The Hull-White Model

- Hull and White (1987) postulate the following *stochastic-volatility* model,

$$
\begin{aligned}
\frac{dS}{S} &= r\,dt + \sqrt{V}\,dW_1, \\
dV &= \mu_{\mathrm{v}} V\,dt + bV\,dW_2.
\end{aligned}
$$

- Above, V is the instantaneous variance.
- They assume μ_{v} depends on V and t (but not S).

The Barone-Adesi–Rasmussen–Ravanelli Model

- Barone-Adesi, Rasmussen, and Ravanelli (2005) postulate the following model,

$$\begin{aligned} \frac{dS}{S} &= \mu\, dt + \sqrt{V}\, dW_1, \\ dV &= \kappa(\theta - V)\, dt + bV\, dW_2. \end{aligned}$$

- Above, W_1 and W_2 are correlated.

The Stein-Stein Model

- E. Stein and J. Stein (1991) postulate the following model,

$$\begin{aligned} \frac{dS}{S} &= r\,dt + V\,dW_1, \\ dV &= \kappa(\mu - V)\,dt + \sigma\,dW. \end{aligned}$$

- Closed-form formulas exist for European calls and puts.[a]

[a] Schöbel & Zhu (1999).

The SABR Model

- Hagan, Kumar, Lesniewski, and Woodward (2002) postulate the following model,

$$\begin{aligned} \frac{dS}{S} &= r\,dt + S^{\theta} V\,dW_1, \\ dV &= bV\,dW_2, \end{aligned}$$

for $0 \leq \theta \leq 1$.

- A nice feature of this model is that the implied volatility surface has a compact approximate closed form.

The Blacher Model

- Blacher (2001) postulates the following model,

$$\begin{aligned}\frac{dS}{S} &= r\,dt + \sigma\left[\,1 + \alpha(S - S_0) + \beta(S - S_0)^2\,\right] dW_1,\\ d\sigma &= \kappa(\theta - \sigma)\,dt + \epsilon\sigma\,dW_2.\end{aligned}$$

- The volatility σ follows a mean-reverting process to level θ.

The Hilliard-Schwartz Model

- Hilliard and Schwartz (1996) postulate the following very general model,

$$\begin{aligned} \frac{dS}{S} &= r\,dt + f(S)V^a\,dW_1, \\ dV &= \mu(V)\,dt + bV\,dW_2, \end{aligned}$$

  for some well-behaved function $f(S)$ and constant a.

- It includes all previously mentioned stochastic-volatility models as special cases.[a]

---

[a] H. Chiu (R98723059) (2012).

Heston's Stochastic-Volatility Model

- Heston (1993) assumes the stock price follows

$$\frac{dS}{S} = (\mu - q)\, dt + \sqrt{V}\, dW_1, \quad (91)$$

$$dV = \kappa(\theta - V)\, dt + \sigma\sqrt{V}\, dW_2. \quad (92)$$

  - V is the instantaneous variance, which follows a square-root process.
  - dW_1 and dW_2 have correlation ρ.
  - The riskless rate r is constant.

- It may be the most popular continuous-time stochastic-volatility model.[a]

[a] Christoffersen, Heston, & Jacobs (2009).

Heston's Stochastic-Volatility Model (continued)

- Heston assumes the market price of risk is $b_2 \sqrt{V}$.
- So $\mu = r + b_2 V$.
- Define

$$\begin{aligned} dW_1^* &= dW_1 + b_2 \sqrt{V}\, dt, \\ dW_2^* &= dW_2 + \rho b_2 \sqrt{V}\, dt, \\ \kappa^* &= \kappa + \rho b_2 \sigma, \\ \theta^* &= \frac{\theta \kappa}{\kappa + \rho b_2 \sigma}. \end{aligned}$$

- dW_1^* and dW_2^* have correlation ρ.

Heston's Stochastic-Volatility Model (continued)

- Under the risk-neutral probability measure Q, both W_1^* and W_2^* are Wiener processes.
- Heston's model becomes, under probability measure Q,

$$\begin{aligned} \frac{dS}{S} &= (r-q)\,dt + \sqrt{V}\,dW_1^*, \\ dV &= \kappa^*(\theta^* - V)\,dt + \sigma\sqrt{V}\,dW_2^*. \end{aligned}$$

Heston's Stochastic-Volatility Model (continued)

- Define

$$
\begin{aligned}
\phi(u,\tau) &= \exp\Big\{\, \imath u(\ln S + (r-q)\,\tau) \\
&\quad + \theta^*\kappa^*\sigma^{-2}\left[(\kappa^* - \rho\sigma u\imath - d)\,\tau - 2\ln\frac{1-ge^{-d\tau}}{1-g} \right] \\
&\quad + \frac{v\sigma^{-2}(\kappa^* - \rho\sigma u\imath - d)\left(1-e^{-d\tau}\right)}{1-ge^{-d\tau}} \Big\}, \\
d &= \sqrt{(\rho\sigma u\imath - \kappa^*)^2 - \sigma^2(-\imath u - u^2)}\,, \\
g &= (\kappa^* - \rho\sigma u\imath - d)/(\kappa^* - \rho\sigma u\imath + d).
\end{aligned}
$$

Heston's Stochastic-Volatility Model (continued)

The formulas for European calls and puts are[a]

$$\begin{aligned} C &= S\left[\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{X^{-\imath u}\phi(u-\imath,\tau)}{\imath u S e^{r\tau}}\right)du\right] \\ &\quad -Xe^{-r\tau}\left[\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{X^{-\imath u}\phi(u,\tau)}{\imath u}\right)du\right], \\ P &= Xe^{-r\tau}\left[\frac{1}{2}-\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{X^{-\imath u}\phi(u,\tau)}{\imath u}\right)du\right], \\ &\quad -S\left[\frac{1}{2}-\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{X^{-\imath u}\phi(u-\imath,\tau)}{\imath u S e^{r\tau}}\right)du\right], \end{aligned}$$

where $\imath = \sqrt{-1}$ and $\mathrm{Re}(x)$ denotes the real part of the complex number x.

---

[a] Contributed by Mr. Chen, Chun-Ying (`D95723006`) on August 17, 2008 and Mr. Liou, Yan-Fu (`R92723060`) on August 26, 2008. See Lord & Kahl (2009) and Cui, Rollin, & Germano (2017) for alternative formulas.

Heston's Stochastic-Volatility Model (concluded)

- For American options, trees are needed.
- They are all $O(n^3)$-sized and do not match all moments.[a]
- An $O(n^{2.5})$-sized 9-jump tree that matches *all* means and variances with valid probabilities is available.[b]
- The size reduces to $O(n^2)$ for knock-out double-barrier options.[c]

---

[a] Nelson & Ramaswamy (1990); Nawalkha & Beliaeva (2007); Leisen (2010); Beliaeva & Nawalkha (2010); M. Chou (`R02723073`) (2015); M. Chou (`R02723073`) & Lyuu (2016).

[b] Z. Lu (`D00922011`) & Lyuu (2018).

[c] Z. Lu (`D00922011`) & Lyuu (2018).

Stochastic-Volatility Models and Further Extensions[a]

- How to explain the October 1987 crash?
  - The Dow Jones Industrial Average fell 22.61% on October 19, 1987 (called the Black Monday).
  - The CBOE S&P 100 Volatility Index (VXO) shot up to 150%, the highest VXO ever recorded.[b]
- Stochastic-volatility models require an implausibly high-volatility level prior to *and* after the crash.
  - Because the processes are continuous.
- Discontinuous jump models *in the asset price* can alleviate the problem somewhat.[c]

---

[a] Eraker (2004).
[b] Caprio (2012).
[c] Merton (1976).

Stochastic-Volatility Models and Further Extensions (continued)

- But if the jump intensity is a constant, it cannot explain the tendency of large movements to cluster over time.
- This assumption also has no impacts on option prices.
- Jump-diffusion models combine both.
  - E.g., add a jump process to Eq. (91) on p. 663.
  - Closed-form formulas exist for GARCH-jump option pricing models.[a]

[a]Liou (R92723060) (2005).

Stochastic-Volatility Models and Further Extensions (concluded)

- But they still do not adequately describe the systematic variations in option prices.[a]
- Jumps *in volatility* are alternatives.[b]
  - E.g., add correlated jump processes to Eqs. (91) *and* Eq. (92) on p. 663.
- Such models allow high level of volatility caused by a jump to volatility.[c]

---

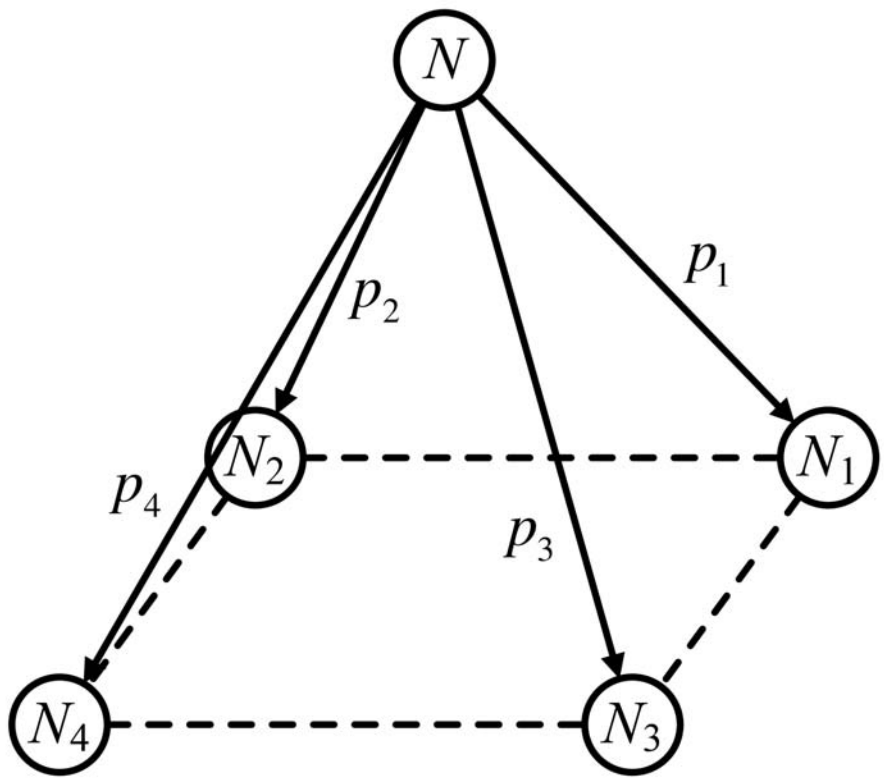
[a] Bates (2000); Pan (2002).
[b] Duffie, Pan, & Singleton (2000).
[c] Eraker, Johnnes, & Polson (2000); Y. Lin (2007); Zhu & Lian (2012).

Why Are Trees for Stochastic-Volatility Models Difficult?

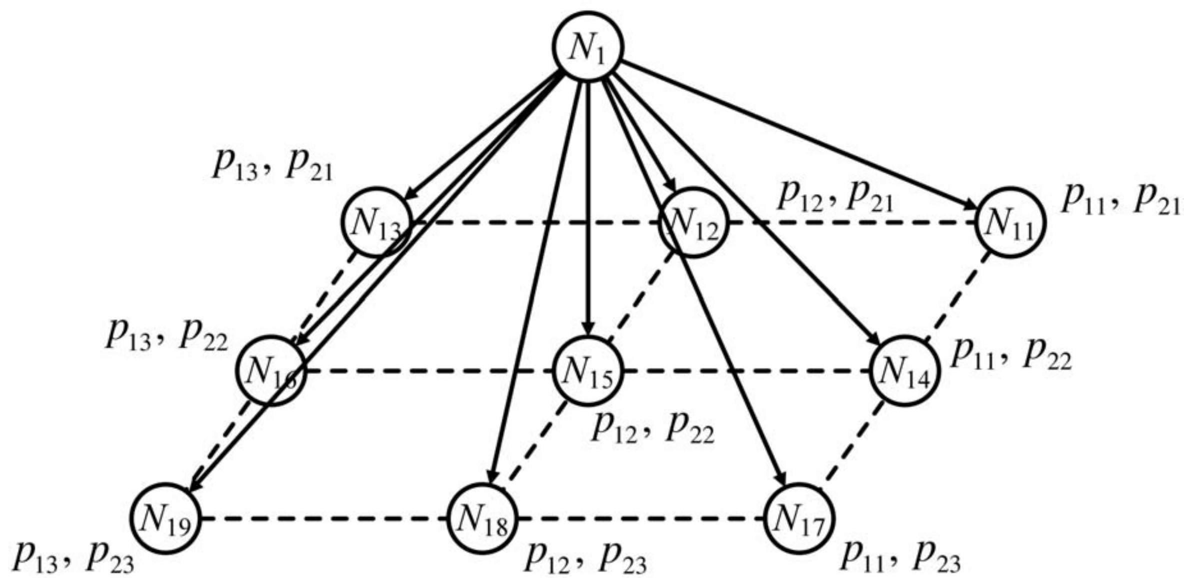
- The CRR tree is 2-dimensional.[a]
- The constant volatility makes the span from any node fixed.
- But a tree for a stochastic-volatility model must be 3-dimensional.
  - Every node is associated with a combination of stock price *and* volatility.

[a]Recall p. 298.

Why Are Trees for Stochastic-Volatility Models Difficult (Binomial Case)?



Why Are Trees for Stochastic-Volatility Models Difficult (Trinomial Case)?



Why Are Trees for Stochastic-Volatility Models Difficult? (concluded)

- *Locally*, the tree looks fine for one time step.
- But the volatility regulates the spans of the nodes on the stock-price plane.
- Unfortunately, those spans differ from node to node because the volatility varies.
- So two time steps from now, the branches will not combine!
- Smart ideas are thus needed.

Complexities of Stochastic-Volatility Models

- A few stochastic-volatility models suffer from subexponential ($c^{\sqrt{n}}$) tree size.
- Examples include the Hull-White (1987), Hilliard-Schwartz (1996), and SABR (2002) models.[a]
- Future research may extend this negative result to more stochastic-volatility models.
  - We suspect many GARCH option pricing models entertain similar problems.[b]

---

[a] H. Chiu (R98723059) (2012).

[b] Y. C. Chen (R95723051) (2008); Y. C. Chen (R95723051), Lyuu, & Wen (D94922003) (2011).

Complexities of Stochastic-Volatility Models (concluded)

- Flexible placement of nodes and removal of low-probability nodes may make the models $O(n^{2.5})$-sized![a]
- Calibration can be computationally hard.
  - Few have tried it on exotic options.[b]
- There are usually several local minima.[c]
  - They will give different prices to options not used in the calibration.
  - But which set capture the smile dynamics?

---

[a] Z. Lu (`D00922011`) & Lyuu (2018).
[b] Ayache, Henrotte, Nassar, & X. Wang (2004).
[c] Ayache (2004).

Continuous-Time Derivatives Pricing

I have hardly met a mathematician
who was capable of reasoning.
— Plato (428 B.C.–347 B.C.)

Fischer [Black] is the only real genius
I've ever met in finance. Other people,
like Robert Merton or Stephen Ross,
are just very smart and quick,
but they think like me.
Fischer came from someplace else entirely.
— John C. Cox, quoted in Mehrling (2005)

Toward the Black-Scholes Differential Equation

- The price of any derivative on a non-dividend-paying stock must satisfy a partial differential equation (PDE).
- The key step is recognizing that the same random process drives both securities.
  - Their prices are perfectly correlated.
- We then figure out the amount of stock such that the gain from it offsets exactly the loss from the derivative.
- The removal of uncertainty forces the portfolio's return to be the riskless rate.
- PDEs allow many numerical methods to be applicable.

Assumptions[a] and Notations

- The stock price follows $dS = \mu S\, dt + \sigma S\, dW$.
- There are no dividends.
- Trading is continuous, and short selling is allowed.
- There are no transactions costs or taxes.
- All securities are infinitely divisible.
- The term structure of riskless rates is flat at r.
- There is unlimited riskless borrowing and lending.
- t is the current time, T is the expiration time, and $\tau \stackrel{\Delta}{=} T - t$.

[a]Derman & Taleb (2005) summarizes criticisms on these assumptions and the replication argument.

Black-Scholes Differential Equation

- Let C be the price of a *simple* derivative[a] on S.
- From Ito's lemma (p. 611),

$$dC = \left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S} dW.$$

  – The same W drives both C and S.
  – Unlike dS/S, the diffusion of dC/C is stochastic!
- Short one derivative and long $\partial C/\partial S$ shares of stock (call it Π).
- By construction,

$$\Pi = -C + S(\partial C/\partial S).$$

[a] Recall p. 435.

Black-Scholes Differential Equation (continued)

- The change in the value of the portfolio at time dt is[a]

$$d\Pi = -dC + \frac{\partial C}{\partial S}\, dS. \tag{93}$$

- Substitute the formulas for dC and dS into the above to yield

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} \right) dt.$$

- As this equation does not involve dW, the portfolio is riskless during dt time: $d\Pi = r\Pi\, dt$.

---

[a]Bergman (1982) and Bartels (1995) argue this is not quite right. But see Macdonald (1997). Mathematically, it is wrong (Bingham & Kiesel, 2004).

Black-Scholes Differential Equation (continued)

- So
$$\left(\frac{\partial C}{\partial t}+\frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 C}{\partial S^2}\right)dt = r\left(C - S\,\frac{\partial C}{\partial S}\right)dt.$$
- Equate the terms to finally obtain[a]
$$\frac{\partial C}{\partial t}+rS\,\frac{\partial C}{\partial S}+\frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 C}{\partial S^2}=rC.$$
- This is a backward equation, which describes the dynamics of a derivative's price *forward* in physical time.

[a]Known as the Feynman-Kac stochastic representation formula.

Black-Scholes Differential Equation (concluded)

- When there is a dividend yield q,

$$\frac{\partial C}{\partial t} + (r-q)\, S\, \frac{\partial C}{\partial S} + \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} = rC. \tag{94}$$

- Dupire's formula (90) for the local-volatility model[a] is simply the dual of this equation:[b]

$$\frac{\partial C}{\partial T} + (r_T - q_T) X \frac{\partial C}{\partial X} - \frac{1}{2}\, \sigma(X,T)^2 X^2 \frac{\partial^2 C}{\partial X^2} = -q_T C.$$

- This is a forward equation, which describes the dynamics of a derivative's price *backward* in maturity time.

---

[a]See p. 641.

[b]Derman & Kani (1997).

Rephrase

- The Black-Scholes differential equation can be expressed in terms of sensitivity numbers,

$$\Theta + rS\Delta + \frac{1}{2}\,\sigma^2 S^2 \Gamma = rC. \tag{95}$$

- Identity (95) leads to an alternative way of computing Θ numerically from Δ and Γ.
- When a portfolio is delta-neutral,

$$\Theta + \frac{1}{2}\,\sigma^2 S^2 \Gamma = rC.$$

  – A definite relation thus exists between Γ and Θ.

Black-Scholes Differential Equation: An Alternative

- Perform the change of variable $V \triangleq \ln S$.
- The option value becomes $U(V, t) \triangleq C(e^V, t)$.
- Furthermore,

$$\begin{aligned} \frac{\partial C}{\partial t} &= \frac{\partial U}{\partial t}, \\ \frac{\partial C}{\partial S} &= \frac{1}{S}\frac{\partial U}{\partial V}, && (96) \\ \frac{\partial^2 C}{\partial^2 S} &= \frac{1}{S^2}\frac{\partial^2 U}{\partial V^2} - \frac{1}{S^2}\frac{\partial U}{\partial V}. && (97) \end{aligned}$$

Black-Scholes Differential Equation: An Alternative (concluded)

- Equations (96) and (97) are alternative ways to calculate delta and gamma.[a]
- They are particularly useful for a tree of *logarithmic* prices.
- The Black-Scholes differential equation (94) on p. 685 becomes

$$\frac{1}{2}\,\sigma^2\,\frac{\partial^2 U}{\partial V^2} + \left(r - q - \frac{\sigma^2}{2}\right)\frac{\partial U}{\partial V} - rU + \frac{\partial U}{\partial t} = 0$$

subject to $U(V, T)$ being the payoff such as $\max(X - e^V, 0)$.

---

[a] See Eqs. (52) on p. 365 and (53) on p. 367.

[Black] got the equation [in 1969] but then was unable to solve it. Had he been a better physicist he would have recognized it as a form of the familiar heat exchange equation, and applied the known solution. Had he been a better mathematician, he could have solved the equation from first principles. Certainly Merton would have known exactly what to do with the equation had he ever seen it.

— Perry Mehrling (2005)

PDEs for Asian Options

- Add the new variable $A(t) \stackrel{\Delta}{=} \int_0^t S(u)\, du$.
- Then the value V of the Asian option satisfies this two-dimensional PDE:[a]

$$\frac{\partial V}{\partial t} + rS\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} + S\,\frac{\partial V}{\partial A} = rV.$$

- The terminal conditions are

$$\begin{aligned} V(T, S, A) &= \max\left(\frac{A}{T} - X, 0\right) \quad \text{for call,} \\ V(T, S, A) &= \max\left(X - \frac{A}{T}, 0\right) \quad \text{for put.} \end{aligned}$$

[a]Kemna & Vorst (1990).

PDEs for Asian Options (continued)

- The two-dimensional PDE produces algorithms similar to that on pp. 447ff.[a]
- But one-dimensional PDEs are available for Asian options.[b]
- For example, Večeř (2001) derives the following PDE for Asian calls:

$$\frac{\partial u}{\partial t} + r\left(1 - \frac{t}{T} - z\right)\frac{\partial u}{\partial z} + \frac{\left(1 - \frac{t}{T} - z\right)^2 \sigma^2}{2}\frac{\partial^2 u}{\partial z^2} = 0$$

with the terminal condition $u(T, z) = \max(z, 0)$.

[a] Barraquand & Pudet (1996).

[b] Rogers & Shi (1995); Večeř (2001); Dubois & Lelièvre (2005).

PDEs for Asian Options (concluded)

- For Asian puts:

$$\frac{\partial u}{\partial t} + r\left(\frac{t}{T} - 1 - z\right)\frac{\partial u}{\partial z} + \frac{\left(\frac{t}{T} - 1 - z\right)^2 \sigma^2}{2}\frac{\partial^2 u}{\partial z^2} = 0$$

  with the same terminal condition.

- One-dimensional PDEs result in highly efficient numerical algorithms.

Hedging

When Professors Scholes and Merton and I
invested in warrants,
Professor Merton lost the most money.
And I lost the least.
— Fischer Black (1938–1995)

Delta Hedge

- Recall the delta (hedge ratio) of a derivative f:

$$\Delta \stackrel{\Delta}{=} \frac{\partial f}{\partial S}.$$

- Thus

$$\Delta f \approx \Delta \times \Delta S$$

  for relatively small changes in the stock price, ΔS.

- A delta-neutral portfolio is hedged as it is immunized against small changes in the stock price.

Delta Hedge (concluded)

- A trading strategy that dynamically maintains a delta-neutral portfolio is called delta hedge.
  - Trading strategies can also be static (or constant).[a]
- Delta changes with the stock price.
- A delta hedge needs to be rebalanced periodically in order to maintain delta neutrality.
- In the limit where the portfolio is adjusted continuously, "perfect" hedge is achieved and the strategy becomes "self-financing."

---

[a] Recall p. 494 for one in hedging the short forward contract with the underlying asset and loans.

Implementing Delta Hedge

- We want to hedge N *short* derivatives.
- Assume the stock pays no dividends.
- The delta-neutral portfolio maintains $N \times \Delta$ shares of stock plus B borrowed dollars such that

$$-N \times f + N \times \Delta \times S - B = 0.$$

- At next rebalancing point when the delta is Δ', buy $N \times (\Delta' - \Delta)$ shares to maintain $N \times \Delta'$ shares.
- Delta hedge is the discrete-time analog of the continuous-time limit.
- It will rarely be self-financing however small Δt is.

Example

- A hedger is *short* 10,000 European calls.
- $S = 50$, $\sigma = 30\%$, and $r = 6\%$.
- This call's expiration is four weeks away, its strike price is \$50, and each call has a current value of $f = 1.76791$.
- As an option covers 100 shares of stock, $N = 1{,}000{,}000$.
- The trader adjusts the portfolio weekly.
- The calls are replicated well if the cumulative cost of trading *stock* is close to the call premium's FV.[a]

[a]This takes the replication viewpoint: One starts with zero dollar.

Example (continued)

- As $\Delta = 0.538560$

  $$N \times \Delta = 538,560$$

  shares are purchased for a total cost of

  $$538{,}560 \times 50 = 26{,}928{,}000$$

  dollars to make the portfolio delta-neutral.

- The trader finances the purchase by borrowing

  $$B = N \times \Delta \times S - N \times f = 25{,}160{,}090$$

  dollars net.[a]

[a] This takes the hedging viewpoint: One starts with the option premium. See Exercise 16.3.2 of the text.

Example (continued)

- At 3 weeks to expiration, the stock price rises to \$51.
- The new call value is $f' = 2.10580$.
- So before rebalancing, the portfolio is worth

$$-N \times f' + 538{,}560 \times 51 - Be^{0.06/52} = 171,622. \qquad (98)$$

- The delta hedge is not self-financing as \$171,622 can be withdrawn.
  - It does *not* replicate the calls perfectly.

Example (continued)

- The magnitude of the tracking error—the variation in the net portfolio value—can be mitigated if adjustments are made more frequently.
- The tracking error over *one* rebalancing act is positive about 68% of the time.
- But its expected value is ~ 0 under the *risk-neutral* probability measure.[a]
  - Of course, the stock price should be sampled under the *real-world* probability measure.[b]

[a] Boyle & Emanuel (1980).

[b] Recall Eq. (93) on p. 683 or see p. 711.

Example (continued)

- The tracking error at maturity is proportional to vega.[a]
- In practice tracking errors will cease to decrease beyond a certain rebalancing frequency.
- With a higher delta $\Delta' = 0.640355$, the trader buys

$$N \times (\Delta' - \Delta) = 101,795$$

shares for \$5,191,545.

- The number of shares is increased to $N \times \Delta' = 640,355$.

[a]Kamal & Derman (1999).

Example (continued)

- The cumulative cost is[a]

$$26{,}928{,}000 \times e^{0.06/52} + 5{,}191{,}545 = 32{,}150{,}634.$$

- The portfolio is again delta-neutral.

[a]We take the replication viewpoint. Under the BOPM, the replicating strategy is self-financing and matches the payoff perfectly.

τ	S (1)	Option value f (2)	Delta Δ (3)	Change in delta (5)	No. shares bought $N\times(5)$ (6)	Cost of shares $(1)\times(6)$ (7)	Cumulative cost FV(8')+(7) (8)
4	50	1.7679	0.53856	—	538,560	26,928,000	26,928,000
3	51	2.1058	0.64036	0.10180	101,795	5,191,545	32,150,634
2	53	3.3509	0.85578	0.21542	215,425	11,417,525	43,605,277
1	52	2.2427	0.83983	−0.01595	−15,955	−829,660	42,825,960
0	54	4.0000	1.00000	0.16017	160,175	8,649,450	51,524,853

The total number of shares is 1,000,000 at expiration (trading takes place at expiration, too).

Example (continued)

- At expiration, the trader has 1,000,000 shares.
- They are exercised against by the in-the-money calls for $50,000,000.
- The trader is left with an obligation of

$$51{,}524{,}853 - 50{,}000{,}000 = 1{,}524{,}853,$$

  which represents the replication cost.
- So if we had started with the PV of $1,524,853, we would have replicated 10,000 such calls in *this* scenario.

Example (concluded)

- The FV of the call premium equals

$$1{,}767{,}910 \times e^{0.06 \times 4/52} = 1{,}776{,}088.$$

- That means the net gain in this scenario is

$$1{,}776{,}088 - 1{,}524{,}853 = 251{,}235$$

if we are hedging 10,000 European calls.

Tracking Error Revisited

- Define the dollar gamma as $S^2\Gamma$.
- The change in value of a delta-hedged *long* option position after a duration of Δt is proportional to the dollar gamma.
- It is about

$$(1/2)S^2\Gamma[\,(\Delta S/S)^2 - \sigma^2\Delta t\,].$$

  – $(\Delta S/S)^2$ is called the daily realized variance.

Tracking Error Revisited (continued)

- In our particular case,

$$S = 50, \Gamma = 0.0957074, \Delta S = 1, \sigma = 0.3, \Delta t = 1/52.$$

- The estimated tracking error is

$$-(1/2)\times 50^2 \times 0.0957074 \times \left[\, (1/50)^2 - (0.09/52) \,\right] = 159,205.$$

- It is very close to our earlier number of 171,622.[a]
- Delta hedge is also called gamma scalping.[b]

---

[a] Recall Eq. (98) on p. 700.

[b] Bennett (2014).

Tracking Error Revisited (continued)

- Let the rebalancing times be $t_1, t_2, \ldots, t_n$.
- Let $\Delta S_i = S_{i+1} - S_i$.
- The total tracking error at expiration is about

$$\sum_{i=0}^{n-1} e^{r(T-t_i)} \frac{S_i^2 \Gamma_i}{2} \left[\left(\frac{\Delta S_i}{S_i} \right)^2 - \sigma^2 \Delta t \right].$$

- The tracking error is clearly path dependent.
- Mathematically,[a]

$$\sum_{i=0}^{n-1} \left(\frac{\Delta S_i}{S_i} \right)^2 \to \sigma^2 T.$$

[a] Protter (2005).

Tracking Error Revisited (concluded)[a]

- The tracking error[b] ϵ_n *over* n *rebalancing acts* has about the same probability of being positive as being negative.
- Subject to certain regularity conditions, the root-mean-square tracking error $\sqrt{E[\,\epsilon_n^2\,]}$ is $O(1/\sqrt{n}\,)$.[c]
- The root-mean-square tracking error increases with σ at first and then decreases.

---

[a] Bertsimas, Kogan, & Lo (2000).

[b] Such as 251,235 on p. 706.

[c] Grannan & Swindle (1996).

Which Probability Measure?[a]

- Should the profit and loss (i.e., tracking error) of a hedging strategy be calculated under the *real-world* or risk-neutral probability measure?
- It is the former.
- But the deltas and option prices should be calculated under the risk-neutral probability measure.
- If whenever we sample the next stock price, backward induction is performed for the delta, it will take a long time to obtain the distribution of the profit and loss.
- How to do it efficiently?[b]

[a] Contributed by Mr. Chiu, Tzu-Hsuan (`R08723061`) on April 9, 2021.
[b] Contributed by Mr. Lu, Zheng-Liang (`D00922011`) in August, 2021.

Delta-Gamma Hedge

- Delta hedge is based on the first-order approximation to changes in the derivative price, Δf, due to changes in the stock price, ΔS.
- When ΔS is not small, the second-order term, gamma $\Gamma \triangleq \partial^2 f / \partial S^2$, helps.
- A delta-gamma hedge is a delta hedge that maintains zero portfolio gamma; it is gamma neutral.
- To meet this extra condition, one more security needs to be brought in.

Delta-Gamma Hedge (concluded)

- Suppose we want to hedge short calls as before.
- A hedging call f_2 is brought in.
- To set up a delta-gamma hedge, we solve

$$\begin{aligned} -N \times f + n_1 \times S + n_2 \times f_2 - B &= 0 \quad \text{(self-financing)}, \\ -N \times \Delta + n_1 + n_2 \times \Delta_2 - 0 &= 0 \quad \text{(delta neutrality)}, \\ -N \times \Gamma + 0 + n_2 \times \Gamma_2 - 0 &= 0 \quad \text{(gamma neutrality)}, \end{aligned}$$

  for n_1, n_2, and B.
  - The gammas of the stock and bond are 0.
- See the numerical example on pp. 231–232 of the text.

Other Hedges

- If volatility changes, delta-gamma hedge may not work well.
- An enhancement is the delta-gamma-vega hedge, which also maintains vega zero portfolio vega.
- To accomplish this, still one more security has to be brought into the process.
- In practice, delta-vega hedge, which may not maintain gamma neutrality, performs better than delta hedge.

Trees

I love a tree more than a man.
— Ludwig van Beethoven (1770–1827)

All those holes and pebbles.
Who could count them?
— James Joyce, *Ulysses* (1922)

And though the holes were rather small,
they had to count them all.
— The Beatles, *A Day in the Life* (1967)

The Combinatorial Method

- The combinatorial method can often cut the running time by an order of magnitude.
- The basic paradigm is to count the number of admissible paths that lead from the root to any terminal node.
- We first used this method in the linear-time algorithm for standard European option pricing on p. 286.
- We will now apply it to price barrier options.

The Reflection Principle[a]

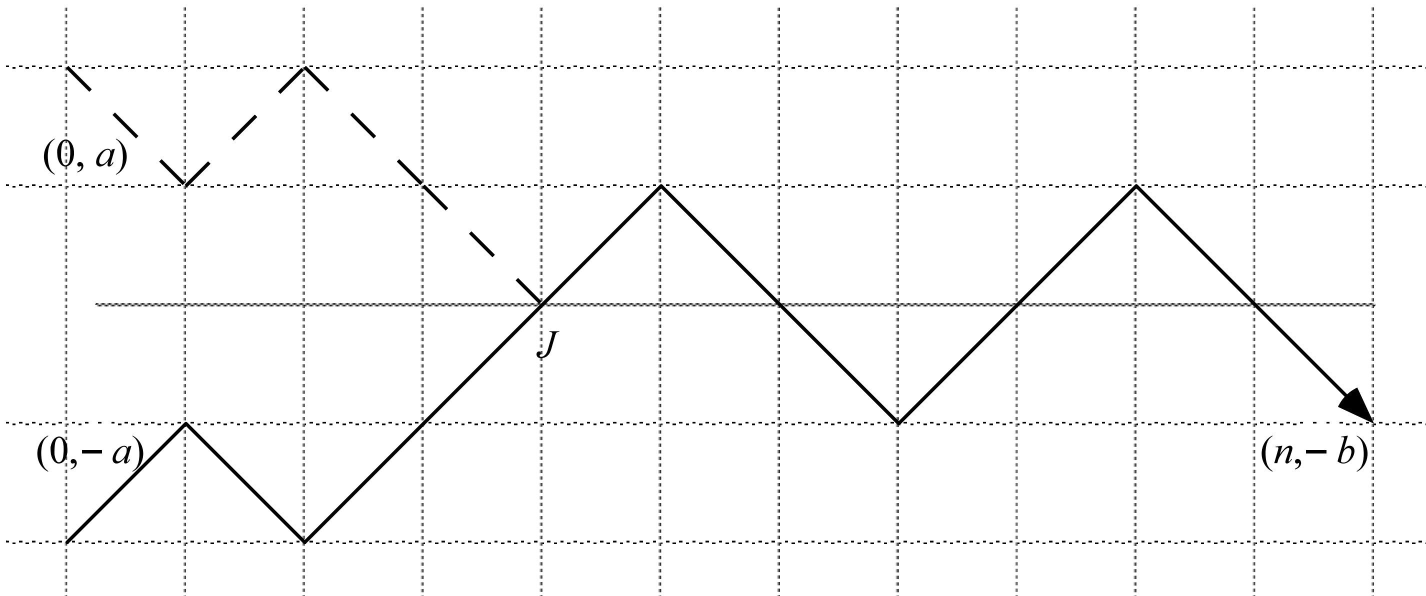
- Imagine a particle at position $(0, -\boldsymbol{a})$ on the integral lattice that is to reach $(n, -\boldsymbol{b})$.
- Without loss of generality, assume $\boldsymbol{a} > 0$ and $\boldsymbol{b} \geq 0$.
- This particle's movement:

$(i, j) \nearrow (i+1, j+1)$ up move $S \to Su$

$(i, j) \searrow (i+1, j-1)$ down move $S \to Sd$

- How many paths touch the x axis?

[a] André (1887).


(0, a)
J
(0,− a)
(n,− b)

The Reflection Principle (continued)

- For a path from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touches the x axis, let J denote the first point this happens.
- Reflect the portion of the path from $(0, -\boldsymbol{a})$ to J.
- A path from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$ is constructed.
- It also hits the x axis at J for the first time.
- The one-to-one mapping shows the number of paths from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touch the x axis equals the number of paths from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$.

The Reflection Principle (concluded)

- A path of this kind has $(n + \boldsymbol{b} + \boldsymbol{a})/2$ down moves and $(n - \boldsymbol{b} - \boldsymbol{a})/2$ up moves.[a]
- Hence there are

$$\binom{n}{\frac{n+\boldsymbol{a}+\boldsymbol{b}}{2}} = \binom{n}{\frac{n-\boldsymbol{a}-\boldsymbol{b}}{2}} \tag{99}$$

such paths for *even* $n + \boldsymbol{a} + \boldsymbol{b}$.

  – Convention: $\binom{n}{k} = 0$ for $k < 0$ or $k > n$.

---

[a] Verify it!

Pricing Barrier Options (Lyuu, 1998)

- Focus on the down-and-in call with barrier $H < X$.
- So $H < S$.
- Define

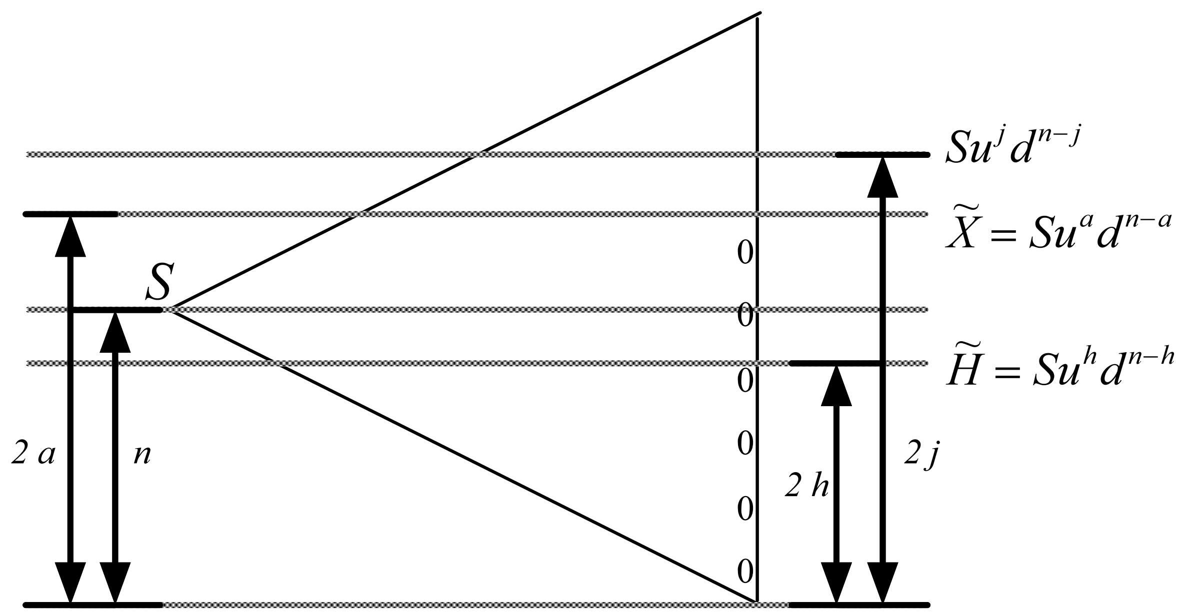
$$
\begin{aligned}
a &\stackrel{\Delta}{=} \left\lceil \frac{\ln\left(X/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rceil = \left\lceil \frac{\ln(X/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rceil, \\
h &\stackrel{\Delta}{=} \left\lfloor \frac{\ln\left(H/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rfloor = \left\lfloor \frac{\ln(H/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rfloor.
\end{aligned}
$$

  - a is such that $\tilde{X} \stackrel{\Delta}{=} Su^a d^{n-a}$ is the *terminal* price that is closest to X from above.
  - h is such that $\tilde{H} \stackrel{\Delta}{=} Su^h d^{n-h}$ is the *terminal* price that is closest to H from below.[a]

[a]So we underestimate the price.

Pricing Barrier Options (continued)

- The true barrier is replaced by the effective barrier $\tilde{H}$ in the binomial model.
- A process with n moves hence ends up in the money if and only if the number of up moves is at least a.
- The price $Su^k d^{n-k}$ is at a distance of $2k$ from the lowest possible price Sd^n on the binomial tree.
  -
$$Su^k d^{n-k} = Sd^{-k} d^{n-k} = Sd^{n-2k}. \tag{100}$$



$Su^j d^{n-j}$
$\widetilde{X} = Su^a d^{n-a}$
S
$\widetilde{H} = Su^h d^{n-h}$
0
0
0
0
0
0
2 a
n
2 h
2 j

Pricing Barrier Options (continued)

- A path from S to the terminal price $Su^j d^{n-j}$ has probability $p^j (1-p)^{n-j}$ of being taken.
- With reference to p. 724, the reflection principle (p. 719) can be applied with

$$\begin{aligned} \boldsymbol{a} &= n - 2h, \\ \boldsymbol{b} &= 2j - 2h, \end{aligned}$$

in Eq. (99) on p. 721 by treating the $\tilde{H}$ line as the x axis.

Pricing Barrier Options (continued)

- Therefore,

$$\binom{n}{\frac{n+(n-2h)+(2j-2h)}{2}} = \binom{n}{n-2h+j}$$

paths hit $\tilde{H}$ in the process for $h \leq n/2$.

- The terminal price $Su^j d^{n-j}$ is reached by a path that hits the effective barrier with probability

$$\binom{n}{n-2h+j} p^j (1-p)^{n-j}, \quad j \leq 2h.$$

Pricing Barrier Options (concluded)

- The option value equals

$$\frac{\sum_{j=a}^{2h} \binom{n}{n-2h+j} p^j (1-p)^{n-j} \left(S u^j d^{n-j} - X\right)}{R^n}. \tag{101}$$

  - $R \stackrel{\Delta}{=} e^{r\tau/n}$ is the riskless return per period.

- It yields a linear-time algorithm.[a]

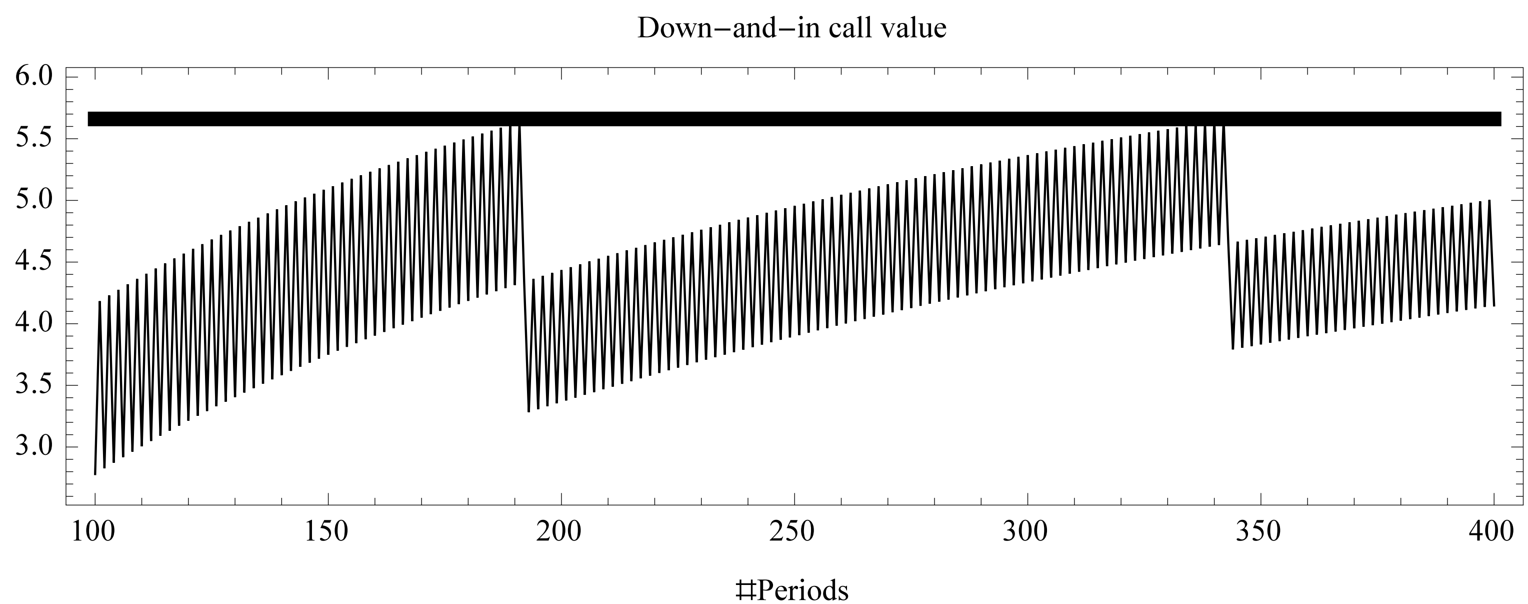
---

[a]Lyuu (1998).

Convergence of BOPM

- Equation (101) results in the same sawtooth-like convergence shown on p. 408 (repeated on next page).
- The reasons are not hard to see.
- The effective barrier $\tilde{H}$ rarely equals the true barrier H.

Convergence of BOPM (continued)



Convergence of BOPM (continued)

- Convergence is actually good if we limit n to certain values—191, for example.
- These values make the true barrier coincide with or just above one of the stock price levels, that is,

$$H \approx Sd^j = Se^{-j\sigma\sqrt{\tau/n}}$$

for some integer j.

- The preferred n's are thus

$$n = \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor, \quad j = 1, 2, 3, \ldots$$

Convergence of BOPM (continued)

- There is only one minor technicality left.
- We picked the effective barrier to be one of the $n+1$ possible *terminal* stock prices.
- However, the effective barrier above, Sd^j, corresponds to a terminal stock price only when $n-j$ is even.[a]
- To close this gap, we decrement n by one, if necessary, to make $n-j$ an even number.

---

[a]This is because $j = n - 2k$ for some k by Eq. (100) on p. 723. Of course we could have adopted the more general form Sd^j $(-n \leq j \leq n)$ for the effective barrier. It makes a good exercise.

Convergence of BOPM (concluded)

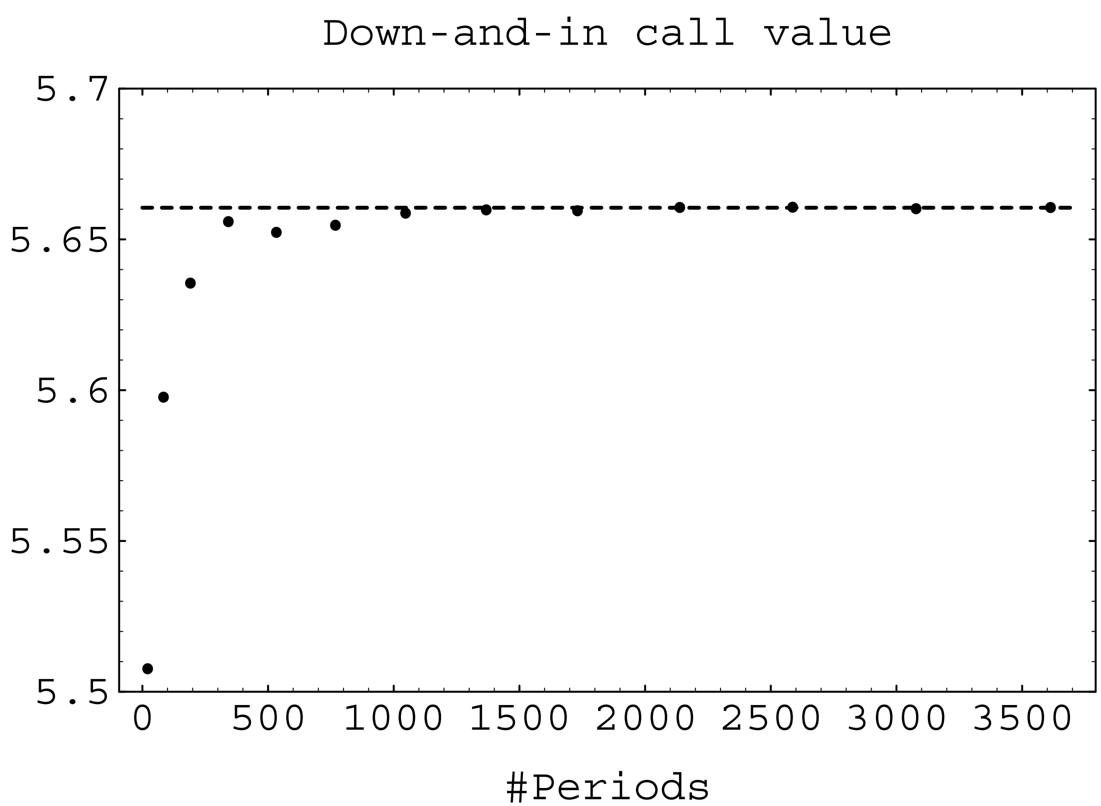
- The preferred n's are now

$$n = \begin{cases} \ell, & \text{if } \ell - j \text{ is even,} \\ \ell - 1, & \text{otherwise,} \end{cases}$$

$j = 1, 2, 3, \ldots,$ where

$$\ell \triangleq \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor.$$

- Evaluate pricing formula (101) on p. 727 only with the n's above.


Down-and-in call value
5.7
5.65
5.6
5.55
5.5
0 500 1000 1500 2000 2500 3000 3500
#Periods

Practical Implications[a]

- This binomial model is $O(1/\sqrt{n})$ convergent in general but $O(1/n)$ convergent when the barrier is matched.[b]
- Now that barrier options can be efficiently priced, we can afford to pick very large n's (p. 735).
- This has profound consequences.[c]

---

[a] Lyuu (1998).

[b] J. Lin (R95221010) (2008); ; J. Lin (R95221010) & Palmer (2013).

[c] See pp. 749ff.

n	Combinatorial method	
	Value	Time (milliseconds)
21	5.507548	0.30
84	5.597597	0.90
191	5.635415	2.00
342	5.655812	3.60
533	5.652253	5.60
768	5.654609	8.00
1047	5.658622	11.10
1368	5.659711	15.00
1731	5.659416	19.40
2138	5.660511	24.70
2587	5.660592	30.20
3078	5.660099	36.70
3613	5.660498	43.70
4190	5.660388	44.10
4809	5.659955	51.60
5472	5.660122	68.70
6177	5.659981	76.70
6926	5.660263	86.90
7717	5.660272	97.20

Practical Implications (concluded)

- Pricing is prohibitively time consuming when $S \approx H$ because
$$n \sim 1/\ln^2(S/H).$$
  - This is called the barrier-too-close problem.
- This observation is indeed true of standard quadratic-time binomial tree algorithms.
- But it no longer applies to linear-time algorithms (see p. 737).

	Barrier at 95.0			Barrier at 99.5			Barrier at 99.9	
n	Value	Time	n	Value	Time	n	Value	Time
	⋮		795	7.47761	8	19979	8.11304	253
2743	2.56095	31.1	3184	7.47626	38	79920	8.11297	1013
3040	2.56065	35.5	7163	7.47682	88	179819	8.11300	2200
3351	2.56098	40.1	12736	7.47661	166	319680	8.11299	4100
3678	2.56055	43.8	19899	7.47676	253	499499	8.11299	6300
4021	2.56152	48.1	28656	7.47667	368	719280	8.11299	8500
True	2.5615			7.4767			8.1130	

(All times in milliseconds.)