Brownian Motion[a]

- Brownian motion is a stochastic process $\{X(t), t \geq 0\}$ with the following properties.

  1. $X(0) = 0$, unless stated otherwise.

  2. for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables

  $$X(t_k) - X(t_{k-1})$$

  for $1 \leq k \leq n$ are independent.[b]

  3. for $0 \leq s < t$, $X(t) - X(s)$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$, where μ and $\sigma \neq 0$ are real numbers.

---

[a] Robert Brown (1773–1858).

[b] So $X(t) - X(s)$ is independent of $X(r)$ for $r \leq s < t$.

Brownian Motion (concluded)

- The existence and uniqueness of such a process is guaranteed by Wiener's theorem.[a]
- This process will be called a (μ, σ) Brownian motion with drift μ and variance σ^2.
- Although Brownian motion is a continuous function of t with probability one, it is almost nowhere differentiable.
- The $(0, 1)$ Brownian motion is called the Wiener process.
- If condition 3 is replaced by "$X(t) - X(s)$ depends only on $t - s$," we have the more general Levy process.[b]

---

[a]Norbert Wiener (1894–1964). He received his Ph.D. from Harvard in 1912.

[b]Paul Levy (1886–1971).

Example

- If $\{ X(t), t \geq 0 \}$ is the Wiener process, then

$$X(t) - X(s) \sim N(0, t - s).$$

- A (μ, σ) Brownian motion $Y = \{ Y(t), t \geq 0 \}$ can be expressed in terms of the Wiener process:

$$Y(t) = \mu t + \sigma X(t). \tag{78}$$

- Note that

$$Y(t + s) - Y(t) \sim N(\mu s, \sigma^2 s).$$

Brownian Motion as Limit of Random Walk

Claim 1 *A* (μ, σ) *Brownian motion is the limiting case of random walk.*

- A particle moves Δx to the right with probability p after Δt time.
- It moves Δx to the left with probability $1 - p$.
- Define

$$X_i \stackrel{\Delta}{=} \begin{cases} +1 & \text{if the } i\text{th move is to the right,} \\ -1 & \text{if the } i\text{th move is to the left.} \end{cases}$$

  - X_i are independent with

$$\text{Prob}[\, X_i = 1\,] = p = 1 - \text{Prob}[\, X_i = -1\,].$$

Brownian Motion as Limit of Random Walk (continued)

- Recall

$$\begin{aligned} E[\,X_i\,] &= 2p-1, \\ \mathrm{Var}[\,X_i\,] &= 1-(2p-1)^2. \end{aligned}$$

- Assume $n \stackrel{\Delta}{=} t/\Delta t$ is an integer.
- Its position at time t is

$$Y(t) \stackrel{\Delta}{=} \Delta x \,(X_1 + X_2 + \cdots + X_n)\,.$$

Brownian Motion as Limit of Random Walk (continued)

- Therefore,

$$E[\,Y(t)\,] = n(\Delta x)(2p-1),$$
$$\mathrm{Var}[\,Y(t)\,] = n(\Delta x)^2\left[\,1-(2p-1)^2\,\right].$$

- With $\Delta x \stackrel{\Delta}{=} \sigma\sqrt{\Delta t}$ and $p \stackrel{\Delta}{=} [\,1+(\mu/\sigma)\sqrt{\Delta t}\,]/2$,[a]

$$\begin{aligned} E[\,Y(t)\,] &= n\sigma\sqrt{\Delta t}\,(\mu/\sigma)\sqrt{\Delta t} = \mu t, \\ \mathrm{Var}[\,Y(t)\,] &= n\sigma^2\Delta t\left[\,1-(\mu/\sigma)^2\Delta t\,\right] \to \sigma^2 t, \end{aligned}$$

as $\Delta t \to 0$.

[a] Identical to Eq. (42) on p. 296!

Brownian Motion as Limit of Random Walk (concluded)

- Thus, $\{ Y(t), t \geq 0 \}$ converges to a (μ, σ) Brownian motion by the central limit theorem.
- Brownian motion with zero drift is the limiting case of symmetric random walk by choosing $\mu = 0$.
- Similarity to the the BOPM: The p is identical to the probability in Eq. (42) on p. 296 and $\Delta x = \ln u$.
- Note that

$$\begin{aligned} & \operatorname{Var}[\, Y(t + \Delta t) - Y(t) \,] \\ = & \operatorname{Var}[\, \Delta x \, X_{n+1} \,] = (\Delta x)^2 \times \operatorname{Var}[\, X_{n+1} \,] \to \sigma^2 \Delta t. \end{aligned}$$

Geometric Brownian Motion

- Let $X \stackrel{\Delta}{=} \{ X(t), t \geq 0 \}$ be a Brownian motion process.
- The process
$$\{ Y(t) \stackrel{\Delta}{=} e^{X(t)}, t \geq 0 \},$$
is called geometric Brownian motion.
- Suppose further that X is a (μ, σ) Brownian motion.
- By assumption, $Y(0) = e^0 = 1$.

Geometric Brownian Motion (concluded)

- $X(t) \sim N(\mu t, \sigma^2 t)$ with moment generating function

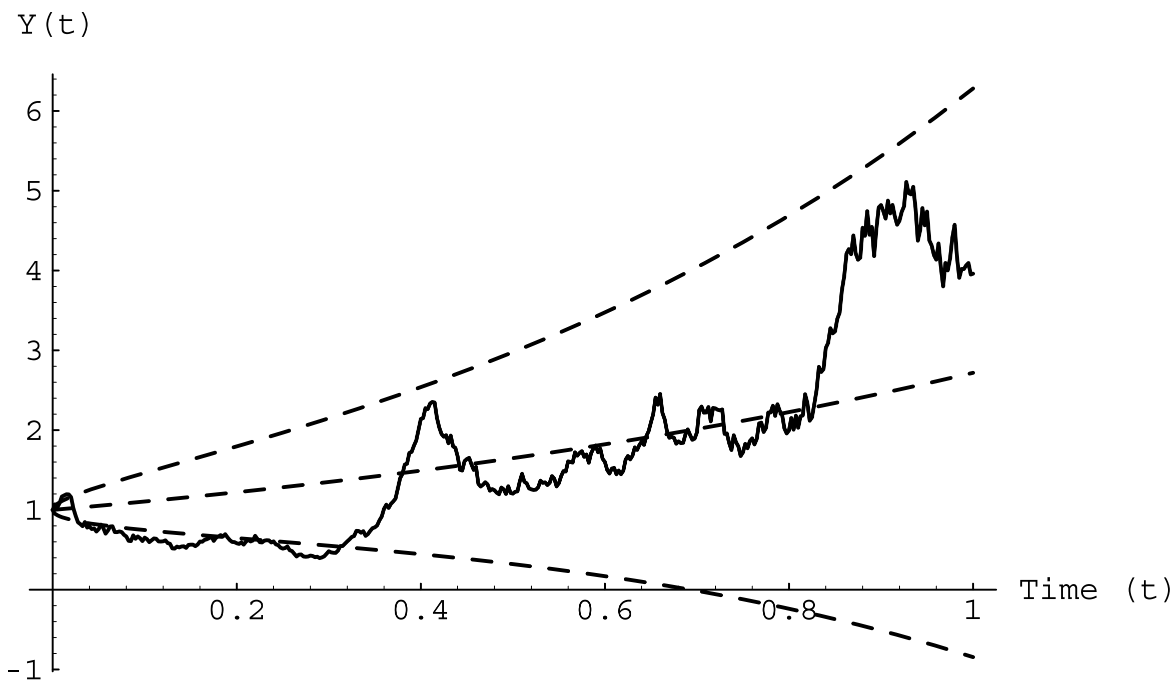
$$E\left[e^{sX(t)}\right] = E\left[\,Y(t)^s\,\right] = e^{\mu ts+(\sigma^2 ts^2/2)}$$

from Eq. (27) on p 171.

- In particular,[a]

$$\begin{aligned} E[\,Y(t)\,] &= e^{\mu t+(\sigma^2 t/2)}, \\ \mathrm{Var}[\,Y(t)\,] &= E\left[\,Y(t)^2\,\right] - E[\,Y(t)\,]^2 \\ &= e^{2\mu t+\sigma^2 t}\left(e^{\sigma^2 t}-1\right). \end{aligned}$$

[a]Recall Eqs. (29) on p. 180.


Y(t)
6
5
4
3
2
1
-1
0.2
0.4
0.6
0.8
1
Time (t)

An Argument for Long-Term Investment[a]

- Suppose the stock follows the geometric Brownian motion

$$S(t) = S(0)\, e^{N(\mu t, \sigma^2 t)} = S(0)\, e^{tN(\mu, \sigma^2/t\,)}, \quad t \geq 0,$$

where $\mu > 0$.

- The annual rate of return has a normal distribution:

$$N\left(\mu, \frac{\sigma^2}{t}\right).$$

- The larger the t, the likelier the return is positive.
- The smaller the t, the likelier the return is negative.

[a] Contributed by Dr. King, Gow-Hsing on April 9, 2015. See `http://www.cb.idv.tw/phpbb3/viewtopic.php?f=7&t=1025`

Continuous-Time Financial Mathematics

A proof is that which convinces a reasonable man;
a rigorous proof is that which convinces an
unreasonable man.
— Mark Kac (1914–1984)

The pursuit of mathematics is a
divine madness of the human spirit.
— Alfred North Whitehead (1861–1947),
Science and the Modern World

Stochastic Integrals

- Use $W \stackrel{\Delta}{=} \{ W(t), t \geq 0 \}$ to denote the Wiener process.
- The goal is to develop integrals of X from a class of stochastic processes,[a]

$$\mathrm{I}_t(X) \stackrel{\Delta}{=} \int_0^t X \, dW, \quad t \geq 0.$$

- $\mathrm{I}_t(X)$ is a random variable called the stochastic integral of X with respect to W.
- The stochastic process $\{ \mathrm{I}_t(X), t \geq 0 \}$ will be denoted by $\int X \, dW$.

---

[a] Kiyoshi Ito (1915–2008).

Stochastic Integrals (concluded)

- Typical requirements for X in financial applications are:
  - $\text{Prob}[\int_0^t X^2(s)\,ds < \infty] = 1$ for all $t \geq 0$ or the stronger $\int_0^t E[X^2(s)]\,ds < \infty$.
  - The information set at time t includes the history of X and W up to that point in time.
  - But it contains nothing about the evolution of X or W after t (nonanticipating, so to speak).
  - The future cannot influence the present.

Ito Integral

- A theory of stochastic integration.
- As with calculus, it starts with step functions.
- A stochastic process $\{ X(t) \}$ is simple if there exist

$$0 = t_0 < t_1 < \cdots$$

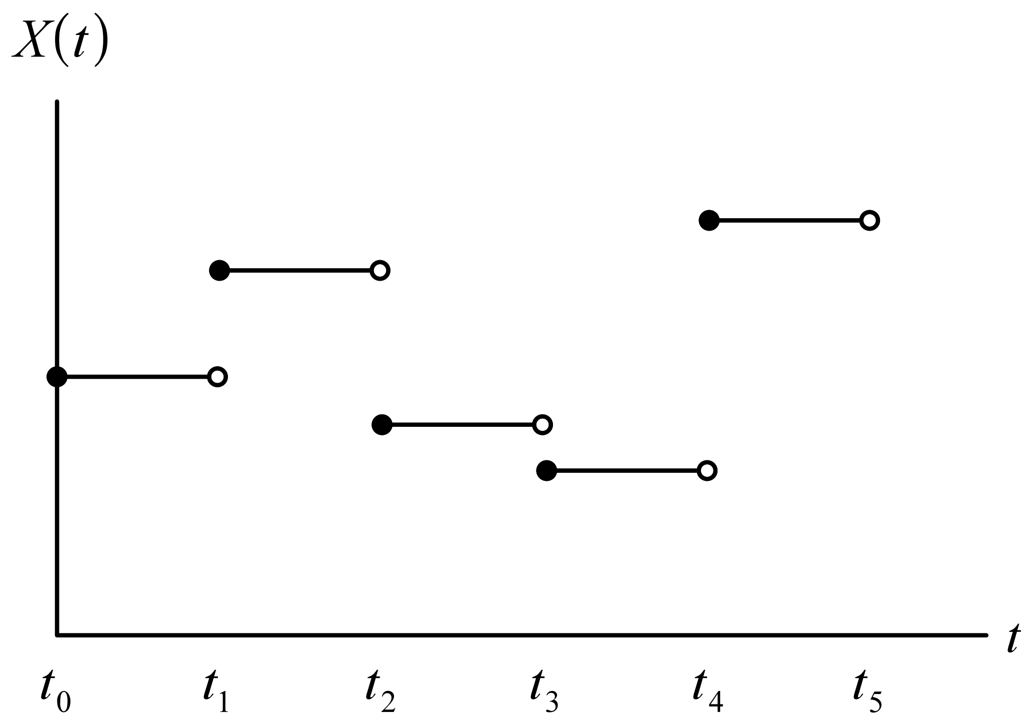
such that

$$X(t) = X(t_{k-1}) \quad \text{for } t \in [\, t_{k-1}, t_k), \, k = 1, 2, \ldots$$

for any realization (see figure on next page).[a]

---

[a]It is right-continuous.



X(t)
t
t0
t1
t2
t3
t4
t5

Ito Integral (continued)

- The Ito integral of a simple process is defined as

$$\mathrm{I}_t(X) \stackrel{\Delta}{=} \sum_{k=0}^{n-1} X(t_k)[\,W(t_{k+1}) - W(t_k)\,], \tag{79}$$

  where $t_n = t$.
  - The integrand X is evaluated at t_k, not t_{k+1}.
- Define the Ito integral of more general processes as a limiting random variable of the Ito integral of simple stochastic processes.

Ito Integral (continued)

- Let $X = \{ X(t), t \geq 0 \}$ be a general stochastic process.
- Then there exists a random variable $\mathrm{I}_t(X)$, unique almost certainly, such that $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ for each sequence of simple stochastic processes $X_1, X_2, \ldots$ such that X_n converges in probability to X.
- If X is continuous with probability one, then $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ as

$$\delta_n \stackrel{\Delta}{=} \max_{1 \leq k \leq n} (t_k - t_{k-1})$$

goes to zero.

Ito Integral (concluded)

- It is a fundamental fact that $\int X\, dW$ is continuous almost surely.
- The following theorem says the Ito integral is a martingale.[a]

Theorem 18 *The Ito integral* $\int X\, dW$ *is a martingale.*

- A corollary is the mean value formula

$$E\left[\int_a^b X\, dW\right] = 0.$$

[a]Exercise 14.1.1 covers simple stochastic processes.

Discrete Approximation and Nonanticipation

- Recall Eq. (79) on p. 592.
- The following *simple* stochastic process $\{\widehat{X}(t)\}$ can be used in place of X to approximate $\int_0^t X\,dW$,

$$\widehat{X}(s) \stackrel{\Delta}{=} X(t_{k-1}) \ \text{ for } \ s \in [\,t_{k-1}, t_k), \ k = 1, 2, \ldots, n.$$

- Note the *nonanticipating* feature of $\widehat{X}$.
  - The information up to time s,

$$\{\,\widehat{X}(t), W(t), 0 \leq t \leq s\,\},$$

    cannot determine the future evolution of X or W.

Discrete Approximation and Nonanticipation (concluded)

- Suppose, unlike Eq. (79) on p. 592, we defined the stochastic integral from

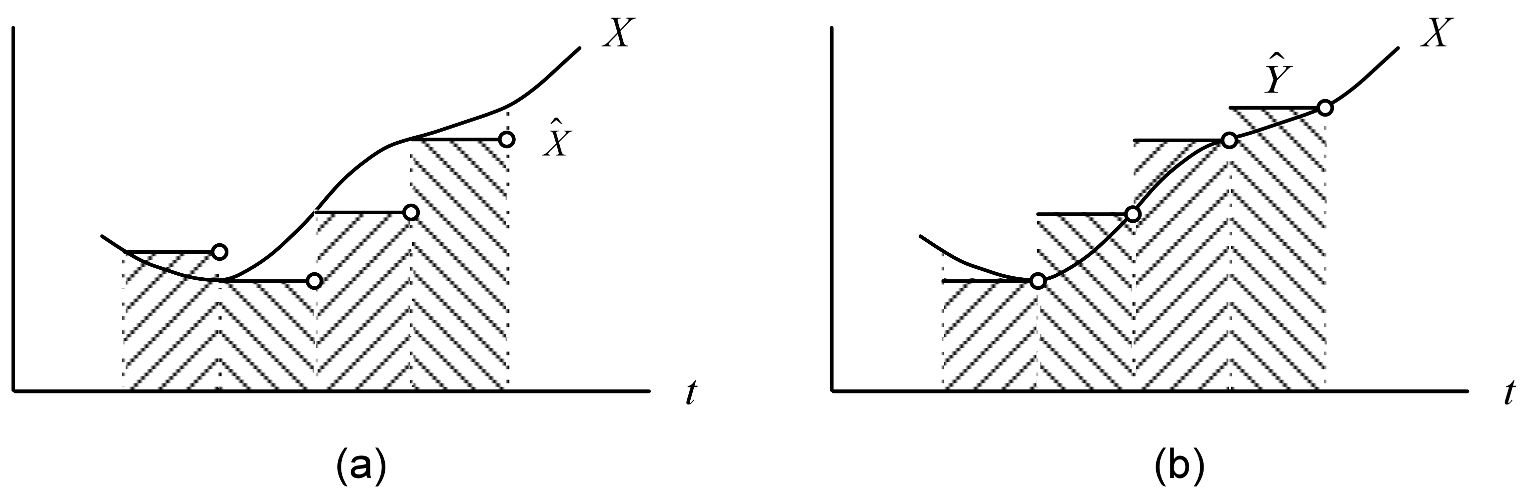
$$\sum_{k=0}^{n-1} X(t_{k+1})[\,W(t_{k+1}) - W(t_k)\,].$$

- Then we would be using the following different simple stochastic process in the approximation,

$$\widehat{Y}(s) \stackrel{\Delta}{=} X(t_k) \quad \text{for} \quad s \in [\,t_{k-1}, t_k), \; k = 1, 2, \ldots, n.$$

- This clearly anticipates the future evolution of X.[a]

[a]See Exercise 14.1.2 for an example where it matters.


X
$\hat{X}$
t
(a)
X
$\hat{Y}$
t
(b)

Ito Process

- The stochastic process $X = \{ X_t, t \geq 0 \}$ that solves

$$X_t = X_0 + \int_0^t a(X_s, s)\, ds + \int_0^t b(X_s, s)\, dW_s, \quad t \geq 0$$

is called an Ito process.

  - X_0 is a scalar starting point.
  - $\{ a(X_t, t) : t \geq 0 \}$ and $\{ b(X_t, t) : t \geq 0 \}$ are stochastic processes satisfying certain regularity conditions.
  - $a(X_t, t)$: the drift.
  - $b(X_t, t)$: the diffusion.

Ito Process (continued)

- Typical regularity conditions are:[a]
  - For all $T > 0$, $x \in \mathbb{R}^n$, and $0 \leq t \leq T$,

    $$| \, a(x,t) \, | + | \, b(x,t) \, | \leq C(1 + | \, x \, |)$$

    for some constant C.[b]
  - (Lipschitz continuity) For all $T > 0$, $x \in \mathbb{R}^n$, and $0 \leq t \leq T$,

    $$| \, a(x,t) - a(y,t) \, | + | \, b(x,t) - b(y,t) \, | \leq D \, | \, x - y \, |$$

    for some constant D.

---

[a] Øksendal (2007).

[b] This condition is not needed in *time-homogeneous* cases, where a and b do not depend on t.

Ito Process (continued)

- A shorthand[a] is the following stochastic differential equation[b] (SDE) for the Ito differential dX_t,

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t. \tag{80}$$

  - Or simply

$$dX_t = a_t\, dt + b_t\, dW_t.$$

  - This is Brownian motion with an *instantaneous* drift a_t and an *instantaneous* variance b_t^2.

- X is a martingale if $a_t = 0$.[c]

---

[a]Paul Langevin (1872–1946) in 1904.

[b]Like any equation, an SDE contains an unknown, the process X_t.

[c]Recall Theorem 18 (p. 594).

Ito Process (concluded)

- From calculus, we would expect $\int_0^t W \, dW = W(t)^2/2$.
- But $W(t)^2/2$ is not a martingale, hence wrong!
- The correct answer is $[\,W(t)^2 - t\,]/2$.
- A popular representation of Eq. (80) is

$$dX_t = a_t \, dt + b_t \sqrt{dt} \, \xi, \tag{81}$$

where $\xi \sim N(0,1)$.

Euler Approximation

- Define $t_n \stackrel{\Delta}{=} n\Delta t$.
- The following approximation follows from Eq. (81),

$$\begin{aligned} & \widehat{X}(t_{n+1}) \\ =& \widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\, \Delta t + b(\widehat{X}(t_n), t_n)\, \Delta W(t_n). \end{aligned} \tag{82}$$

- It is called the Euler or Euler-Maruyama method.
- Recall that $\Delta W(t_n)$ should be interpreted as

$$W(t_{n+1}) - W(t_n),$$

not $W(t_n) - W(t_{n-1})$![a]

[a] Recall Eq. (79) on p. 592.

Euler Approximation (concluded)

- With the Euler method, one can obtain a sample path

$$\widehat{X}(t_1), \widehat{X}(t_2), \widehat{X}(t_3), \ldots$$

  from a sample path

$$W(t_0), W(t_1), W(t_2), \ldots .$$

- Under mild conditions, $\widehat{X}(t_n)$ converges to $X(t_n)$.

More Discrete Approximations

- Under fairly loose regularity conditions, Eq. (82) on p. 602 can be replaced by

$$\begin{aligned}&\widehat{X}(t_{n+1})\\=&\widehat{X}(t_n)+a(\widehat{X}(t_n),t_n)\,\Delta t+b(\widehat{X}(t_n),t_n)\sqrt{\Delta t}\,Y(t_n).\end{aligned}$$

  - $Y(t_0), Y(t_1), \ldots$ are independent and identically distributed with zero mean and unit variance.

More Discrete Approximations (concluded)

- An even simpler discrete approximation scheme:

$$\begin{aligned}&\widehat{X}(t_{n+1})\\=&\widehat{X}(t_n)+a(\widehat{X}(t_n),t_n)\,\Delta t+b(\widehat{X}(t_n),t_n)\sqrt{\Delta t}\,\xi.\end{aligned}$$

  - $\text{Prob}[\,\xi=1\,]=\text{Prob}[\,\xi=-1\,]=1/2$.
  - Note that $E[\,\xi\,]=0$ and $\text{Var}[\,\xi\,]=1$.

- This is a binomial model.

- As Δt goes to zero, $\widehat{X}$ converges to X.[a]

---

[a] He (1990).

Trading and the Ito Integral

- Consider an Ito process

$$d\boldsymbol{S}_t = \mu_t \, dt + \sigma_t \, dW_t.$$

  – $\boldsymbol{S}_t$ is the vector of security prices at time t.

- Let $\boldsymbol{\phi}_t$ be a trading strategy denoting the quantity of each type of security held at time t.

  – Hence the stochastic process $\boldsymbol{\phi}_t \boldsymbol{S}_t$ is the value of the portfolio $\boldsymbol{\phi}_t$ at time t.

- $\boldsymbol{\phi}_t \, d\boldsymbol{S}_t \stackrel{\Delta}{=} \boldsymbol{\phi}_t(\mu_t \, dt + \sigma_t \, dW_t)$ represents the change in the value from security price changes occurring at time t.

Trading and the Ito Integral (concluded)

- The equivalent Ito integral,

$$G_T(\boldsymbol{\phi}) \stackrel{\Delta}{=} \int_0^T \boldsymbol{\phi}_t \, d\boldsymbol{S}_t = \int_0^T \boldsymbol{\phi}_t \mu_t \, dt + \int_0^T \boldsymbol{\phi}_t \sigma_t \, dW_t,$$

measures the gains realized by the trading strategy over the period $[\,0, T\,]$.

Ito's Lemma[a]

A smooth function of an Ito process is itself an Ito process.

Theorem 19 *Suppose* $f : R \to R$ *is twice continuously differentiable and* $dX = a_t\, dt + b_t\, dW$*. Then* $f(X)$ *is the Ito process,*

$$\begin{aligned} & f(X_t) \\ = \; & f(X_0) + \int_0^t f'(X_s)\, a_s\, ds + \int_0^t f'(X_s)\, b_s\, dW \\ & + \frac{1}{2} \int_0^t f''(X_s)\, b_s^2\, ds \end{aligned}$$

for $t \geq 0$.

---

[a] Ito (1944).

Ito's Lemma (continued)

- In differential form, Ito's lemma becomes

$$\begin{aligned}
& df(X) \\
= \quad & f'(X)\, a\, dt + f'(X)\, b\, dW + \frac{1}{2}\, f''(X)\, b^2\, dt \qquad (83) \\
= \quad & \left[f'(X)\, a + \boxed{\tfrac{1}{2}\, f''(X)\, b^2} \right] dt + f'(X)\, b\, dW.
\end{aligned}$$

- Compared with calculus, the interesting part is the third term on the right-hand side of Eq. (83).

- A convenient formulation of Ito's lemma is

$$df(X) = f'(X)\, dX + \frac{1}{2}\, f''(X)(dX)^2. \qquad (84)$$

Ito's Lemma (continued)

- We are supposed to multiply out $(dX)^2 = (a\,dt + b\,dW)^2$ symbolically according to

$\times$	dW	dt
dW	dt	0
dt	0	0

  - The $(dW)^2 = dt$ entry is justified by a known result.
- Hence $(dX)^2 = (a\,dt + b\,dW)^2 = b^2\,dt$ in Eq. (84).
- This form is easy to remember because of its similarity to the Taylor expansion.

Ito's Lemma (continued)

Theorem 20 (Higher-Dimensional Ito's Lemma) *Let* $W_1, W_2, \ldots, W_n$ *be independent Wiener processes and* $X \triangleq (X_1, X_2, \ldots, X_m)$ *be a vector process. Suppose* $f : R^m \to R$ *is twice continuously differentiable and* X_i *is an Ito process with* $dX_i = a_i\, dt + \sum_{j=1}^{n} b_{ij}\, dW_j$. *Then* $df(X)$ *is an Ito process with the differential,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X)\, dX_i\, dX_k,$$

where $f_i \triangleq \partial f / \partial X_i$ *and* $f_{ik} \triangleq \partial^2 f / \partial X_i \partial X_k$.

Ito's Lemma (continued)

- The multiplication table for Theorem 20 is

$\times$	dW_i	dt
dW_k	$\delta_{ik}\, dt$	0
dt	0	0

in which

$$\delta_{ik} = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{otherwise.} \end{cases}$$

Ito's Lemma (continued)

- In applying the higher-dimensional Ito's lemma, usually one of the variables, say X_1, is time t and $dX_1 = dt$.
- In this case, $b_{1j} = 0$ for all j and $a_1 = 1$.
- As an example, let

$$dX_t = a_t\,dt + b_t\,dW_t.$$

- Consider the process $f(X_t, t)$.

Ito's Lemma (continued)

- Then

$$
\begin{aligned}
& df \\
= \quad & \frac{\partial f}{\partial X_t}\, dX_t + \frac{\partial f}{\partial t}\, dt + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2}\, (dX_t)^2 \\
= \quad & \frac{\partial f}{\partial X_t}\, (a_t\, dt + b_t\, dW_t) + \frac{\partial f}{\partial t}\, dt \\
& + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2}\, (a_t\, dt + b_t\, dW_t)^2 \\
= \quad & \left(\frac{\partial f}{\partial X_t}\, a_t + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2}\, b_t^2 \right) dt + \frac{\partial f}{\partial X_t}\, b_t\, dW_t. \quad (85)
\end{aligned}
$$

Ito's Lemma (continued)

Theorem 21 (Alternative Ito's Lemma) *Let $W_1, W_2, \ldots, W_m$ be Wiener processes and $X \stackrel{\Delta}{=} (X_1, X_2, \ldots, X_m)$ be a vector process. Suppose $f : R^m \to R$ is twice continuously differentiable and X_i is an Ito process with $dX_i = a_i\, dt + b_i\, dW_i$. Then $df(X)$ is the following Ito process,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X)\, dX_i\, dX_k.$$

Ito's Lemma (concluded)

- The multiplication table for Theorem 21 is

$\times$	dW_i	dt
dW_k	$\rho_{ik}\, dt$	0
dt	0	0

- Above, ρ_{ik} denotes the correlation between dW_i and dW_k.

Geometric Brownian Motion

- Consider geometric Brownian motion

$$Y(t) \stackrel{\Delta}{=} e^{X(t)}.$$

  - $X(t)$ is a (μ, σ) Brownian motion.
  - By Eq. (78) on p. 577,

$$dX = \mu\, dt + \sigma\, dW.$$

- Note that

$$\begin{aligned} \frac{\partial Y}{\partial X} &= Y, \\ \frac{\partial^2 Y}{\partial X^2} &= Y. \end{aligned}$$

Geometric Brownian Motion (continued)

- Ito's formula (83) on p. 609 implies

$$\begin{aligned} dY &= Y\, dX + (1/2)\, Y\, (dX)^2 \\ &= Y\, (\mu\, dt + \sigma\, dW) + (1/2)\, Y\, (\mu\, dt + \sigma\, dW)^2 \\ &= Y\, (\mu\, dt + \sigma\, dW) + (1/2)\, Y \sigma^2\, dt. \end{aligned}$$

- Hence

$$\frac{dY}{Y} = \left(\mu + \sigma^2/2\right) dt + \sigma\, dW. \tag{86}$$

- The annualized *instantaneous* rate of return is $\mu + \sigma^2/2$ (not μ).[a]

[a] Consistent with Lemma 10 (p. 301).

Geometric Brownian Motion (continued)

- Alternatively, from Eq. (78) on p. 577,

$$X_t = X_0 + \mu t + \sigma W_t,$$

admits an explicit (strong) solution.

- Hence

$$Y_t = Y_0 \, e^{\mu t + \sigma W_t}, \tag{87}$$

a strong solution to the SDE (86) where $Y_0 = e^{X_0}$.

Geometric Brownian Motion (concluded)

- On the other hand, suppose

$$\frac{dY}{Y} = \mu\, dt + \sigma\, dW.$$

- Then $X(t) \stackrel{\Delta}{=} \ln Y(t)$ follows

$$dX = \left(\mu - \sigma^2/2\right) dt + \sigma\, dW.$$

Exponential Martingale

- The Ito process
$$dX_t = b_t X_t \, dW_t$$
is a martingale.[a]
- It is called an exponential martingale.
- By Ito's formula (83) on p. 609,
$$X(t) = X(0) \exp\left[-\frac{1}{2} \int_0^t b_s^2 \, ds + \int_0^t b_s \, dW_s \right].$$

[a] Recall Theorem 18 (p. 594).

Product of Geometric Brownian Motion Processes

- Let
$$\begin{aligned}\frac{dY}{Y} &= a\,dt + b\,dW_Y,\\ \frac{dZ}{Z} &= f\,dt + g\,dW_Z.\end{aligned}$$
- Assume dW_Y and dW_Z have correlation ρ.
- Consider the Ito process
$$U \stackrel{\Delta}{=} YZ.$$

Product of Geometric Brownian Motion Processes (continued)

- Apply Ito's lemma (Theorem 21 on p. 615):

$$\begin{aligned} dU &= Z\,dY + Y\,dZ + dY\,dZ \\ &= ZY(a\,dt + b\,dW_Y) + YZ(f\,dt + g\,dW_Z) \\ &\quad + YZ(a\,dt + b\,dW_Y)(f\,dt + g\,dW_Z) \\ &= U(a + f + bg\rho)\,dt + Ub\,dW_Y + Ug\,dW_Z. \end{aligned}$$

- The product of correlated geometric Brownian motion processes thus remains geometric Brownian motion.

Product of Geometric Brownian Motion Processes (continued)

- Note that

$$\begin{aligned} Y &= \exp\left[\left(a - b^2/2\right) dt + b\, dW_Y\right], \\ Z &= \exp\left[\left(f - g^2/2\right) dt + g\, dW_Z\right], \\ U &= \exp\left[\,\left(a + f - \left(b^2 + g^2\right)/2\right) dt + b\, dW_Y + g\, dW_Z\,\right]. \end{aligned}$$

- The strong solutions are:

$$\begin{aligned} Y(t) &= \exp\left[\left(a - b^2/2\right) t + b\, W_Y(t)\right], \\ Z(t) &= \exp\left[\left(f - g^2/2\right) t + g\, W_Z(t)\right], \\ U(t) &= \exp\left[\,\left(a + f - \left(b^2 + g^2\right)/2\right) t + b\, dW_Y + g\, W_Z(t)\,\right]. \end{aligned}$$

Product of Geometric Brownian Motion Processes (concluded)

- $\ln U$ is Brownian motion with a mean equal to the sum of the means of $\ln Y$ and $\ln Z$.
- This holds even if Y and Z are correlated.
- Finally, $\ln Y$ and $\ln Z$ have correlation ρ.

Quotients of Geometric Brownian Motion Processes

- Suppose Y and Z are drawn from p. 622.
- Let
$$U \triangleq Y/Z.$$
- We now show that[a]
$$\frac{dU}{U} = (a - f + g^2 - b g \rho)\, dt + b\, dW_Y - g\, dW_Z. \tag{88}$$
- Keep in mind that dW_Y and dW_Z have correlation ρ.

[a]Exercise 14.3.6 of the textbook is erroneous.

Quotients of Geometric Brownian Motion Processes (concluded)

- The multidimensional Ito's lemma (Theorem 21 on p. 615) can be employed to show that

$$
\begin{aligned}
& dU \\
= \;& (1/Z)\, dY - (Y/Z^2)\, dZ - (1/Z^2)\, dY\, dZ + (Y/Z^3)\, (dZ)^2 \\
= \;& (1/Z)(aY\, dt + bY\, dW_Y) - (Y/Z^2)(f Z\, dt + gZ\, dW_Z) \\
& -(1/Z^2)(bgY Z \rho\, dt) + (Y/Z^3)(g^2 Z^2\, dt) \\
= \;& U(a\, dt + b\, dW_Y) - U(f\, dt + g\, dW_Z) \\
& -U(bg\rho\, dt) + U(g^2\, dt) \\
= \;& U(a - f + g^2 - bg\rho)\, dt + Ub\, dW_Y - Ug\, dW_Z.
\end{aligned}
$$

Forward Price

- Suppose S follows
$$\frac{dS}{S} = \mu\, dt + \sigma\, dW.$$
- Consider functional $F(S, t) \stackrel{\Delta}{=} S e^{y(T-t)}$ for constants y and T.
- As F is a function of two variables, we need the various partial derivatives of $F(S, t)$ with respect to S and t.
- Note that in partial differentiation with respect to one variable, other variables are held constant.[a]

[a]Contributed by Mr. Sun, Ao (`R05922147`) on April 26, 2017.

Forward Prices (continued)

- Now,

$$\begin{aligned}\frac{\partial F}{\partial S} &= e^{y(T-t)},\\ \frac{\partial^2 F}{\partial S^2} &= 0,\\ \frac{\partial F}{\partial t} &= -ySe^{y(T-t)}.\end{aligned}$$

- Then[a]

$$\begin{aligned} dF &= e^{y(T-t)}\, dS - ySe^{y(T-t)}\, dt\\ &= Se^{y(T-t)}\,(\mu\, dt + \sigma\, dW) - ySe^{y(T-t)}\, dt\\ &= F(\mu - y)\, dt + F\sigma\, dW.\end{aligned}$$

[a]One can also prove it by Eq. (85) on p. 614.

Forward Prices (concluded)

- Thus F follows
$$\frac{dF}{F} = (\mu - y)\, dt + \sigma\, dW.$$
- This result has applications in forward and futures contracts.
- In Eq. (60) on p. 490, $\mu = r = y$.
- So
$$\frac{dF}{F} = \sigma\, dW,$$
a martingale.[a]

[a]It is consistent with p. 566. Furthermore, it explains why Black's formulas (68)–(69) on p. 518 use the same volatility σ as the stock's.

Ornstein-Uhlenbeck (OU) Process

- The OU process:

$$dX = -\kappa X\, dt + \sigma\, dW,$$

  where $\kappa, \sigma \geq 0$.

- For $t_0 \leq s \leq t$ and $X(t_0) = x_0$, it is known that

$$\begin{aligned}
E[\, X(t)\,] &= e^{-\kappa(t-t_0)}\, E[\, x_0\,], \\
\mathrm{Var}[\, X(t)\,] &= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(t-t_0)}\right) + e^{-2\kappa(t-t_0)}\, \mathrm{Var}[\, x_0\,], \\
\mathrm{Cov}[\, X(s), X(t)\,] &= \frac{\sigma^2}{2\kappa}\, e^{-\kappa(t-s)}\left[\, 1 - e^{-2\kappa(s-t_0)}\,\right] \\
&\quad + e^{-\kappa(t+s-2t_0)}\, \mathrm{Var}[\, x_0\,].
\end{aligned}$$

Ornstein-Uhlenbeck Process (continued)

- $X(t)$ is normally distributed if x_0 is a constant or normally distributed.
  - $E[\,x_0\,] = x_0$ and $\text{Var}[\,x_0\,] = 0$ if x_0 is a constant.
- X is said to be a normal process.
- The OU process has the following mean-reverting property if $\kappa > 0$.
  - When $X > 0$, X is pulled toward zero.
  - When $X < 0$, it is pulled toward zero again.

Ornstein-Uhlenbeck Process (continued)

- A generalized version:

$$dX = \kappa(\mu - X)\,dt + \sigma\,dW,$$

where $\kappa, \sigma \geq 0$.

- Given $X(t_0) = x_0$, a constant, it is known that

$$\begin{aligned} E[\,X(t)\,] &= \mu + (x_0 - \mu)\,e^{-\kappa(t-t_0)}, \\ \mathrm{Var}[\,X(t)\,] &= \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa(t-t_0)}\right], \end{aligned} \tag{89}$$

for $t_0 \leq t$.

Ornstein-Uhlenbeck Process (concluded)

- The mean and standard deviation are roughly μ and $\sigma/\sqrt{2\kappa}$, respectively.
- For large t, the probability of $X < 0$ is extremely unlikely in any finite time interval when $\mu > 0$ is large relative to $\sigma/\sqrt{2\kappa}$.
- The process is mean-reverting.
  - X tends to move toward μ.
  - Useful for modeling term structure, stock price volatility, and stock price return.[a]

[a] See Knutson, Wimmer, Kuhnen, & Winkielman (2008) for the biological basis for mean reversion in financial decision making.

Square-Root Process

- Suppose X is an OU process.
- Consider
$$V \triangleq X^2.$$
- Ito's lemma says V has the differential,
$$\begin{aligned} dV &= 2X\,dX + (dX)^2 \\ &= 2\sqrt{V}\,(-\kappa\sqrt{V}\,dt + \sigma\,dW) + \sigma^2\,dt \\ &= \left(-2\kappa V + \sigma^2\right) dt + 2\sigma\sqrt{V}\,dW, \end{aligned}$$

a square-root process.

Square-Root Process (continued)

- In general, the square-root process has the SDE,

$$dX = \kappa(\mu - X)\, dt + \sigma\sqrt{X}\, dW,$$

  where $\kappa, \sigma > 0$, $\mu \geq 0$, and $X(0) \geq 0$ is a constant.

- Like the OU process, it possesses mean reversion: X tends to move toward μ, but the volatility is proportional to $\sqrt{X}$ instead of a constant.

Square-Root Process (continued)

- When X hits zero and $\mu \geq 0$, the probability is one that it will not move below zero.
  - Zero is a reflecting boundary.
- Hence, the square-root process is a good candidate for modeling interest rates.[a]
- The OU process, in contrast, allows negative interest rates.[b]
- The two processes are related.[c]

---

[a] Cox, Ingersoll, & Ross (1985).
[b] Some rates did go negative in Europe in 2015.
[c] Recall p. 635.

Square-Root Process (concluded)

- The random variable $2cX(t)$ follows the noncentral chi-square distribution,[a]

$$\chi\left(\frac{4\kappa\mu}{\sigma^2}, 2cX(0)\, e^{-\kappa t}\right),$$

where $c \triangleq (2\kappa/\sigma^2)(1-e^{-\kappa t})^{-1}$ and $\mu > 0$.

- Given $X(0) = x_0$, a constant,

$$\begin{aligned} E[\, X(t)\,] &= x_0 e^{-\kappa t} + \mu\left(1-e^{-\kappa t}\right), \\ \mathrm{Var}[\, X(t)\,] &= x_0\, \frac{\sigma^2}{\kappa}\, \left(e^{-\kappa t} - e^{-2\kappa t}\right) + \mu\, \frac{\sigma^2}{2\kappa}\, \left(1-e^{-\kappa t}\right)^2, \end{aligned}$$

for $t \geq 0$.

[a]William Feller (1906–1970) in 1951.

Modeling Stock Prices

- The most popular stochastic model for stock prices has been the geometric Brownian motion,

$$\frac{dS}{S} = \mu\, dt + \sigma\, dW.$$

- The logarithmic price $X \stackrel{\Delta}{=} \ln S$ follows

$$dX = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma\, dW$$

by Eq. (86) on p. 618.

Local-Volatility Models

- The deterministic-volatility model for "smile" posits

$$\frac{dS}{S} = (r_t - q_t)\, dt + \sigma(S, t)\, dW,$$

  where instantaneous volatility $\sigma(S, t)$ is called the local-volatility function.[a]

  – "The most popular model after Black-Scholes is a local volatility model as it is the only completely consistent volatility model."

- A (weak) solution exists if $S\sigma(S, t)$ is continuous and grows at most linearly in S and t.[b]

[a] Derman & Kani (1994); Dupire (1994).
[b] Skorokhod (1961); Achdou & Pironneau (2005).

Local-Volatility Models (continued)

- One needs to recover the local volatility surface $\sigma(S, t)$ from the implied volatility surface.
- Theoretically,[a]

$$\sigma(X,T)^2 = 2\,\frac{\frac{\partial C}{\partial T} + (r_T - q_T)X\,\frac{\partial C}{\partial X} + q_T C}{X^2\,\frac{\partial^2 C}{\partial X^2}}. \tag{90}$$

  - C is the call price at time $t = 0$ (today) with strike price X and time to maturity T.
  - $\sigma(X, T)$ is the local volatility that will prevail at *future time* T and *stock price* $S_T = X$.

[a]Dupire (1994); Andersen & Brotherton-Ratcliffe (1998).

Local-Volatility Models (continued)

- For more general models, this equation gives the expectation as seen from today, under the risk-neural probability, of the instantaneous variance at time T given that $S_T = X$.[a]
- In practice, the $\sigma(S, t)^2$ derived by Dupire's formula (90) may have spikes, vary wildly, or even be negative.
- The term $\partial^2 C / \partial X^2$ in the denominator often results in numerical instability.

[a] Derman & Kani (1997); R. W. Lee (2001); Derman & M. B. Miller (2016).

Local-Volatility Models (continued)

- Denote the implied volatility surface by $\Sigma(X, T)$ and the local volatility surface by $\sigma(S, t)$.
- The relation between $\Sigma(X, T)$ and $\sigma(X, T)$ is[a]

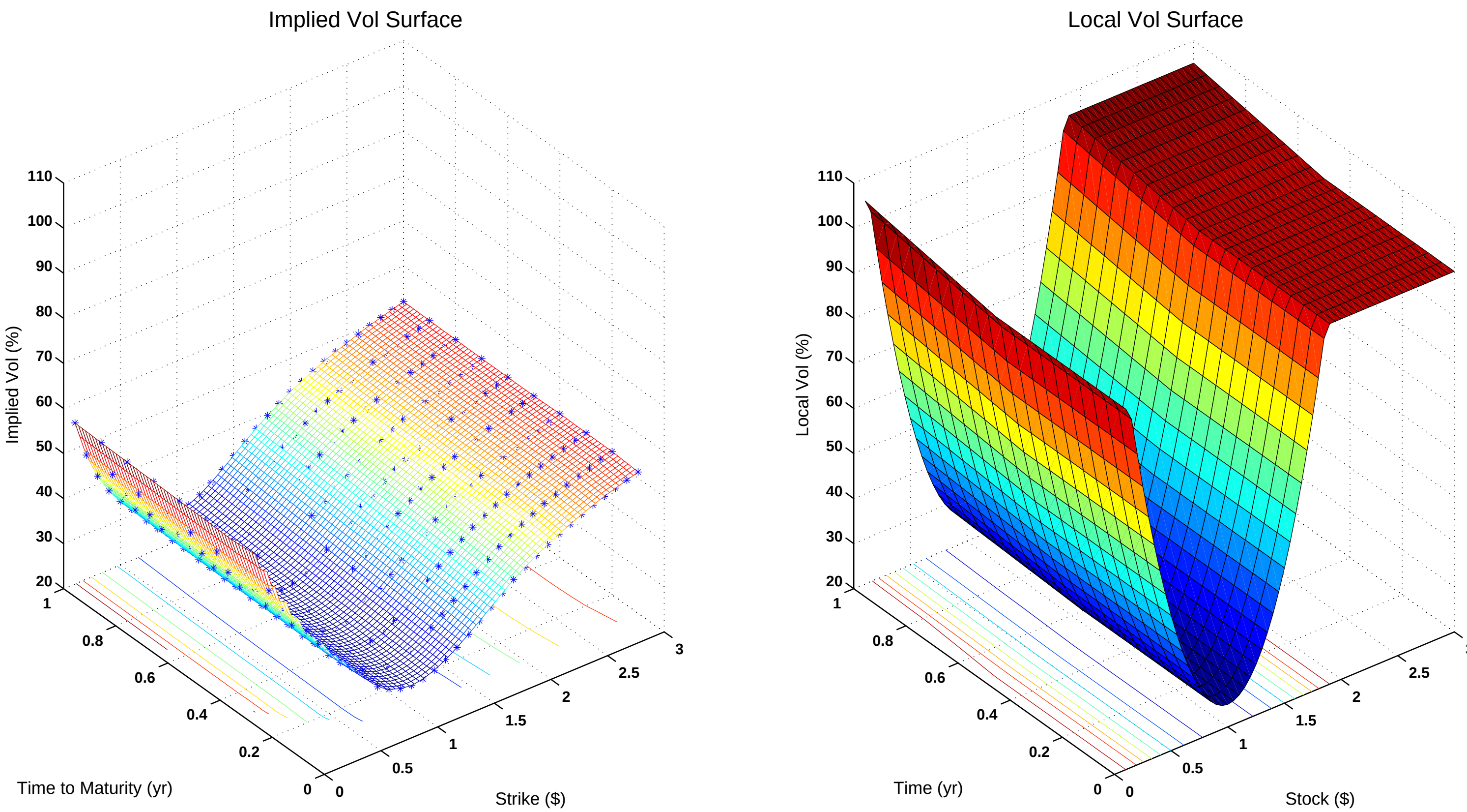
$$
\begin{aligned}
\sigma(X,T)^2 &= \frac{\Sigma^2 + 2\Sigma\tau\left[\frac{\partial\Sigma}{\partial T} + (r_T - q_T)X\frac{\partial\Sigma}{\partial X}\right]}{\left(1 - \frac{Xy}{\Sigma}\frac{\partial\Sigma}{\partial X}\right)^2 + X\Sigma\tau\left[\frac{\partial\Sigma}{\partial X} - \frac{X\Sigma\tau}{4}\left(\frac{\partial\Sigma}{\partial X}\right)^2 + X\frac{\partial^2\Sigma}{\partial X^2}\right]}, \\
\tau &\stackrel{\Delta}{=} T - t, \\
y &\stackrel{\Delta}{=} \ln(X/S_t) + \int_t^T (q_s - r_s)\,ds.
\end{aligned}
$$

[a] Andreasen (1996); Andersen & Brotherton-Ratcliffe (1998); Gatheral (2003); Wilmott (2006); Kamp (2009).

Local-Volatility Models (continued)

- Although this version may be more stable than Eq. (90) on p. 641, it is expected to suffer from the same problems.
- Small changes to the implied volatility surface may produce big changes to the local volatility surface.

Implied and Local Volatility Surfaces[a]



[a]Contributed by Mr. Lok, U Hou (D99922028) on April 5, 2014.

Local-Volatility Models (continued)

- In reality, option prices only exist for a finite set of maturities and strike prices.
- Hence interpolation and extrapolation may be needed to construct the volatility surface.[a]
- But then some implied volatility surfaces generate option prices that allow arbitrage opportunities.[b]

---

[a] Doing it to the option prices produces worse results (Li, 2000/2001).
[b] See Rebonato (2004) for an example.

Local-Volatility Models (concluded)

- There exist conditions for a set of option prices to be arbitrage-free.[a]
- Some adopt parameterized implied volatility surfaces that guarantee freedom from certain arbitrages.[b]
- For some vanilla equity options, the Black-Scholes model seems better than the local-volatility model in predictive power.[c]
- The exact opposite is concluded for hedging in equity index markets![d]

---

[a] Kahalé (2004); Davis & Hobson (2007).
[b] Gatheral & Jacquier (2014).
[c] Dumas, Fleming, & Whaley (1998).
[d] Crépey (2004); Derman & M. B. Miller (2016).

Local-Volatility Models: Popularity

- Hirsa and Neftci (2014), "most traders and firms actively utilize this [local-volatility] model."
- Bennett (2014), "Of all the four volatility regimes, [sticky local volatility] is arguably the most realistic and fairly prices skew."
- Derman & M. B. Miller (2016), "Right or wrong, local volatility models have become popular and ubiquitousin modeling the smile."

Implied Trees

- The trees for the local-volatility model are called implied trees.[a]
- Their construction requires option prices at all strike prices and maturities.
  - That is, an implied volatility surface.
- The local volatility model does *not* imply that the implied tree must combine.
- Exponential-sized implied trees exist.[b]

---

[a]Derman & Kani (1994); Dupire (1994); Rubinstein (1994).

[b]Charalambousa, Christofidesb, & Martzoukosa (2007); Gong & Xu (2019).

Implied Trees (continued)

- How to construct a valid implied tree with efficiency has been open for a long time.[a]
  - Reasons may include: noise and nonsynchrony in data, arbitrage opportunities in the smoothed and interpolated/extrapolated implied volatility surface, wrong model, wrong algorithms, nonlinearity, instability, etc.
- Inversion is an ill-posed numerical problem.[b]

---

[a] Rubinstein (1994); Derman & Kani (1994); Derman, Kani, & Chriss (1996); Jackwerth & Rubinstein (1996); Jackwerth (1997); Coleman, Kim, Li, & Verma (2000); Li (2000/2001); Rebonato (2004); Moriggia, Muzzioli, & Torricelli (2009).

[b] Ayache, Henrotte, Nassar, & X. Wang (2004).

Implied Trees (continued)

- It is finally solved for separable local volatilities.[a]
  - The local-volatility function $\sigma(S, t)$ is separable[b] if

$$\sigma(S, t) = \sigma_1(S)\,\sigma_2(t).$$

- A solution is also available for *any* upper- and lower-bounded σ.[c]

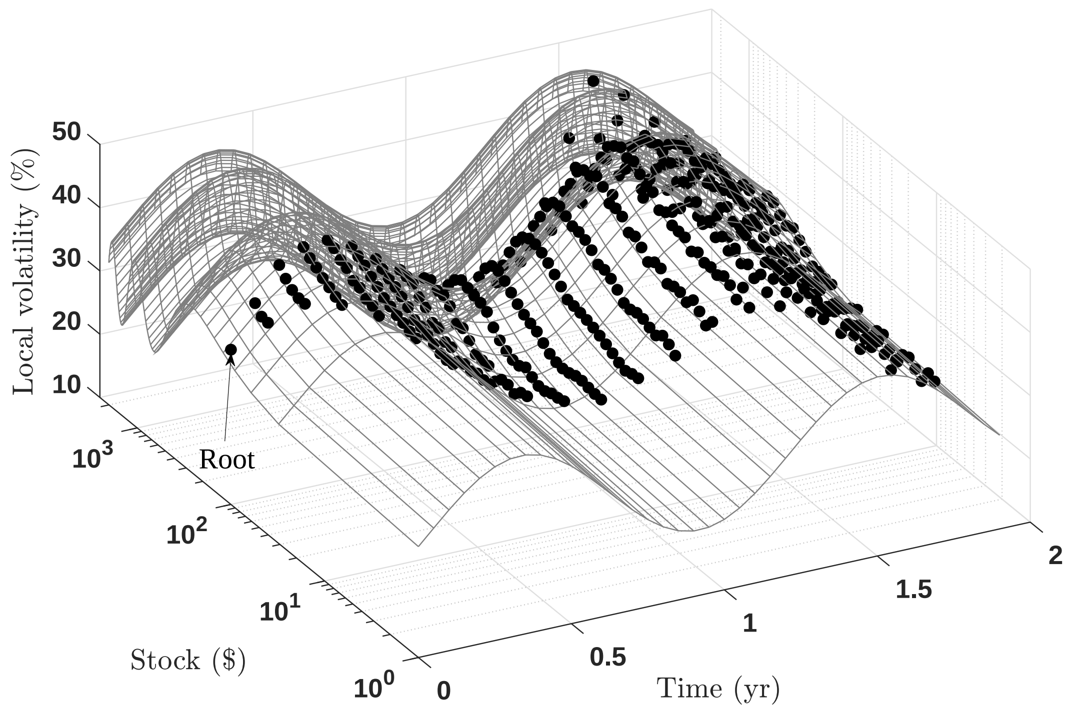
---

[a] Lok (`D99922028`) & Lyuu (2015, 2016, 2017).

[b] Brace, Gątarek, & Musiela (1997); Rebonato (2004).

[c] Lok (`D99922028`) & Lyuu (2016, 2017, 2020).

Implied Trees[a] (concluded)

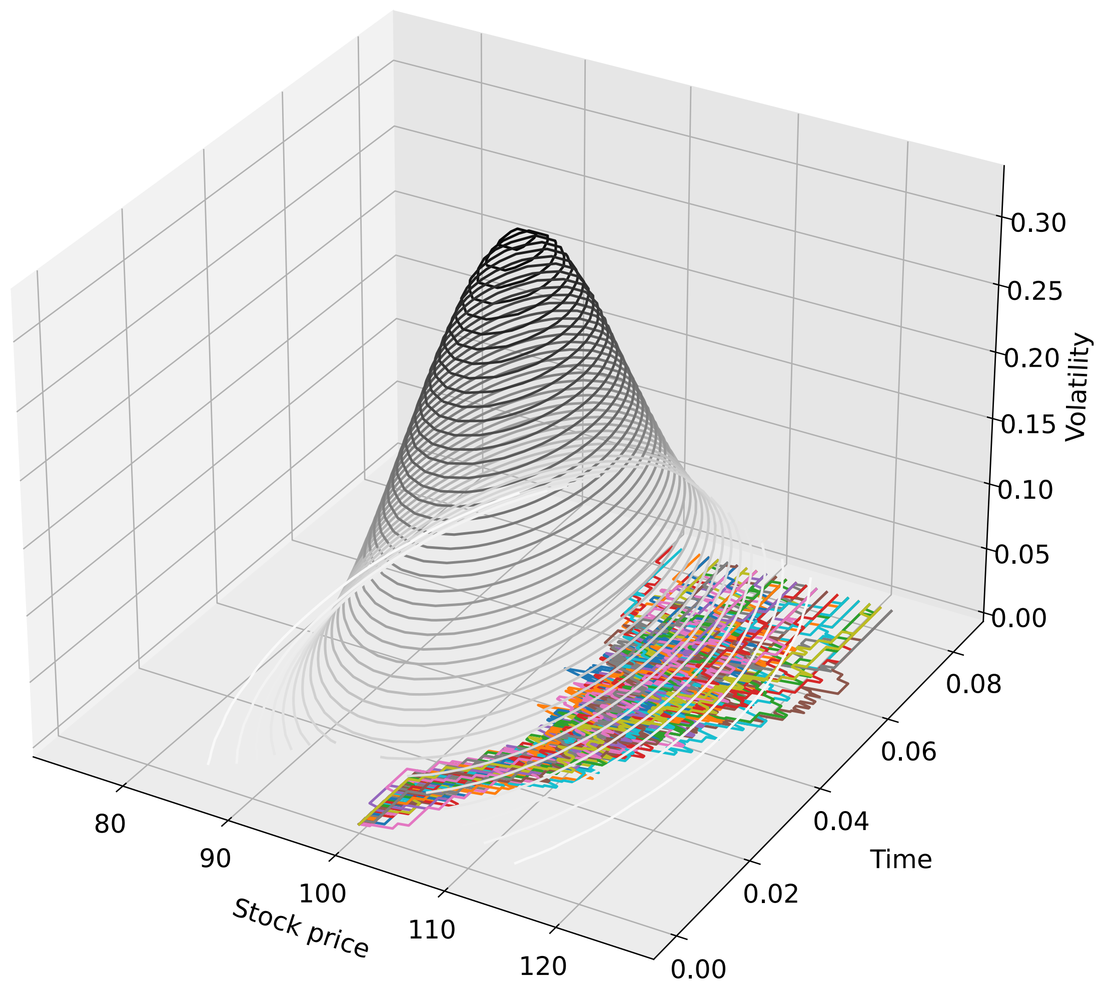


[a]Plot supplied by Prof. Lok, U Hou (`D99922028`) on May 4, 2019.

Delta Hedge under the Local-Volatility Model

- Delta by the implied tree differs from delta by the Black-Scholes model's implied volatility.
  - The latter is by formula (46) or (47) (p. 343) after calculating the implied volatility from the same option price by the implied tree.
- Hence the profits and losses of their delta hedges will differ.
- The next plot shows the best 100 out of 100,000 random paths where the implied tree delta outperforms the Black-Scholes delta.[a]

[a]In terms of profits and losses. Plot supplied by Mr. Chiu, Tzu-Hsuan (`R08723061`) on November 20, 2021. We are hedging a long call.



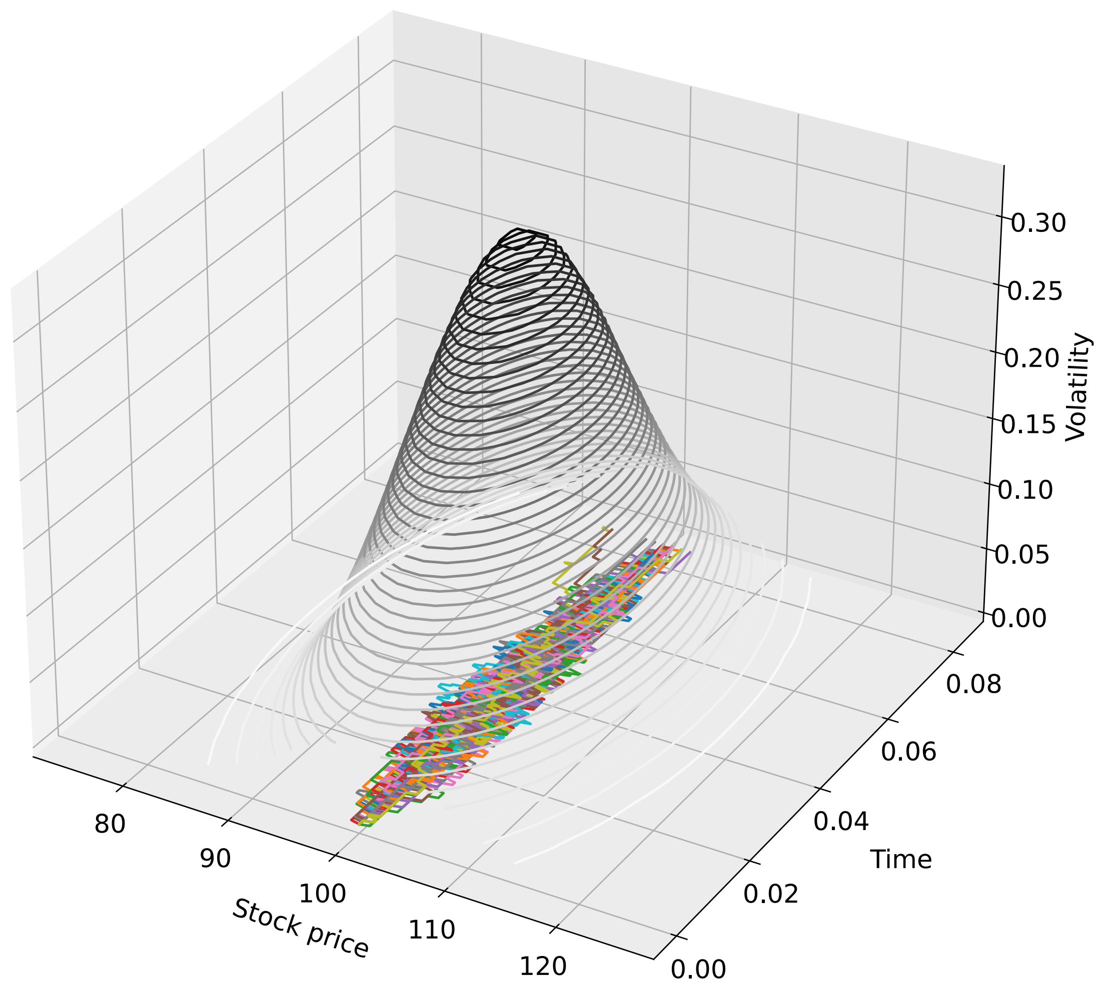
0.30
0.25
0.20
0.15
0.10
0.05
0.00
Volatility
80
90
100
110
120
Stock price
0.00
0.02
0.04
0.06
0.08
Time

Delta Hedge under the Local-Volatility Model (concluded)

- The next plot shows the best 100 out of 100,000 random paths where the Black-Scholes delta outperforms the implied tree delta.[a]

---

[a]Plot supplied by Mr. Chiu, Tzu-Hsuan (`R08723061`) on November 20, 2021. We are again hedging a long call.



0.30
0.25
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0.15
0.10
0.05
0.00
Volatility
80
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120
Stock price
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0.02
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0.06
0.08
Time