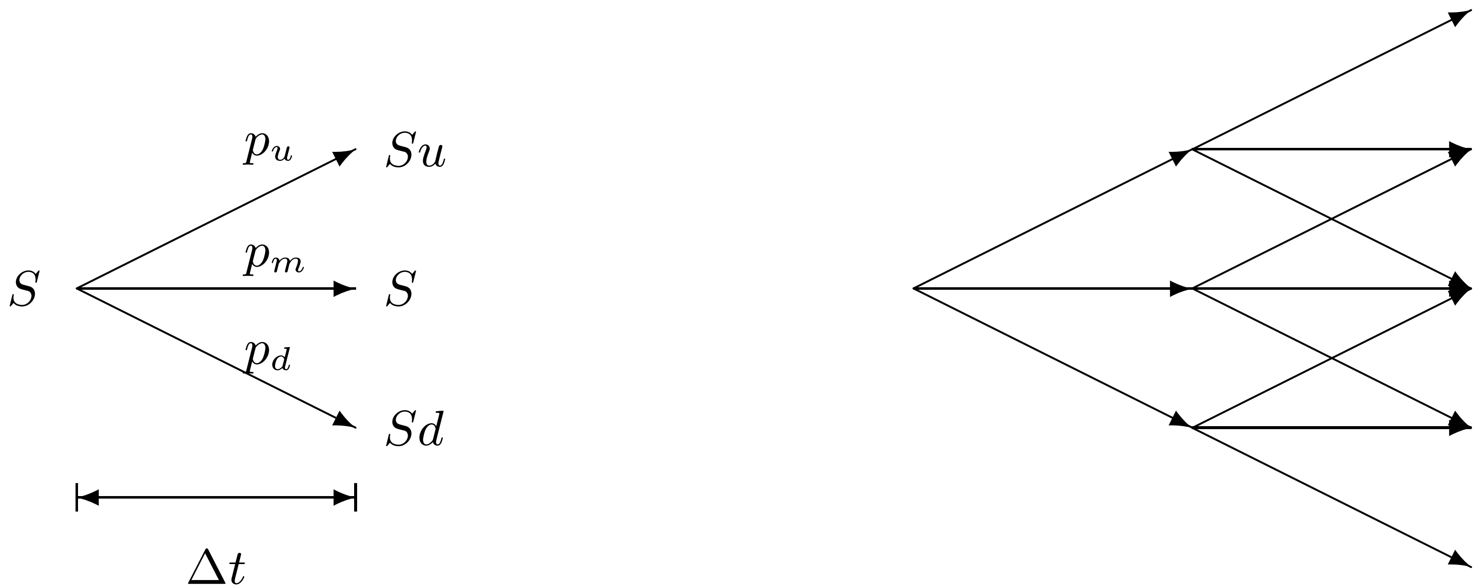
Trinomial Tree

- Set up a trinomial approximation to the geometric Brownian motion[a]
$$\frac{dS}{S} = r\,dt + \sigma\,dW.$$
- The three stock prices at time Δt are S, Su, and Sd, where $ud = 1$.
- Let the mean and variance of the stock price be SM and S^2V, respectively.

[a]Parkinson (1977); Boyle (1986).


p_u
Su
p_m
S
S
p_d
Sd
Δt

Trinomial Tree (continued)

- By Eqs. (29) on p. 180,

$$\begin{aligned} M &\stackrel{\Delta}{=} e^{r\Delta t}, \\ V &\stackrel{\Delta}{=} M^2(e^{\sigma^2 \Delta t} - 1). \end{aligned}$$

- Impose the matching of mean and that of variance:

$$\begin{aligned} 1 &= p_u + p_m + p_d, \\ SM &= [\, p_u u + p_m + (p_d/u) \,]\, S, \\ S^2 V &= p_u(Su - SM)^2 + p_m(S - SM)^2 + p_d(Sd - SM)^2. \end{aligned}$$

Trinomial Tree (continued)

- Use linear algebra to verify that

$$
\begin{aligned}
p_u &= \frac{u\left(V+M^2-M\right)-(M-1)}{(u-1)\left(u^2-1\right)}, \\
p_d &= \frac{u^2\left(V+M^2-M\right)-u^3(M-1)}{(u-1)\left(u^2-1\right)}.
\end{aligned}
$$

  – We must also make sure the probabilities lie between 0 and 1.

Trinomial Tree (concluded)

- There are countless variations.
- But all converge to the Black-Scholes option pricing model.[a]
- The trinomial model has a linear-time algorithm for European options.[b]

---

[a] Madan, Milne, & Shefrin (1989).
[b] T. Chen (`R94922003`) (2007).

A Trinomial Tree

- Use $u = e^{\lambda\sigma\sqrt{\Delta t}}$, where $\lambda \geq 1$ is a tunable parameter.
- Then
$$\begin{aligned} p_u &\to \frac{1}{2\lambda^2} + \frac{\left(r + \sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}, \\ p_d &\to \frac{1}{2\lambda^2} - \frac{\left(r - 2\sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}. \end{aligned}$$
- A nice choice for λ is $\sqrt{\pi/2}$.[a]

[a] Omberg (1988).

Barrier Options Revisited

- BOPM introduces a specification error by replacing the barrier with a nonidentical effective barrier.
- The trinomial model solves the problem by adjusting λ so that the barrier is hit exactly.[a]
- When
$$Se^{-h\lambda\sigma\sqrt{\Delta t}} = H,$$
it takes h down moves to go from S to H, if h is an integer.
- Then
$$h = \frac{\ln(S/H)}{\lambda\sigma\sqrt{\Delta t}}.$$

[a] Ritchken (1995).

Barrier Options Revisited (continued)

- This is easy to achieve by adjusting λ.
- Typically, we find the smallest $\lambda \geq 1$ such that h is an integer.[a]
  - Such a λ may not exist for very small n's.[b]
- Toward that end, we find the *largest* integer $j \geq 1$ that satisfies $\frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}} \geq 1$ to be the h.
- Then let

$$\lambda = \frac{\ln(S/H)}{h\sigma\sqrt{\Delta t}}.$$

[a] Why must $\lambda \geq 1$?

[b] This is not hard to check.

Barrier Options Revisited (continued)

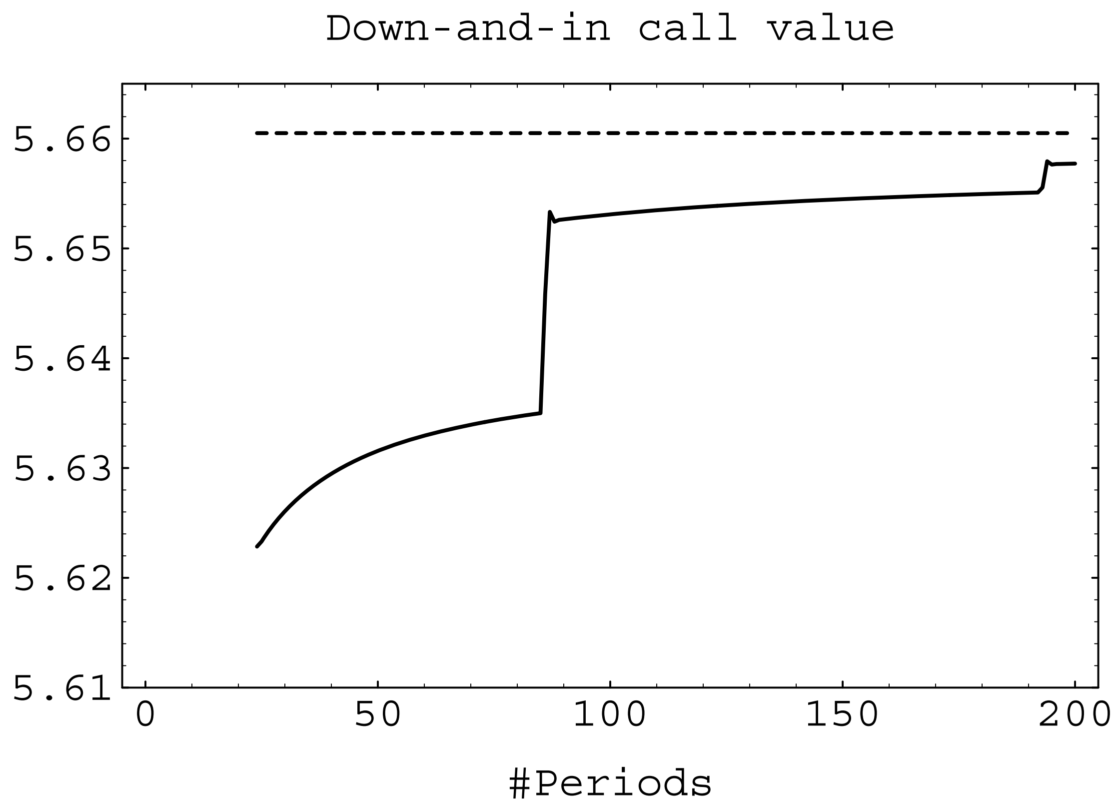
- Alternatively, pick
$$h = \left\lfloor \frac{\ln(S/H)}{\sigma\sqrt{\Delta t}} \right\rfloor.$$
- Make sure $h \geq 1$.
- Then let
$$\lambda = \frac{\ln(S/H)}{h\sigma\sqrt{\Delta t}}.$$

Barrier Options Revisited (concluded)

- This done, one of the layers of the trinomial tree coincides with the barrier.
- The following probabilities may be used,

$$
\begin{aligned}
p_u &= \frac{1}{2\lambda^2} + \frac{\mu' \sqrt{\Delta t}}{2\lambda\sigma}, \\
p_m &= 1 - \frac{1}{\lambda^2}, \\
p_d &= \frac{1}{2\lambda^2} - \frac{\mu' \sqrt{\Delta t}}{2\lambda\sigma}.
\end{aligned}
$$

  - $\mu' \stackrel{\Delta}{=} r - (\sigma^2/2)$.


Down-and-in call value
5.66
5.65
5.64
5.63
5.62
5.61
0
50
100
150
200
#Periods

Algorithms Comparison[a]

- So which algorithm is better, binomial or trinomial?
- Algorithms are often compared based on the n value at which they "converge."
  - The one with the smallest n wins.
- So giraffes are faster than cheetahs because they take fewer strides to travel the same distance!
- Performance must be based on actual running times, not n.[b]

[a] Lyuu (1998).
[b] Patterson & Hennessy (1994).

Algorithms Comparison (continued)

- Pages 729 and 748 seem to show the trinomial model converges at a smaller n than BOPM.
- It is in this sense when people say trinomial models converge faster than binomial ones.
- But does it make the trinomial model better then?

Algorithms Comparison (concluded)

- The linear-time binomial tree algorithm actually performs better than the trinomial one.
- See the next page, expanded from p. 735.
- The barrier-too-close problem is also too hard for a quadratic-time trinomial tree algorithm.[a]
  - See pp. 762ff for an alternative solution.

---
[a] Lyuu (1998).

n	Combinatorial method		Trinomial tree algorithm	
	Value	Time	Value	Time
21	5.507548	0.30		
84	5.597597	0.90	5.634936	35.0
191	5.635415	2.00	5.655082	185.0
342	5.655812	3.60	5.658590	590.0
533	5.652253	5.60	5.659692	1440.0
768	5.654609	8.00	5.660137	3080.0
1047	5.658622	11.10	5.660338	5700.0
1368	5.659711	15.00	5.660432	9500.0
1731	5.659416	19.40	5.660474	15400.0
2138	5.660511	24.70	5.660491	23400.0
2587	5.660592	30.20	5.660493	34800.0
3078	5.660099	36.70	5.660488	48800.0
3613	5.660498	43.70	5.660478	67500.0
4190	5.660388	44.10	5.660466	92000.0
4809	5.659955	51.60	5.660454	130000.0
5472	5.660122	68.70		
6177	5.659981	76.70		

(All times in milliseconds.)

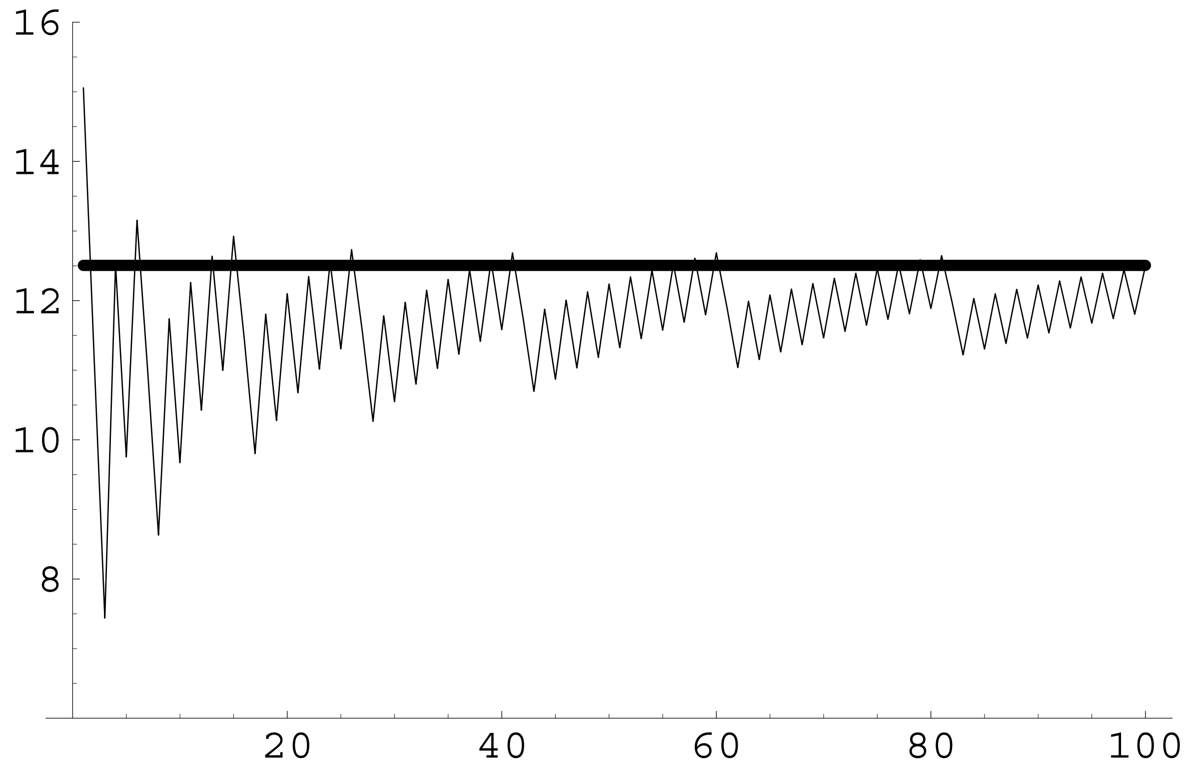
Double-Barrier Options

- Double-barrier options are barrier options with two barriers $L < H$.
  - They make up "less than 5% of the light exotic market."[a]
- Assume $L < S < H$.
- The binomial model produces oscillating option values (see plot on next page).[b]

---

[a] Bennett (2014).

[b] Chao (R86526053) (1999); Dai (B82506025, R86526008, D8852600) & Lyuu (2005).


16
14
12
10
8
20
40
60
80
100

Double-Barrier Options (concluded)

- The combinatorial method yields a linear-time algorithm.[a]
- This binomial model is $O(1/\sqrt{n})$ convergent in general.[b]
- If the barriers L and H depend on time, we have moving-barrier options.[c]

---

[a] See p. 241 of the textbook.
[b] Gobet (1999).
[c] Rogers & Zane (1998).

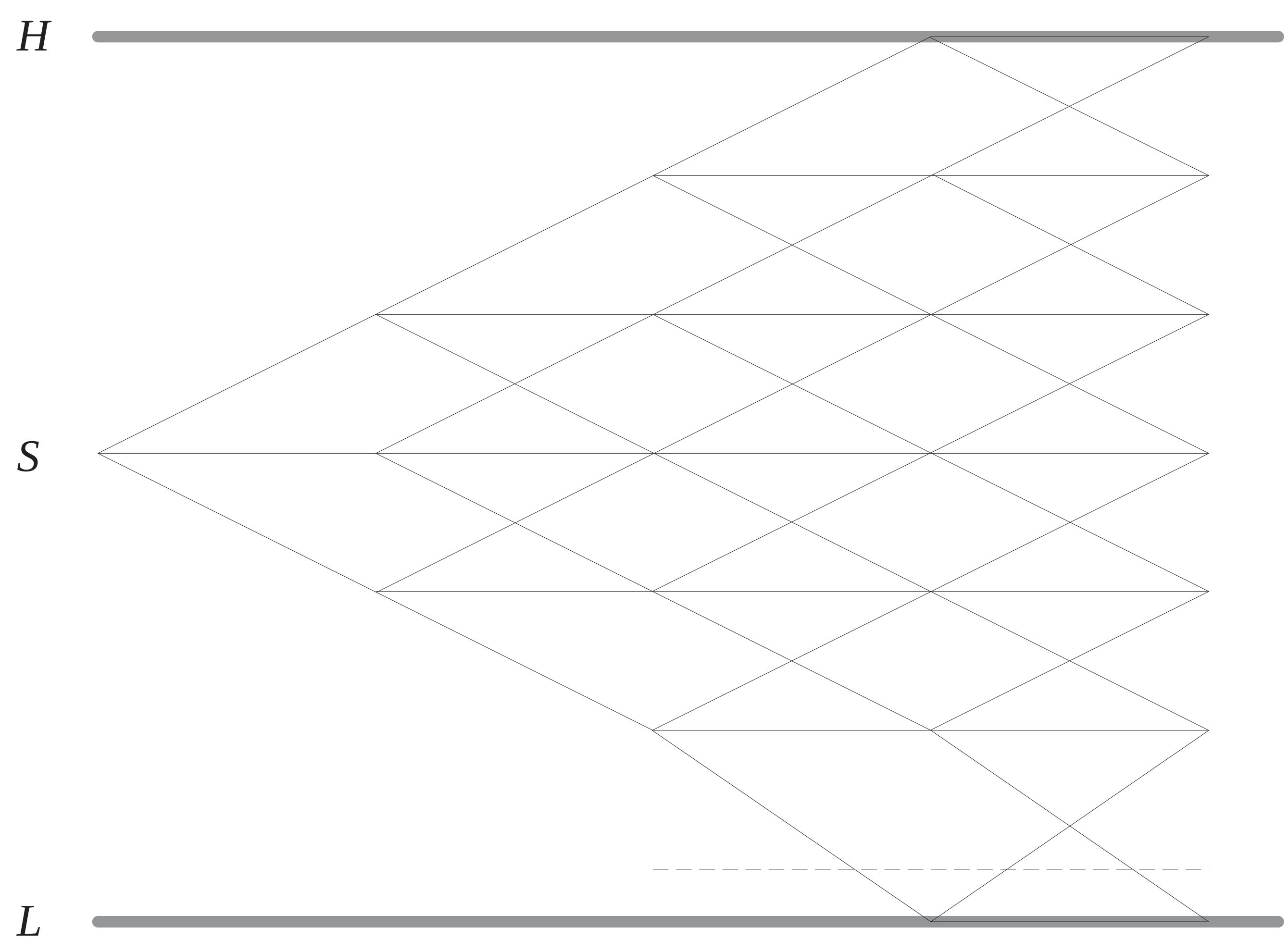
Double-Barrier Knock-Out Options

- We knew how to pick the λ so that one of the layers of the trinomial tree coincides with one barrier, say H.
- This choice, however, does not guarantee that the other barrier, L, is also hit.
- One way to handle this problem is to *lower* the layer of the tree just above L to coincide with L.[a]
  - More general ways to make the trinomial model hit both barriers are available.[b]

---

[a] Ritchken (1995); Hull (1999).

[b] Hsu (`R7526001`, `D89922012`) & Lyuu (2006). Dai (`B82506025`, `R86526008`, `D8852600`) & Lyuu (2006) combine binomial and trinomial trees to derive an $O(n)$-time algorithm for double-barrier options (see pp. 762ff).


H
S
L

Double-Barrier Knock-Out Options (continued)

- The probabilities of the nodes on the layer above L must be adjusted.
- Let ℓ be the positive integer such that

$$Sd^{\ell+1} < L < Sd^{\ell}.$$

- Hence the layer of the tree just above L has price Sd^{ℓ}.[a]

[a] You probably cannot do the same thing for binomial models (why?). Thanks to a lively discussion on April 25, 2012.

Double-Barrier Knock-Out Options (concluded)

- Define $\gamma > 1$ as the number satisfying

$$L = Sd^{\ell-1} e^{-\gamma\lambda\sigma\sqrt{\Delta t}}.$$

  - The prices between the barriers are (from low to high)

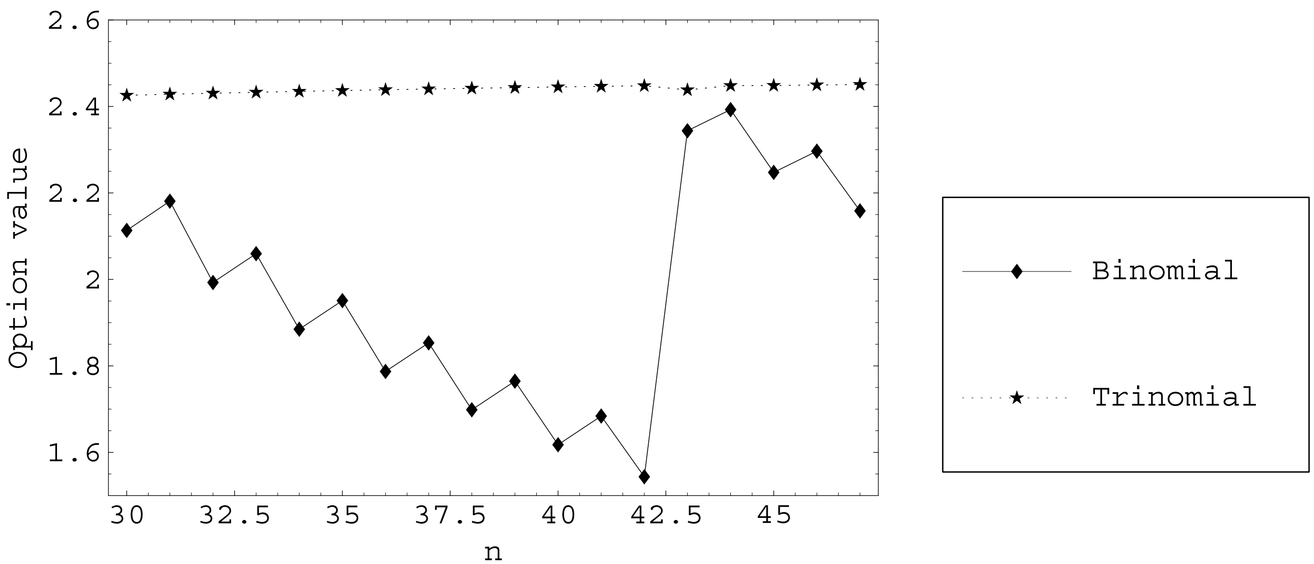
$$L, Sd^{\ell-1}, \ldots, Sd^2, Sd, S, Su, Su^2, \ldots, Su^{h-1}, Su^h = H.$$

- The probabilities for the nodes with price equal to $Sd^{\ell-1}$ are

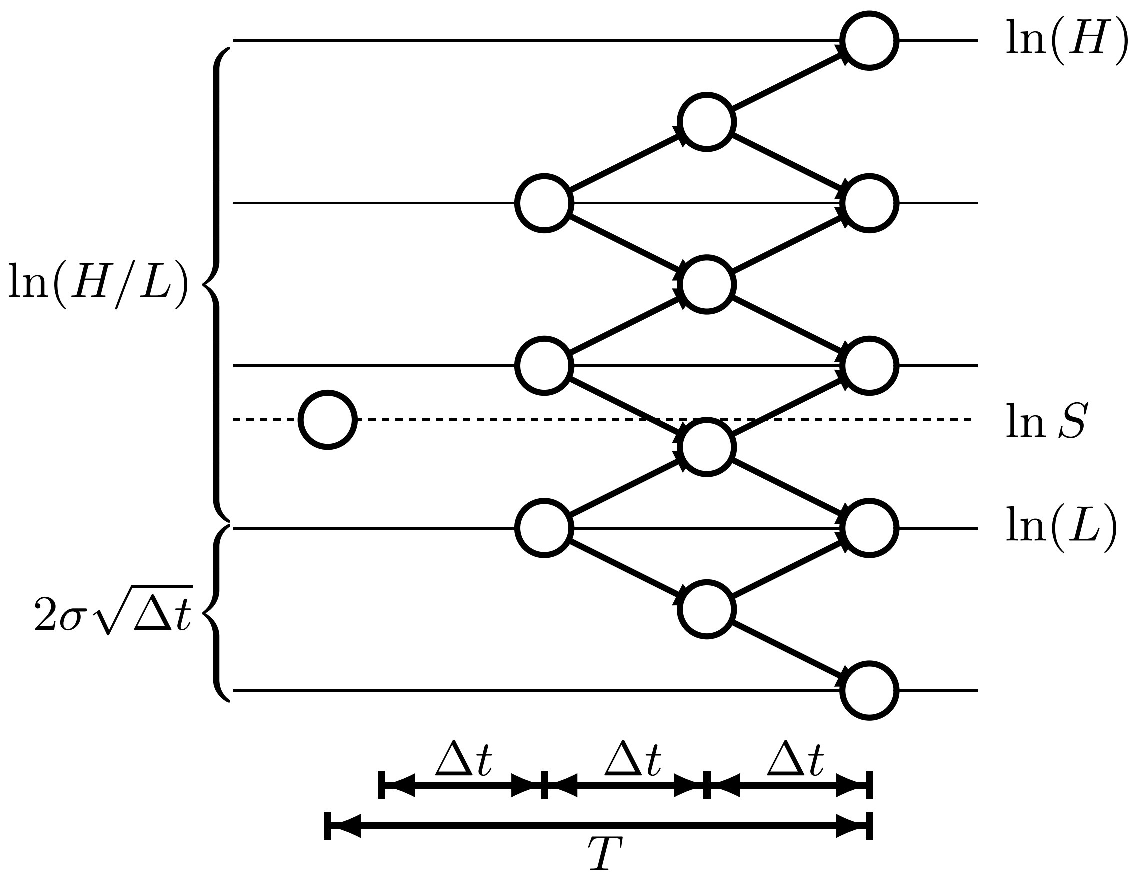
$$p'_u = \frac{b + a\gamma}{1+\gamma}, \quad p'_d = \frac{b - a}{\gamma + \gamma^2}, \quad \text{and} \quad p'_m = 1 - p'_u - p'_d,$$

where $a \stackrel{\Delta}{=} \mu' \sqrt{\Delta t}/(\lambda\sigma)$ and $b \stackrel{\Delta}{=} 1/\lambda^2$.

Convergence: Binomial vs. Trinomial



Ideas for Binomial Trees To Handle Two Barriers



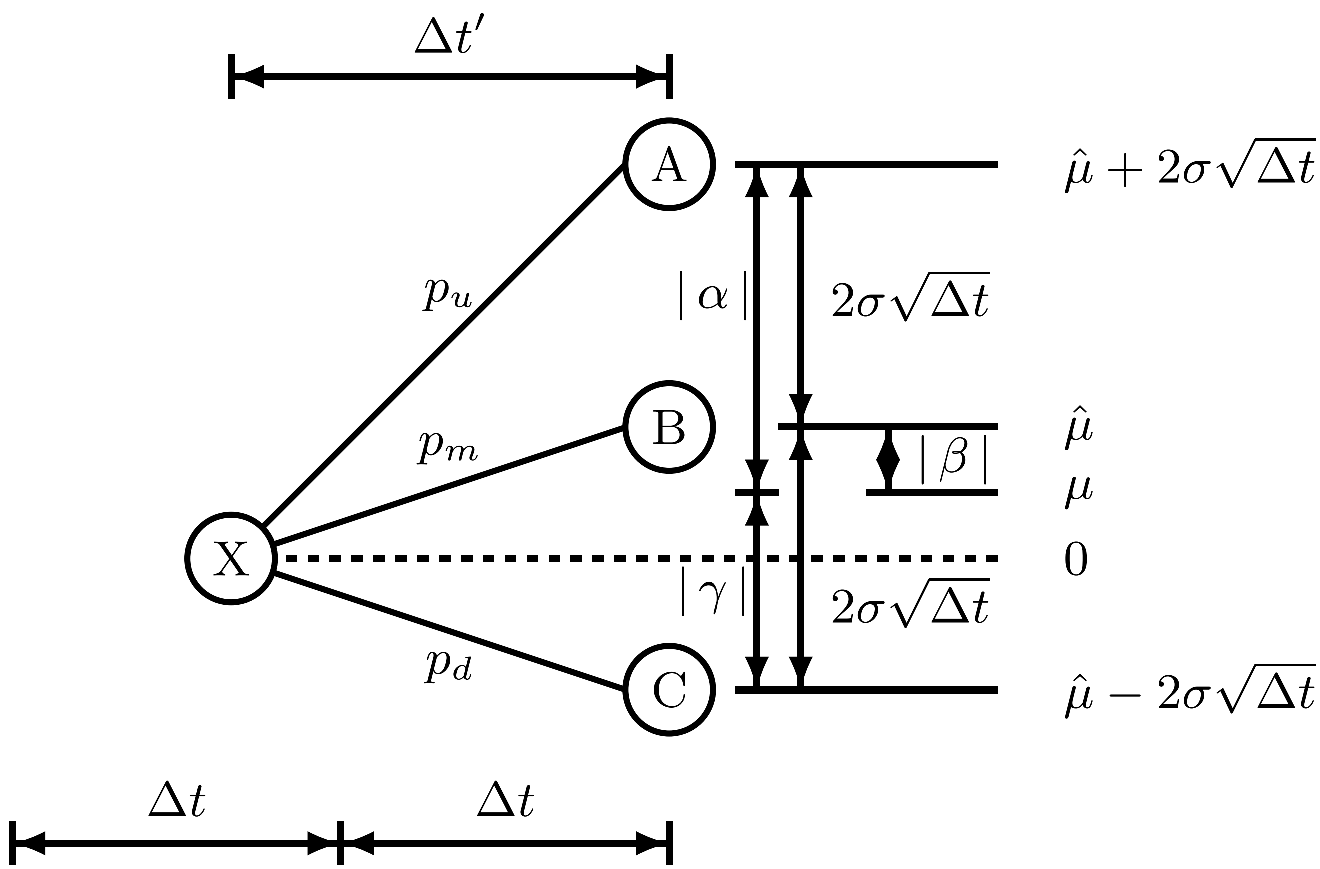
The Binomial-Trinomial Tree

- Append a trinomial structure to a binomial tree can lead to improved convergence and efficiency.[a]
- The resulting tree is called the binomial-trinomial tree.[b]
- Suppose a *binomial* tree will be built with Δt as the duration of one period.
- Node X at time t needs to pick three nodes on the binomial tree at time $t + \Delta t'$ as its successor nodes.
  - Later, $\Delta t \leq \Delta t' < 2\Delta t$.

[a] Dai (B82506025, R86526008, D8852600) & Lyuu (2006, 2008, 2010).

[b] The idea first emerged in a hotel in Muroran, Hokkaido, Japan, in May of 2005.

The Binomial-Trinomial Tree (continued)



The Binomial-Trinomial Tree (continued)

- These three nodes should guarantee:
  1. The mean and variance of the stock price are matched.
  2. The branching probabilities are between 0 and 1.
- Let S be the stock price at node X.
- Use $s(z)$ to denote the stock price at node z.

The Binomial-Trinomial Tree (continued)

- Recall that the expected value of the logarithmic return $\ln(S_{t+\Delta t'}/S)$ at time $t + \Delta t'$ equals[a]

$$\mu \stackrel{\Delta}{=} \left(r - \frac{\sigma^2}{2} \right) \Delta t'. \tag{102}$$

- Its variance equals

$$\text{Var} \stackrel{\Delta}{=} \sigma^2 \Delta t'. \tag{103}$$

- Let node B be the node whose logarithmic return $\hat{\mu} \stackrel{\Delta}{=} \ln(s(\text{B})/S)$ is closest to μ among all the nodes at time $t + \Delta t'$.

[a]Recall p. 301.

The Binomial-Trinomial Tree (continued)

- The middle branch from node X will end at node B.
- The two nodes A and C, which bracket node B, are the destinations of the other two branches from node X.
- Recall that adjacent nodes on the binomial tree are spaced at $2\sigma\sqrt{\Delta t}$ apart.
- Review the illustration on p. 763.

The Binomial-Trinomial Tree (continued)

- The three branching probabilities from node X are obtained through matching the mean and variance of the logarithmic return $\ln(S_{t+\Delta t'}/S)$.
- Recall that
$$\hat{\mu} \stackrel{\Delta}{=} \ln\left(s(B)/S\right)$$
is the logarithmic return of the middle node B.
- Let α, β, and γ be the differences between μ and the three logarithmic returns
$$\ln(s(\text{A})/S), \ln(s(\text{B})/S), \ln(s(\text{C})/S),$$
in that order.

The Binomial-Trinomial Tree (continued)

- In other words,

$$\alpha \stackrel{\Delta}{=} \hat{\mu} + 2\sigma\sqrt{\Delta t} - \mu = \beta + 2\sigma\sqrt{\Delta t}\,, \tag{104}$$

$$\beta \stackrel{\Delta}{=} \hat{\mu} - \mu, \tag{105}$$

$$\gamma \stackrel{\Delta}{=} \hat{\mu} - 2\sigma\sqrt{\Delta t} - \mu = \beta - 2\sigma\sqrt{\Delta t}\,. \tag{106}$$

- The three branching probabilities p_u, p_m, p_d then satisfy

$$p_u\alpha + p_m\beta + p_d\gamma = 0, \tag{107}$$

$$p_u\alpha^2 + p_m\beta^2 + p_d\gamma^2 = \mathrm{Var}, \tag{108}$$

$$p_u + p_m + p_d = 1. \tag{109}$$

The Binomial-Trinomial Tree (concluded)

- Equation (107) matches the mean (102) of the logarithmic return $\ln(S_{t+\Delta t'}/S)$ on p. 765.
- Equation (108) matches its variance (103) on p. 765.
- The three probabilities can be proved to lie between 0 and 1 by Cramer's rule.

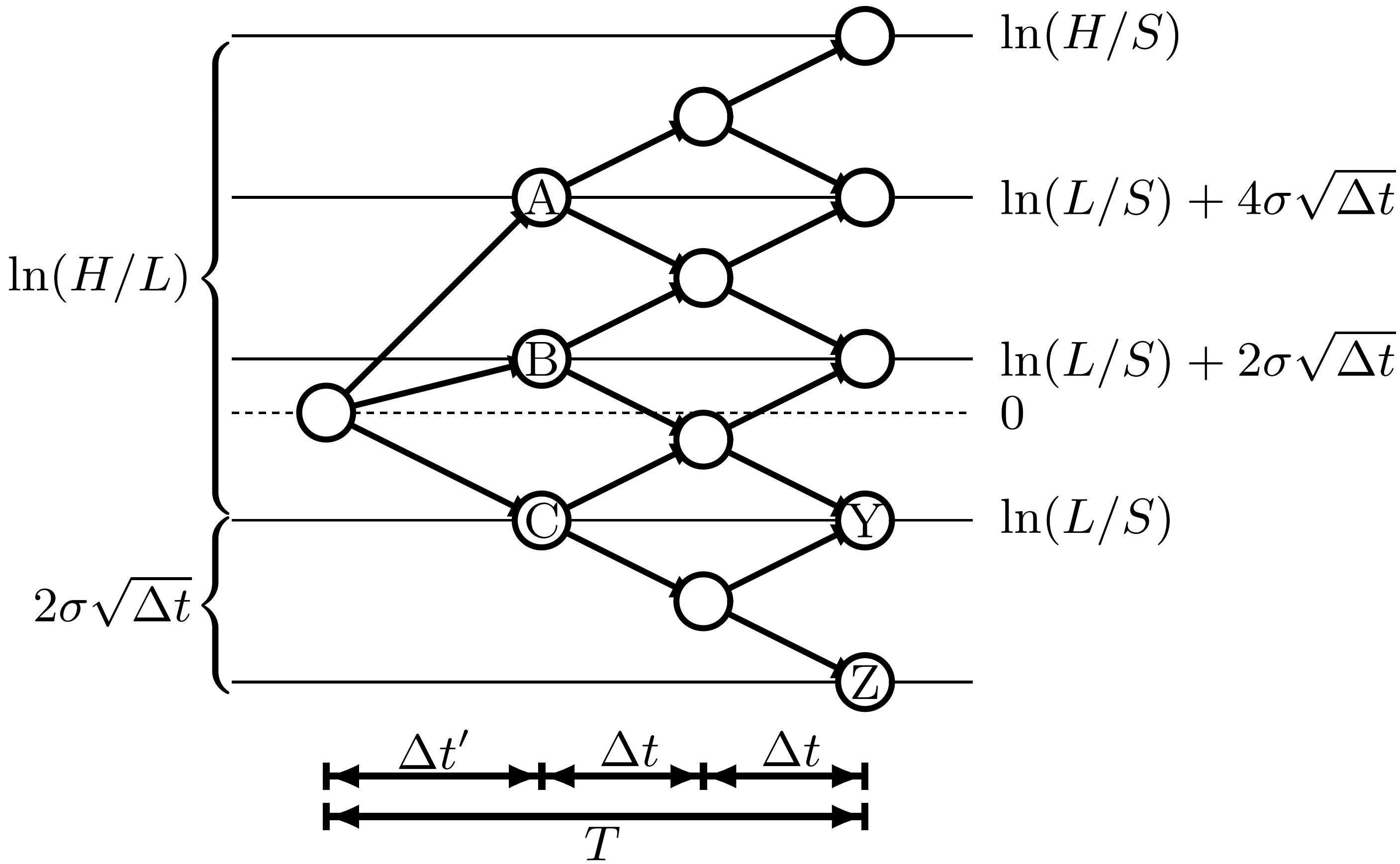
Pricing Double-Barrier Options

- Consider a double-barrier option with two barriers L and H, where $L < S < H$.
- We need to make each barrier coincide with a layer of the binomial tree for better convergence.
- The idea is to choose a Δt such that

$$\frac{\ln(H/L)}{2\sigma\sqrt{\Delta t}} \tag{110}$$

  is a positive integer.
  - The distance between two adjacent nodes such as nodes Y and Z in the figure on p. 771 is $2\sigma\sqrt{\Delta t}$.

Pricing Double-Barrier Options (continued)



Pricing Double-Barrier Options (continued)

- Suppose that the goal is a tree with $\sim m$ periods.
- Suppose we pick $\Delta\tau \triangleq T/m$ for the length of each period.
- There is no guarantee that $\frac{\ln(H/L)}{2\sigma\sqrt{\Delta\tau}}$ is an integer.
- We will pick the largest Δt that does not exceed, $\Delta\tau$ and makes $\frac{\ln(H/L)}{2\sigma\sqrt{\Delta t}}$ some integer κ.
- In other words, we select

$$\Delta t = \left(\frac{\ln(H/L)}{2\kappa\sigma}\right)^2,$$

where $\kappa = \left\lceil \frac{\ln(H/L)}{2\sigma\sqrt{\Delta\tau}} \right\rceil$.

Pricing Double-Barrier Options (continued)

- We now proceed to build the binomial-trinomial tree.
- Start with the binomial part.
- Lay out the nodes from the low barrier L upward.
- Automatically, a layer coincides with the high barrier H.
- It is unlikely that Δt divides T.
- The initial stock price is also unlikely to be on a layer of the binomial tree.[a]

[a] Recall p. 771.

Pricing Double-Barrier Options (continued)

- The binomial-trinomial structure can address this problem as follows.
- Between time 0 and time T, the binomial tree spans $\lfloor T/\Delta t \rfloor$ periods.
- Keep only the last $\lfloor T/\Delta t \rfloor - 1$ periods and let the first period have a duration equal to

$$\Delta t' \stackrel{\Delta}{=} T - \left(\left\lfloor \frac{T}{\Delta t} \right\rfloor - 1 \right) \Delta t.$$

- Then these $\lfloor T/\Delta t \rfloor$ periods span T years.
- It is easy to verify that $\Delta t \leq \Delta t' < 2\Delta t$.

Pricing Double-Barrier Options (continued)

- Start with the root node at time 0 and at a price with logarithmic return $\ln(S/S) = 0$.
- Find the three nodes on the binomial tree at time $\Delta t'$ as described earlier.
- Calculate the three branching probabilities to them.
- Grow the binomial tree from these three nodes until time T to obtain a binomial-trinomial tree with $\lfloor T/\Delta t \rfloor$ periods.
- Review the illustration on p. 771.

Pricing Double-Barrier Options (continued)

- Now the binomial-trinomial tree can be used to price double-barrier options by backward induction.
- That takes quadratic time.
- But a linear-time algorithm exists for double-barrier options on the *binomial* tree.[a]
- Apply that algorithm to price the double-barrier option's prices at the three nodes at time $\Delta t'$.
  - That is, nodes A, B, and C on p. 771.
- Then calculate their expected discounted value at the root node.

[a]See p. 241 of the textbook; Chao (`R86526053`) (1999); Dai (`B82506025`, `R86526008`, `D8852600`) & Lyuu (2008).

Pricing Double-Barrier Options (continued)

- The overall running time is only linear!
- Binomial trees have troubles pricing barrier options.[a]
- Even pit against the trinomial tree, the binomial-trinomial tree converges faster and smoother.[b]
- In fact, the binomial-trinomial tree has an error of $O(1/n)$ for single-barrier options.[c]
- It has an error of $O(1/n^{1-a})$ for any $0 < a < 1$ for double-barrier options.[d]

---

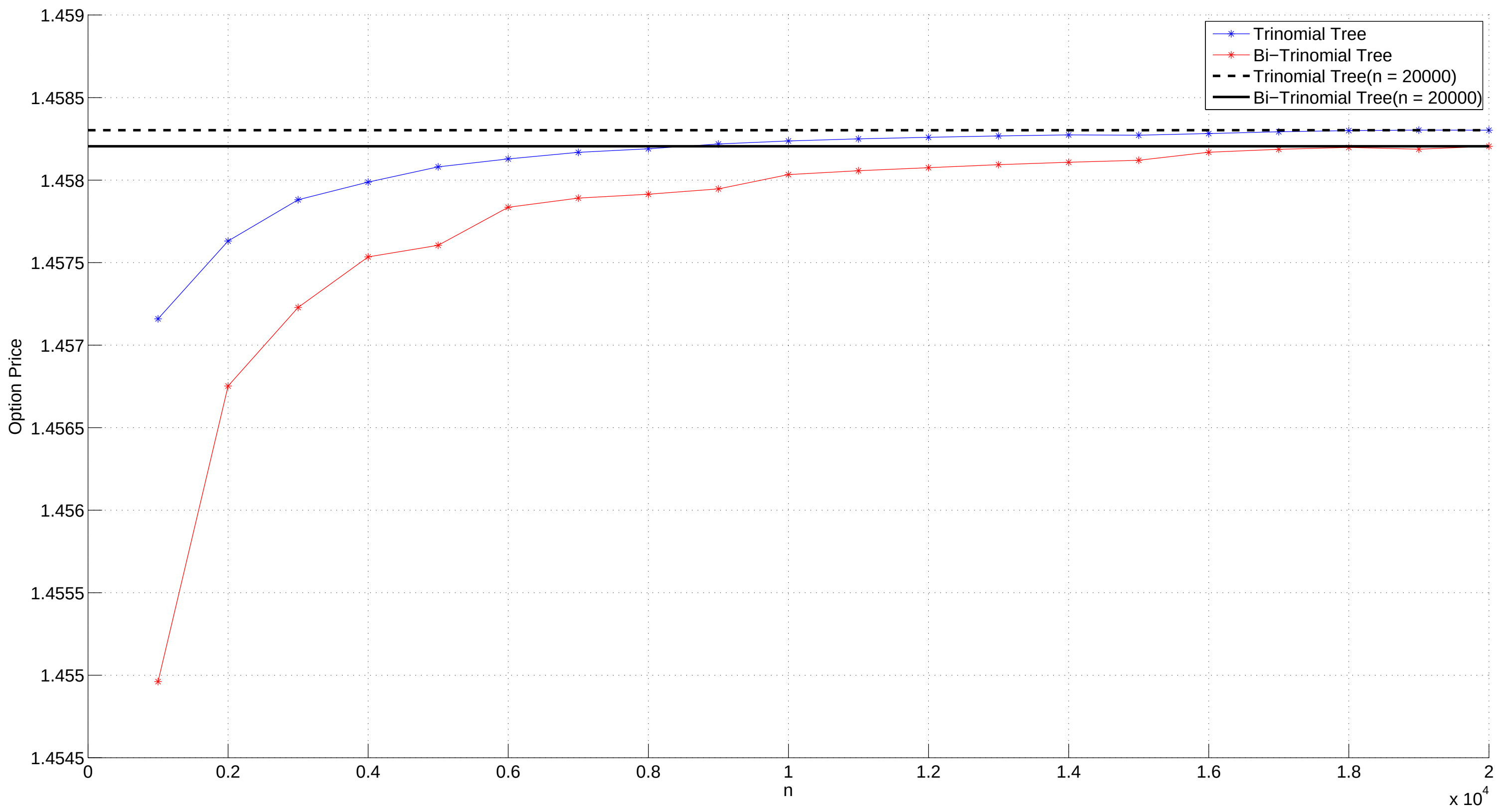
[a] See p. 408, p. 754, and p. 760.

[b] See p. 778 and p. 779.

[c] Lyuu & Palmer (2010).

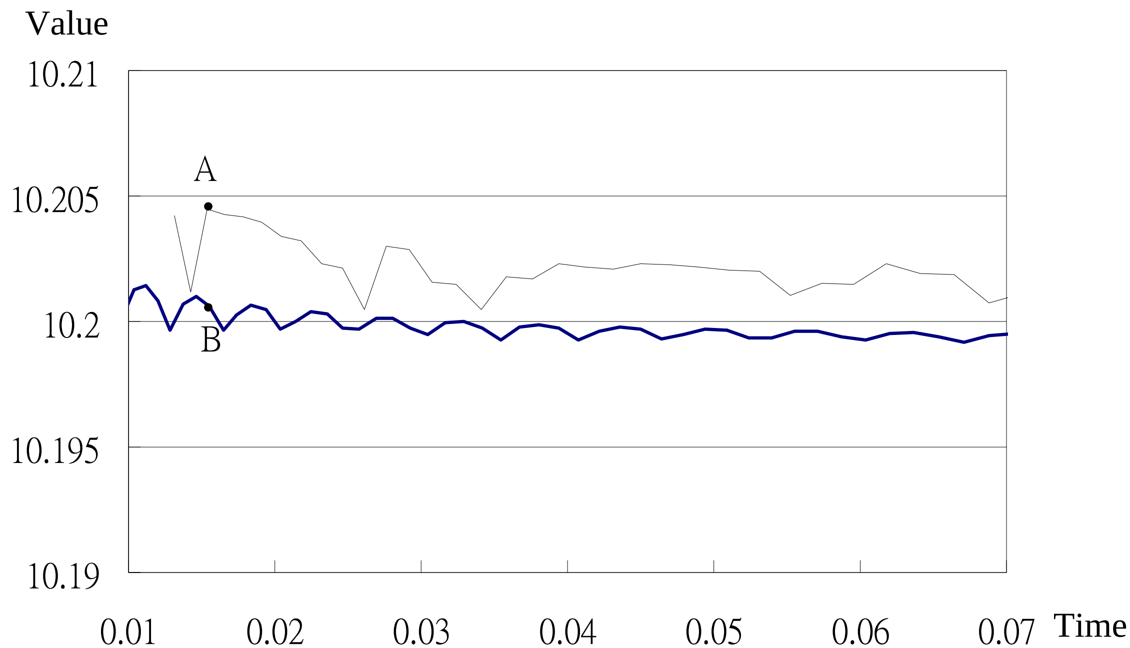
[d] Appolloni, Gaudenziy, & Zanette (2014).

Pricing Double-Barrier Options[a] (continued)



[a] Generated by Mr. Lin, Ying-Hung (R01723029) on June 6, 2014.

Pricing Double-Barrier Options (concluded)



The thin line denotes the double-barrier option prices computed by the trinomial tree against the running time in seconds (such as point A). The thick line denotes those computed by the binomial-trinomial tree (such as point B).

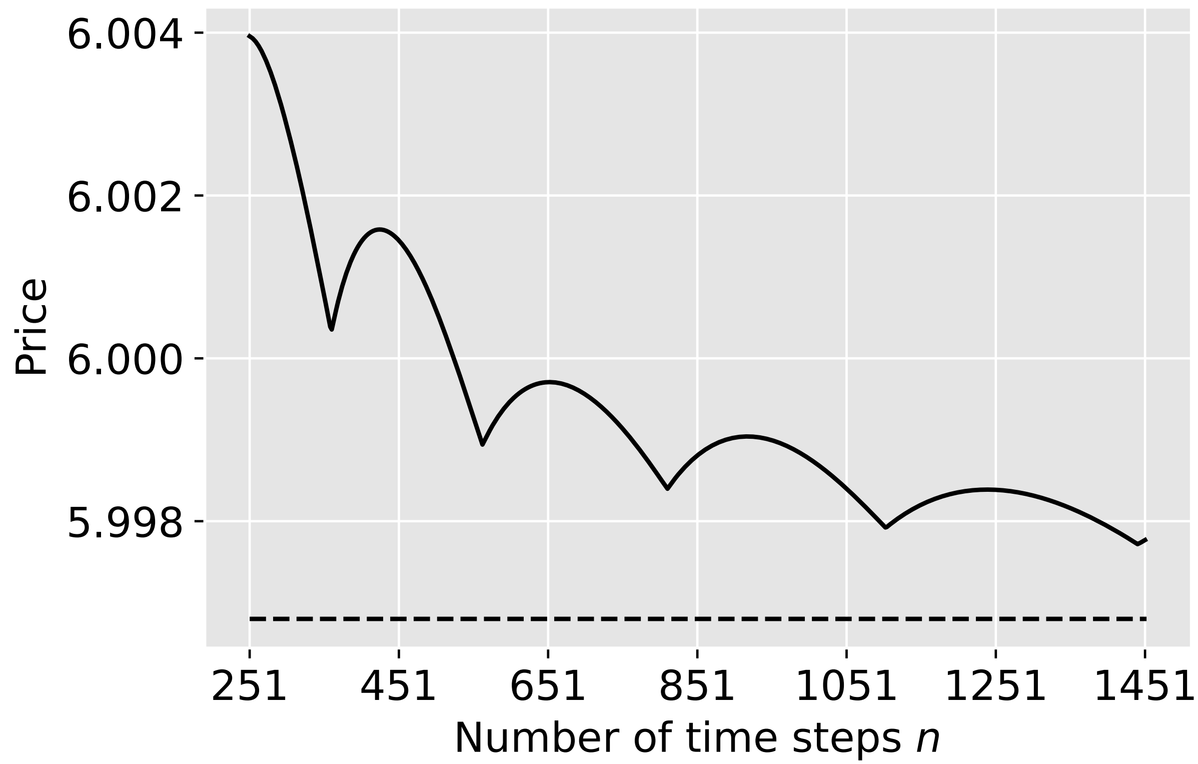
The Barrier-Too-Close Problem (p. 736) Revisited

- Our idea solves it even if one barrier is very close to S.
  - It runs in linear time, unlike an earlier quadratic-time solution with trinomial trees (pp. 744ff).
  - Unlike an earlier solution using combinatorics (p. 727), now the choice of n is not that restricted.
- So it combines the strengths of binomial and trinomial trees.
- This holds for single-barrier options too.
- Here is how.

The Barrier-Too-Close Problem Revisited (continued)

- We can build the tree treating S as if it were a second barrier.
- So both H and S are matched.
- Alternatively, we can pick $\Delta\tau \stackrel{\Delta}{=} T/m$ as our length of a period Δt without the subsequent adjustment.[a]
- Then build the tree from the price H down.
- So H is matched.
- The initial price S is automatically matched by the *trinomial* structure.

[a] There is no second barrier to match!



Plot of convergence of a down-and-out call to 5.9968 supplied by Mr. Lu, Yu-Ming (`R06723032`, `D08922008`) on February 26. 2021.

The Barrier-Too-Close Problem Revisited (concluded)

- The earlier trinomial tree is impractical as it needs a very large n when the barrier H is very close to S.[a]
  - It needs at least one *up* move to connect S to H as its middle branch is flat.
  - But when $S \approx H$, that up move must take a very small step, necessitating a small Δt.
- Our trinomial structure's middle branch is *not* required to be flat.
- So S can be connected to H via the middle branch, and the need of a very large n disappears!

---

[a]Recall the table on p. 737.

Pricing Discrete and Moving Barrier Options

- Barrier options whose barrier is monitored only at discrete times are called discrete barrier options.
- They are less common than the continuously monitored versions for single stocks.[a]
- The main difficulty with pricing discrete barrier options lies in matching the monitored *times*.
- Here is why.

---

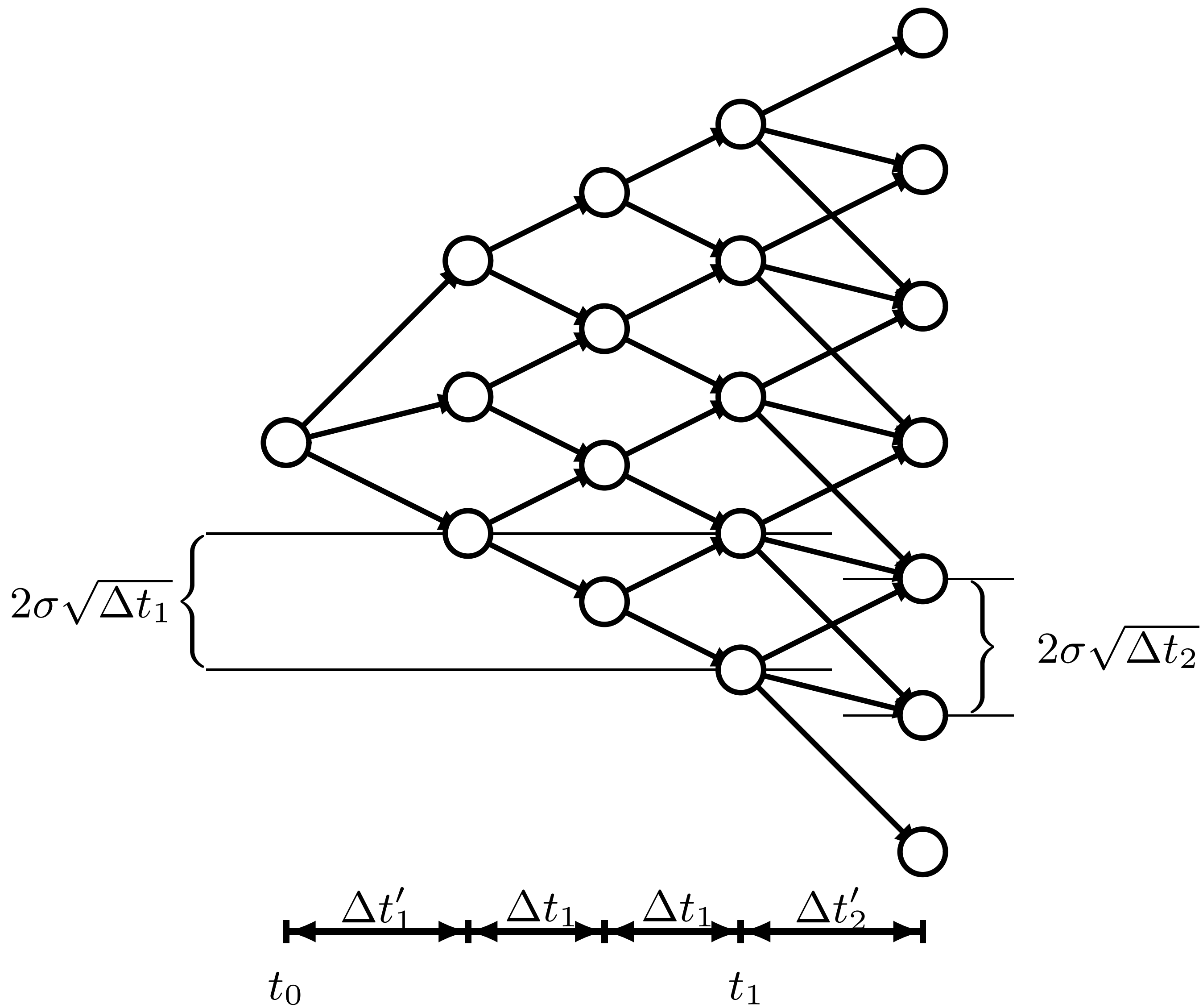
[a] Bennett (2014).

Pricing Discrete and Moving Barrier Options (continued)

- Suppose each period has a duration of Δt and the $\ell > 1$ monitored times are

$$t_0 = 0, t_1, t_2, \ldots, t_\ell = T.$$

- It is unlikely that *all* monitored times coincide with the end of a period on the tree, or Δt divides t_i for all i.
- The binomial-trinomial tree can handle discrete options with ease, however.
- Simply build a binomial-trinomial tree from time 0 to time t_1, followed by one from time t_1 to time t_2, and so on until time t_ℓ.


$2\sigma\sqrt{\Delta t_1}$
$2\sigma\sqrt{\Delta t_2}$
$\Delta t_1'$
Δt_1
Δt_1
$\Delta t_2'$
t_0
t_1

Pricing Discrete and Moving Barrier Options (concluded)

- This procedure works even if each t_i is associated with a distinct barrier or if each window $[\,t_i, t_{i+1})$ has its own continuously monitored barrier or double barriers.
- Pricing in both scenarios can actually be done in time $O[\ell n \ln(n/\ell)\,]$.[a]
- For typical discrete barriers, placing barriers midway between two price levels on the tree may increase accuracy.[b]

---

[a] Y. Lu (R06723032, D08922008) & Lyuu (2021).
[b] Steiner & Wallmeier (1999); Tavella & Randall (2000).

Options on a Stock That Pays Known Dividends

- Many ad hoc assumptions have been postulated for option pricing with known dividends.[a]
  1. The one we saw earlier[b] models the stock price minus the present value of the anticipated dividends as following geometric Brownian motion.
  2. One can also model the stock price plus the forward values of the dividends as following geometric Brownian motion.

---

[a]Frishling (2002).

[b]On p. 325.

Options on a Stock That Pays Known Dividends (continued)

- Realistic models assume:
  - The stock price decreases by the amount of the dividend paid at the ex-dividend date.
  - The dividend is part cash and part yield (i.e., $\alpha(t)S_0 + \beta(t)S_t$), for practitioners.[a]
- The stock price follows geometric Brownian motion between adjacent ex-dividend dates.
- But they result in exponential-sized binomial trees.[b]
- The binomial-trinomial tree can avoid this problem in most cases.

---

[a] Henry-Labordère (2009).

[b] Recall p. 324.

Options on a Stock That Pays Known Dividends (continued)

- Suppose that the known dividend is D dollars and the ex-dividend date is at time t.
- So there are $m \stackrel{\Delta}{=} t/\Delta t$ periods between time 0 and the ex-dividend date.[a]
- To avoid negative stock prices, we need to make sure the lowest stock price at time t is at least D, i.e.,

$$Se^{-(t/\Delta t)\sigma\sqrt{\Delta t}} \geq D.$$

  – Or,

$$\Delta t \geq \left[\frac{t\sigma}{\ln(S/D)} \right]^2.$$

[a]That is, m is an integer input and $\Delta t \stackrel{\Delta}{=} t/m$.

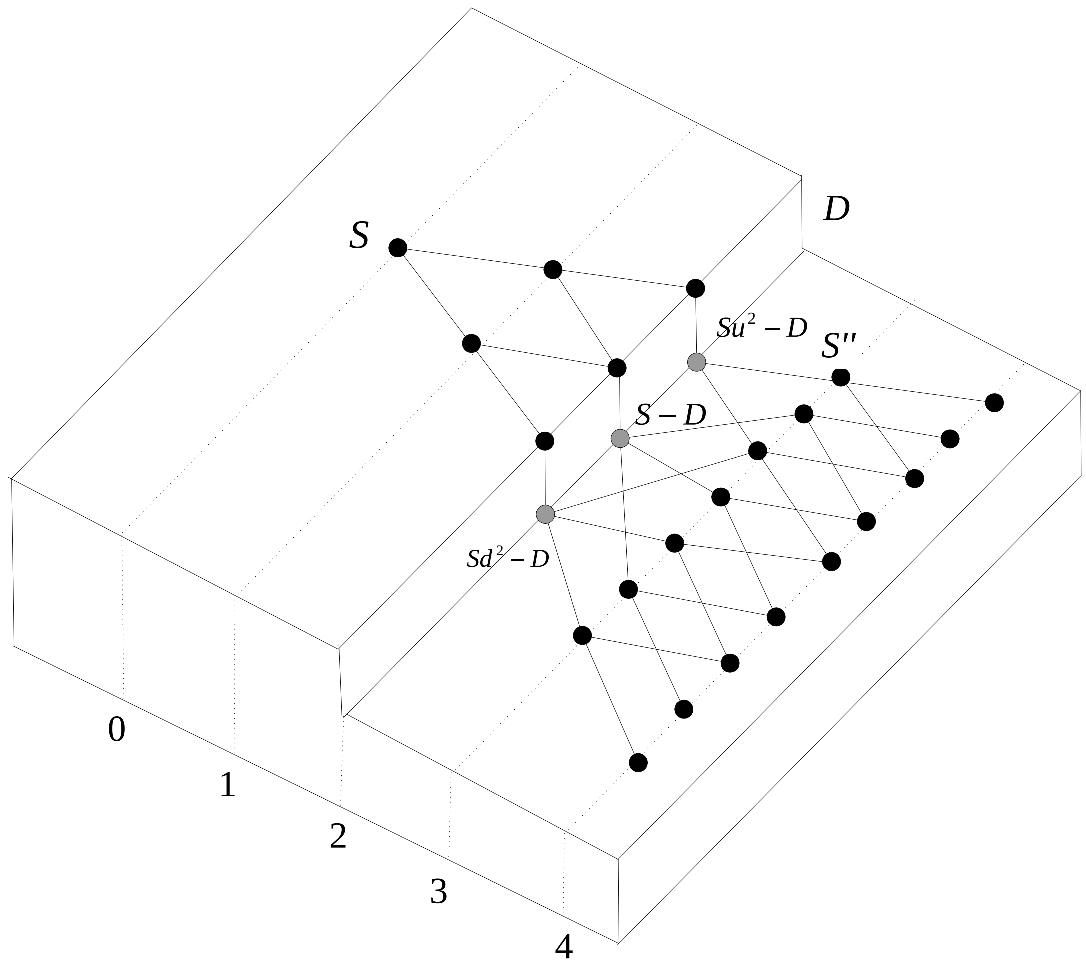
Options on a Stock That Pays Known Dividends (continued)

- Build a CRR tree from time 0 to time t as before.
- Subtract D from all the stock prices on the tree at time t to represent the price drop on the ex-dividend date.
- Assume the top node's price equals S'.
  - As usual, its two successor nodes will have prices $S'u$ and $S'u^{-1}$.
- The remaining nodes' successor nodes at time $t + \Delta t$ will choose from prices

$$S'u, S', S'u^{-1}, S'u^{-2}, S'u^{-3}, \dots ,$$

same as the CRR tree.

A Stair Tree



Options on a Stock That Pays Known Dividends (continued)

- For each node at time t below the top node, we build the trinomial connection.
- Note that the binomial-trinomial structure remains valid in the special case when $\Delta t' = \Delta t$ on p. 763.

Options on a Stock That Pays Known Dividends (concluded)

- Hence the construction can be completed.
- From time $t + \Delta t$ onward, the standard binomial tree will be used until the maturity date or the next ex-dividend date when the procedure can be repeated.
- The resulting tree is called the stair tree.[a]

[a] Dai (B82506025, R86526008, D8852600) & Lyuu (2004); Dai (B82506025, R86526008, D8852600) (2009).

Other Applications of Binomial-Trinomial Trees

- Pricing guaranteed minimum withdrawal benefits.[a]
- Option pricing with stochastic volatilities.[b]
- Efficient Parisian option pricing.[c]
- Defaultable bond pricing.[d]
- Implied barrier.[e]

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[a] H. Wu (R96723058) (2009).

[b] C. Huang (R97922073) (2010).

[c] Y. Huang (R97922081) (2010).

[d] Dai (B82506025, R86526008, D8852600), Lyuu, & C. Wang (F95922018) (2009, 2010, 2014).

[e] Y. Lu (R06723032, D08922008) (2019); Y. Lu (R06723032, D08922008) & Lyuu (2021).

Mean Tracking[a]

- The general idea behind the binomial-trinomial tree on pp. 762ff is very powerful.
- One finds the sucessor middle node as the one closest to the mean.
- The two flanking successor nodes are then spaced at $c\sigma\sqrt{\Delta t}$ from the middle node for a suitably large $c > 0$.
- The resulting trinomial structure are then guaranteed to have valid branching probabilities.

[a]Lyuu & C. Wu (R90723065) (2003, 2005).

Default Boundary as Implied Barrier

- Under the structural model,[a] the default boundary is modeled as a barrier.[b]
- The constant barrier can be inferred from the closed-form formula given the firm's market capitalization, etc.[c]
- More generally, the moving barrier can be inferred from the term structure of default probabilities with the binomial-trinomial tree.[d]

---

[a] Recall p. 373.

[b] Black & Cox (1976).

[c] Brockman & Turtle (2003).

[d] Y. Lu (R06723032, D08922008) (2019); Y. Lu (R06723032, D08922008) & Lyuu (2021).

Default Boundary as Implied Barrier (continued)

- This barrier is called the implied barrier.[a]
- If the barrier is a step function,[b] the implied barrier can be obtained in $O(n \ln n)$ time with an error of $O(1/n)$.[c]
- The next plot shows the convergence of the implied barrier (as a percentage of the initial stock price).[d]
  - The implied barrier is already very good with $n = 1$!

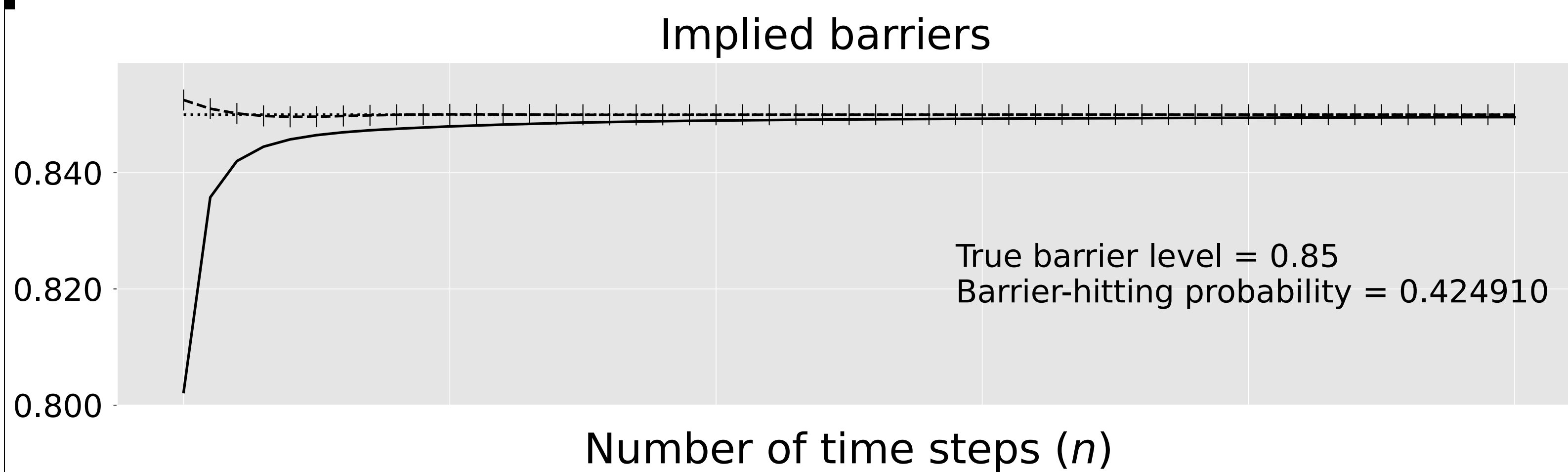
[a]Brockman & Turtle (2003).

[b]The option is then called a rolling option.

[c]Y. Lu (`R06723032`, `D08922008`) & Lyuu (2021). This is linear convergence.

[d]Plot supplied by Mr. Lu, Yu-Ming (`R06723032`, `D08922008`) on November 20. 2021.

Default Boundary as Implied Barrier (continued)

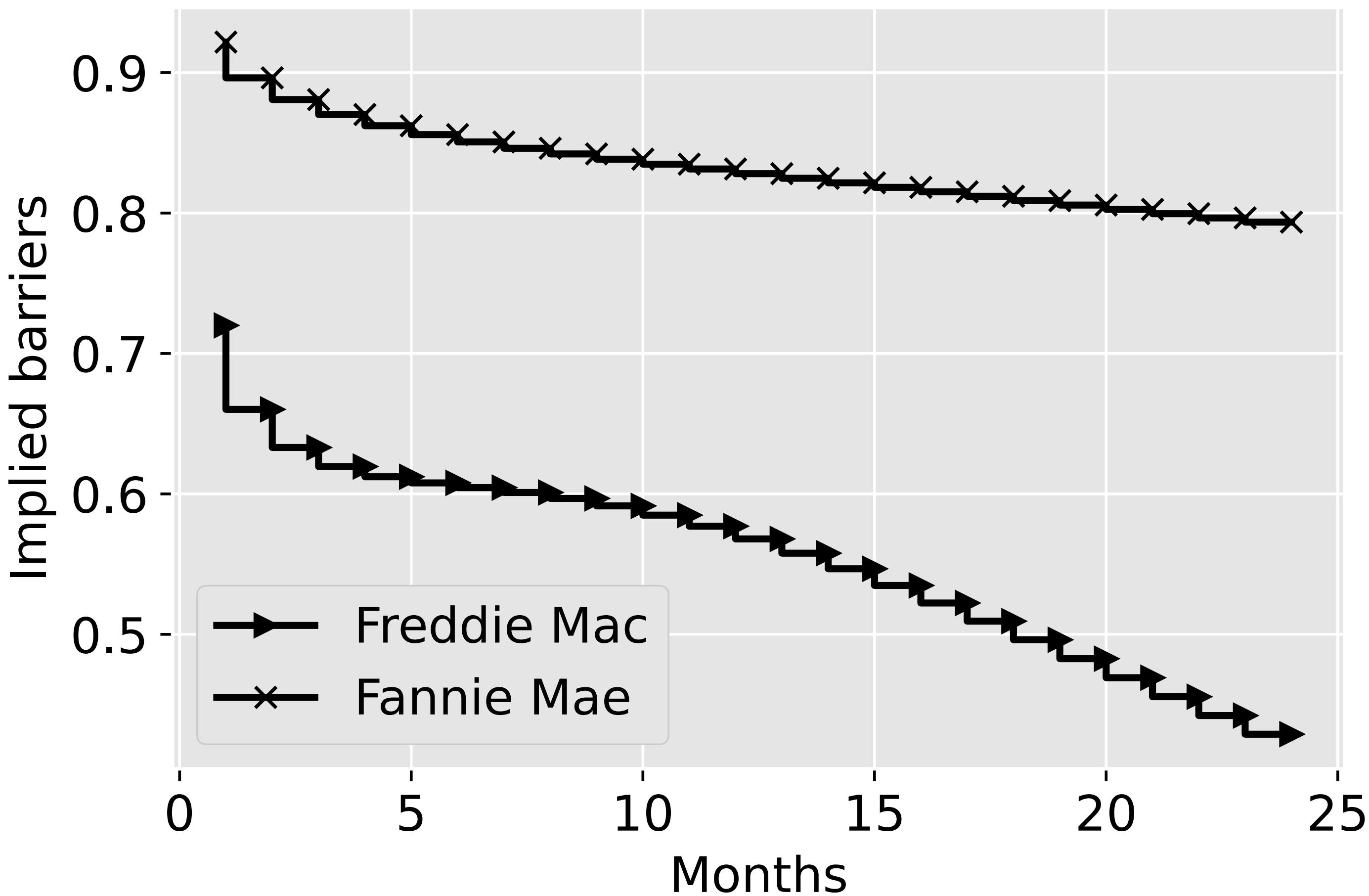


$1 \leq n \leq 101.$

Default Boundary as Implied Barrier (concluded)

- The next plot shows the implied barriers of Freddie Mac and Fannie Mae as of February 2008 (as percentages of the initial asset values).[a]

[a]Plot supplied by Mr. Lu, Yu-Ming (`R06723032`, `D08922008`) on February 26. 2021.



Implied barriers
0.9
0.8
0.7
0.6
0.5
0
5
10
15
20
25
Months
Freddie Mac
Fannie Mae

Time-Varying Double Barriers under Time-Dependent Volatility[a]

- More general models allow a time-varying $\sigma(t)$ (p. 316).
- Let the two barriers $L(t)$ and $H(t)$ be functions of time.[b]
  - Exponential functions are popular.[c]
- Still, we can price double-barrier options in $O(n^2)$ time or less with trinomial trees.
- Continuously monitored double-barrier knock-out options with time-varying barriers are called hot dog options.[d]

---

[a] Y. Zhang (R05922052) (2019).

[b] So the barriers are continuously monitored.

[c] C. Chou (R97944012) (2010); C. I. Chen (R98922127) (2011).

[d] El Babsiri & Noel (1998).

General Local-Volatility Models and Their Trees

- Consider the general local-volatility model

$$\frac{dS}{S} = (r_t - q_t)\, dt + \sigma(S, t)\, dW,$$

where $L \leq \sigma(S, t) \leq U$ for some positive L and U.

- This model has a unique (weak) solution.[a]

- The positive lower bound is justifiable because prices fluctuate.

[a] Achdou & Pironneau (2005).

General Local-Volatility Models and Their Trees (continued)

- The upper-bound assumption is also reasonable.
- Even on October 19, 1987, the CBOE S&P 100 Volatility Index (VXO) was about 150%, the highest ever.[a]
- An efficient quadratic-sized tree for this range-bounded model is straightforward.[b]
- Pick any $\sigma' \geq U$.
- Grow the trinomial tree with the node spacing $\sigma' \sqrt{\Delta t}$.[c]
- The branching probabilities are guaranteed to be valid.

[a] Caprio (2012).
[b] Lok (D99922028) & Lyuu (2016, 2017, 2020).
[c] Haahtela (2010).

General Local-Volatility Models and Their Trees (concluded)

- The same idea can be applied to price double-barrier options.
- Pick any

$$\sigma' \geq \max \left[\max_{S, 0 \leq t \leq T} \sigma(S, t), \sqrt{2}\, \sigma(S_0, 0) \right].$$

- Grow the trinomial tree with the node spacing $\sigma' \sqrt{\Delta t}$.
- Apply the mean-tracking idea to the first period and Eqs. (104)–(109) on p. 768 to obtain the probabilities

Merton's Jump-Diffusion Model

- Empirically, stock returns tend to have fat tails, inconsistent with the Black-Scholes model's assumptions.
- Stochastic volatility and jump processes have been proposed to address this problem.
- Merton's (1976) jump-diffusion model is our focus.

Merton's Jump-Diffusion Model (continued)

- This model superimposes a jump component on a diffusion component.
- The diffusion component is the familiar geometric Brownian motion.
- The jump component is composed of lognormal jumps driven by a Poisson process.
  - It models the *rare* but *large* changes in the stock price because of the arrival of important news.[a]

---

[a] Derman & M. B. Miller (2016), "There is no precise, universally accepted definition of a jump, but it usually comes down to magnitude, duration, and frequency."

Merton's Jump-Diffusion Model (continued)

- Let S_t be the stock price at time t.
- The risk-neutral jump-diffusion process for the stock price follows[a]

$$\frac{dS_t}{S_t} = (r - \lambda \bar{k})\, dt + \sigma\, dW_t + k\, dq_t. \tag{111}$$

- Above, σ denotes the volatility of the diffusion component.

[a] Derman & M. B. Miller (2016), "[M]ost jump-diffusion models simply assume risk-neutral pricing without convincing justification."

Merton's Jump-Diffusion Model (continued)

- The jump event is governed by a compound Poisson process q_t with intensity λ, where k denotes the magnitude of the *random* jump.
  - The distribution of k obeys

$$\ln(1+k) \sim N\left(\gamma, \delta^2\right)$$

    with mean $\bar{k} \stackrel{\Delta}{=} E\left(k\right) = e^{\gamma+\delta^2/2} - 1$.
  - Note that $k > -1$.
  - Note also that k is not related to dt.
- The model with $\lambda = 0$ reduces to the Black-Scholes model.

Merton's Jump-Diffusion Model (continued)

- The solution to Eq. (111) on p. 808 is

$$S_t = S_0 e^{(r - \lambda \bar{k} - \sigma^2/2)\, t + \sigma W_t} U(n(t)), \tag{112}$$

where

$$U(n(t)) = \prod_{i=0}^{n(t)} (1 + k_i)\,.$$

  - k_i is the magnitude of the ith jump with $\ln(1 + k_i) \sim N(\gamma, \delta^2)$.
  - $k_0 = 0$.
  - $n(t)$ is a Poisson process with intensity λ.

Merton's Jump-Diffusion Model (concluded)

- Recall that $n(t)$ denotes the number of jumps that occur up to time t.
- It is known that $E[\,n(t)\,] = \text{Var}[\,n(t)\,] = \lambda t$.
- As $k_i > -1$, stock prices will stay positive.
- The geometric Brownian motion, the lognormal jumps, and the Poisson process are assumed to be independent.

Tree for Merton's Jump-Diffusion Model[a]

- Define the S-logarithmic return of the stock price S' as

$$\ln(S'/S).$$

- Define the logarithmic distance between stock prices S' and S as

$$|\ln(S') - \ln(S)| = |\ln(S'/S)|.$$

[a]Dai (B82506025, R86526008, D8852600), C. Wang (F95922018), Lyuu, & Y. Liu (2010).

Tree for Merton's Jump-Diffusion Model (continued)

- Take the logarithm of Eq. (112) on p. 810:

$$M_t \triangleq \ln\left(\frac{S_t}{S_0}\right) = X_t + Y_t, \tag{113}$$

where

$$X_t \triangleq \left(r - \lambda\bar{k} - \frac{\sigma^2}{2}\right) t + \sigma W_t, \tag{114}$$

$$Y_t \triangleq \sum_{i=0}^{n(t)} \ln\left(1 + k_i\right). \tag{115}$$

- It decomposes the S_0-logarithmic return of S_t into the diffusion component X_t and the jump component Y_t.

Tree for Merton's Jump-Diffusion Model (continued)

- Motivated by decomposition (113) on p. 813, the tree construction divides each period into a diffusion phase followed by a jump phase.
- In the diffusion phase, X_t is approximated by the BOPM.
- So X_t makes an up move to $X_t + \sigma\sqrt{\Delta t}$ with probability p_u or a down move to $X_t - \sigma\sqrt{\Delta t}$ with probability p_d.

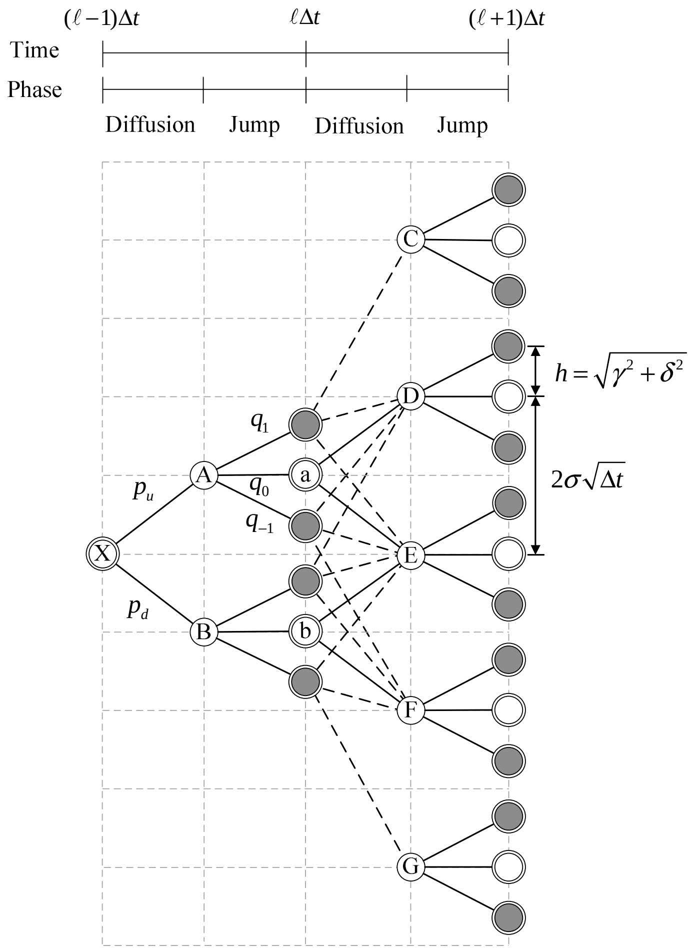
Tree for Merton's Jump-Diffusion Model (continued)

- According to BOPM,

$$
\begin{aligned}
p_u &= \frac{e^{\mu \Delta t} - d}{u - d}, \\
p_d &= 1 - p_u,
\end{aligned}
$$

  except that $\mu = r - \lambda \bar{k}$ here.

- The diffusion component gives rise to diffusion nodes.
- They are spaced at $2\sigma\sqrt{\Delta t}$ apart such as the white nodes A, B, C, D, E, F, and G on p. 816.



White nodes are *diffusion nodes*. Gray nodes are *jump nodes*. In the diffusion phase, the solid black lines denote the binomial structure of BOPM; the dashed lines denote the trinomial structure. Only the double-circled nodes will remain after the construction. Note that a and b are diffusion nodes because no jump occurs in the jump phase.

Tree for Merton's Jump-Diffusion Model (continued)

- In the jump phase, $Y_{t+\Delta t}$ is approximated by moves from *each* diffusion node to $2m$ jump nodes that match the first $2m$ moments of the lognormal jump.
- The m jump nodes above the diffusion node are spaced at $h \stackrel{\Delta}{=} \sqrt{\gamma^2 + \delta^2}$ apart.
- Note that h is independent of Δt.

Tree for Merton's Jump-Diffusion Model (concluded)

- The same holds for the m jump nodes below the diffusion node.
- The gray nodes at time $\ell \Delta t$ on p. 816 are jump nodes.
  - We set $m = 1$ on p. 816.
- The size of the tree is $O(n^{2.5})$.