifies how it evolves.

The Heath-Jarrow-Morton (HJM) Model[a]

- This influential model is a forward rate model.
- The HJM model specifies the initial forward rate curve and the forward rate volatility structure.
  - The volatility structure describes the volatility of each forward rate for a given maturity date.
- Like the Black-Scholes option pricing model, neither risk preference assumptions nor the drifts of forward rates are needed.

[a]Heath, Jarrow, & Morton (1992).

The HJM Model (continued)

- Within a finite-time horizon $[\,0, U\,]$, we take as given the time-zero forward rate curve $f(0, T)$ for $T \in [\,0, U\,]$.
- Since this curve is used as the boundary value at $t = 0$, perfect fit to the observed term structure is *automatic*.
- The forward rates are driven by k stochastic factors.

The HJM Model (continued)

- Specifically the forward rate movements are governed by the stochastic process,

$$df(t,T) = \mu(t,T)\,dt + \sum_{i=1}^{k} \sigma_i(t,T)\,dW_i, \tag{177}$$

  where μ and σ_i may depend on the past history of the independent Wiener processes $W_1, W_2, \ldots, W_k$.

- One-factor models seem to perform better than multifactor models empirically, at least for *pricing* short-dated options.[a]

[a] Amin & Morton (1994).

The HJM Model (continued)

- But two-factor models perform better in *hedging* caps and floors.[a]

- Kamakura (2019) has a 10-factor[b] (14-factor[c]) HJM model for the U.S. Treasuries (German bonds, respectively).

- A unique equivalent martingale measure π can be established under which the prices of interest rate derivatives do not depend on the market prices of risk.

---

[a] Gupta & Subrahmanyam (2001, 2005).

[b] See `http://www.kamakuraco.com/KamakuraReleasesNewStochasticVolatilityModel`

[c] See `http://www.kamakuraco.com/KamakuraReleases14FactorHeathJarrowandMorto.`

The HJM Model (continued)

Theorem 23 (1) *For all* $0 < t \leq T$,

$$\mu(t,T) = \sum_{i=1}^{k} \sigma_i(t,T) \int_t^T \sigma_i(t,u)\, du \tag{178}$$

holds under π *almost surely.* (2) *The bond price dynamics under* π *is given by*

$$\frac{dP(t,T)}{P(t,T)} = r(t)\, dt - \sum_{i=1}^{k} \sigma_{\mathrm{p},i}(t,T)\, dW_i, \tag{179}$$

where $\sigma_{\mathrm{p},i}(t,T) \equiv \int_t^T \sigma_i(t,u)\, du$.

The HJM Model (concluded)

- Hence choosing the volatility functions $\sigma_i(t, T)$ of the forward rate dynamics under π uniquely determines the drift parameters under π and the prices of all claims.

The Use of the HJM Model

- Take the one-factor model,

$$df(t,T) = \mu(t,T)\,dt + \sigma(t,T)\,dW_t.$$

- To use the HJM model, we first pick $\sigma(t,T)$.
- This is the modeling part.
- The drift parameters are then determined by Eq. (178) on p. 1256.
- Now fetch today's forward rate curve $\{\, f(0,T), T \geq 0 \,\}$ and integrate it to obtain the forward rates,

$$f(t,T) = f(0,T) + \int_0^t \mu(s,T)\,ds + \int_0^t \sigma(s,T)\,dW_s.$$

The Use of the HJM Model (concluded)

- Compute the future bond prices by

$$P(t,T) = e^{-\int_t^T f(t,s)\,ds}$$

  if necessary.

- European-style derivatives can be priced by simulating many paths and taking average.

Short Rate under the HJM Model

- From Eq. (26.19) of the textbook, the short rate follows the following SDE,

$$\begin{aligned} dr(t) &= \frac{\partial f(0,t)}{\partial t}\, dt \\ &\quad + \left[\int_0^t \left(\sigma_{\text{p}}(s,t)\, \frac{\partial \sigma(s,t)}{\partial t} + \sigma(s,t)^2 \right) ds \right] dt \\ &\quad + \left(\int_0^t \frac{\partial \sigma(s,t)}{\partial t}\, dW_s \right) dt + \sigma(t,t)\, dW_t. \quad (180) \end{aligned}$$

- Since the second and the third terms on the right-hand side depend on the history of σ_{p} and/or dW, they can make r non-Markovian.

Short Rate under the HJM Model (concluded)

- If $\sigma_{\rm p}(t,T) = \sigma(T-t)$ for a constant σ, the short rate process r becomes Markovian.
- Then Eq. (180) on p. 1260 is reduced to

$$dr = \left(\frac{\partial f(0,t)}{\partial t} + \sigma^2 t \right) dt + \sigma\, dW.$$

- This is the continuous-time Ho-Lee model (162) on p. 1182.[a]
- See Carverhill (1994) and Jeffrey (1995) for conditions for the short rate to be Markovian.

[a]See p. 392 of the textbook.

The Alternative HJM Model

- Alternatively, we can start with the bond process under π:

$$\frac{dP(t,T)}{P(t,T)} = r(t)\,dt + \sum_{i=1}^{k} \sigma_{\mathrm{p},i}(t,T)\,dW_i. \tag{181}$$

- Then[a]

$$\begin{aligned} df(t,T) &= \sum_{i=1}^{k} \sigma_{\mathrm{p},i}(t,T)\,\frac{\partial \sigma_{\mathrm{p},i}(t,T)}{\partial T}\,dt \\ &\quad - \sum_{i=1}^{k} \frac{\partial \sigma_{\mathrm{p},i}(t,T)}{\partial T}\,dW_i. \end{aligned}$$

[a]Carverhill (1995); Musiela & Rutkowski (1997); Hull (1999).

Gaussian HJM Models[a]

- A nonstochastic volatility depends on only t and T.
- When the forward rate volatilities $\sigma_i(t, T)$ are nonstochastic, we have a Gaussian HJM model.
- For Gaussian HJM models, the bond price volatilities $\sigma_{\mathrm{p},i}(t, T)$ must also be nonstochastic.
- The forward rates have a normal distribution, whereas the bond prices have a lognormal distribution.

[a]Musiela & Rutkowski (1997).

Gaussian HJM Models (concluded)

- $\sigma(t, T) = \sigma$: The Ho-Lee model (162) on p. 1182 obtains.
- $\sigma(t, T) = \sigma e^{-a(T-t)}$: The Hull-White model (166) on p. 1210 obtains.
- $\sigma(t, T) = \sigma_0 + \sigma_1 (T - t)$: The linear absolute model.[a]
- $\sigma(t, T) = \sigma \, [\, \gamma(T - t) + 1 \,] \, e^{-(\lambda/2)(T-t)}$: The Mercurio-Moraleda (2000) model.

---

[a] Gupta & Subrahmanyam (2001, 2005).

Local-Volatility HJM Models[a]

- If the forward rate volatilities $\sigma_i(t, T, f(t, T))$ depend on t, T, and $f(t, T)$ only, we have a local-volatility HJM model.
- The same term may also apply to HJM models whose bond price volatilities $\sigma_{\mathrm{p},i}(t, T, P(t, T))$ depend on t, T, and $P(t, T)$ only.

---

[a]Brigo & Mercurio (2006).

Local-Volatility HJM Models (continued)

- The (nearly) proportional volatility model:[a]

$$\sigma(t, T, f(t, T)) = \sigma_0 \min(\kappa, f(t, T)), \quad \sigma_0, \kappa > 0.$$

- The proportional volatility model:[b]

$$\sigma(t, T, f(t, T)) = \sigma_0 f(t, T). \tag{182}$$

- The linear proportional model:[c]

$$\sigma(t, T, f(t, T)) = [\, \sigma_0 + \sigma_1 (T - t) \,]\, f(t, T).$$

---

[a] Heath, Jarrow, & Morton (1992); Jarrow (1996). The large positive constant κ prevents explosion in finite time.

[b] Gupta & Subrahmanyam (2001, 2005).

[c] Gupta & Subrahmanyam (2001, 2005).

Local-Volatility HJM Models (continued)

- Exponentially dampened volatility proportional to the short rate:[a]

$$\sigma(t,T) = \sigma f(t,t)\, e^{-a(T-t)}.$$

- The Ritchken-Sankarasubramanian (1995) model:[b]

$$\sigma(t,T) = \sigma(t,t)\, e^{-\int_t^T \kappa(x)\, dx}.$$

  – For example,[c]

$$\sigma(t,t) = \sigma r(t)^{\gamma}.$$

---

[a] Grant & Vora (1999).

[b] The short rate volatility $\sigma(t,t)$ may depend on the short rate $r(t)$.

[c] Ritchken & Sankarasubramanian (1995); Li, Ritchken, & Sankarasubramanian (1995).

Local-Volatility HJM Models (concluded)

- A model attributed to Ian Cooper (1993):[a]

$$\sigma_{\mathrm{p}}(t, T, P(t, T)) = \psi(t) \ln P(t, T)$$

in Eq. (181) on p. 1262:

---

[a]Rebonato (1996). It is equivalent to the proportional volatility model (182) when $\psi(t)$ is a constant.

Trees for HJM Models

- Obtain today's forward rate curve:

$$f(0,0), f(0,\Delta t), f(0,2\Delta t), f(0,3\Delta t), \ldots, f(0,T).$$

- For binomial trees, generate the two forward rate curves at time Δt:

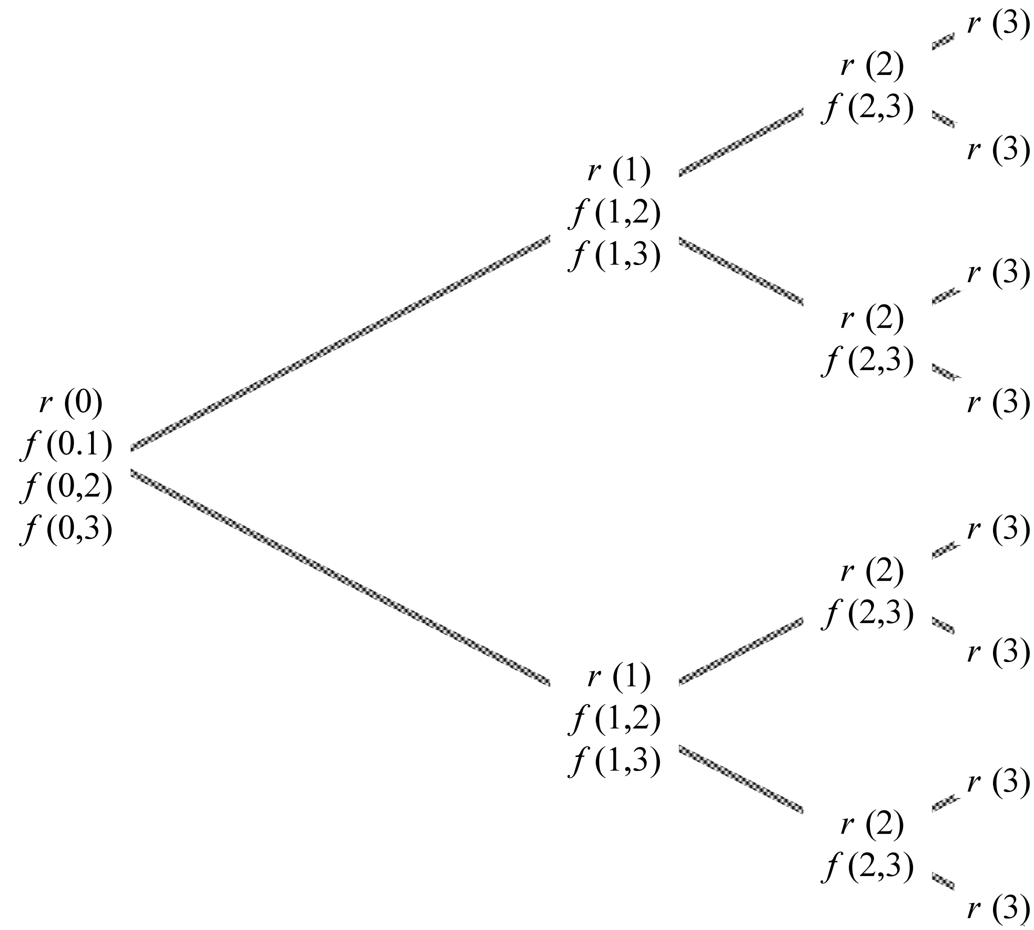
$$f_{\mathrm{u}}(\Delta t,\Delta t), f_{\mathrm{u}}(\Delta t,2\Delta t), f_{\mathrm{u}}(\Delta t,3\Delta t), \ldots, f_{\mathrm{u}}(\Delta t,T),$$
$$f_{\mathrm{d}}(\Delta t,\Delta t), f_{\mathrm{d}}(\Delta t,2\Delta t), f_{\mathrm{d}}(\Delta t,3\Delta t), \ldots, f_{\mathrm{d}}(\Delta t,T).$$

  by Eq. (177) on p. 1254 with $\mu(t,T)$ from Eq. (178) on p. 1256.

Trees for HJM Models (continued)

- Iterate until the maturity $t \leq T$ of the derivative.
- A straightforward implementation of the HJM model results in noncombining trees.
  - For a binomial tree with n time steps, $O(2^n)$ nodes for one-factor HJM models; $O(3^n)$ or $O(4^n)$ for two-factor models.[a]

[a]Clewlow & Strickland (1998); Hull (1999); Nawalkha, Beliaeva, & Soto (2007).


r (0)
f (0.1)
f (0,2)
f (0,3)
r (1)
f (1,2)
f (1,3)
r (1)
f (1,2)
f (1,3)
r (2)
f (2,3)
r (2)
f (2,3)
r (2)
f (2,3)
r (2)
f (2,3)
r (3)
r (3)
r (3)
r (3)
r (3)
r (3)
r (3)
r (3)

Trees for HJM Models (continued)

- Jarrow (1996): "a large number of time steps is not always essential for obtaining good approximations."
- Rebonato (1996): "it is difficult to see how a five-year cap with quarterly resets (let alone an option thereon) could be priced using [10 or 12 time steps]."
- Some trees are not analyzed.[a]

[a] Brace (1996); Gątarek & Kołakowski (2003); Ferris (2012).

Trees for HJM Models (concluded)

- Nawalkha & J. Zhang (2004) has a combining tree for the proportional volatility model with a positive lower bound.
  - It is described in Nawalkha, Beliaeva, & Soto (2007) but not published.
- For Gaussian HJM models, $O(n^2)$ nodes may suffice.[a]

[a] Lok (D99922028), Lu (D00922011), & Lyuu (2020); Lyuu (2019).

Introduction to Mortgage-Backed Securities

Anyone stupid enough to promise to be
responsible for a stranger's debts
deserves to have his own property
held to guarantee payment.
— Proverbs 27:13

I'm not putting my money in real estate.
I prefer bonds.
— Margaret Mitchell (1900–1949),
Gone with the Wind (1936)

Mortgages

- A mortgage is a loan secured by the collateral of real estate property.
- Suppose the borrower (the mortgagor) defaults, that is, fails to make the contractual payments.
- The lender (the mortgagee) can foreclose the loan by seizing the property.

Mortgage-Backed Securities

- A mortgage-backed security (MBS) is a bond backed by an undivided interest in a pool of mortgages.[a]
- MBSs traditionally enjoy high returns, wide ranges of products, high credit quality, and liquidity.
- The mortgage market has witnessed tremendous innovations in product design.
- The collapse of MBSs also triggered the financial crisis of 2008.

[a]They can be traced to 1880s (Levy, 2012).

Mortgage-Backed Securities (concluded)

- The complexity of the products and the prepayment option require advanced models and software techniques.
  - In fact, the mortgage market probably could not have operated efficiently without them.[a]
- They also consume lots of computing power.
- Our focus will be on residential mortgages.
- But the underlying principles are applicable to other types of assets.

---

[a]Merton (1994).

Types of MBSs

- An MBS is issued with pools of mortgage loans as the collateral.
- The cash flows of the mortgages making up the pool naturally reflect upon those of the MBS.
- There are three basic types of MBSs:
  1. Mortgage pass-through security (MPTS).
  2. Collateralized mortgage obligation (CMO).
  3. Stripped mortgage-backed security (SMBS).

Problems Investing in Mortgages

- The MBS sector is one of the largest in the debt market.[a]
- Individual mortgages are unattractive for many investors.
- Often at hundreds of thousands of U.S. dollars or more, they demand too much investment.
- Most investors lack the resources and knowledge to assess the credit risk involved.

---

[a]See p. 3 of the textbook. In the U.S., the outstanding balance was US$9.3 trillion as of 2017 vs. the US Treasury's US$14.5 trillion and corporate debt's US$9.0 trillion (SIFMA, 2018). The residential property is valued at around US$34 trillion in 2020 (*The Economist*, 2020).

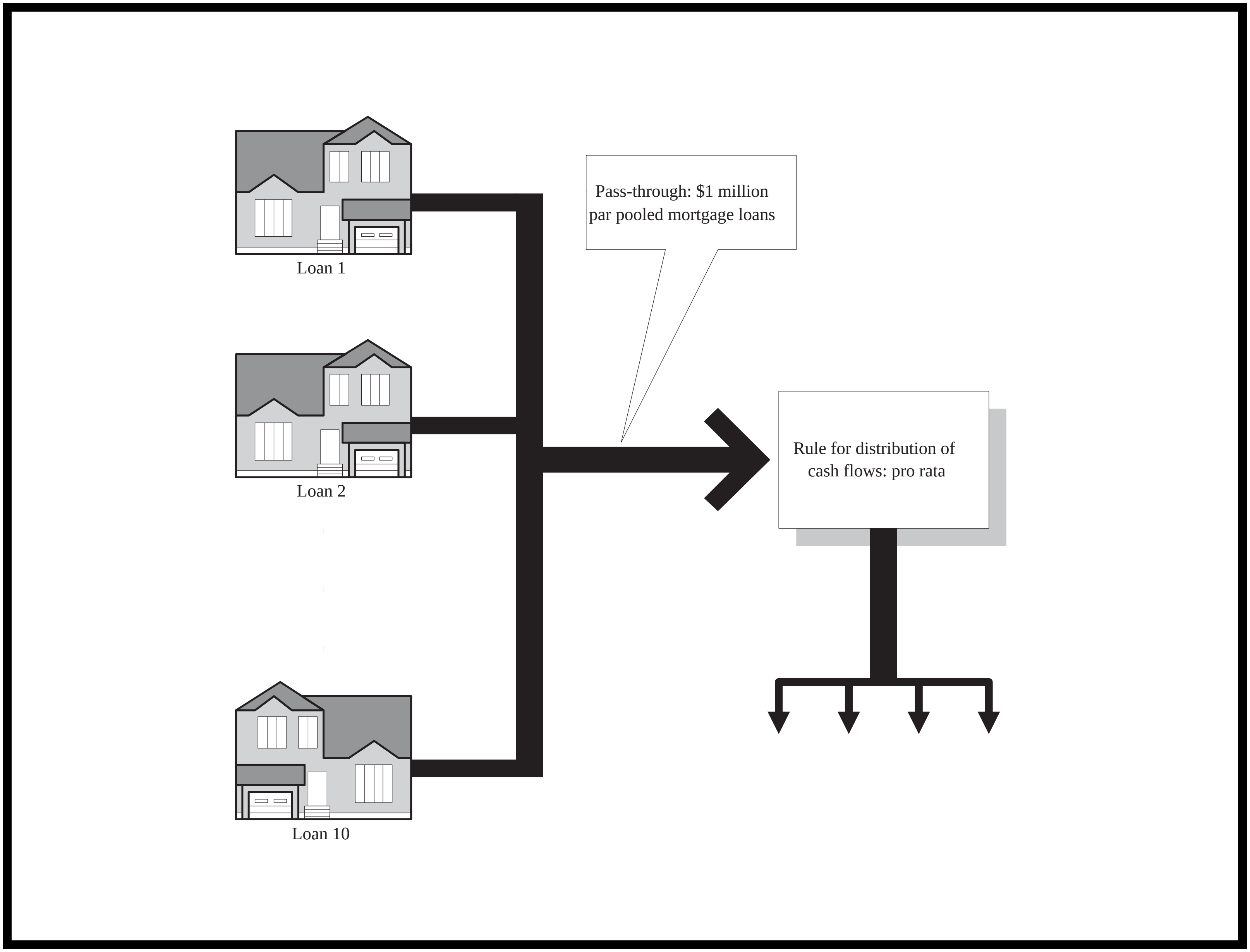
Problems Investing in Mortgages (concluded)

- Recall that a traditional mortgage is fixed rate, level payment, and fully amortized.
- So the percentage of principal and interest (P&I) varying from month to month, creating accounting headaches.
- Prepayment levels fluctuate with a host of factors.
- That makes the size and the timing of the cash flows unpredictable.

Mortgage Pass-Throughs[a]

- The simplest kind of MBS.
- Payments from the underlying mortgages are passed from the mortgage holders through the servicing agency, after a fee is subtracted.
- They are distributed to the security holder on a pro rata basis.
  - The holder of a $25,000 certificate from a $1 million pool is entitled to 2½% (or 1/40th) of the cash flow.
- Because of higher marketability, a pass-through is easier to sell than its individual loans.

[a] First issued by Ginnie Mae (Government National Mortgage Association) in 1970.


Pass-through: $1 million
par pooled mortgage loans
Loan 1
Loan 2
Loan 10
Rule for distribution of
cash flows: pro rata

Collateralized Mortgage Obligations (CMOs)

- A pass-through exposes the investor to the total prepayment risk.
- Such risk is undesirable from an asset/liability perspective.
- To deal with prepayment uncertainty, CMOs were created.[a]
- Mortgage pass-throughs have a single maturity and are backed by individual mortgages.

---

[a]In June 1983 by Freddie Mac (Federal Home Loan Mortgage Corporation), bailed out by the U.S. government in 2008, with the help of First Boston, which was acquired by Credit Suisse in 1990.

Collateralized Mortgage Obligations (CMOs) (continued)

- CMOs are *multiple*-maturity, *multi*class debt instruments collateralized by pass-throughs, stripped mortgage-backed securities, and whole loans.
- The total prepayment risk is now divided among classes of bonds called classes or tranches.[a]
- The principal, scheduled and prepaid, is allocated on a *prioritized* basis so as to redistribute the prepayment risk among the tranches in an unequal way.

[a] *Tranche* is a French word for "slice."

Collateralized Mortgage Obligations (CMOs) (concluded)

- CMOs were the first successful attempt to alter mortgage cash flows in a security form that attracts a wide range of investors
  - The outstanding balance of agency CMOs was US$1.1 trillion as of the first quarter of 2015.[a]

[a]SIFMA (2015).

Sequential Tranche Paydown

- In the sequential tranche paydown structure, Class A receives principal paydown and prepayments before Class B, which in turn does it before Class C, and so on.
- Each tranche thus has a different effective maturity.
- Each tranche may even have a different coupon rate.

An Example

- Consider a two-tranche sequential-pay CMO backed by \$1,000,000 of mortgages with a 12% coupon and 6 months to maturity.
- The cash flow pattern for each tranche with zero prepayment and zero servicing fee is shown on p. 1289.
- The calculation can be carried out first for the Total columns, which make up the amortization schedule.
- Then the cash flow is allocated.
- Tranche A is retired after 4 months, and tranche B starts principal paydown at the end of month 4.

CMO Cash Flows without Prepayments

Month	Interest			Principal			Remaining principal		
	A	B	Total	A	B	Total	A	B	Tota
							500,000	500,000	1,000,0
1	5,000	5,000	10,000	162,548	0	162,548	337,452	500,000	837,4
2	3,375	5,000	8,375	164,173	0	164,173	173,279	500,000	673,2
3	1,733	5,000	6,733	165,815	0	165,815	7,464	500,000	507,4
4	75	5,000	5,075	7,464	160,009	167,473	0	339,991	339,9
5	0	3,400	3,400	0	169,148	169,148	0	170,843	170,8
6	0	1,708	1,708	0	170,843	170,843	0	0	
Total	10,183	25,108	35,291	500,000	500,000	1,000,000			

The total monthly payment is \$172,548. Month-i numbers reflect the ith monthly payment.

Another Example

- When prepayments are present, the calculation is only slightly more complex.
- Suppose the single monthly mortality (SMM) per month is 5%.
- This means the prepayment amount is 5% of the *remaining* principal.
- The remaining principal at month i *after* prepayment then equals the scheduled remaining principal as computed by Eq. (7) on p. 56 times $(0.95)^i$.
- This done for all the months, the interest payment at any month is the remaining principal of the previous month times 1%.

Another Example (continued)

- The prepayment amount equals the remaining principal times 0.05/0.95.
  - The division by 0.95 yields the remaining principal *before* prepayment.
- Page 1292 tabulates the cash flows of the same two-tranche CMO under 5% SMM.

Another Example (continued)

Month	Interest			Principal			Remaining principal		
	A	B	Total	A	B	Total	A	B	Total
							500,000	500,000	1,000,00
1	5,000	5,000	10,000	204,421	0	204,421	295,579	500,000	795,57
2	2,956	5,000	7,956	187,946	0	187,946	107,633	500,000	607,63
3	1,076	5,000	6,076	107,633	64,915	172,548	0	435,085	435,08
4	0	4,351	4,351	0	158,163	158,163	0	276,922	276,92
5	0	2,769	2,769	0	144,730	144,730	0	132,192	132,19
6	0	1,322	1,322	0	132,192	132,192	0	0	
Total	9,032	23,442	32,474	500,000	500,000	1,000,000			

Month-i numbers reflect the ith monthly payment.

Another Example (continued)

- For instance, the total principal payment at month one, \$204,421, can be verified as follows.
- The *scheduled* remaining principal is \$837,452 from p. 1289.
- The remaining principal is hence

$$837452 \times 0.95 = 795579.$$

- That makes the total principal payment

$$1000000 - 795579 = 204421.$$

Another Example (concluded)

- As tranche A's remaining principal is \$500,000, all 204,421 dollars go to tranche A.
- Incidentally, the prepayment is

$$837452 \times 5\% = 41873.$$

  - Aalternatively, $795579 \times 0.05/0.95 = 41873$.
- Tranche A is retired after 3 months, and tranche B starts principal paydown at the end of month 3.

Stripped Mortgage-Backed Securities (SMBSs)[a]

- The principal and interest are divided between the PO strip and the IO strip.
- In the scenarios on p. 1288 and p. 1290:
  - The IO strip receives all the interest payments under the Interest/Total column.
  - The PO strip receives all the principal payments under the Principal/Total column.

[a]They were created in February 1987 when Fannie Mae issued its Trust 1 stripped MBS. Fannie Mae was bailed out by the U.S. government in 2008.

Stripped Mortgage-Backed Securities (SMBSs) (concluded)

- These new instruments allow investors to better exploit anticipated changes in interest rates.[a]
- The collateral for an SMBS is a pass-through.
- CMOs and SMBSs are usually called derivative MBSs.

---

[a]See p. 357 of the textbook.

Prepayments

- The prepayment option sets MBSs apart from other fixed-income securities.
- The exercise of options on most securities is expected to be "rational."
- This kind of "rationality" is weakened when it comes to the homeowner's decision to prepay.
- For example, even when the prevailing mortgage rate exceeds the mortgage's loan rate, some loans are prepaid.

Prepayment Risk

- Prepayment risk is the uncertainty in the amount and timing of the principal prepayments in the pool of mortgages that collateralize the security.
- This risk can be divided into contraction risk and extension risk.[a]
- Contraction risk is the risk of having to reinvest the prepayments at a rate lower than the coupon rate when interest rates decline.

[a] Similar to mortality risk and longevity risk in life insurance.

Prepayment Risk (continued)

- Extension risk is due to the slowdown of prepayments when interest rates climb, making the investor earn the security's lower coupon rate rather than the market's higher rate.
- Prepayments can be in whole or in part.
  - The former is called liquidation.
  - The latter is called curtailment.

Prepayment Risk (concluded)

- The holder of a pass-through security is exposed to the total prepayment risk associated with the underlying pool of mortgage loans.
- CMOs are designed to alter the distribution of that risk among investors.
- They contributed to the subprime mortgage crisis of 2008.

Other Risks

- Investors in mortgages are exposed to at least three other risks.
  - Interest rate risk is inherent in any fixed-income security.
  - Credit risk is the risk of loss from default.
    - For privately insured mortgage, the risk is related to the credit rating of the company that insures the mortgage.
  - Liquidity risk is the risk of loss if the investment must be sold quickly.

Prepayment: Causes

Prepayments have at least five components.

Home sale ("housing turnover"). The sale of a home generally leads to the prepayment of mortgage because of the full payment of the remaining principal.

Refinancing. Mortgagors can refinance their home mortgage at a lower mortgage rate. This is the most volatile component of prepayment and constitutes the bulk of it when prepayments are extremely high.

Default. Caused by foreclosure and subsequent liquidation of a mortgage. Relatively minor in most cases.[a]

[a] Except the subprime mortgage crisis that started the financial crisis of 2007–2008 (Zuckerman, 2010; Lewis, 2011).

Prepayment: Causes (concluded)

Curtailment. As the extra payment above the scheduled payment, curtailment applies to the principal and shortens the maturity of fixed-rate loans. Its contribution to prepayments is minor.

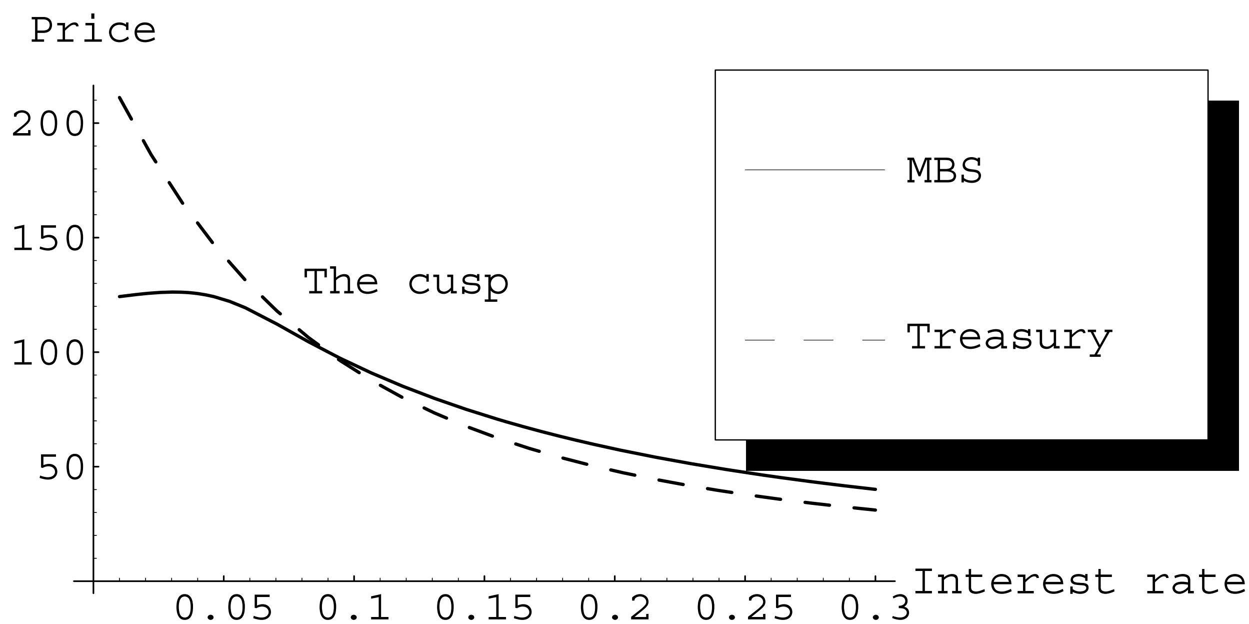
Full payoff (liquidation). There is evidence that many mortgagors pay off their mortgage completely when it is very seasoned and the remaining balance is small. Full payoff can also be due to natural disasters.

Prepayment: Characteristics

- Prepayments usually increase as the mortgage ages — first at an increasing rate and then at a decreasing rate.
- They are higher in the spring and summer and lower in the fall and winter.
- They vary by the geographic locations of the underlying properties.
- They increase when interest rates drop but with a time lag.

Prepayment: Characteristics (continued)

- If prepayments were higher for some time because of high refinancing rates, they tend to slow down.
  - Perhaps, homeowners who do not prepay when rates have been low for a prolonged time tend never to prepay.
- Plot on p. 1306 illustrates the typical price/yield curves of the Treasury and pass-through.



Price compression occurs as yields fall through a threshold.

The cusp represents that point.

Prepayment: Characteristics (concluded)

- As yields fall and the pass-through's price moves above a certain price, it flattens and then follows a downward slope.
- This phenomenon is called the price compression of premium-priced MBSs.
- It demonstrates the negative convexity of such securities.