Foundations of Term Structure Modeling

[Meriwether] scoring especially high marks in mathematics — an indispensable subject for a bond trader. — Roger Lowenstein, When Genius Failed (2000) [The] fixed-income traders I knew seemed smarter than the equity trader $[\cdots]$ there's no competitive edge to being smart in the equities business[.] — Emanuel Derman, My Life as a Quant (2004)

Bond market terminology was designed less to convey meaning than to bewilder outsiders. — Michael Lewis, *The Big Short* (2011)

Terminology

- A period denotes a unit of elapsed time.
 - Viewed at time t, the next time instant refers to time t + dt in the continuous-time model and time t + 1 in the discrete-time case.
- Bonds will be assumed to have a par value of one unless stated otherwise.
- The time unit for continuous-time models will usually be measured by the year.

Standard Notations

The following notation will be used throughout.

- t: a point in time.
- r(t): the one-period riskless rate prevailing at time t for repayment one period later.^a
- P(t,T): the present value at time t of one dollar at time T.

^aAlternatively, the instantaneous spot rate, or short rate, at time t.

Standard Notations (continued)

- r(t,T): the (T-t)-period interest rate prevailing at time t stated on a per-period basis and compounded once per period.^a
- F(t, T, M): the forward price at time t of a forward contract that delivers at time T a zero-coupon bond maturing at time $M \ge T$.

^aIn other words, the (T-t)-period spot rate at time t.

Standard Notations (concluded)

- f(t, T, L): the L-period forward rate at time T implied at time t stated on a per-period basis and compounded once per period.
- f(t,T): the one-period or instantaneous forward rate at time T as seen at time t stated on a per period basis and compounded once per period.
 - It is f(t, T, 1) in the discrete-time model and f(t, T, dt) in the continuous-time model.
 - Note that f(t,t) equals the short rate r(t).

Fundamental Relations

• The price of a zero-coupon bond equals

$$P(t,T) = \begin{cases} (1+r(t,T))^{-(T-t)}, & \text{in discrete time,} \\ e^{-r(t,T)(T-t)}, & \text{in continuous time.} \end{cases}$$
(145)

- r(t,T) as a function of T defines the spot rate curve at time t.
- By definition,

$$f(t,t) = \begin{cases} r(t,t+1), & \text{in discrete time,} \\ r(t,t), & \text{in continuous time.} \end{cases}$$

• Forward prices and zero-coupon bond prices are related:

$$F(t, T, M) = \frac{P(t, M)}{P(t, T)}, \quad T \le M.$$
 (146)

- The forward price equals the future value at time T of the underlying asset.^a
- The above identity holds whether the model is discrete-time or continuous-time.

^aSee Exercise 24.2.1 of the textbook for proof.

• Forward rates and forward prices are related definitionally by

$$f(t,T,L) = \left(\frac{1}{F(t,T,T+L)}\right)^{1/L} - 1 = \left(\frac{P(t,T)}{P(t,T+L)}\right)^{1/L} - 1$$
(147)

in discrete time.

• The analog under simple compounding is

$$f(t, T, L) = \frac{1}{L} \left(\frac{P(t, T)}{P(t, T + L)} - 1 \right).$$

• In continuous time,

$$f(t,T,L) = -\frac{\ln F(t,T,T+L)}{L} = \frac{\ln(P(t,T)/P(t,T+L))}{L}$$
(148)

by Eq. (146) on p. 1095.

• Furthermore,

$$f(t,T,\Delta t) = \frac{\ln(P(t,T)/P(t,T+\Delta t))}{\Delta t} \to -\frac{\partial \ln P(t,T)}{\partial T}$$
$$= -\frac{\partial P(t,T)/\partial T}{P(t,T)}.$$

• So

$$f(t,T) \stackrel{\Delta}{=} -\frac{\partial \ln P(t,T)}{\partial T} = -\frac{\partial P(t,T)/\partial T}{P(t,T)}, \quad t \le T.$$
(149)

• Because the above identity is equivalent to

$$P(t,T) = e^{-\int_t^T f(t,s) \, ds},\tag{150}$$

the spot rate curve is

$$r(t,T) = \frac{\int_t^T f(t,s) \, ds}{T-t}$$

• The discrete analog to Eq. (150) is

$$P(t,T) = \frac{1}{(1+r(t))(1+f(t,t+1))\cdots(1+f(t,T-1))}$$

• The short rate and the market discount function are related by

$$r(t) = -\left.\frac{\partial P(t,T)}{\partial T}\right|_{T=t}$$

Risk-Neutral Pricing

- Assume the local expectations theory.
- The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.

- For all t + 1 < T,

$$\frac{E_t[P(t+1,T)]}{P(t,T)} = 1 + r(t).$$
(151)

Relation (151) in fact follows from the risk-neutral valuation principle.^a

^aRecall Theorem 17 on p. 566.

Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability π .
- Equation (151) on p. 1100 can also be expressed as

 $E_t[P(t+1,T)] = F(t,t+1,T).$

- Verify that with, e.g., Eq. (146) on p. 1095.

• Hence the forward price for the next period is an unbiased estimator of the expected bond price.^a

 But the forward rate is *not* an unbiased estimator of the expected future short rate.^b

^aUnder the local expectations theory. ^bRecall p. 1049.

Risk-Neutral Pricing (continued)

• Rewrite Eq. (151) on p. 1100 as

$$\frac{E_t^{\pi}[P(t+1,T)]}{1+r(t)} = P(t,T).$$
(152)

 It says the current market discount function equals the expected market discount function one period from now discounted by the short rate.

Risk-Neutral Pricing (concluded)

• Apply the above equality iteratively to obtain

$$P(t,T) = E_t^{\pi} \left[\frac{P(t+1,T)}{1+r(t)} \right]$$

= $E_t^{\pi} \left[\frac{E_{t+1}^{\pi} \left[P(t+2,T) \right]}{(1+r(t))(1+r(t+1))} \right] = \cdots$
= $E_t^{\pi} \left[\frac{1}{(1+r(t))(1+r(t+1))\cdots(1+r(T-1))} \right]$

Continuous-Time Risk-Neutral Pricing

- In continuous time, the local expectations theory implies $P(t,T) = E_t \left[e^{-\int_t^T r(s) \, ds} \right], \quad t < T. \quad (153)$
- Note that $e^{\int_t^T r(s) ds}$ is the bank account process, which denotes the rolled-over money market account.

Interest Rate Swaps

- Consider an interest rate swap made at time t (now) with payments to be exchanged at times t_1, t_2, \ldots, t_n .
- For simplicity, assume $t_{i+1} t_i$ is a fixed constant Δt for all *i*, and the notional principal is one dollar.
- The fixed rate is c per annum.
- The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.
- The payoff at time t_{i+1} for the fixed-rate payer is $(f_i c) \Delta t$.



Interest Rate Swaps (continued)

- Simple rates are adopted here.
- Hence f_i satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$

- If $t < t_0$, we have a forward interest rate swap.
- The ordinary swap corresponds to $t = t_0$.

Interest Rate Swaps (continued)

• The value of the swap at time t is thus

$$\sum_{i=1}^{n} E_{t}^{\pi} \left[e^{-\int_{t}^{t_{i}} r(s) \, ds} (f_{i-1} - c) \, \Delta t \right]$$

$$= \sum_{i=1}^{n} E_{t}^{\pi} \left[e^{-\int_{t}^{t_{i}} r(s) \, ds} \left(\frac{1}{P(t_{i-1}, t_{i})} - (1 + c\Delta t) \right) \right]$$

$$= \sum_{i=1}^{n} E_{t}^{\pi} \left[e^{-\int_{t}^{t_{i}} r(s) \, ds} \left(e^{\int_{t_{i-1}}^{t_{i}} r(s) \, ds} - (1 + c\Delta t) \right) \right]$$

$$= \sum_{i=1}^{n} \left[P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_{i}) \right]$$

$$= P(t, t_{0}) - P(t, t_{n}) - c\Delta t \sum_{i=1}^{n} P(t, t_{i}).$$

Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple PV calculations.

Swap Rate

• The swap rate, which gives the swap zero value, equals

$$S_n(t) \stackrel{\Delta}{=} \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^n P(t, t_i) \,\Delta t}.$$
 (154)

- The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.
- For an ordinary swap, $P(t, t_0) = 1$.
- The swap rate is called a forward swap rate if $t_0 > t$.

The Term Structure Equation $^{\rm a}$

- Let us start with the zero-coupon bonds and the money market account.
- Let the zero-coupon bond price P(r, t, T) follow

$$\frac{dP}{P} = \mu_p \, dt + \sigma_p \, dW.$$

• At time t, short one unit of a bond maturing at time s_1 and buy α units of a bond maturing at time s_2 .

^aVasicek (1977).

• The net wealth change follows

 $-dP(r,t,s_1) + \alpha \, dP(r,t,s_2)$

$$= (-P(r,t,s_1) \mu_p(r,t,s_1) + \alpha P(r,t,s_2) \mu_p(r,t,s_2)) dt + (-P(r,t,s_1) \sigma_p(r,t,s_1) + \alpha P(r,t,s_2) \sigma_p(r,t,s_2)) dW.$$

• Pick

$$\alpha \stackrel{\Delta}{=} \frac{P(r, t, s_1) \,\sigma_p(r, t, s_1)}{P(r, t, s_2) \,\sigma_p(r, t, s_2)}.$$

• Then the net wealth has no volatility and must earn the riskless return:

$$\frac{-P(r,t,s_1)\,\mu_p(r,t,s_1) + \alpha P(r,t,s_2)\,\mu_p(r,t,s_2)}{-P(r,t,s_1) + \alpha P(r,t,s_2)} = r.$$

• Simplify the above to obtain

$$\frac{\sigma_p(r,t,s_1)\,\mu_p(r,t,s_2) - \sigma_p(r,t,s_2)\,\mu_p(r,t,s_1)}{\sigma_p(r,t,s_1) - \sigma_p(r,t,s_2)} = r.$$

• This becomes

$$\frac{\mu_p(r,t,s_2) - r}{\sigma_p(r,t,s_2)} = \frac{\mu_p(r,t,s_1) - r}{\sigma_p(r,t,s_1)}$$

after rearrangement.

• Since the above equality holds for any s_1 and s_2 ,

$$\frac{\mu_p(r,t,s) - r}{\sigma_p(r,t,s)} \stackrel{\Delta}{=} \lambda(r,t)$$
(155)

for some λ independent of the bond maturity s.

- As $\mu_p = r + \lambda \sigma_p$, all assets are expected to appreciate at a rate equal to the sum of the short rate and a constant times the asset's volatility.
- The term $\lambda(r, t)$ is called the market price of risk.
- The market price of risk must be the same for all bonds to preclude arbitrage opportunities.

• Assume a Markovian short rate model,

$$dr = \mu(r, t) \, dt + \sigma(r, t) \, dW.$$

- Then the bond price process is also Markovian.
- By Eq. (14.15) on p. 202 of the textbook,

$$\mu_p = \left[-\frac{\partial P}{\partial T} + \mu(r,t) \frac{\partial P}{\partial r} + \frac{\sigma(r,t)^2}{2} \frac{\partial^2 P}{\partial r^2} \right] / P,$$
(156)

$$\sigma_p = \sigma(r, t) \,\frac{\partial P}{\partial r} / P,\tag{156'}$$

subject to
$$P(\cdot, T, T) = 1$$
.

• Substitute μ_p and σ_p into Eq. (155) on p. 1114 to obtain

$$-\frac{\partial P}{\partial T} + \left[\mu(r,t) - \lambda(r,t)\,\sigma(r,t)\right]\frac{\partial P}{\partial r} + \frac{1}{2}\,\sigma(r,t)^2\,\frac{\partial^2 P}{\partial r^2} = rP.$$
(157)

- This is called the term structure equation.
- It applies to all interest rate derivatives: The differences are the terminal and boundary conditions.
- Once P is available, the spot rate curve emerges via

$$r(t,T) = -\frac{\ln P(t,T)}{T-t}.$$

Numerical Examples

• Assume this spot rate curve:

Year	1	2
Spot rate	4%	5%

• Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:



Numerical Examples (continued)

- No real-world probabilities are given.
- The prices of one- and two-year zero-coupon bonds are, respectively,

$$100/1.04 = 96.154,$$

 $100/(1.05)^2 = 90.703.$

• They follow the binomial processes on p. 1119.



Numerical Examples (continued)

- The pricing of derivatives can be simplified by assuming investors are risk-neutral.
- Suppose all securities have the same expected one-period rate of return, the riskless rate.
- Then

$$(1-p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\%,$$

where p denotes the risk-neutral probability of a down move in rates.

Numerical Examples (concluded)

- Solving the equation leads to p = 0.319.
- Interest rate contingent claims can be priced under this probability.

Numerical Examples: Fixed-Income Options

• A one-year European call on the two-year zero with a \$95 strike price has the payoffs,



• To solve for the option value C, we replicate the call by a portfolio of x one-year and y two-year zeros.
Numerical Examples: Fixed-Income Options (continued)

• This leads to the simultaneous equations,

$$x \times 100 + y \times 92.593 = 0.000,$$

 $x \times 100 + y \times 98.039 = 3.039.$

- They give x = -0.5167 and y = 0.5580.
- Consequently,

$$C = x \times 96.154 + y \times 90.703 \approx 0.93$$

to prevent arbitrage.

Numerical Examples: Fixed-Income Options (continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.

Numerical Examples: Fixed-Income Options (concluded)

- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth

$$C = \frac{(1-p) \times 0 + p \times 3.039}{1.04} \approx 0.93,$$

the same as before.

• This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.

Numerical Examples: Futures and Forward Prices

• A one-year futures contract on the one-year rate has a payoff of 100 - r, where r is the one-year rate at maturity:



• As the futures price F is the expected future payoff,^a

$$F = (1 - p) \times 92 + p \times 98 = 93.914.$$

^aSee Exercise 13.2.11 of the textbook or p. 567.

Numerical Examples: Futures and Forward Prices (concluded)

• The forward price for a one-year forward contract on a one-year zero-coupon bond is^a

90.703/96.154 = 94.331%.

• The forward price exceeds the futures price.^b

^aBy Eq. (146) on p. 1095. ^bUnlike the nonstochastic case on p. 509.

Equilibrium Term Structure Models

The nature of modern trade is to give to those who have much and take from those who have little. — Walter Bagehot (1867), *The English Constitution*

8. What's your problem? Any moron can understand bond pricing models.
— Top Ten Lies Finance Professors Tell Their Students

Introduction

- We now survey equilibrium models.
- Recall that the spot rates satisfy

$$r(t,T) = -\frac{\ln P(t,T)}{T-t}$$

by Eq. (145) on p. 1094.

- Hence the discount function P(t,T) suffices to establish the spot rate curve.
- All models to follow are short rate models.
- Unless stated otherwise, the processes are risk-neutral.

The Vasicek Model $^{\rm a}$

• The short rate follows

$$dr = \beta(\mu - r) \, dt + \sigma \, dW.$$

- The short rate is pulled to the long-term mean level μ at rate β .
- Superimposed on this "pull" is a normally distributed stochastic term σdW .
- Since the process is an Ornstein-Uhlenbeck process,

$$E[r(T) | r(t) = r] = \mu + (r - \mu) e^{-\beta(T-t)}$$

from Eq. (89) on p. 634.

^aVasicek (1977). Vasicek co-founded KMV, which was sold to Moody's for USD\$210 million in 2002.

The Vasicek Model (continued)

• The price of a zero-coupon bond paying one dollar at maturity can be shown to be

$$P(t,T) = A(t,T) e^{-B(t,T) r(t)}, \qquad (158)$$

where

$$A(t,T) = \begin{cases} \exp\left[\frac{(B(t,T) - T + t)(\beta^2 \mu - \sigma^2/2)}{\beta^2} - \frac{\sigma^2 B(t,T)^2}{4\beta}\right], & \text{if } \beta \neq 0, \\\\ \exp\left[\frac{\sigma^2 (T - t)^3}{6}\right], & \text{if } \beta = 0, \end{cases}$$

and

$$B(t,T) = \begin{cases} \frac{1-e^{-\beta(T-t)}}{\beta}, & \text{if } \beta \neq 0, \\ T-t, & \text{if } \beta = 0. \end{cases}$$

The Vasicek Model (continued)

- If $\beta = 0$, then P goes to infinity as $T \to \infty$.
- Sensibly, P goes to zero as $T \to \infty$ if $\beta \neq 0$.
- But even if $\beta \neq 0$, P may exceed one for a finite T.
- The long rate $r(t, \infty)$ is the constant

$$\mu - \frac{\sigma^2}{2\beta^2},$$

independent of the current short rate.

The Vasicek Model (concluded)

• The spot rate volatility structure is the curve

$$\sigma \, \frac{\partial r(t,T)}{\partial r} = \frac{\sigma B(t,T)}{T-t}.$$

- As it depends only on T t not on t by itself, the same curve is maintained for any future time t.
- When β > 0, the curve tends to decline with maturity.
 The long rate's volatility is zero unless β = 0.
- The speed of mean reversion, β, controls the shape of the curve.
- Higher β leads to greater attenuation of volatility with maturity.



The Vasicek Model: Options on Zeros^a

- Consider a European call with strike price X expiring at time T on a zero-coupon bond with par value \$1 and maturing at time s > T.
- Its price is given by

$$P(t,s) N(x) - XP(t,T) N(x - \sigma_v).$$

^aJamshidian (1989).

The Vasicek Model: Options on Zeros (concluded)Above

$$\begin{aligned} x &\triangleq \frac{1}{\sigma_v} \ln\left(\frac{P(t,s)}{P(t,T)X}\right) + \frac{\sigma_v}{2}, \\ \sigma_v &\equiv v(t,T) B(T,s), \\ v(t,T)^2 &\triangleq \begin{cases} \frac{\sigma^2 \left[1 - e^{-2\beta(T-t)}\right]}{2\beta}, & \text{if } \beta \neq 0 \\ \sigma^2(T-t), & \text{if } \beta = 0 \end{cases} \end{aligned}$$

• By the put-call parity, the price of a European put is $XP(t,T) N(-x+\sigma_v) - P(t,s) N(-x).$

${\sf Binomial}\ {\sf Vasicek^a}$

• Consider a binomial model for the short rate in the time interval [0,T] divided into n identical pieces.

• Let
$$\Delta t \stackrel{\Delta}{=} T/n$$
 and^b

$$p(r) \stackrel{\Delta}{=} \frac{1}{2} + \frac{\beta(\mu - r)\sqrt{\Delta t}}{2\sigma}$$

• The following binomial model converges to the Vasicek model,^c

$$r(k+1) = r(k) + \sigma \sqrt{\Delta t} \ \xi(k), \quad 0 \le k < n.$$

^aNelson & Ramaswamy (1990).

^bThe same form as Eq. (42) on p. 296 for the BOPM.

^cSame as the CRR tree except that the probabilities vary here.

Binomial Vasicek (continued)

• Above, $\xi(k) = \pm 1$ with

$$\operatorname{Prob}[\xi(k) = 1] = \begin{cases} p(r(k)), & \text{if } 0 \le p(r(k)) \le 1 \\ 0, & \text{if } p(r(k)) < 0, \\ 1, & \text{if } 1 < p(r(k)). \end{cases}$$

- Observe that the probability of an up move, p, is a decreasing function of the interest rate r.
- This is consistent with mean reversion.

Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its constant volatility, σ .

The Cox-Ingersoll-Ross Model^a

• It is the following square-root short rate model:

$$dr = \beta(\mu - r) dt + \sigma \sqrt{r} dW.$$
 (159)

- The diffusion differs from the Vasicek model by a multiplicative factor \sqrt{r} .
- The parameter β determines the speed of adjustment.
- If r(0) > 0, then the short rate can reach zero only if

$$2\beta\mu < \sigma^2.$$

- This is called the Feller (1951) condition.

• See text for the bond pricing formula.

^aCox, Ingersoll, & Ross (1985).

Binomial CIR

- We want to approximate the short rate process in the time interval [0, T].
- Divide it into n periods of duration $\Delta t \stackrel{\Delta}{=} T/n$.
- Assume $\mu, \beta \ge 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will *not* combine.

Binomial CIR (continued)

• Instead, consider the transformed process^a

$$x(r) \stackrel{\Delta}{=} 2\sqrt{r}/\sigma$$

• By Ito's lemma (p. 609),

$$dx = m(x) \, dt + dW,$$

where

$$m(x) \stackrel{\Delta}{=} 2\beta \mu / (\sigma^2 x) - (\beta x/2) - 1/(2x).$$

- This new process has a *constant* volatility.
- Thus its binomial tree combines.

^aSee pp. 1153ff for justification.

Binomial CIR (continued)

- Construct the combining tree for r as follows.
- First, construct a tree for x.
- Then transform each node of the tree into one for r via the inverse transformation (see next page)

$$r = f(x) \stackrel{\Delta}{=} \frac{x^2 \sigma^2}{4}.$$

• But when $x \approx 0$ (so $r \approx 0$), the moments may not be matched well.^a

^aNawalkha & Beliaeva (2007).



Binomial CIR (continued)

• The probability of an up move at each node r is

$$p(r) \stackrel{\Delta}{=} \frac{\beta(\mu - r)\,\Delta t + r - r^-}{r^+ - r^-}$$

 $-r^+ \stackrel{\Delta}{=} f(x + \sqrt{\Delta t})$ denotes the result of an up move from r.

$$-r^{-} \stackrel{\Delta}{=} f(x - \sqrt{\Delta t})$$
 the result of a down move.

• Finally, set the probability p(r) to one as r goes to zero to make the probability stay between zero and one.

Binomial CIR (concluded)

• It can be shown that

$$p(r) = \left(\beta\mu - \frac{\sigma^2}{4}\right)\sqrt{\frac{\Delta t}{r}} - B\sqrt{r\Delta t} + C,$$

for some $B \ge 0$ and C > 0.^a

- If $\beta \mu (\sigma^2/4) \ge 0$, the up-move probability p(r) decreases if and only if short rate r increases.
- Even if $\beta \mu (\sigma^2/4) < 0$, p(r) tends to decrease as r increases and decrease as r declines.
- This phenomenon agrees with mean reversion.

^aThanks to a lively class discussion on May 28, 2014.

Numerical Examples

• Consider the process,

$$0.2\,(0.04 - r)\,dt + 0.1\sqrt{r}\,dW,$$

for the time interval [0,1] given the initial rate r(0) = 0.04.

- We shall use $\Delta t = 0.2$ (year) for the binomial approximation.
- See p. 1149(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.



Numerical Examples (concluded)

- Consider the node which is the result of an up move from the root.
- Since the root has $x = 2\sqrt{r(0)}/\sigma = 4$, this particular node's x value equals $4 + \sqrt{\Delta t} = 4.4472135955$.
- Use the inverse transformation to obtain the short rate $\frac{x^2 \times (0.1)^2}{4} \approx 0.0494442719102.$
- Once the short rates are in place, computing the probabilities is easy.
- Convergence is quite good.^a

^aSee p. 369 of the textbook.

Trinomial CIR

- The binomial CIR tree does not have the degree of freedom to match the mean and variance exactly.
- It actually fails to match them at very low x.
- A trinomial tree for the CIR model with O(n^{1.5}) nodes that matches the mean and variance exactly is recently obtained using the ideas on pp. 803ff and others.^a

^aZ. Lu (D00922011) & Lyuu (2018); H. Huang (R03922103) (2019).



A General Method for Constructing Binomial Models $^{\rm a}$

• We are given a continuous-time process,

$$dy = \alpha(y, t) \, dt + \sigma(y, t) \, dW.$$

- Need to make sure the binomial model's drift and diffusion converge to the above process.
- Set the probability of an up move to

$$\frac{\alpha(y,t)\,\Delta t + y - y_{\rm d}}{y_{\rm u} - y_{\rm d}}$$

• Here
$$y_{\rm u} \stackrel{\Delta}{=} y + \sigma(y, t) \sqrt{\Delta t}$$
 and $y_{\rm d} \stackrel{\Delta}{=} y - \sigma(y, t) \sqrt{\Delta t}$
represent the two rates that follow the current rate y

^aNelson & Ramaswamy (1990).

A General Method (continued)

- The displacements are identical, at $\sigma(y,t)\sqrt{\Delta t}$.
- But the binomial tree may not combine as

$$\sigma(y,t)\sqrt{\Delta t} - \sigma(y_{\rm u},t+\Delta t)\sqrt{\Delta t}$$

$$\neq -\sigma(y,t)\sqrt{\Delta t} + \sigma(y_{\rm d},t+\Delta t)\sqrt{\Delta t}$$

in general.

• When $\sigma(y, t)$ is a constant independent of y, equality holds and the tree combines.

A General Method (continued)

• To achieve this, define the transformation

$$x(y,t) \stackrel{\Delta}{=} \int^{y} \sigma(z,t)^{-1} dz.$$

• Then x follows

$$dx = m(y,t) \, dt + dW$$

for some m(y,t).^a

• The diffusion term is now a constant, and the binomial tree for x combines.

^aSee Exercise 25.2.13 of the textbook.

A General Method (concluded)

- The transformation is unique.^a
- The probability of an up move remains

$$\frac{\alpha(y(x,t),t)\,\Delta t + y(x,t) - y_{\mathrm{d}}(x,t)}{y_{\mathrm{u}}(x,t) - y_{\mathrm{d}}(x,t)},$$

where y(x,t) is the inverse transformation of x(y,t)from x back to y.

• Note that

$$y_{\rm u}(x,t) \stackrel{\Delta}{=} y(x + \sqrt{\Delta t}, t + \Delta t),$$

$$y_{\rm d}(x,t) \stackrel{\Delta}{=} y(x - \sqrt{\Delta t}, t + \Delta t).$$

^aH. Chiu (**R98723059**) (2012).



• The transformation is

$$\int^r (\sigma\sqrt{z})^{-1} dz = \frac{2\sqrt{r}}{\sigma}$$

for the CIR model.

• The transformation is

$$\int^{S} (\sigma z)^{-1} dz = \frac{\ln S}{\sigma}$$

for the Black-Scholes model $dS = \mu S dt + \sigma S dW$.

• The familiar BOPM and CRR discretize $\ln S$ not S.

On One-Factor Short Rate Models

- By using only the short rate, they ignore other rates on the yield curve.
- Such models also restrict the volatility to be a function of interest rate *levels* only.
- The prices of all bonds move in the same direction at the same time (their magnitudes may differ).
- The returns on all bonds thus become highly correlated.
- In reality, there seems to be a certain amount of independence between short- and long-term rates.
On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.
- Derivatives whose values depend on the correlation structure will be mispriced.
- The calibrated models may not generate term structures as concave as the data suggest.
- The term structure empirically changes in slope and curvature as well as makes parallel moves.
- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.

On One-Factor Short Rate Models (concluded)

- Multifactor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.
- But they are much harder to think about and work with.
- They also take much more computer time—the curse of dimensionality.
- These practical concerns limit the use of multifactor models to two- or three-factor ones.^a

^aKamakura(2019)hasa10-factorHJMmodelfortheU.S.Treasuries(seehttp://www.kamakuraco.com/KamakuraReleasesNewStochasticVolatilityHodel.aspx).

Options on Coupon $\mathsf{Bonds}^{\mathrm{a}}$

- Assume the market discount function P is a monotonically decreasing function of the short rate r.
 Such as a one-factor short rate model.
- The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.
- Consider a European call expiring at time T on a bond with par value \$1.
- Let X denote the strike price.

^aJamshidian (1989).

Options on Coupon Bonds (continued)

- The bond has cash flows c_1, c_2, \ldots, c_n at times t_1, t_2, \ldots, t_n , where $t_i > T$ for all i.
- The payoff for the option is

$$\max\left\{\left[\sum_{i=1}^{n} c_i P(r(T), T, t_i)\right] - X, 0\right\}.$$

 At time T, there is a unique value r* for r(T) that renders the coupon bond's price equal the strike price X.

Options on Coupon Bonds (continued)

• This r^* can be obtained by solving

$$X = \sum_{i=1}^{n} c_i P(r, T, t_i)$$

numerically for r.

• Let

$$X_i \stackrel{\Delta}{=} P(r^*, T, t_i),$$

the value at time T of a zero-coupon bond with par value \$1 and maturing at time t_i if $r(T) = r^*$.

• Note that $P(r, T, t_i) \ge X_i$ if and only if $r \le r^*$.

Options on Coupon Bonds (concluded)
As
$$X = \sum_{i} c_i X_i$$
, the option's payoff equals

$$\max\left\{ \left[\sum_{i=1}^{n} c_i P(r(T), T, t_i) \right] - \left[\sum_{i=1}^{n} c_i X_i \right], 0 \right\}$$

$$= \sum_{i=1}^{n} c_i \times \max(P(r(T), T, t_i) - X_i, 0).$$

- Thus the call is a package of n options on the underlying zero-coupon bond.
- Why can't we do the same thing for Asian options?^a

^aContributed by Mr. Yang, Jui-Chung (D97723002) on May 20, 2009.

No-Arbitrage Term Structure Models

How much of the structure of our theories really tells us about things in nature, and how much do we contribute ourselves? — Arthur Eddington (1882–1944)

How can I apply this modelif I don't understand it?— Edward I. Altman (2019)

Motivations

- Recall the difficulties facing equilibrium models mentioned earlier.
 - They usually require the estimation of the market price of risk.^a
 - They cannot fit the market term structure.
 - But consistency with the market is often mandatory in practice.

^aRecall p. 1114.

No-Arbitrage Models

- No-arbitrage models utilize the full information of the term structure.
- They accept the observed term structure as consistent with an unobserved and unspecified equilibrium.
- From there, arbitrage-free movements of interest rates or bond prices over time are modeled.
- By definition, the market price of risk must be reflected in the current term structure; hence the resulting interest rate process is risk-neutral.

No-Arbitrage Models (concluded)

- No-arbitrage models can specify the dynamics of zero-coupon bond prices, forward rates, or the short rate.
- Bond price and forward rate models are usually non-Markovian (path dependent).
- In contrast, short rate models are generally constructed to be explicitly Markovian (path independent).
- Markovian models are easier to handle computationally.

The Ho-Lee $\mathsf{Model}^{\mathrm{a}}$

- The short rates at any given time are evenly spaced.
- Let *p* denote the risk-neutral probability that the short rate makes an up move.
- We shall adopt continuous compounding.

^aT. Ho & S. B. Lee (1986).



The Ho-Lee Model (continued)

- The Ho-Lee model starts with zero-coupon bond prices $P(t, t+1), P(t, t+2), \ldots$ at time t identified with the root of the tree.
- Let the discount factors in the next period be

 $P_{\rm d}(t+1,t+2), P_{\rm d}(t+1,t+3), \dots, \qquad \text{if short rate moves down,}$ $P_{\rm u}(t+1,t+2), P_{\rm u}(t+1,t+3), \dots, \qquad \text{if short rate moves up.}$

• By backward induction, it is not hard to see that for $n \geq 2,^{a}$

$$P_{\rm u}(t+1,t+n) = P_{\rm d}(t+1,t+n) e^{-(v_2+\dots+v_n)}.$$
(160)

^aSee p. 376 of the textbook.

The Ho-Lee Model (continued)

• It is also not hard to check that the *n*-period zero-coupon bond has yields

$$y_{d}(n) \stackrel{\Delta}{=} -\frac{\ln P_{d}(t+1,t+n)}{n-1}$$

$$y_{u}(n) \stackrel{\Delta}{=} -\frac{\ln P_{u}(t+1,t+n)}{n-1} = y_{d}(n) + \frac{v_{2} + \dots + v_{n}}{n-1}$$

• The volatility of the yield to maturity for this bond is therefore

$$\kappa_n \stackrel{\Delta}{=} \sqrt{py_u(n)^2 + (1-p)y_d(n)^2 - [py_u(n) + (1-p)y_d(n)]^2} \\ = \sqrt{p(1-p)} (y_u(n) - y_d(n)) \\ = \sqrt{p(1-p)} \frac{v_2 + \dots + v_n}{n-1}.$$

The Ho-Lee Model (concluded)

• In particular, the short rate volatility is determined by taking n = 2:

$$\sigma = \sqrt{p(1-p)} v_2. \tag{161}$$

• The volatility of the short rate therefore equals

$$\sqrt{p(1-p)} \left(r_{\rm u} - r_{\rm d} \right),$$

where $r_{\rm u}$ and $r_{\rm d}$ are the two successor rates.^a

^aContrast this with the lognormal model (138) on p. 1033.

The Ho-Lee Model: Volatility Term Structure

• The volatility term structure is composed of

 $\kappa_2, \kappa_3, \ldots$

- The volatility structure is supplied by the market.
- For the Ho-Lee model, it is independent of

 r_2, r_3, \ldots

- It is easy to compute the v_i s from the volatility structure, and vice versa.^a
- The r_i s can be computed by forward induction.

^aReview p. 1173.

The Ho-Lee Model: Bond Price Process

• In a risk-neutral economy, the initial discount factors satisfy^a

 $P(t,t+n) = [pP_{u}(t+1,t+n) + (1-p)P_{d}(t+1,t+n)]P(t,t+1).$

• Combine the above with Eq. (160) on p. 1172 and assume p = 1/2 to obtain^b

$$P_{\rm d}(t+1,t+n) = \frac{P(t,t+n)}{P(t,t+1)} \frac{2 \times \exp[v_2 + \dots + v_n]}{1 + \exp[v_2 + \dots + v_n]},$$
$$P_{\rm u}(t+1,t+n) = \frac{P(t,t+n)}{P(t,t+1)} \frac{2}{1 + \exp[v_2 + \dots + v_n]}.$$

^aRecall Eq. (152) on p. 1102.

^bIn the limit, only the volatility matters; the first formula is similar to multiple logistic regression.

The Ho-Lee Model: Bond Price Process (concluded)

- The bond price tree combines.^a
- Suppose all v_i equal some constant v and $\delta \stackrel{\Delta}{=} e^v > 0$.
- Then

$$P_{\rm d}(t+1,t+n) = \frac{P(t,t+n)}{P(t,t+1)} \frac{2\delta^{n-1}}{1+\delta^{n-1}},$$

$$P_{\rm u}(t+1,t+n) = \frac{P(t,t+n)}{P(t,t+1)} \frac{2}{1+\delta^{n-1}}.$$

- Short rate volatility $\sigma = v/2$ by Eq. (161) on p. 1174.
- Price derivatives by taking expectations under the risk-neutral probability.

^aSee Exercise 26.2.3 of the textbook.

Calibration

- The Ho-Lee model can be calibrated in $O(n^2)$ time using state prices.
- But it can actually be calibrated in O(n) time.^a
 - Derive the v_i 's in linear time.
 - Derive the r_i 's in linear time.

^aSee Programming Assignment 26.2.6 of the textbook.

The Ho-Lee Model: Yields and Their Covariances

• The one-period rate of return of an n-period zero-coupon bond is^a

$$r(t,t+n) \stackrel{\Delta}{=} \ln\left(\frac{P(t+1,t+n)}{P(t,t+n)}\right).$$

• Its two possible value are

$$\ln \frac{P_{\rm d}(t+1,t+n)}{P(t,t+n)}$$
 and $\ln \frac{P_{\rm u}(t+1,t+n)}{P(t,t+n)}$.

• Thus the variance of return is^b

Var[
$$r(t, t+n)$$
] = $p(1-p)[(n-1)v]^2 = (n-1)^2\sigma^2$.

^aSo r(t, t + n) does not mean the *n*-period spot rate at time *t* here. ^bRecall that σ is the short rate volatility by Eq. (161) on p. 1174.

The Ho-Lee Model: Yields and Their Covariances (concluded)

• The covariance between r(t, t+n) and r(t, t+m) is^a

$$(n-1)(m-1)\,\sigma^2.$$

- As a result, the correlation between any two one-period rates of return is one.
- Strong correlation between rates is inherent in all one-factor Markovian models.

^aSee Exercise 26.2.7 of the textbook.

The Ho-Lee Model: Short Rate Process

• The continuous-time limit of the Ho-Lee model is^a

$$dr = \theta(t) \, dt + \sigma \, dW. \tag{162}$$

- This is Vasicek's model with the mean-reverting drift replaced by a deterministic, time-dependent drift.
- A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying,

$$dr = \theta(t) \, dt + \sigma(t) \, dW.$$

• This corresponds to the discrete-time model in which v_i are not all identical.

^aSee Exercise 26.2.10 of the textbook.

The Ho-Lee Model: Some Problems

- Future (nominal) interest rates may be negative.
- The short rate volatility is independent of the rate level.
- It has all the problems associated with a one-factor model.^a

^aRecall pp. 1158ff. See T. Ho & S. B. Lee (2004) for a multifactor Ho-Lee model.

Problems with No-Arbitrage Models in General

- Interest rate movements should reflect shifts in the model's state variables (factors) not its parameters.
- Model *parameters*, such as the drift θ(t) in the continuous-time Ho-Lee model, should be stable over time.
- But in practice, no-arbitrage models capture yield curve shifts through the recalibration of parameters.
 - A new model is thus born every day.

Problems with No-Arbitrage Models in General (concluded)

- This in effect says the model estimated at some time does not describe the term structure of interest rates and their volatilities at other times.
- Consequently, a model's intertemporal behavior is suspect, and using it for hedging and risk management may be unreliable.

The Black-Derman-Toy Model^a

- This model is extensively used by practitioners.
- The BDT short rate process is the lognormal binomial interest rate process described on pp. 1029ff.^b
- The volatility structure^c is given by the market.
- From it, the short rate volatilities (thus v_i) are determined together with the baseline rates r_i .

^aBlack, Derman, & Toy (BDT) (1990), but essentially finished in 1986 according to Mehrling (2005).

^bRepeated on next page. ^cRecall Eq. (144) on p. 1080.



The Black-Derman-Toy Model (concluded)

- Our earlier binomial interest rate tree, in contrast, assumes v_i are given a priori.
- Lognormal models preclude negative short rates.

The BDT Model: Volatility Structure

- The volatility structure defines the yield volatilities of zero-coupon bonds of various maturities.
- Let the yield volatility of the *i*-period zero-coupon bond be denoted by κ_i .
- $P_{\rm u}$ is the price of the *i*-period zero-coupon bond one period from now if the short rate makes an up move.
- $P_{\rm d}$ is the price of the *i*-period zero-coupon bond one period from now if the short rate makes a down move.

The BDT Model: Volatility Structure (concluded)

• Corresponding to these two prices are the following yields to maturity,

$$y_{\rm u} \stackrel{\Delta}{=} P_{\rm u}^{-1/(i-1)} - 1,$$
$$y_{\rm d} \stackrel{\Delta}{=} P_{\rm d}^{-1/(i-1)} - 1.$$

• The yield volatility is defined as^a

$$\kappa_i \stackrel{\Delta}{=} \frac{\ln(y_{\rm u}/y_{\rm d})}{2}$$

^aRyecall Eq. (144) on p. 1080.

The BDT Model: Calibration

- The inputs to the BDT model are riskless zero-coupon bond yields and their volatilities.
- For economy of expression, all numbers are period based.
- Suppose inductively that we have calculated

$$(r_1, v_1), (r_2, v_2), \ldots, (r_{i-1}, v_{i-1}).$$

- They define the binomial tree up to time i 2 (thus period i 1).^a
- Thus the spot rates up to time i 1 have been matched.

^aRecall that (r_{i-1}, v_{i-1}) defines i-1 short rates at time i-2, which are applicable to period i-1.

- We now proceed to calculate r_i and v_i to extend the tree to cover period i.
- Assume the price of the *i*-period zero can move to $P_{\rm u}$ or $P_{\rm d}$ one period from now.
- Let y denote the current *i*-period spot rate, which is known.
- In a risk-neutral economy,

$$\frac{P_{\rm u} + P_{\rm d}}{2(1+r_i)} = \frac{1}{(1+y)^i}.$$
(163)

• Obviously, $P_{\rm u}$ and $P_{\rm d}$ are functions of the unknown r_i and v_i .

- Viewed from now, the future (i-1)-period spot rate at time 1 is uncertain.
- Recall that y_u and y_d represent the spot rates at the up node and the down node, respectively.^a
- With κ_i^2 denoting their variance, we have

$$\kappa_i = \frac{1}{2} \ln \left(\frac{{P_{\rm u}}^{-1/(i-1)} - 1}{{P_{\rm d}}^{-1/(i-1)} - 1} \right).$$
(164)

^aRecall p. 1189.

- Solving Eqs. (163)–(164) for r_i and v_i with backward induction takes $O(i^2)$ time.
 - That leads to a cubic-time algorithm.
- We next employ forward induction to derive a quadratic-time calibration algorithm.^a
- Forward induction figures out, by moving *forward* in time, how much \$1 at a node contributes to the price.^b
- This number is called the state price and is the price of the claim that pays \$1 at that node and zero elsewhere.

^aW. J. Chen (**R84526007**) & Lyuu (1997); Lyuu (1999). ^bReview p. 1057(a).

- Let the unknown baseline rate for period i be $r_i = r$.
- Let the unknown multiplicative ratio be $v_i = v$.
- Let the state prices at time i-1 be

$$P_1, P_2, \ldots, P_i$$
.

• They correspond to rates

$$r, rv, \ldots, rv^{i-1}$$

for period i, respectively.

• One dollar at time *i* has a present value of $f(r,v) \stackrel{\Delta}{=} \frac{P_1}{1+r} + \frac{P_2}{1+rv} + \frac{P_3}{1+rv^2} + \dots + \frac{P_i}{1+rv^{i-1}}.$
The BDT Model: Calibration (continued)

• By Eq. (164) on p. 1192, the yield volatility is

$$g(r,v) \stackrel{\Delta}{=} \frac{1}{2} \ln \left(\frac{\left(\frac{P_{\mathrm{u},1}}{1+rv} + \frac{P_{\mathrm{u},2}}{1+rv^2} + \dots + \frac{P_{\mathrm{u},i-1}}{1+rv^{i-1}}\right)^{-1/(i-1)} - 1}{\left(\frac{P_{\mathrm{d},1}}{1+r} + \frac{P_{\mathrm{d},2}}{1+rv} + \dots + \frac{P_{\mathrm{d},i-1}}{1+rv^{i-2}}\right)^{-1/(i-1)} - 1} \right)$$

- Above, $P_{u,1}, P_{u,2}, \ldots$ denote the state prices at time i-1 of the subtree rooted at the up node.^a
- And $P_{d,1}, P_{d,2}, \ldots$ denote the state prices at time i-1of the subtree rooted at the down node.^b

^aLike r_2v_2 on p. 1186. ^bLike r_2 on p. 1186.

The BDT Model: Calibration (concluded)

- Note that every node maintains three state prices: $P_i, P_{u,i}, P_{d,i}$.
- Now solve

$$f(r,v) = \frac{1}{(1+y)^i},$$

$$g(r,v) = \kappa_i,$$

for $r = r_i$ and $v = v_i$.

• This $O(n^2)$ -time algorithm appears on p. 382 of the textbook.

Calibrating the BDT Model with the Differential Tree (in seconds)^{\rm a}

Number	Running	Number	Running	Number	Running
of years	time	of years	time	of years	time
3000	398.880	39000	8562.640	75000	26182.080
6000	1697.680	42000	9579.780	78000	28138.140
9000	2539.040	45000	10785.850	81000	30230.260
12000	2803.890	48000	11905.290	84000	32317.050
15000	3149.330	51000	13199.470	87000	34487.320
18000	3549.100	54000	14411.790	90000	36795.430
21000	3990.050	57000	15932.370	120000	63767.690
24000	4470.320	60000	17360.670	150000	98339.710
27000	5211.830	63000	19037.910	180000	140484.180
30000	5944.330	66000	20751.100	210000	190557.420
33000	6639.480	69000	22435.050	240000	249138.210
36000	7611.630	72000	24292.740	270000	313480.390

75MHz Sun SPARCstation 20, one period per year.

^aLyuu (1999).

The BDT Model: Continuous-Time Limit

• The continuous-time limit of the BDT model is^a

$$d\ln r = \left(\theta(t) + \frac{\sigma'(t)}{\sigma(t)}\ln r\right) dt + \sigma(t) dW.$$

- The short rate volatility σ(t) should be a declining function of time for the model to display mean reversion.
 That makes σ'(t) < 0.
- In particular, constant $\sigma(t)$ will not attain mean reversion.

^aJamshidian (1991).

The Black-Karasinski Model^a

• The BK model stipulates that the short rate follows

$$d\ln r = \kappa(t)(\theta(t) - \ln r) dt + \sigma(t) dW.$$

- This explicitly mean-reverting model depends on time through $\kappa(\cdot)$, $\theta(\cdot)$, and $\sigma(\cdot)$.
- The BK model hence has one more degree of freedom than the BDT model.
- The speed of mean reversion $\kappa(t)$ and the short rate volatility $\sigma(t)$ are independent.

^aBlack & Karasinski (1991).

The Black-Karasinski Model: Discrete Time

- The discrete-time version of the BK model has the same representation as the BDT model.
- To maintain a combining binomial tree, however, requires some manipulations.
- The next plot illustrates the ideas in which

$$\begin{array}{rcl} t_2 & \stackrel{\Delta}{=} & t_1 + \Delta t_1, \\ t_3 & \stackrel{\Delta}{=} & t_2 + \Delta t_2. \end{array}$$



The Black-Karasinski Model: Discrete Time (continued)

• Note that

 $\ln r_{\rm d}(t_2) = \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1)) \Delta t_1 - \sigma(t_1) \sqrt{\Delta t_1}, \\ \ln r_{\rm u}(t_2) = \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1)) \Delta t_1 + \sigma(t_1) \sqrt{\Delta t_1}.$

• To make sure an up move followed by a down move coincides with a down move followed by an up move,

$$\ln r_{\rm d}(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_{\rm d}(t_2)) \Delta t_2 + \sigma(t_2)\sqrt{\Delta t_2},$$

= $\ln r_{\rm u}(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_{\rm u}(t_2)) \Delta t_2 - \sigma(t_2)\sqrt{\Delta t_2}.$

The Black-Karasinski Model: Discrete Time (continued)

• They imply

$$\kappa(t_2) = \frac{1 - (\sigma(t_2)/\sigma(t_1))\sqrt{\Delta t_2/\Delta t_1}}{\Delta t_2}.$$
(165)

• So from Δt_1 , we can calculate the Δt_2 that satisfies the combining condition and then iterate.

$$-t_0 \to \Delta t_1 \to t_1 \to \Delta t_2 \to t_2 \to \Delta t_3 \to \dots \to T$$

(roughly).^a

^aAs $\kappa(t), \theta(t), \sigma(t)$ are independent of r, the Δt_i will not depend on r either.

The Black-Karasinski Model: Discrete Time (concluded)

• Unequal durations Δt_i are often necessary to ensure a combining tree.^a

^aAmin (1991); C. I. Chen (**R98922127**) (2011); Lok (**D99922028**) & Lyuu (2016, 2017).

Problems with Lognormal Models in General

- Lognormal models such as BDT and BK share the problem that $E^{\pi}[M(t)] = \infty$ for any finite t if they model the continuously compounded rate.^a
- So periodically compounded rates should be modeled.^b
- Another issue is computational.
- Lognormal models usually do not admit of analytical solutions to even basic fixed-income securities.
- As a result, to price short-dated derivatives on long-term bonds, the tree has to be built over the life of the underlying asset instead of the life of the derivative.

^aHogan & Weintraub (1993).

^bSandmann & Sondermann (1993).

Problems with Lognormal Models in General (concluded)

- This problem can be somewhat mitigated by adopting variable-duration time steps.^a
 - Use a fine time step up to the maturity of the short-dated derivative.
 - Use a coarse time step beyond the maturity.
- A down side of this procedure is that it has to be tailor-made for each derivative.
- Finally, empirically, interest rates do not follow the lognormal distribution.

^aHull & White (1993).