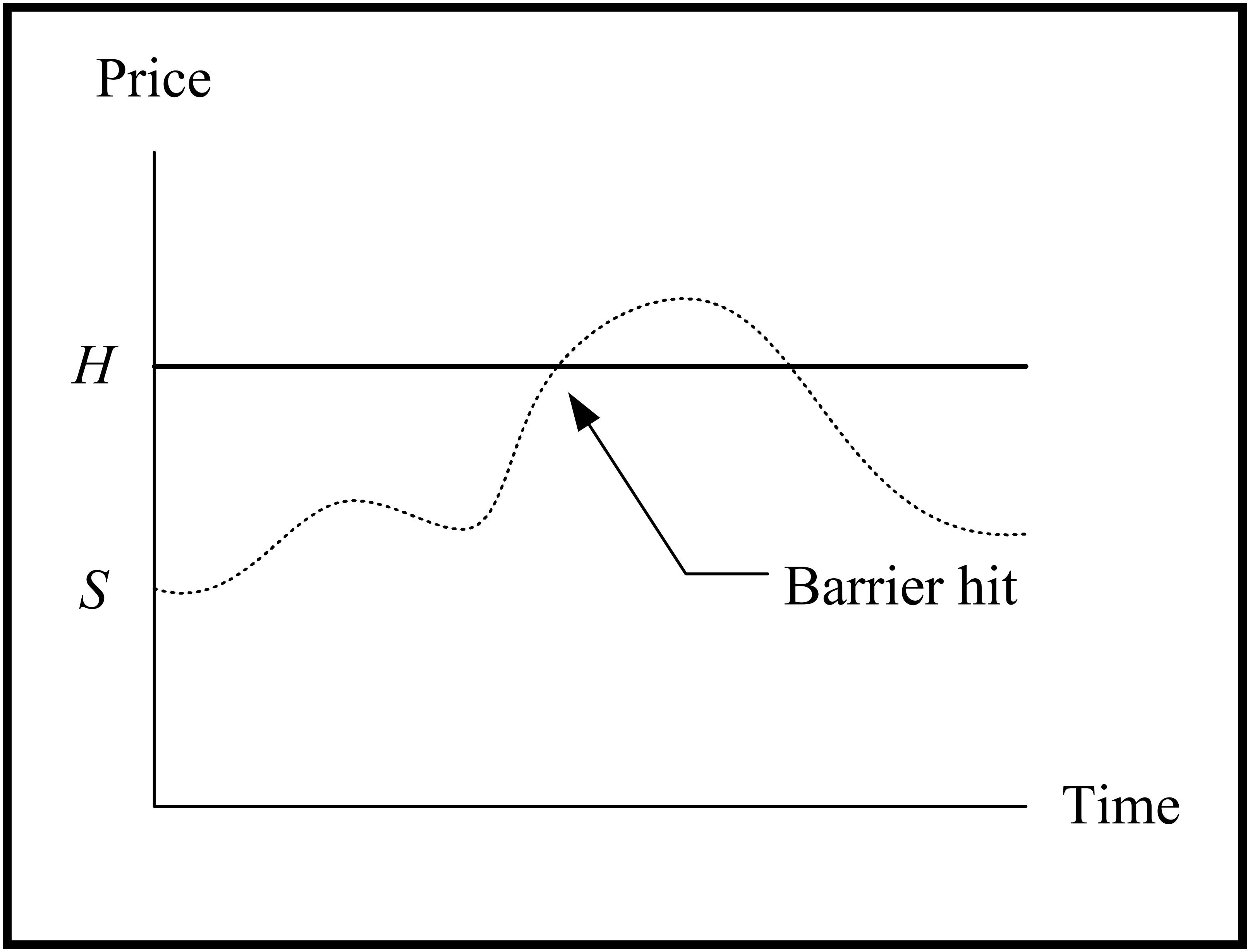
Barrier Options[a]

- Their payoff depends on whether the underlying asset's price reaches a certain price level H *throughout its life*.
- A knock-out (KO) option is an ordinary European option which ceases to exist if the barrier H is reached by the price of its underlying asset.
- A call knock-out option is sometimes called a down-and-out option if $H < S$.
- A put knock-out option is sometimes called an up-and-out option when $H > S$.

[a] A former MBA student in finance told me on March 26, 2004, that she did not understand why I covered barrier options until she started working in a bank. She was working for Lehman Brothers in Hong Kong as of April, 2006.


Price
H
S
Barrier hit
Time

Barrier Options (concluded)

- A knock-in (KI) option comes into existence if a certain barrier is reached.
- A down-and-in option is a call knock-in option that comes into existence only when the barrier is reached and $H < S$.
- An up-and-in is a put knock-in option that comes into existence only when the barrier is reached and $H > S$.
- Formulas exist for all the possible barrier options mentioned above.[a]

[a] Haug (2006).

A Formula for Down-and-In Calls[a]

- Assume $X \geq H$.
- The value of a European down-and-in call on a stock paying a dividend yield of q is

$$Se^{-q\tau} \left(\frac{H}{S}\right)^{2\lambda} N(x) - Xe^{-r\tau} \left(\frac{H}{S}\right)^{2\lambda-2} N(x - \sigma\sqrt{\tau}), \tag{56}$$

  - $x \stackrel{\Delta}{=} \frac{\ln(H^2/(SX)) + (r-q+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$.
  - $\lambda \stackrel{\Delta}{=} (r - q + \sigma^2/2)/\sigma^2$.
- A European down-and-out call can be priced via the in-out parity (see text).

[a]Merton (1973). See Exercise 17.1.6 of the textbook for a proof.

A Formula for Up-and-In Puts[a]

- Assume $X \leq H$.
- The value of a European up-and-in put is

$$X e^{-r\tau} \left(\frac{H}{S}\right)^{2\lambda-2} N(-x + \sigma\sqrt{\tau}) - S e^{-q\tau} \left(\frac{H}{S}\right)^{2\lambda} N(-x).$$

- Again, a European up-and-out put can be priced via the in-out parity.

[a] Merton (1973).

Barrier Options: Popularity

- Knock-out options were issued in the U.S. in 1967.[a]
- Knock-in puts are the most popular barrier options.[b]
- Knock-out puts are the second most popular barrier options.[c]
- Knock-out calls are the most popular among barrier call options.[d]

---

[a] Cox & Rubinstein (1985).
[b] Bennett (2014).
[c] Bennett (2014).
[d] Bennett (2014).

Are American Options Barrier Options?[a]

- American options are barrier options with the exercise boundary as the barrier and the payoff as the rebate?
- One salient difference is that the exercise boundary must be found by backward induction.
- It cannot be specified in an arbitrary way.

[a]Contributed by Mr. Yang, Jui-Chung (D97723002) on March 25, 2009.

Are American Options Barrier Options? (concluded)

- In contrast, the barrier in a barrier option is fixed by a contract.[a]
  - The option remains European-style, without the right to early exercise.[b]
- One can also have American barrier options.
  - Need to specify whether one can exercise the option early if the stock price has not touched the barrier.[c]

---

[a] Cox & Rubinstein (1985).

[b] Contributed by Ms. Chen, Sin-Huei (Amber) (P00922005) on March 31, 2021.

[c] Contributed by Mr. Lu, Yu-Ming (R06723032, D08922008) on March 31, 2021.

Interesting Observations

- Assume $H < X$.
- Replace S in the Merton pricing formula Eq. (44) on p. 333 for the call with H^2/S.
  - Equation (56) on p. 398 for the down-and-in call becomes Eq. (44) when $r - q = \sigma^2/2$.
  - Equation (56) becomes S/H times Eq. (44) when $r - q = 0$.

Interesting Observations (concluded)

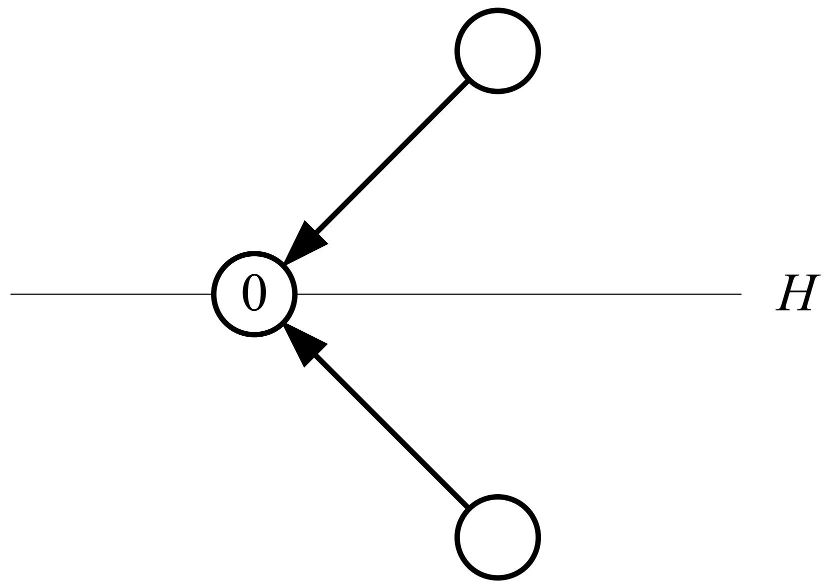
- Replace S in the pricing formula for the down-and-in call, Eq. (56), with H^2/S.
  - Equation (56) becomes Eq. (44) when $r - q = \sigma^2/2$.
  - Equation (56) becomes H/S times Eq. (44) when $r - q = 0$.[a]
- Why?

---

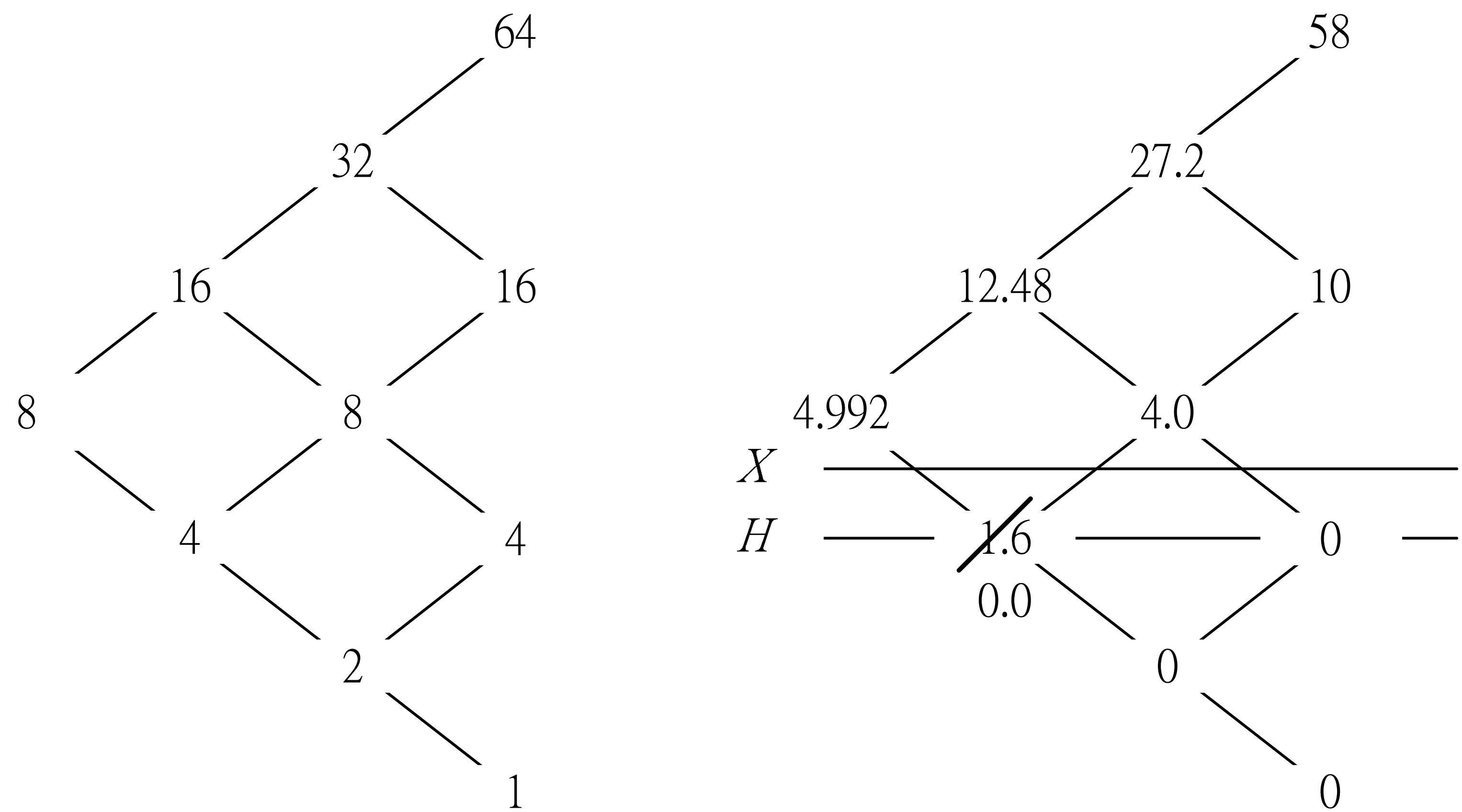
[a]Contributed by Mr. Chou, Ming-Hsin (R02723073) on April 24, 2014.

Binomial Tree Algorithms

- Barrier options can be priced by binomial tree algorithms.
- Below is for the down-and-out option.



- Pricing down-and-in options is subtler.



$S = 8$, $X = 6$, $H = 4$, $R = 1.25$, $u = 2$, and $d = 0.5$.

Backward-induction: $C = (0.5 \times C_u + 0.5 \times C_d)/1.25$.

Binomial Tree Algorithms (continued)

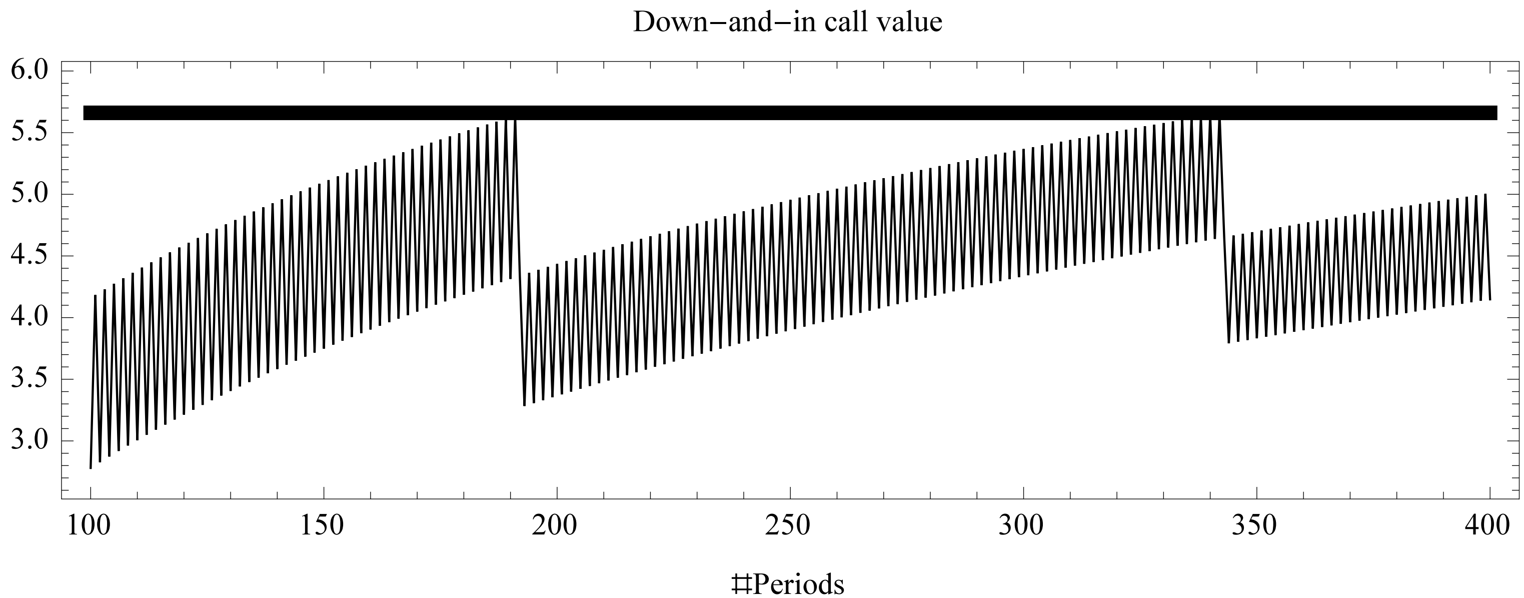
- But convergence is erratic because H is not at a price level on the tree (see plot on next page).[a]
  - The barrier H is moved lower (or higher, if you so choose) to a close-by node price.
  - This "effective barrier" thus changes as n increases.
- In fact, the binomial tree is $O(1/\sqrt{n})$ convergent.[b]
- Solutions will be presented later.

---

[a] Boyle & Lau (1994).

[b] Tavella & Randall (2000); J. Lin (`R95221010`) (2008); J. Lin (`R95221010`) & Palmer (2013).

Binomial Tree Algorithms (concluded)[a]



[a]Lyuu (1998).

Other Types of Barrier Options[a]

- Partial barrier options.
- Forward-starting barrier options.
- Window barrier options.
- Rolling barrier options.
- Moving barrier options.

---

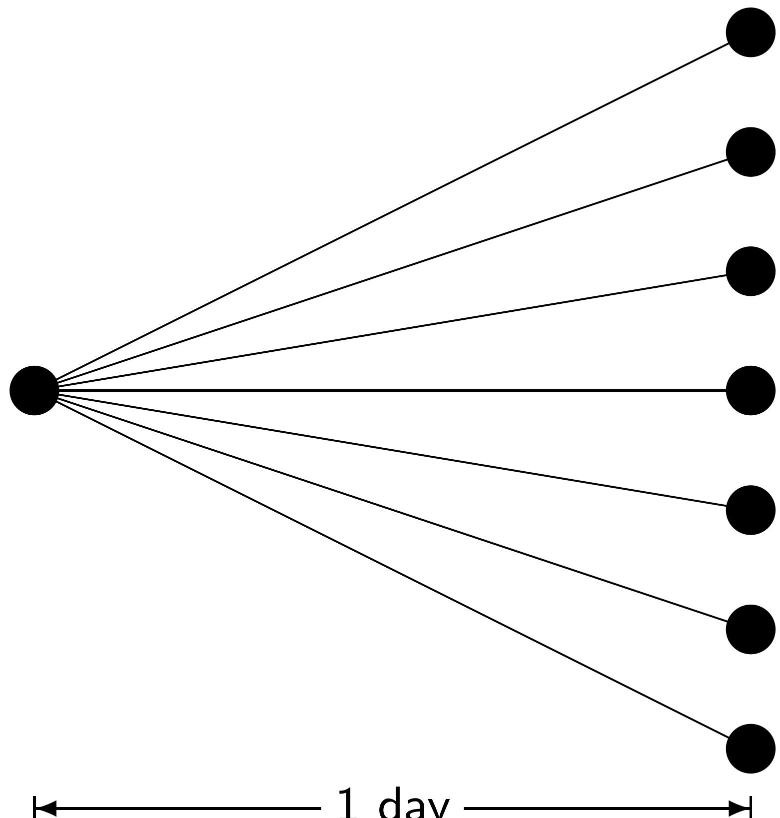
[a] Armtrong (2001); Carr & A. Chou (1997); Davydov & Linetsky (2001/2002); Haug (1998).

Daily Monitoring

- Many barrier options monitor the barrier only for daily *closing prices*.
- If so, only nodes at the end of a day need to check for the barrier condition.
- We can even remove intraday nodes to create a multinomial tree.
  - A node is then followed by $d + 1$ nodes if each day is partitioned into d periods.
- Does this save time or space?[a]

[a]Contributed by Ms. Chen, Tzu-Chun (`R94922003`) and others on April 12, 2006.

A Heptanomial Tree (6 Periods Per Day)



Discrete Monitoring vs. Continuous Monitoring

- Discrete barriers are more expensive for knock-out options than continuous ones.
- But discrete barriers are less expensive for knock-in options than continuous ones.
- Discrete barriers are far less popular than continuous ones for individual stocks.[a]
- They are equally popular for indices.[b]

---

[a] Bennett (2014).
[b] Bennett (2014).

Data! data! data!

— Arthur Conan Doyle (1892),

The Adventures of Sherlock Holmes

Foreign Currencies

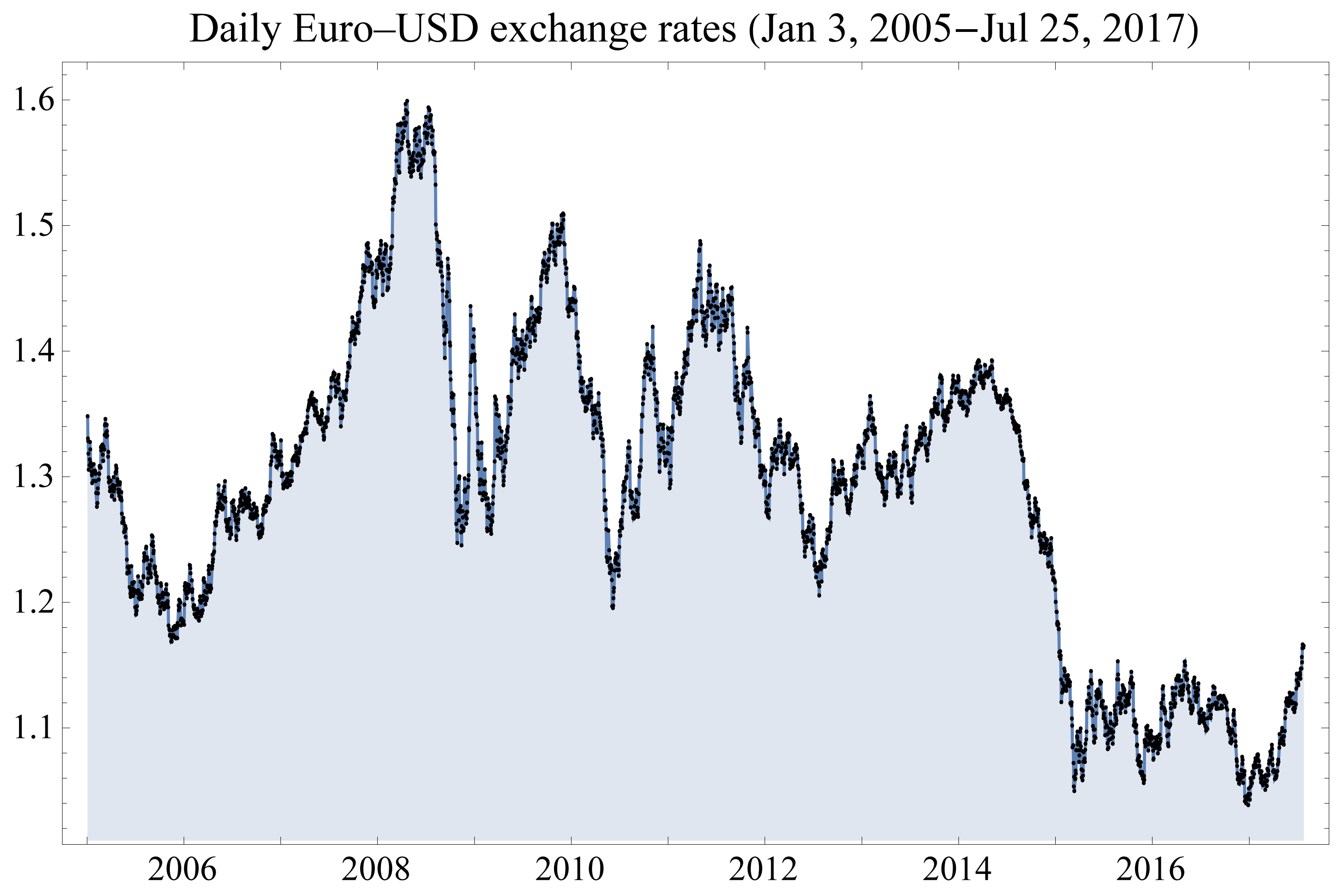
- S denotes the spot exchange rate in domestic/foreign terms.
  - By that we mean the number of domestic currencies per unit of foreign currency.[a]
- σ denotes the volatility of the exchange rate.
- r denotes the domestic interest rate.
- $\hat{r}$ denotes the foreign interest rate.

[a]The market convention is the opposite: A/B $= x$ means one unit of currency A (the reference currency or base currency) is equal to x units of currency B (the counter-value currency).

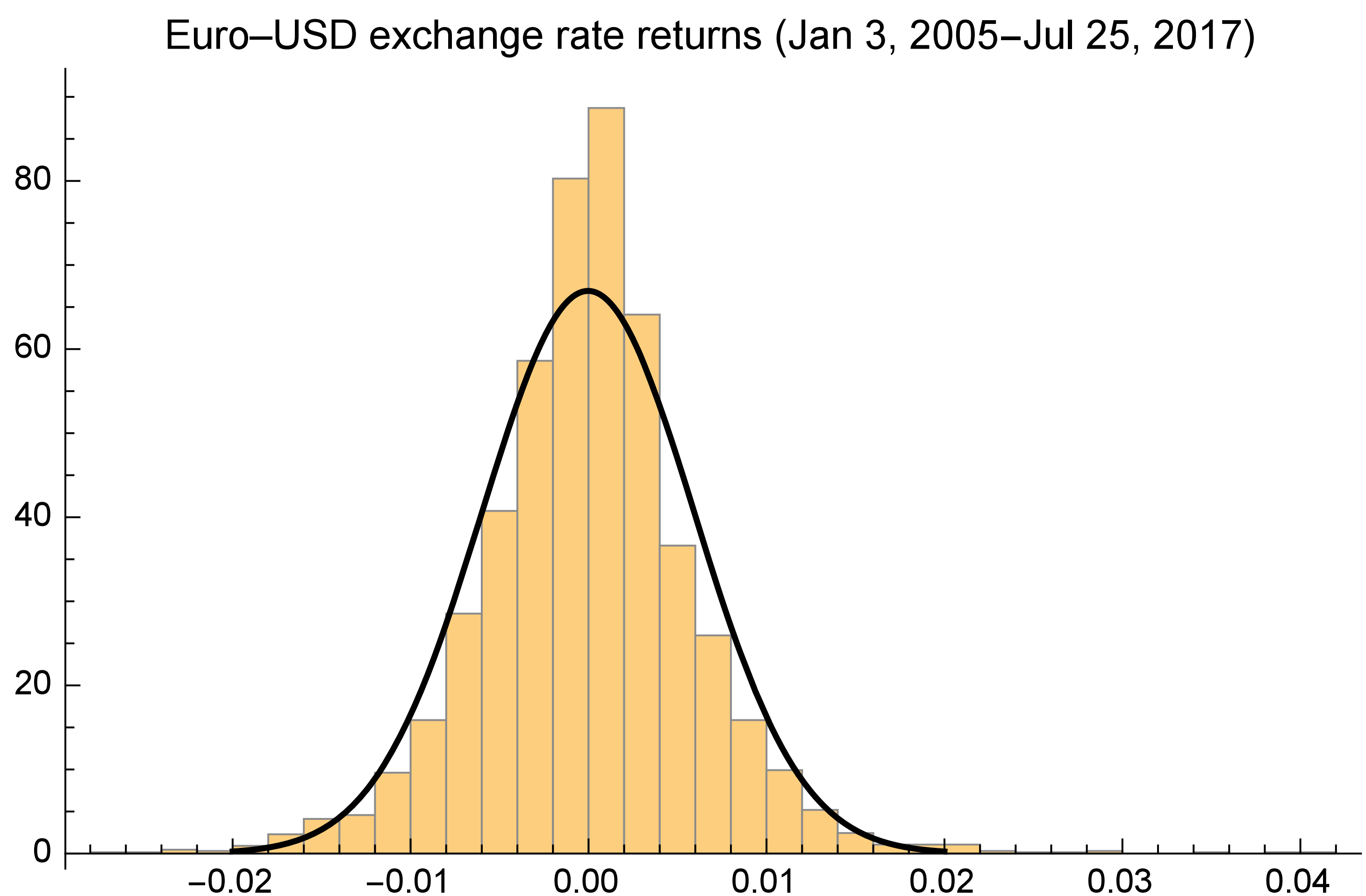
Foreign Currencies (concluded)

- A foreign currency is analogous to a stock paying a known dividend yield.
  - Foreign currencies pay a "continuous dividend yield" equal to $\hat{r}$ in the foreign currency.

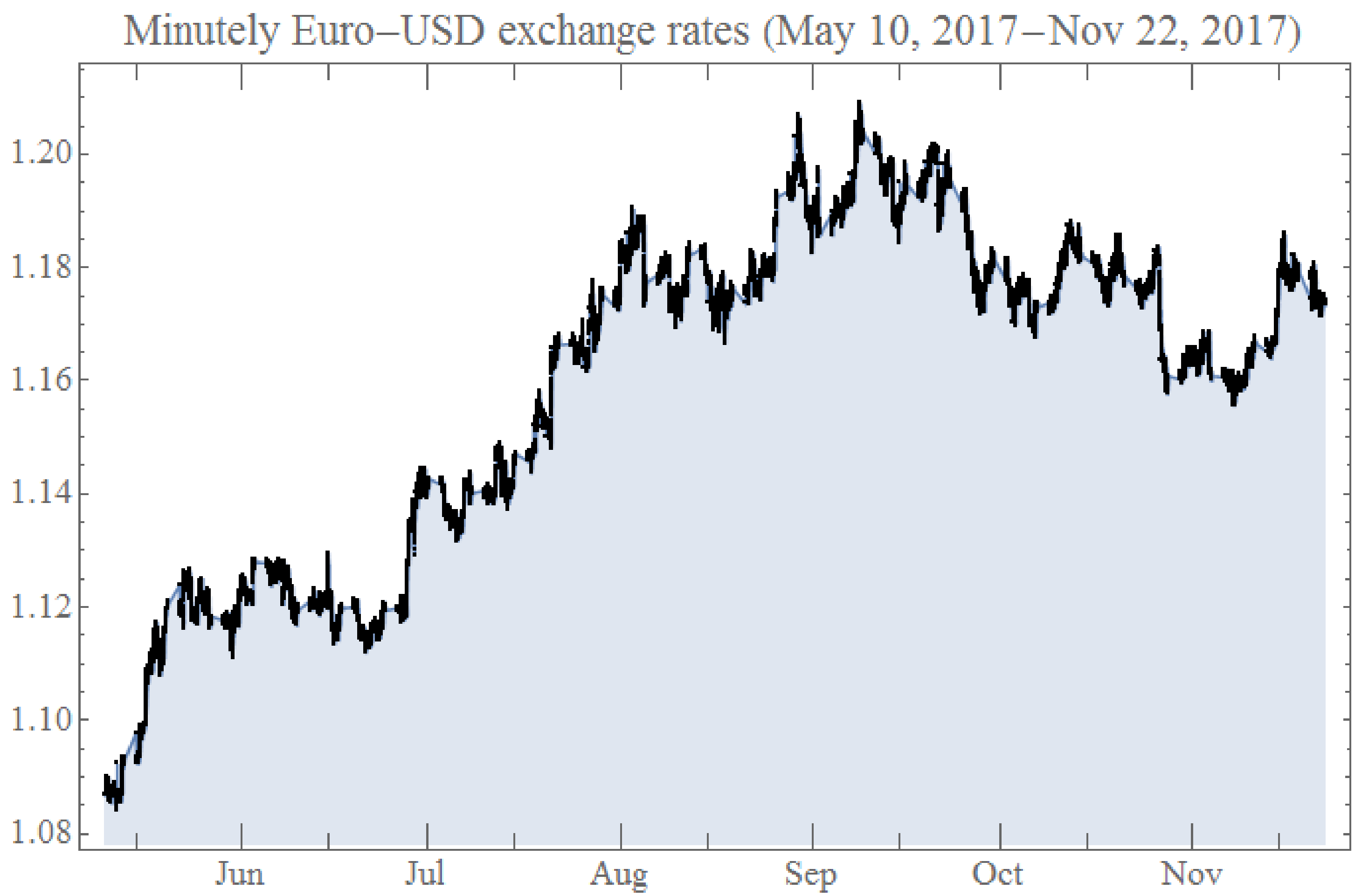
Time Series of the Daily Euro–USD Exchange Rate



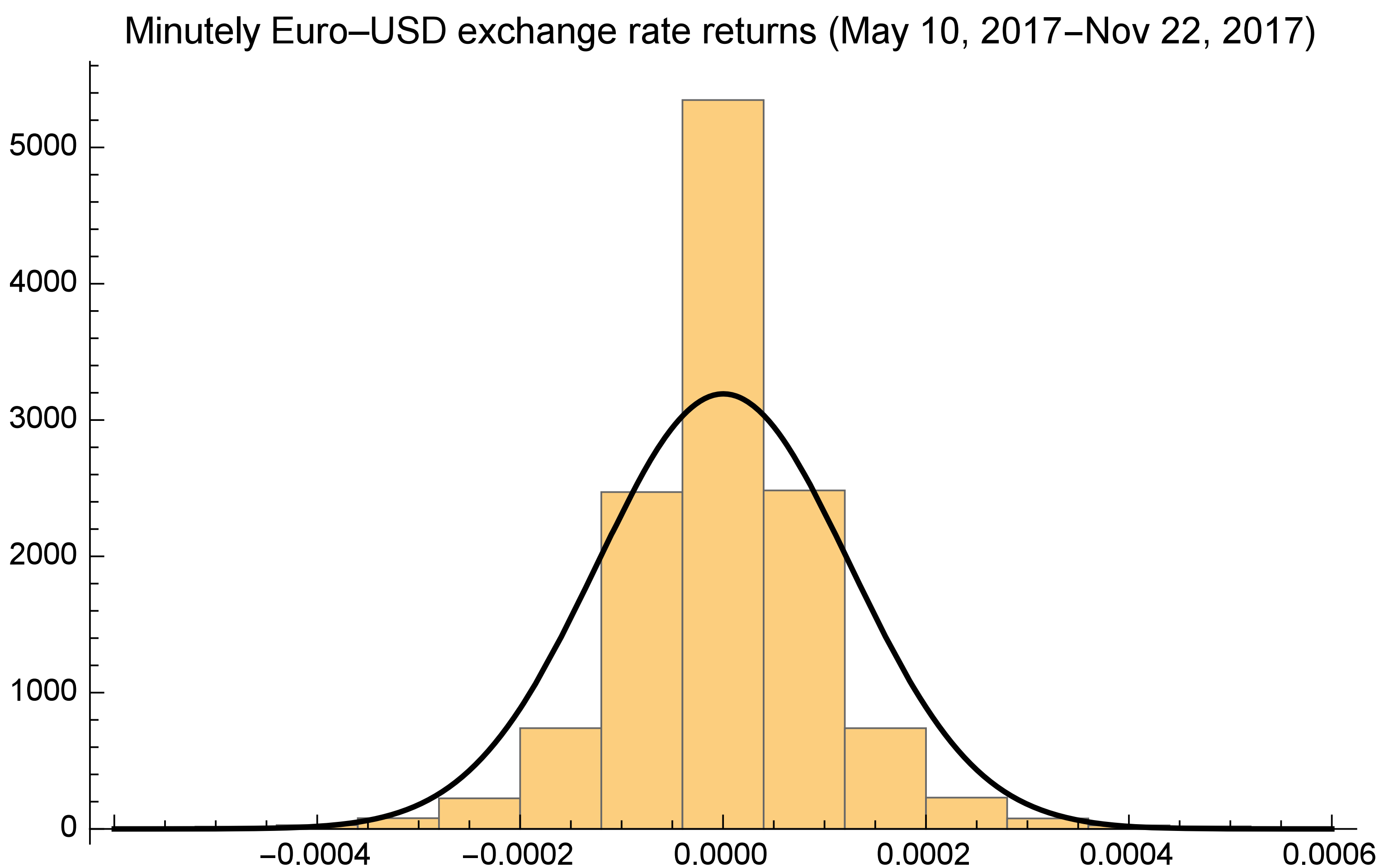
Distribution of the Daily Euro–USD Exchange Rate



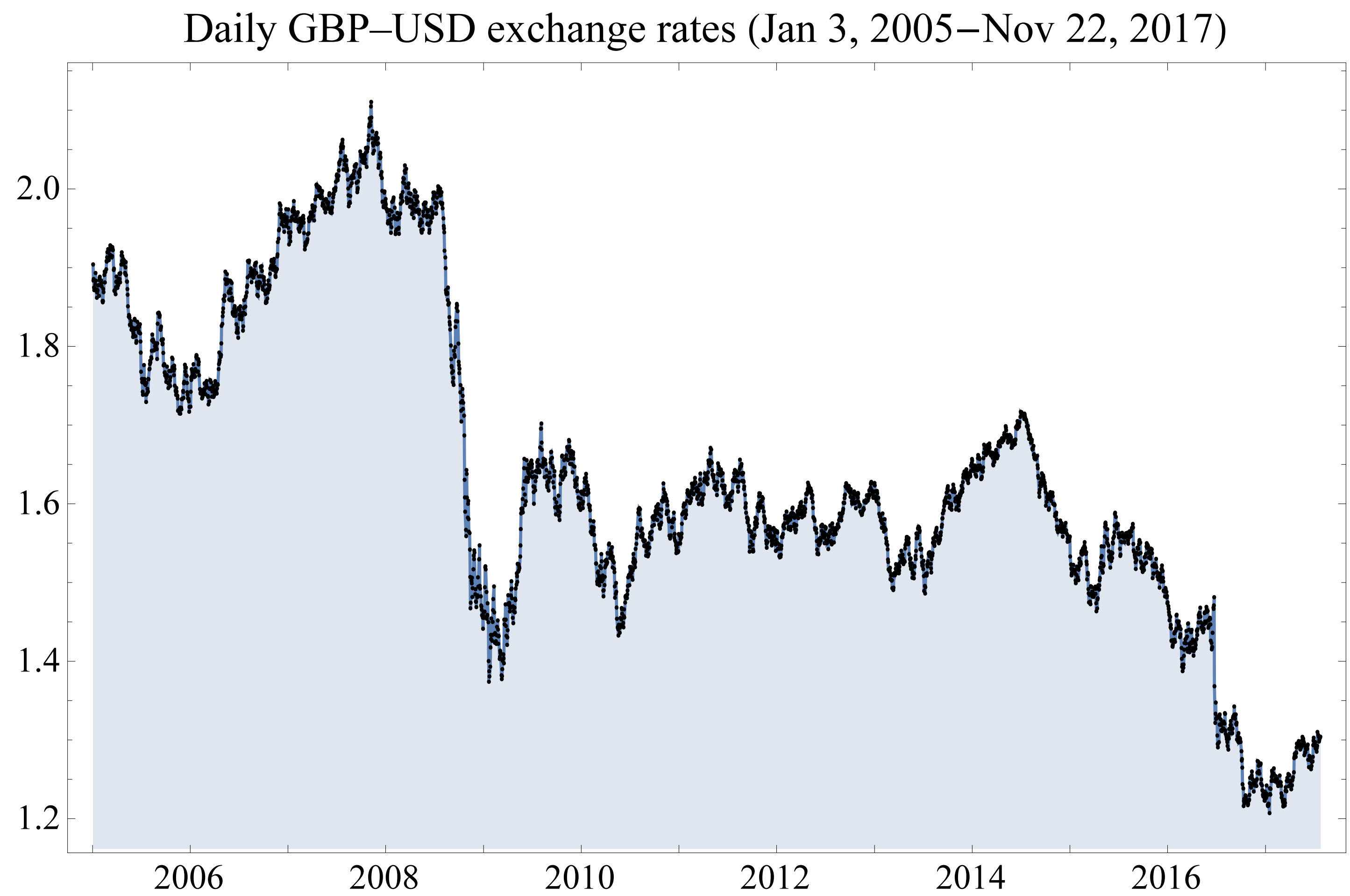
Time Series of the Minutely Euro–USD Exchange Rate



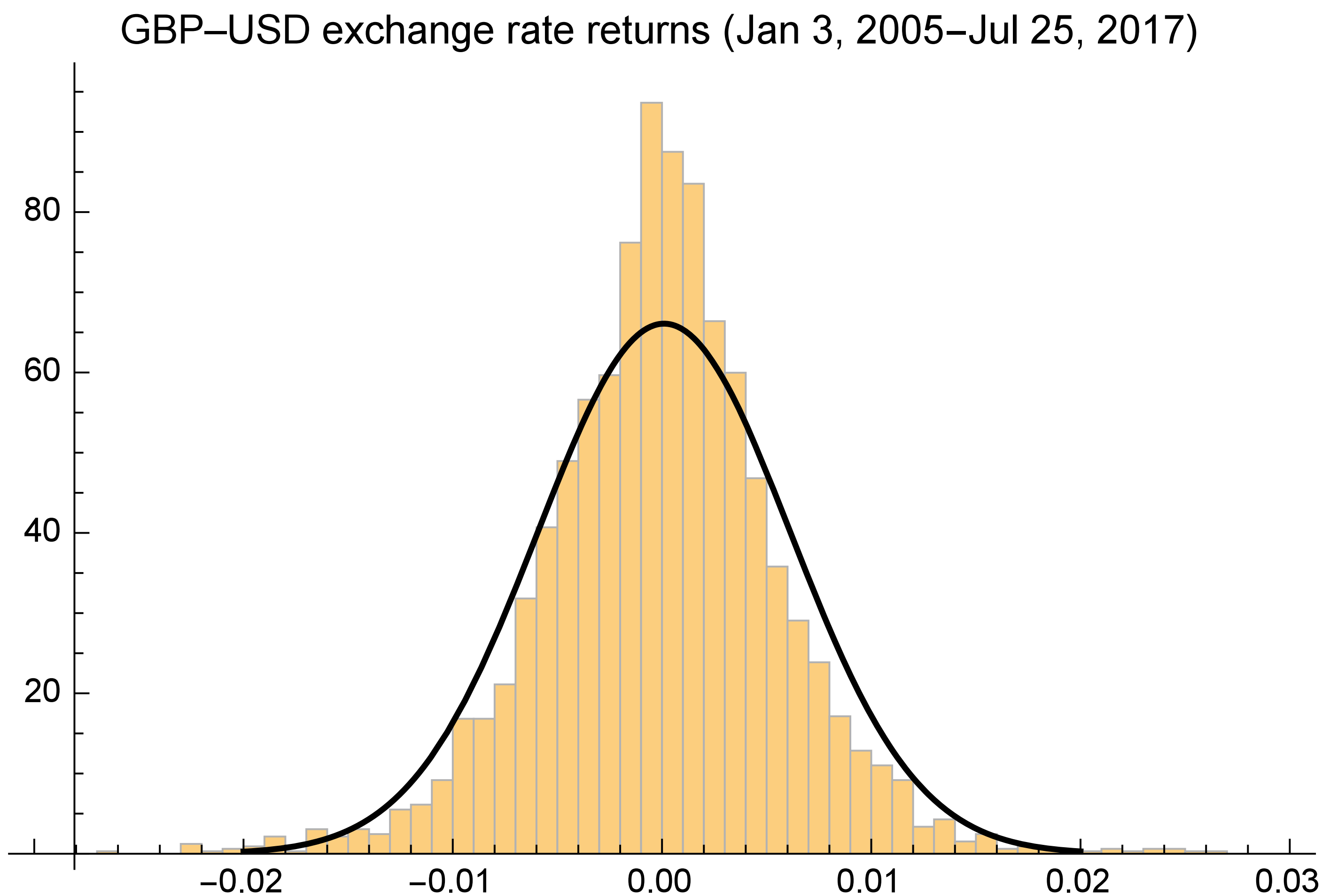
Distribution of the Minutely Euro–USD Exchange Rate



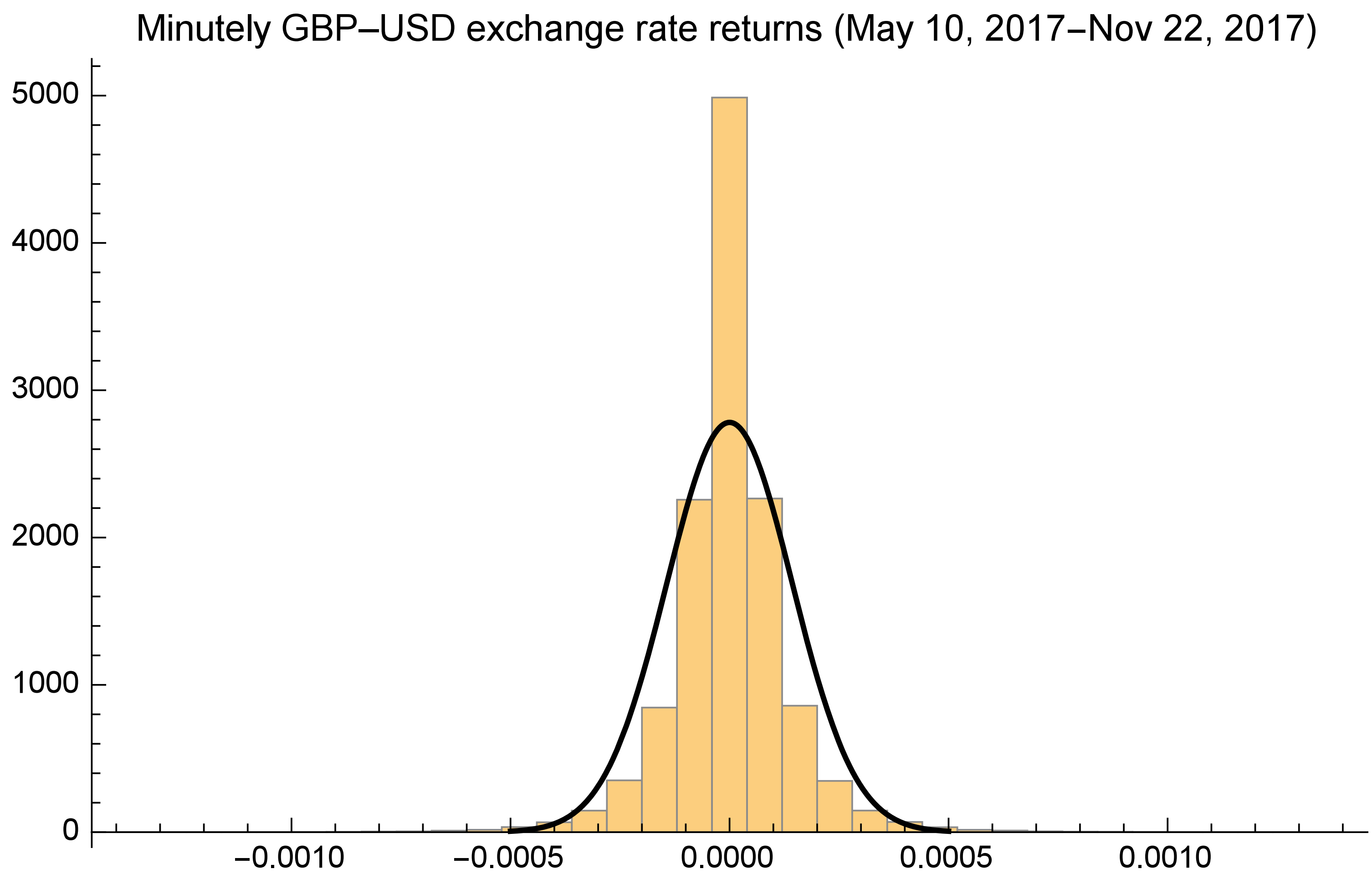
Time Series of the Daily GBP–USD Exchange Rate



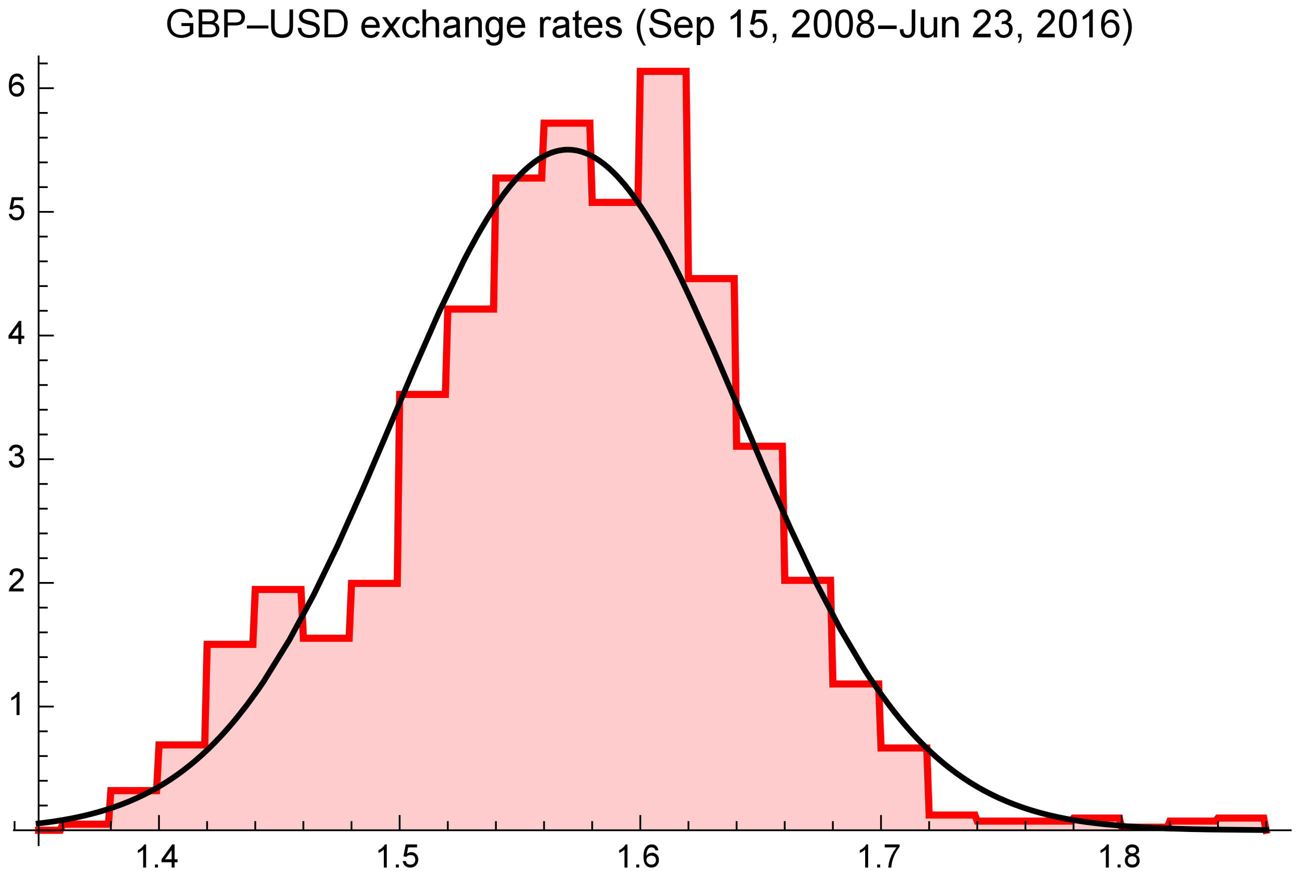
Distribution of the Daily GBP–USD Exchange Rate



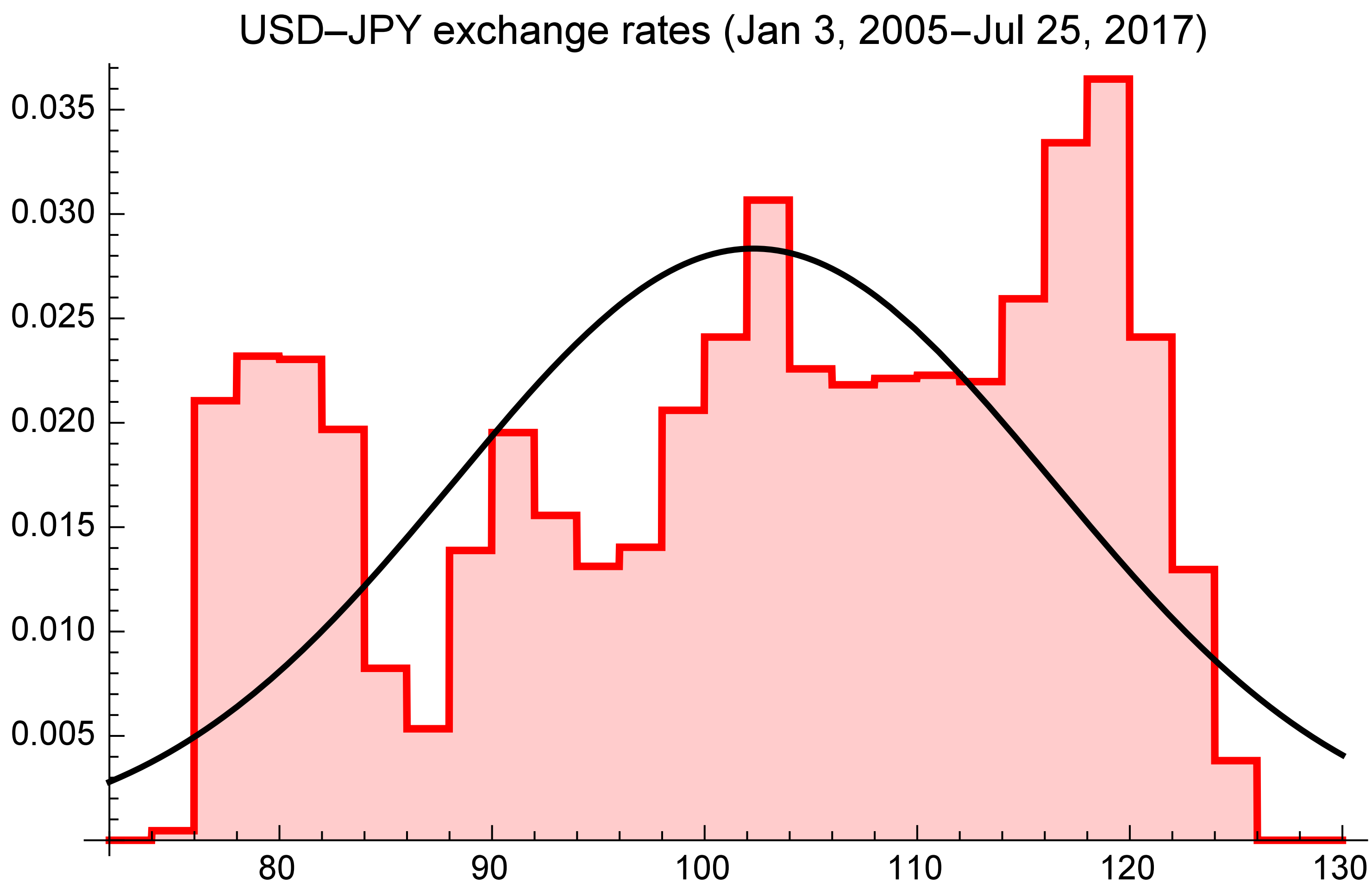
Distribution of the Minutely GBP–USD Exchange Rate



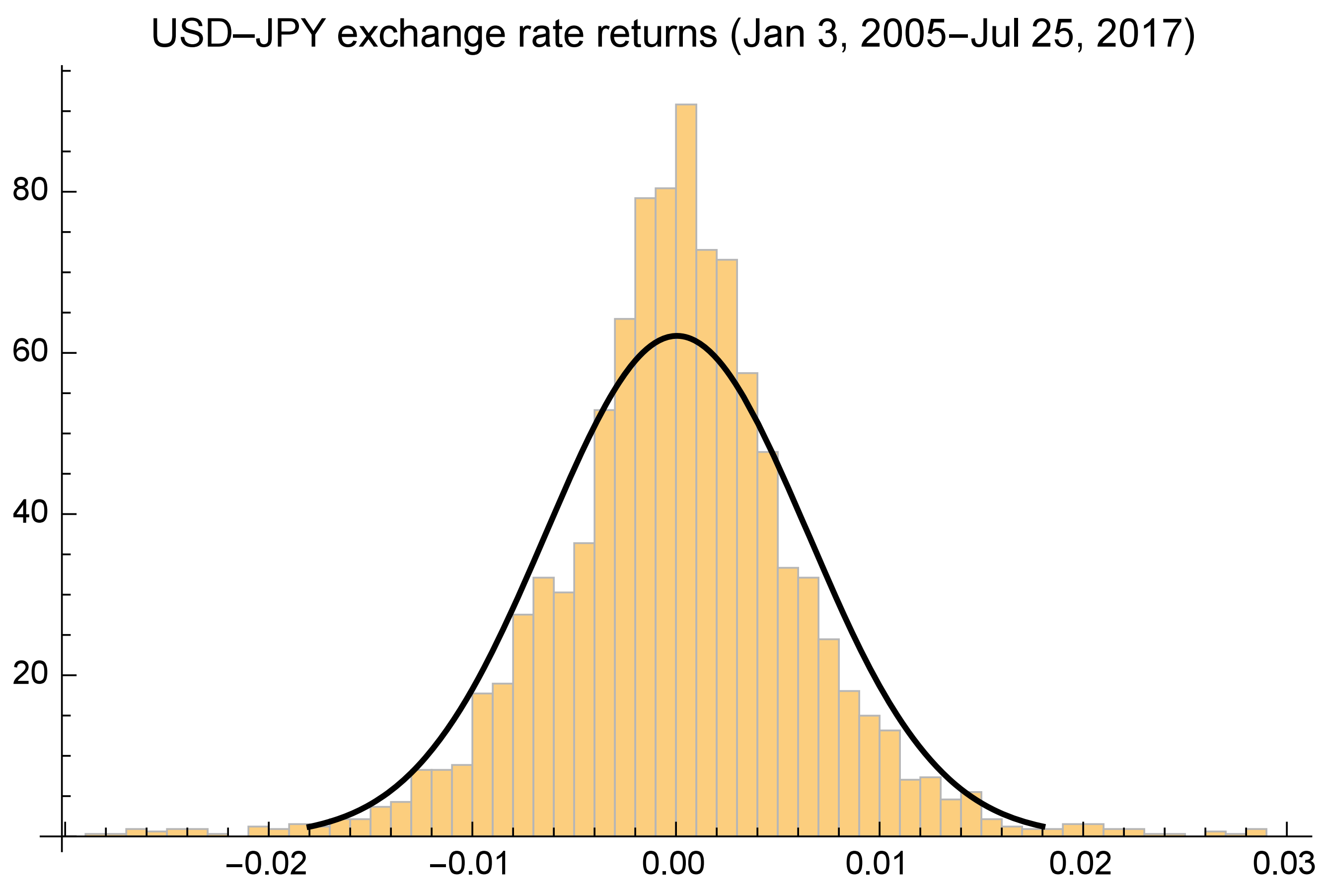
Distribution of the GBP–USD Exchange Rate (after the Collapse of Lehman Brothers and before Brexit)



Distribution of the Daily JPY–USD Exchange Rate



Distribution of the Daily JPY–USD Exchange Rate (concluded)



Foreign Exchange Options

- In 2000 the total notional volume of foreign exchange options was US$13 trillion.[a]
  - 38.5% were vanilla calls and puts with a maturity less than one month.
  - 52.5% were vanilla calls and puts with a maturity between one and 18 months.
  - 4% were barrier options.
  - 1.5% were vanilla calls and puts with a maturity more than 18 months.
  - 1% were binary options (recall p. 207 or see p. 873).
  - 0.7% were Asian options (see p. 438).

[a]Lipton (2002).

Foreign Exchange Options (continued)

- Foreign exchange options are settled via delivery of the underlying currency.
- A primary use of foreign exchange (or forex) options is to hedge currency risk.
- Consider a U.S. company expecting to receive 100 million Japanese yen in March 2000.
- Those 100 million Japanese yen will be exchanged for U.S. dollars.

Foreign Exchange Options (continued)

- The contract size for the Japanese yen option is JPY6,250,000.
- The company purchases

$$\frac{100{,}000{,}000}{6{,}250{,}000} = 16$$

  puts on the Japanese yen with a strike of \$.0088/JPY1 and an exercise month in March 2000.
- This put is in the money if the JPY-USD exchange rate drops below 0.0088.

Foreign Exchange Options (continued)

- These puts provide the company the right to sell 100,000,000 Japanese yen for

$$100{,}000{,}000 \times .0088 = 880{,}000$$

  U.S. dollars.

- Note that these puts are equivalent to the right to buy 880,000 U.S. dollars with 100,000,000 Japanese yen.
  - From this angle, they become *calls*.

Foreign Exchange Options (concluded)

- Assume the exchange rate S is lognormally distributed.
- The formulas derived for stock index options in Eqs. (44) on p. 333 apply with the dividend yield equal to $\hat{r}$:

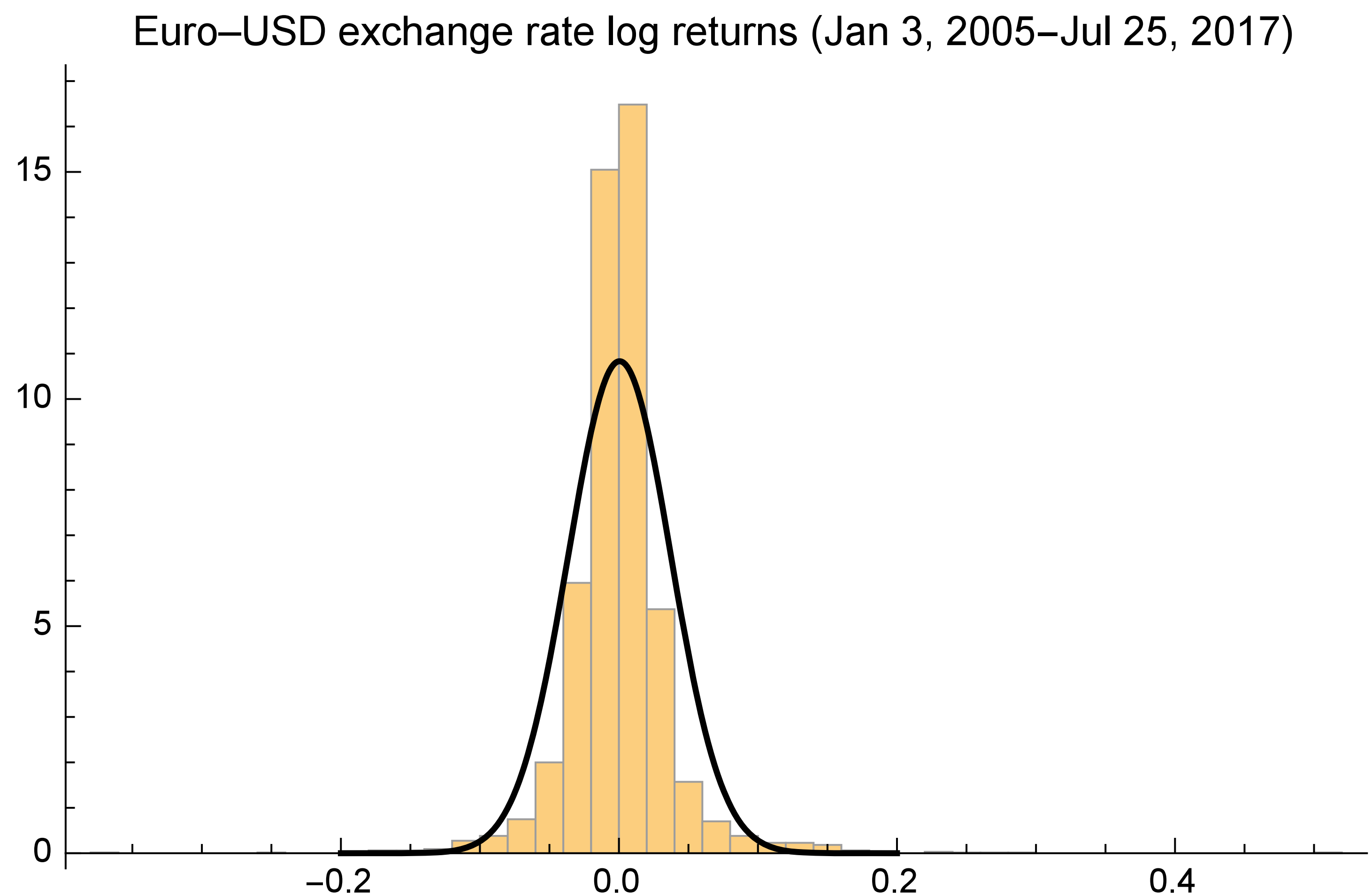
$$C = Se^{-\hat{r}\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{57}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-\hat{r}\tau} N(-x). \tag{57'}$$

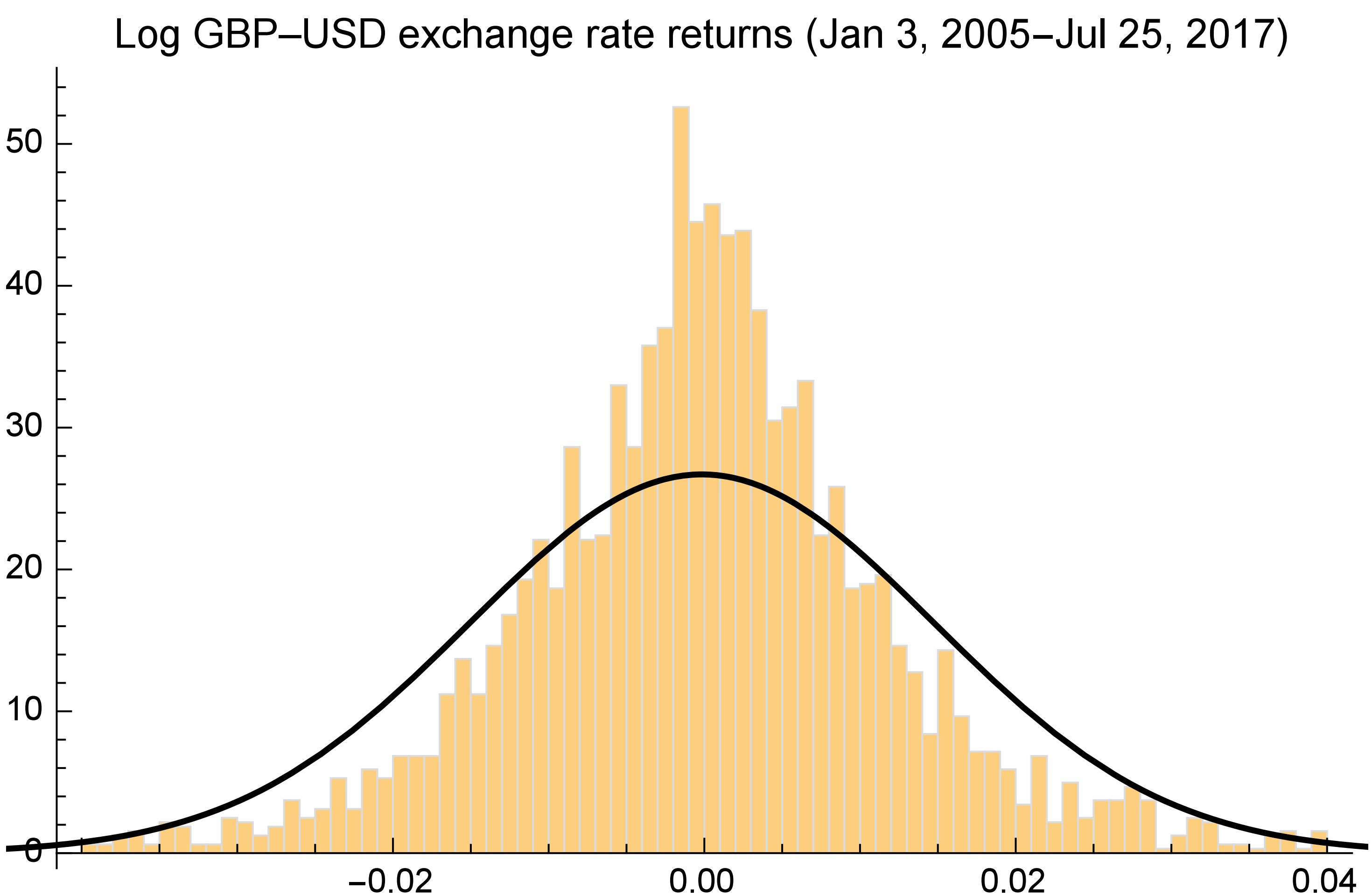
  – Above,

$$x \stackrel{\Delta}{=} \frac{\ln(S/X) + (r - \hat{r} + \sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}.$$

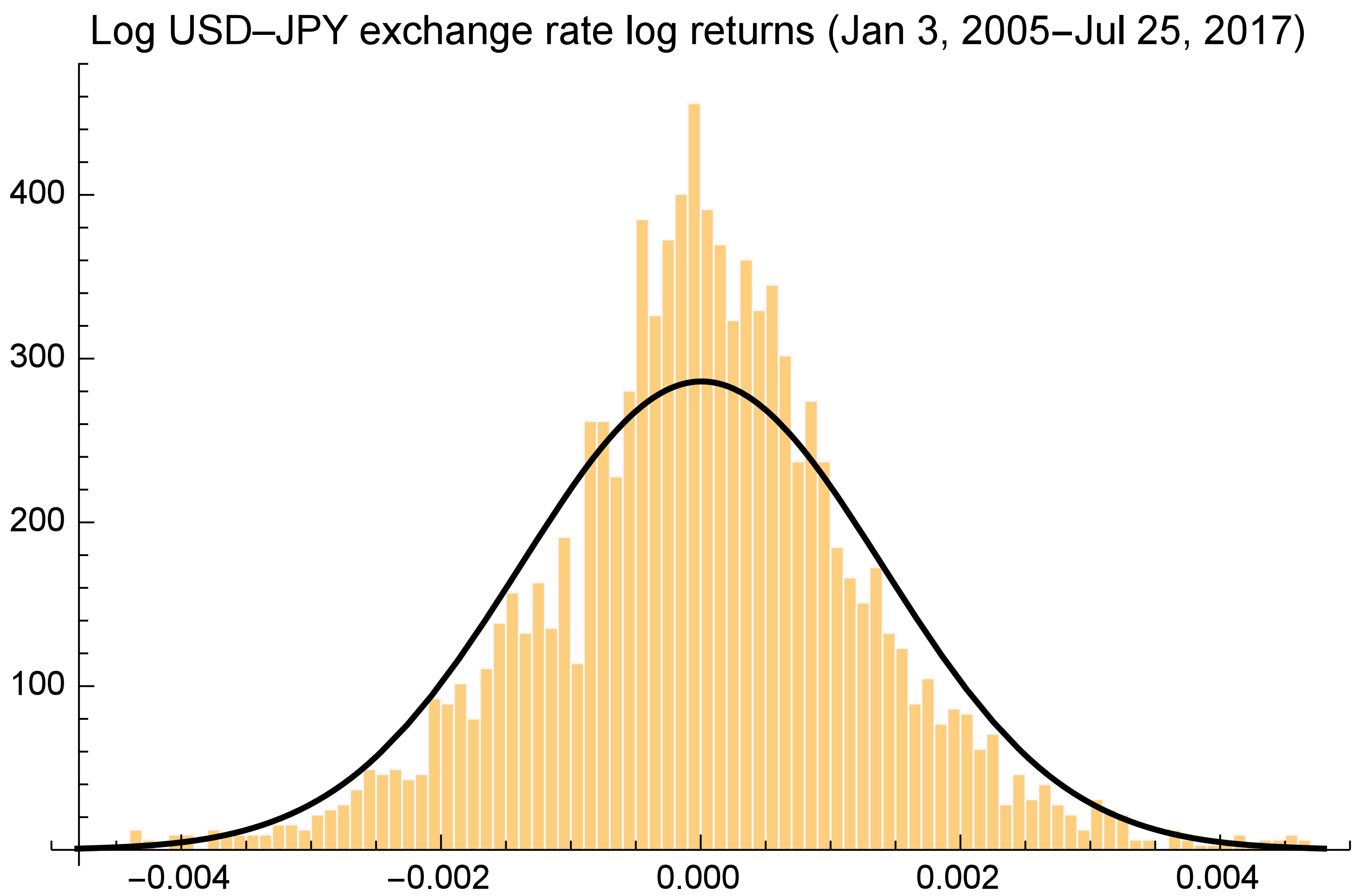
Distribution of the Logarithmic Euro–USD Exchange Rate



Distribution of the Logarithmic GBP–USD Exchange Rate



Distribution of the Logarithmic JPY–USD Exchange Rate



Bar the roads!

Bar the paths!

Wert thou to flee from here, wert thou

to find all the roads of the world,

the way thou seekst

the path to that thou'dst find not[.]

— Richard Wagner (1813–1883), *Parsifal*

Path-Dependent Derivatives

- Let $S_0, S_1, \ldots, S_n$ denote the prices of the underlying asset over the life of the option.
- S_0 is the known price at time zero.
- S_n is the price at expiration.
- The standard European call has a terminal value depending only on the last price, $\max(S_n - X, 0)$.
- Its value thus depends only on the underlying asset's terminal price regardless of how it gets there.[a]

[a] Called simple claims (Björk, 2009).

Path-Dependent Derivatives (continued)

- Some derivatives are path-dependent in that their terminal payoff depends *explicitly* on the path.
- The (arithmetic) average-rate call has this terminal value:
$$\max\left(\frac{1}{n+1}\sum_{i=0}^{n} S_i - X, 0\right).$$
- The average-rate put's terminal value is given by
$$\max\left(X - \frac{1}{n+1}\sum_{i=0}^{n} S_i, 0\right).$$

Path-Dependent Derivatives (continued)

- Average-rate options are also called Asian options.
- They are very popular.[a]
- They are useful hedging tools for firms that will make a stream of purchases over a time period because the costs are likely to be linked to the average price.
- They are mostly European.
- The averaging clause is also common in convertible bonds and structured notes.

[a]As of the late 1990s, the outstanding volume was in the range of 5–10 billion U.S. dollars (Nielsen & Sandmann, 2003).

Path-Dependent Derivatives (continued)

- A lookback call option on the minimum has a terminal payoff of
$$S_n - \min_{0 \leq i \leq n} S_i.$$
- A lookback put on the maximum has a terminal payoff of
$$\max_{0 \leq i \leq n} S_i - S_n.$$

Path-Dependent Derivatives (concluded)

- The fixed-strike lookback option[a] provides a payoff of
  - $\max(\max_{0 \leq i \leq n} S_i - X, 0)$ for the call.
  - $\max(X - \min_{0 \leq i \leq n} S_i, 0)$ for the put.
- Lookback calls and puts on the average (instead of a constant X) are called average-strike options.

[a] Also called forward lookback option.

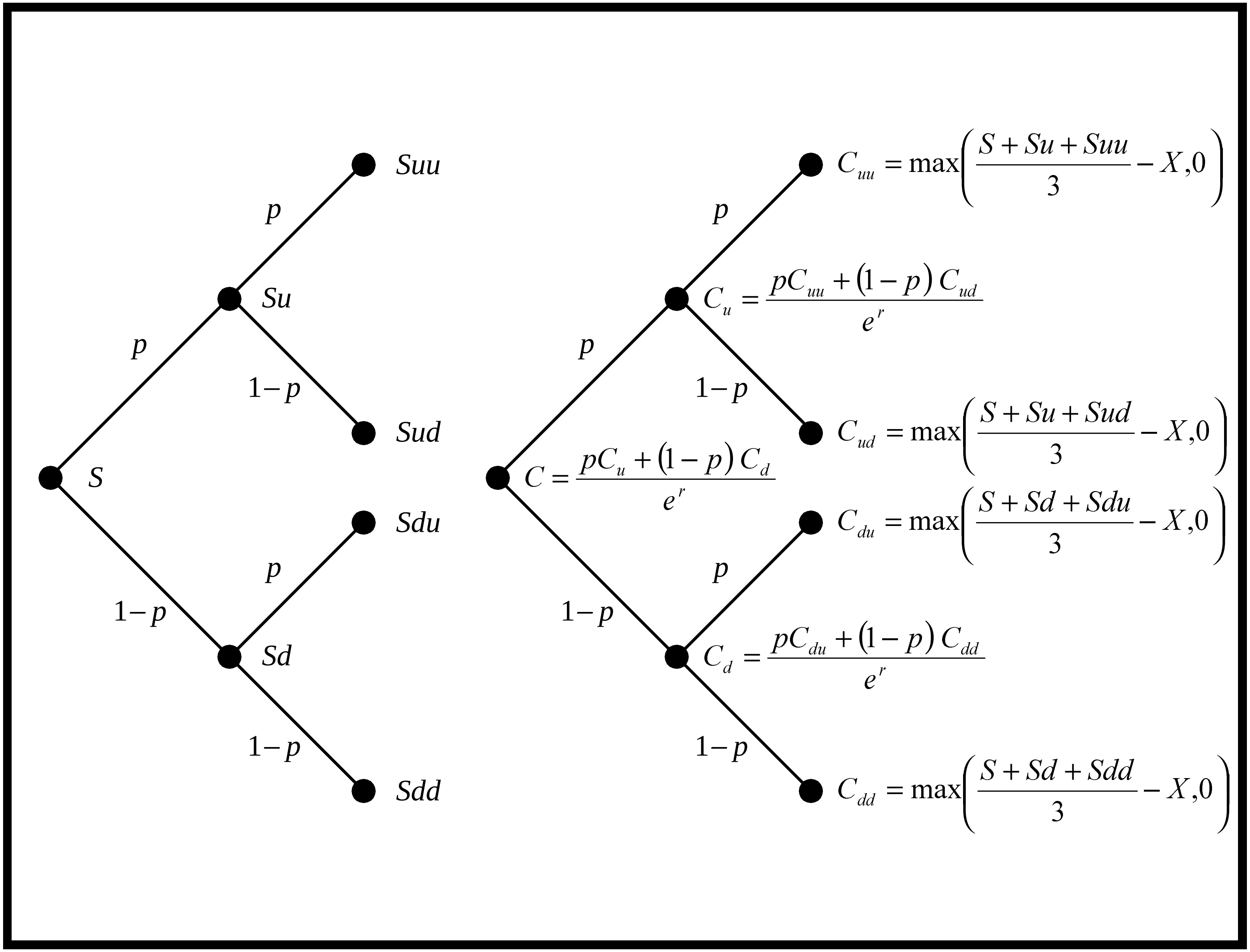
Average-Rate Options

- Average-rate options are notoriously hard to price.
- The binomial tree for the averages does not combine (see next page).
- A naive algorithm enumerates the 2^n paths for an n-period binomial tree and then averages the payoffs.
- But the complexity is exponential.[a]
- The Monte Carlo method[b] and approximation algorithms are some of the alternatives left.

---

[a] Dai (`B82506025`, `R86526008`, `D8852600`) & Lyuu (2007) reduce it to $2^{O(\sqrt{n})}$.

[b] See pp. 858ff.



Suu
Su
Sud
S
Sdu
Sd
Sdd
p
1−p
$C_{uu} = \max\left(\frac{S+Su+Suu}{3} - X, 0\right)$
$C_u = \frac{pC_{uu} + (1-p)\,C_{ud}}{e^r}$
$C_{ud} = \max\left(\frac{S+Su+Sud}{3} - X, 0\right)$
$C = \frac{pC_u + (1-p)\,C_d}{e^r}$
$C_{du} = \max\left(\frac{S+Sd+Sdu}{3} - X, 0\right)$
$C_d = \frac{pC_{du} + (1-p)\,C_{dd}}{e^r}$
$C_{dd} = \max\left(\frac{S+Sd+Sdd}{3} - X, 0\right)$

States and Their Transitions

- The tuple

$$(i, S, P)$$

captures the state[a] for the Asian option.

  - i: the time.
  - S: the prevailing stock price.
  - P: the running sum.[b]

[a] A "sufficient statistic," if you will.

[b] When the average is a moving average, a different technique is needed (C. Kao (R89723057) & Lyuu, 2003).

States and Their Transitions (concluded)

- For the binomial model, the state transition is:

$$
\begin{array}{ccl}
 & & (i+1, Su, P+Su), \quad \text{for the up move} \\
 & \nearrow & \\
(i, S, P) & & \\
 & \searrow & \\
 & & (i+1, Sd, P+Sd), \quad \text{for the down move}
\end{array}
$$

- This leads to an exponential-time algorithm.

Pricing Some Path-Dependent Options

- Not all path-dependent derivatives are hard to price.
  - Barrier options are easy to price.
- When averaging is done *geometrically*, the option payoffs are

$$\max\left((S_0 S_1 \cdots S_n)^{1/(n+1)} - X, 0\right),$$
$$\max\left(X - (S_0 S_1 \cdots S_n)^{1/(n+1)}, 0\right).$$

Pricing Some Path-Dependent Options (concluded)

- The limiting analytical solutions are the Black-Scholes formulas:[a]

$$C = S e^{-q_{\mathrm{a}} \tau} N(x) - X e^{-r\tau} N(x - \sigma_{\mathrm{a}} \sqrt{\tau}), \tag{58}$$

$$P = X e^{-r\tau} N(-x + \sigma_{\mathrm{a}} \sqrt{\tau}) - S e^{-q_{\mathrm{a}} \tau} N(-x), \tag{58'}$$

  - With the volatility set to $\sigma_{\mathrm{a}} \stackrel{\Delta}{=} \sigma / \sqrt{3}$.
  - With the dividend yield set to $q_{\mathrm{a}} \stackrel{\Delta}{=} (r + q + \sigma^2/6)/2$.
  - $x \stackrel{\Delta}{=} \frac{\ln(S/X) + \left(r - q_{\mathrm{a}} + \sigma_{\mathrm{a}}^2/2\right)\tau}{\sigma_{\mathrm{a}} \sqrt{\tau}}$.

[a] Angus (1999).

An Approximate Formula for Asian Calls[a]

$$C = e^{-r\tau}\left[\frac{S}{\tau}\int_0^\tau e^{\mu t+\sigma^2 t/2} N\left(\frac{-\gamma+(\sigma t/\tau)(\tau-t/2)}{\sqrt{\tau/3}}\right) dt \right.$$
$$\left. -XN\left(\frac{-\gamma}{\sqrt{\tau/3}}\right)\right],$$

where

- $\mu \stackrel{\Delta}{=} r-\sigma^2/2$.
- γ is the unique value that satisfies

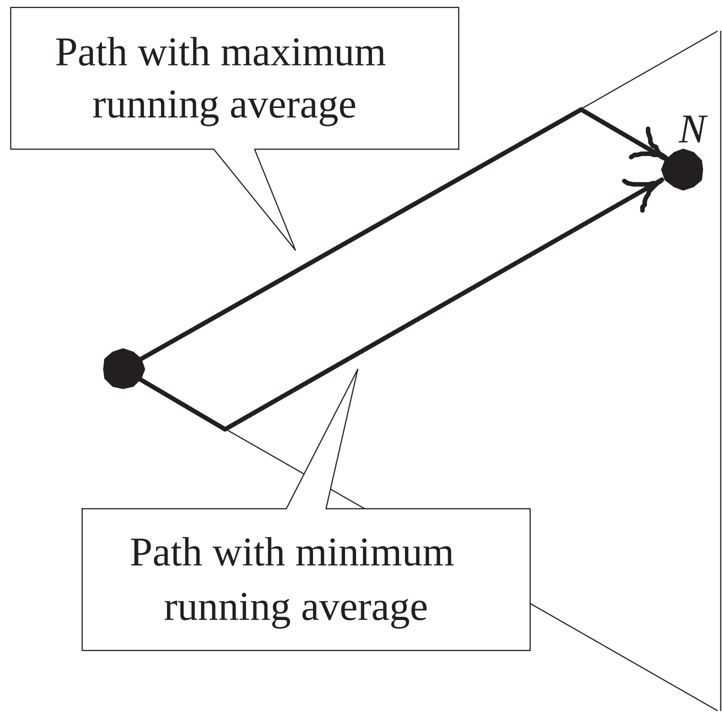
$$\frac{S}{\tau}\int_0^\tau e^{3\gamma\sigma t(\tau-t/2)/\tau^2+\mu t+\sigma^2[\,t-(3t^2/\tau^3)(\tau-t/2)^2\,]/2}\, dt = X.$$

[a]Rogers & Shi (1995); Thompson (1999); K. Chen (R92723061) (2005); K. Chen (R92723061) & Lyuu (2006).

Approximation Algorithm for Asian Options

- Based on the BOPM.
- Consider a node at time j with the underlying asset price equal to $S_0 u^{j-i} d^i$.
- Name such a node $N(j, i)$.
- The running sum $\sum_{m=0}^{j} S_m$ at this node has a maximum value of

$$
\begin{aligned}
& S_0(1 + \overbrace{u + u^2 + \cdots + u^{j-i} + u^{j-i} d + \cdots + u^{j-i} d^i}^{j}) \\
= {} & S_0 \, \frac{1 - u^{j-i+1}}{1-u} + S_0 u^{j-i} d \, \frac{1 - d^i}{1 - d}.
\end{aligned}
$$


Path with maximum
running average
N
Path with minimum
running average

Approximation Algorithm for Asian Options (continued)

- Divide this value by $j+1$ and call it $A_{\max}(j,i)$.
- Similarly, the running sum has a minimum value of

$$S_0(1+\overbrace{d+d^2+\cdots+d^i+d^iu+\cdots+d^iu^{j-i}}^{j})$$

$$= S_0\,\frac{1-d^{i+1}}{1-d}+S_0d^iu\,\frac{1-u^{j-i}}{1-u}.$$

- Divide this value by $j+1$ and call it $A_{\min}(j,i)$.
- $A_{\min}$ and $A_{\max}$ are running averages.

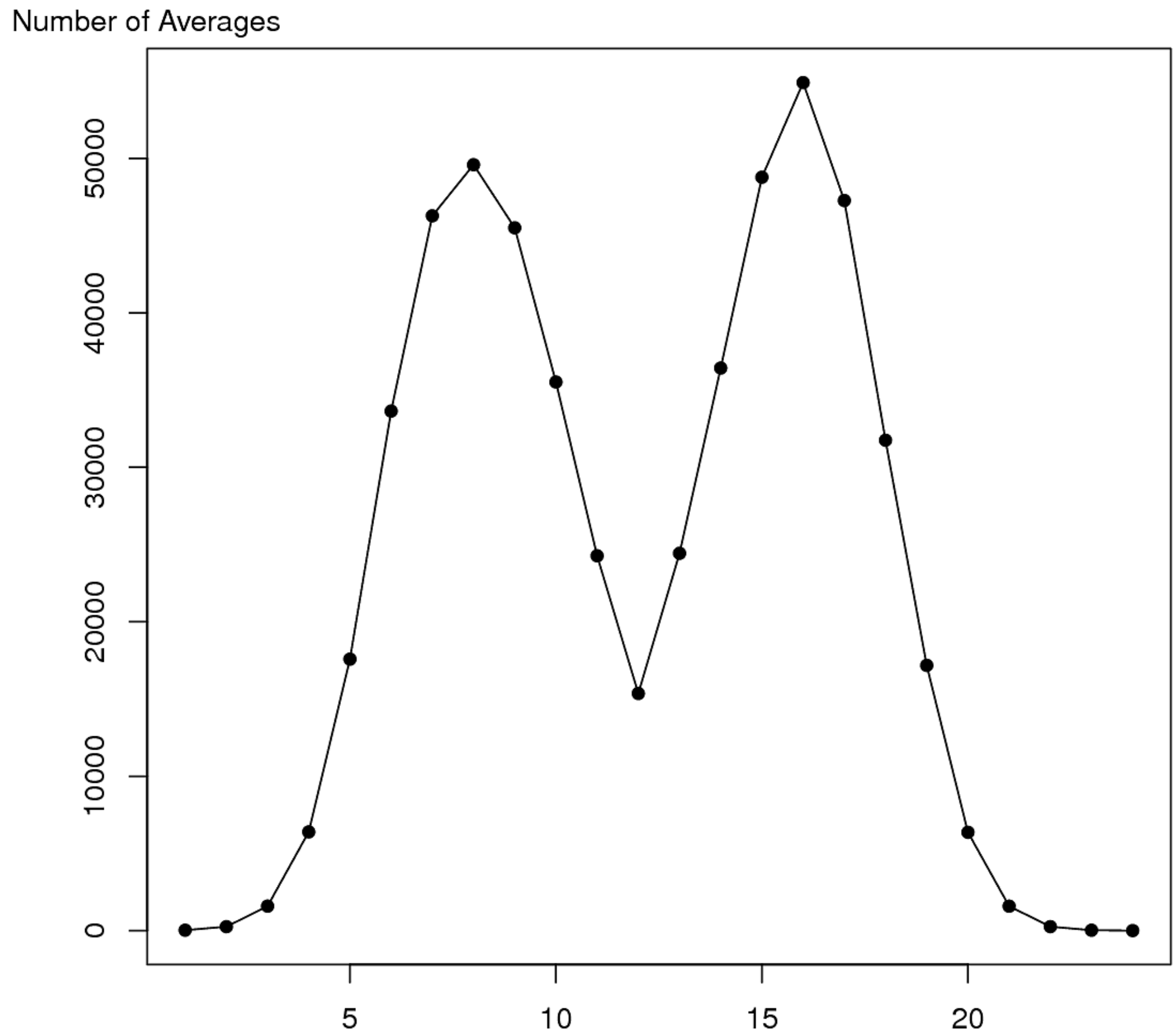
Approximation Algorithm for Asian Options (continued)

- The number of paths to $N(j, i)$ are far too many: $\binom{j}{i}$.
  - For example,

$$\binom{j}{j/2} \sim 2^j \sqrt{2/(\pi j)}\,.$$

- The number of distinct running averages for the nodes at any given time step n seems to be bimodal for n big enough.[a]
  - In the plot on the next page, $u = 5/4$ and $d = 4/5$.

[a]Contributed by Mr. Liu, Jun (`R99944027`) on April 15, 2014.

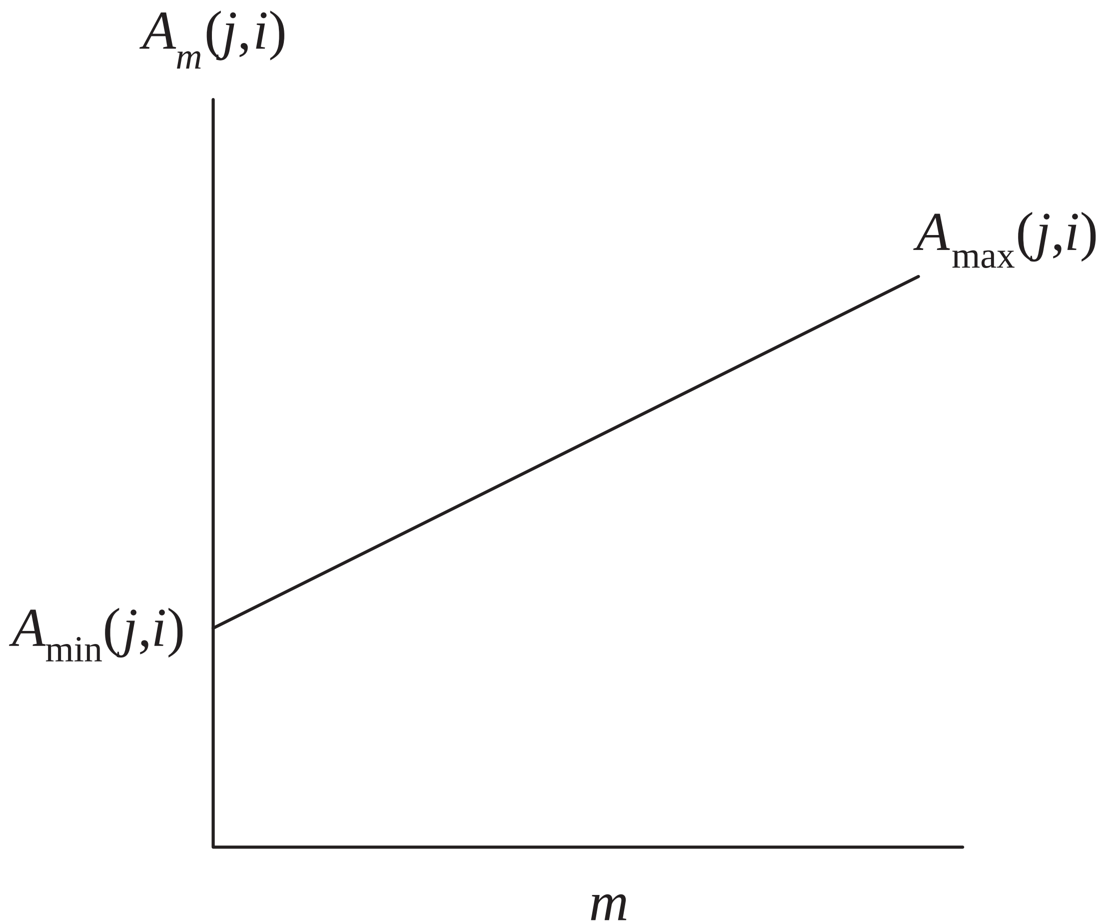

Number of Averages
0
10000
20000
30000
40000
50000
5
10
15
20

Approximation Algorithm for Asian Options (continued)

- But all averages must lie between $A_{\min}(j,i)$ and $A_{\max}(j,i)$.
- Pick $k+1$ equally spaced values in this range and treat them as the true and only running averages:

$$A_m(j,i) \triangleq \left(\frac{k-m}{k}\right) A_{\min}(j,i) + \left(\frac{m}{k}\right) A_{\max}(j,i)$$

for $m = 0, 1, \ldots, k$.


$A_m(j,i)$
$A_{\max}(j,i)$
$A_{\min}(j,i)$
m

Approximation Algorithm for Asian Options (continued)

- Such "bucketing" or "binning " introduces errors, but it works reasonably well in practice.[a]
- A better alternative picks values whose logarithms are equally spaced.[b]
- Still other alternatives are possible, considering the distribution of averages on p. 452.

[a] Hull & White (1993); Ritchken, Sankarasubramanian, & Vijh (1993).
[b] Called log-linear interpolation.

Approximation Algorithm for Asian Options (continued)

- Backward induction calculates the option values at each node for the $k+1$ running averages.
- Suppose the current node is $N(j, i)$ and the running average is a.
- Assume the next node is $N(j+1, i)$, after an up move.
- As the asset price there is $S_0 u^{j+1-i} d^i$, we seek the option value corresponding to the new running average

$$A_{\mathrm{u}} \triangleq \frac{(j+1)\, a + S_0 u^{j+1-i} d^i}{j+2}.$$

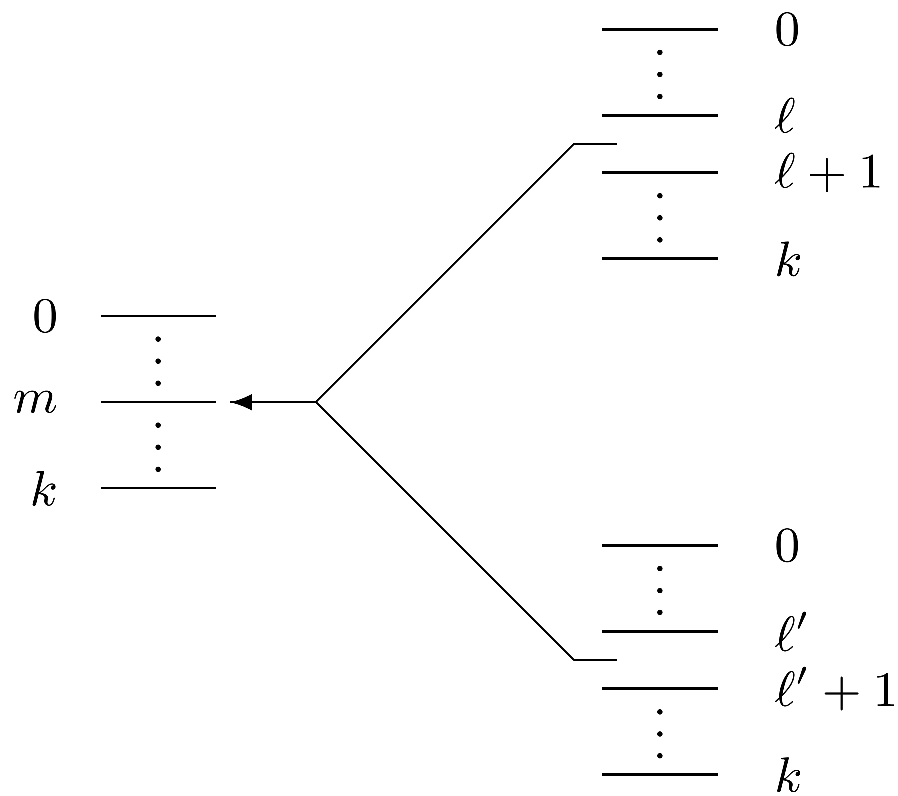
Approximation Algorithm for Asian Options (continued)

- But A_{u} is not likely to be one of the $k+1$ running averages at $N(j+1,i)$!
- Find the 2 running averages that bracket it:

$$A_\ell(j+1,i) \leq A_{\mathrm{u}} < A_{\ell+1}(j+1,i).$$

- In "most" cases, the fastest way to nail ℓ is via

$$\ell = \left\lfloor \frac{A_{\mathrm{u}} - A_{\min}(j+1,i)}{[\,A_{\max}(j+1,i) - A_{\min}(j+1,i)\,]/k} \right\rfloor.$$


0
ℓ
$\ell+1$
k
0
m
k
0
ℓ'
$\ell'+1$
k

Approximation Algorithm for Asian Options (continued)

- But watch out for the rare case where

$$A_{\mathrm{u}} = A_{\ell}(j+1, i)$$

  for some ℓ.

- Also watch out for the case where

$$A_{\mathrm{u}} = A_{\max}(j, i).$$

- Finally, watch out for the degenerate case where $A_0(j+1, i) = \cdots = A_k(j+1, i)$.
  - It happens to the two extreme paths.

Approximation Algorithm for Asian Options (continued)

- Express A_{u} as a linearly interpolated value of the two running averages,

$$A_{\mathrm{u}} = x A_\ell(j+1, i) + (1-x)\, A_{\ell+1}(j+1, i), \quad 0 < x \leq 1.$$

- Obtain the approximate option value given the running average A_{u} via

$$C_{\mathrm{u}} \stackrel{\Delta}{=} x C_\ell(j+1, i) + (1-x)\, C_{\ell+1}(j+1, i).$$

  – $C_\ell(t, s)$ denotes the option value at node $N(t, s)$ with running average $A_\ell(t, s)$.

Approximation Algorithm for Asian Options (continued)

- This interpolation introduces the second source of error.
  - Alternatives to linear interpolation exist.
- The same steps are repeated for the down node $N(j+1, i+1)$ to obtain another approximate option value C_{d}.
- Finally obtain the option value as

$$[\,pC_{\mathrm{u}} + (1-p)\,C_{\mathrm{d}}\,]\,e^{-r\Delta t}.$$

Approximation Algorithm for Asian Options (continued)

- No interpolation is needed for the calculations at time step $n - 1$.[a]
  - The running time is $O(kn^2)$.
    * There are $O(n^2)$ nodes.
    * Each node has $O(k)$ buckets.
  - The option values are simply (for calls):

$$\begin{aligned} C_{\mathrm{u}} &= \max(A_{\mathrm{u}} - X, 0), \\ C_{\mathrm{d}} &= \max(A_{\mathrm{d}} - X, 0). \end{aligned}$$

  - That saves $O(nk)$ calculations.

[a]Not n. Contributed by Mr. Chen, Shih-Hang (`R02723031`) on April 9, 2014.

Approximation Algorithm for Asian Options (concluded)

- Arithmetic average-rate options were assumed to be newly issued: no historical average to deal with.
- This problem can be easily addressed.[a]
- How about the Greeks?[b]

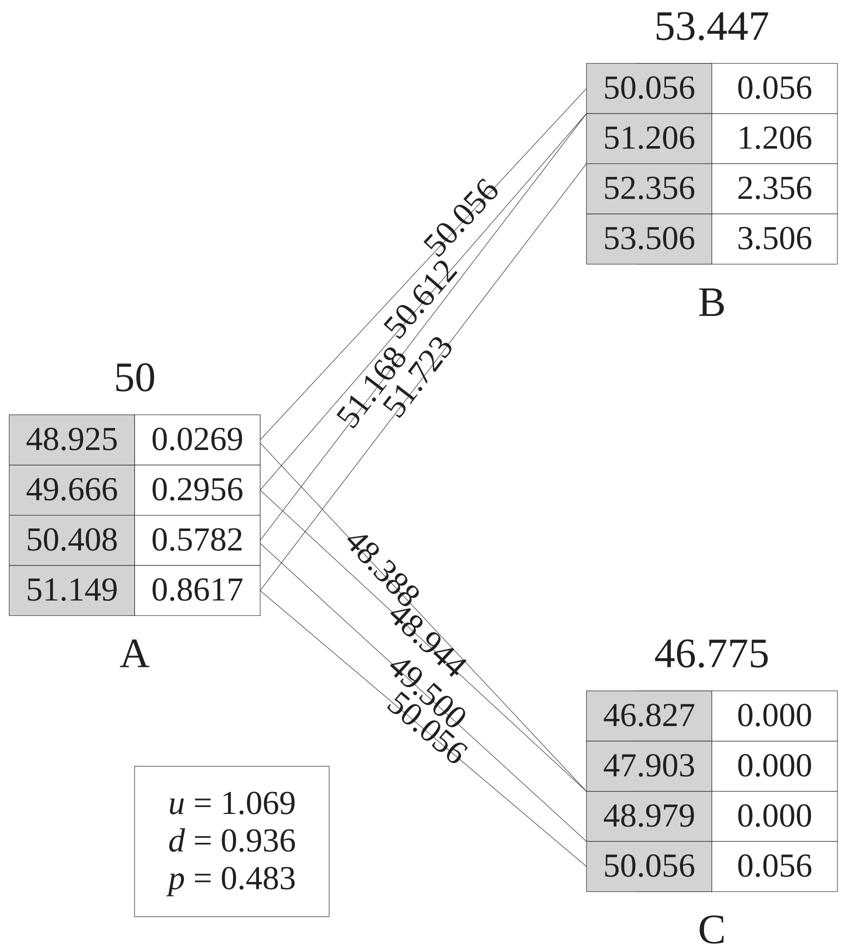
---

[a] See Exercise 11.7.4 of the textbook.

[b] Thanks to lively class discussions on March 31, 2004, and April 9, 2014.

A Numerical Example

- Consider a European arithmetic average-rate call with strike price 50.
- Assume zero interest rate in order to dispense with discounting.
- The minimum running average at node A in the figure on p. 465 is 48.925.
- The maximum running average at node A in the same figure is 51.149.


53.447
50.056 0.056
51.206 1.206
52.356 2.356
53.506 3.506
B
50
48.925 0.0269
49.666 0.2956
50.408 0.5782
51.149 0.8617
A
50.056
50.612
51.168
51.723
48.388
48.944
49.500
50.056
46.775
46.827 0.000
47.903 0.000
48.979 0.000
50.056 0.056
C
u = 1.069
d = 0.936
p = 0.483

A Numerical Example (continued)

- Each node picks $k = 3$ for 4 equally spaced running averages.
- The same calculations are done for node A's successor nodes B and C.
- Suppose node A is 2 periods from the root node.
- Consider the up move from node A with running average 49.666.

A Numerical Example (continued)

- Because the stock price at node B is 53.447, the new running average will be

$$\frac{3 \times 49.666 + 53.447}{4} \approx 50.612.$$

- With 50.612 lying between 50.056 and 51.206 at node B, we solve

$$50.612 = x \times 50.056 + (1 - x) \times 51.206$$

to obtain $x \approx 0.517$.

A Numerical Example (continued)

- The option value corresponding to running average 50.056 at node B is 0.056.
- The option values corresponding to running average 51.206 at node B is 1.206.
- Their contribution to the option value corresponding to running average 49.666 at node A is weighted linearly as

$$x \times 0.056 + (1 - x) \times 1.206 \approx 0.611.$$

A Numerical Example (continued)

- Now consider the down move from node A with running average 49.666.
- Because the stock price at node C is 46.775, the new running average will be
$$\frac{3 \times 49.666 + 46.775}{4} \approx 48.944.$$
- With 48.944 lying between 47.903 and 48.979 at node C, we solve
$$48.944 = x \times 47.903 + (1 - x) \times 48.979$$
to obtain $x \approx 0.033$.

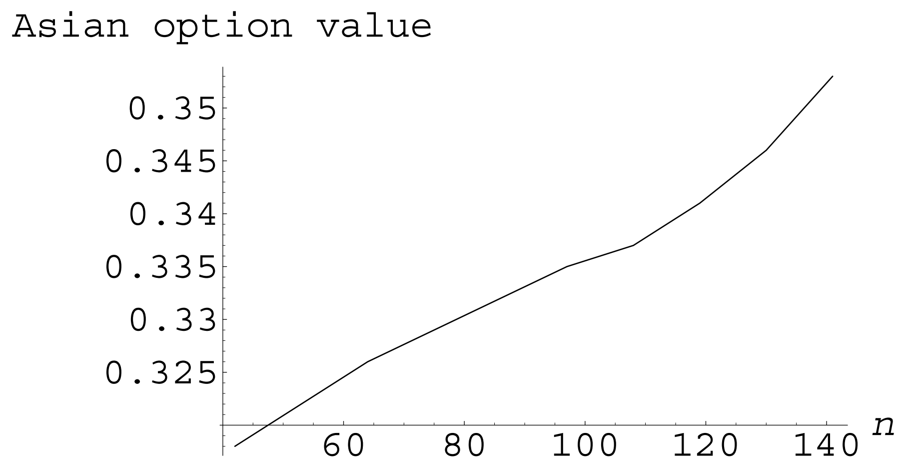
A Numerical Example (concluded)

- The option values corresponding to running averages 47.903 and 48.979 at node C are both 0.0.
- Their contribution to the option value corresponding to running average 49.666 at node A is 0.0.
- Finally, the option value corresponding to running average 49.666 at node A equals

$$p \times 0.611 + (1 - p) \times 0.0 \approx 0.2956,$$

  where $p = 0.483$.
- The remaining three option values at node A can be computed similarly.

Convergence Behavior of the Approximation Algorithm with $k = 50000$[a]



[a] Dai (B82506025, R86526008, D8852600) & Lyuu (2002).

Remarks on Asian Option Pricing

- Asian option pricing is an active research area.
- The above algorithm overestimates the "true" value.[a]
- To guarantee convergence, k needs to grow with n at least.[b]
- There is a convergent approximation algorithm[c] that does away with interpolation with a running time of

$$2^{O(\sqrt{n}\,)}.$$

---

[a] Dai (B82506025, R86526008, D8852600), G. Huang (F83506075), & Lyuu (2002).

[b] Dai (B82506025, R86526008, D8852600), G. Huang (F83506075), & Lyuu (2002).

[c] Dai (B82506025, R86526008, D8852600) & Lyuu (2002, 2004).

Remarks on Asian Option Pricing (continued)

- There is an $O(kn^2)$-time algorithm with an error bound of $O(Xn/k)$ from the naive $O(2^n)$-time binomial tree algorithm in the case of European Asian options.[a]
  - k can be varied for trade-off between time and accuracy.
  - If we pick $k = O(n^2)$, then the error is $O(1/n)$, and the running time is $O(n^4)$.

[a]Aingworth, Motwani (1962–2009), & Oldham (2000).

Remarks on Asian Option Pricing (continued)

- Another approximation algorithm reduces the error to $O(X\sqrt{n}/k)$.[a]
  - It varies the number of buckets per node.
  - If we pick $k = O(n)$, the error is $O(n^{-0.5})$.
  - If we pick $k = O(n^{1.5})$, then the error is $O(1/n)$, and the running time is $O(n^{3.5})$.
- Under "reasonable assumptions," an $O(n^2)$-time algorithm with an error bound of $O(1/n)$ exists.[b]

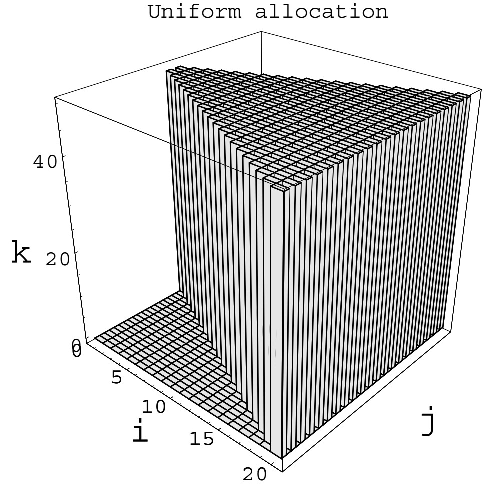
---

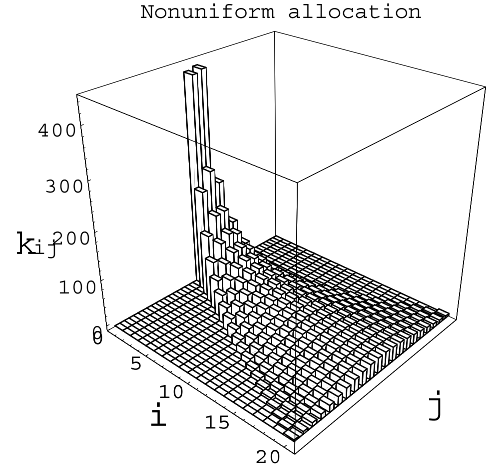
[a] Dai (B82506025, R86526008, D8852600), G. Huang (F83506075), & Lyuu (2002).

[b] Hsu (R7526001, D89922012) & Lyuu (2004).

Remarks on Asian Option Pricing (concluded)

- The basic idea is a nonuniform allocation of running averages instead of a uniform k.
- It strikes a tight balance between error and complexity.





A Grand Comparison[a]

[a]Hsu (R7526001, D89922012) & Lyuu (2004); J. E. Zhang (2001,2003); K. Chen (R92723061) & Lyuu (2006).

X	σ	r	Exact	AA2	AA3	Hsu-Lyuu	Chen-Lyuu
95	0.05	0.05	7.1777275	7.1777244	7.1777279	7.178812	7.177726
100			2.7161745	2.7161755	2.7161744	2.715613	2.716168
105			0.3372614	0.3372601	0.3372614	0.338863	0.337231
95		0.09	8.8088392	8.8088441	8.8088397	8.808717	8.808839
100			4.3082350	4.3082253	4.3082331	4.309247	4.308231
105			0.9583841	0.9583838	0.9583841	0.960068	0.958331
95		0.15	11.0940944	11.0940964	11.0940943	11.093903	11.094094
100			6.7943550	6.7943510	6.7943553	6.795678	6.794354
105			2.7444531	2.7444538	2.7444531	2.743798	2.744406
90	0.10	0.05	11.9510927	11.9509331	11.9510871	11.951610	11.951076
100			3.6413864	3.6414032	3.6413875	3.642325	3.641344
110			0.3312030	0.3312563	0.3311968	0.331348	0.331074
90		0.09	13.3851974	13.3851165	13.3852048	13.385563	13.385190
100			4.9151167	4.9151388	4.9151177	4.914254	4.915075
110			0.6302713	0.6302538	0.6302717	0.629843	0.630064
90		0.15	15.3987687	15.3988062	15.3987860	15.398885	15.398767
100			7.0277081	7.0276544	7.0277022	7.027385	7.027678
110			1.4136149	1.4136013	1.4136161	1.414953	1.413286

A Grand Comparison (concluded)

X	σ	r	Exact	AA2	AA3	Hsu-Lyuu	Chen-Lyuu
90	0.20	0.05	12.5959916	12.5957894	12.5959304	12.596052	12.595602
100			5.7630881	5.7631987	5.7631187	5.763664	5.762708
110			1.9898945	1.9894855	1.9899382	1.989962	1.989242
90		0.09	13.8314996	13.8307782	13.8313482	13.831604	13.831220
100			6.7773481	6.7775756	6.7773833	6.777748	6.776999
110			2.5462209	2.5459150	2.5462598	2.546397	2.545459
90		0.15	15.6417575	15.6401370	15.6414533	15.641911	15.641598
100			8.4088330	8.4091957	8.4088744	8.408966	8.408519
110			3.5556100	3.5554997	3.5556415	3.556094	3.554687
90	0.30	0.05	13.9538233	13.9555691	13.9540973	13.953937	13.952421
100			7.9456288	7.9459286	7.9458549	7.945918	7.944357
110			4.0717942	4.0702869	4.0720881	4.071945	4.070115
90		0.09	14.9839595	14.9854235	14.9841522	14.984037	14.982782
100			8.8287588	8.8294164	8.8289978	8.829033	8.827548
110			4.6967089	4.6956764	4.6969698	4.696895	4.694902
90		0.15	16.5129113	16.5133090	16.5128376	16.512963	16.512024
100			10.2098305	10.2110681	10.2101058	10.210039	10.208724
110			5.7301225	5.7296982	5.7303567	5.730357	5.728161

Forwards, Futures, Futures Options, Swaps

Summon the nations to come to the trial.
Which of their gods can predict the future?
— Isaiah 43:9

The sure fun of the evening
outweighed the uncertain treasure[.]
— Mark Twain (1835–1910),
The Adventures of Tom Sawyer

Terms

- r will denote the riskless interest rate.
- The current time is t.
- The maturity date is T.
- The remaining time to maturity is $\tau \stackrel{\Delta}{=} T - t$ (years).
- The spot price is S.
- The spot price at maturity is S_T.
- The delivery price is X.

Terms (concluded)

- The forward or futures price is F for a newly written contract.
- The value of the contract is f.
- A price with a subscript t usually refers to the price at time t.
- Continuous compounding will be assumed.

Forward Contracts

- *Long* forward contracts are for the purchase of the underlying asset for a certain delivery price on a specific time.
  - Foreign currencies, bonds, corn, etc.
- Ideal for hedging purposes.
- A farmer enters into a forward contract with a food processor to deliver 100,000 bushels of corn for \$2.5 per bushel on September 27, 1995.[a]
- The farmer is assured of a buyer at an acceptable price.
- The processor knows the cost of corn in advance.

[a] The farmer assumes a *short* position.

Forward Contracts (concluded)

- A forward agreement limits both risk and rewards.
  - If the spot price of corn rises on the delivery date, the farmer will miss the opportunity of extra profits.
  - If the price declines, the processor will be paying more than it would.
- Either side has an incentive to default.
- Other problems: The food processor may go bankrupt, the farmer can go bust, the farmer might not be able to harvest 100,000 bushels of corn because of bad weather, the cost of growing corn may skyrocket, etc.

Spot and Forward Exchange Rates

- Let S denote the spot exchange rate.
- Let F denote the forward exchange rate one year from now (both in domestic/foreign terms).
- r_{f} denotes the annual interest rate of the foreign currency.
- r_{ℓ} denotes the annual interest rate of the local currency.
- Arbitrage opportunities will arise unless these four numbers satisfy an equation.

Interest Rate Parity[a]

$$\frac{F}{S} = e^{r_\ell - r_\mathrm{f}}. \tag{59}$$

- A holder of the local currency can do either of:
  - Lend the money in the domestic market to receive e^{r_ℓ} one year from now.
  - Convert local currency for foreign currency, lend for 1 year in foreign market, and convert foreign currency into local currency at the fixed forward exchange rate, F, by selling forward foreign currency now.

[a]Keynes (1923). John Maynard Keynes (1883–1946) was one of the greatest economists in history. The parity broke down in late 2008 (Mancini-Griffoli & Ranaldo, 2013).

Interest Rate Parity (concluded)

- No money changes hand in entering into a forward contract.
- One unit of local currency will hence become $Fe^{r_{\mathrm{f}}}/S$ one year from now in the 2nd case.
- If $Fe^{r_{\mathrm{f}}}/S > e^{r_\ell}$, an arbitrage profit can result from borrowing money in the domestic market and lending it in the foreign market.
- If $Fe^{r_{\mathrm{f}}}/S < e^{r_\ell}$, an arbitrage profit can result from borrowing money in the foreign market and lending it in the domestic market.

Forward Price

- The payoff of a forward contract at maturity is

$$S_T - X.$$

  - Contrast that with call's payoff

$$\max(S_T - X, 0).$$

- Forward contracts do not involve any initial cash flow.
- The forward price F is the delivery price X which makes the forward contract zero valued.
  - That is,

$$f = 0 \quad \text{when} \quad X = F.$$

Forward Price (continued)

Forward Price (concluded)

- The delivery price cannot change because it is written in the contract.
- But the forward price may change after the contract comes into existence.
- So although the value of a forward contract, f, is 0 at the outset, it will fluctuate thereafter.
  - This value is enhanced when the spot price climbs.
  - It is depressed when the spot price declines.
- The forward price also varies with the maturity of the contract.

Forward Price: Underlying Pays No Income

Lemma 12 *For a forward contract on an underlying asset providing no income,*

$$F = Se^{r\tau}. \tag{60}$$

- If $F > Se^{r\tau}$:
  - Borrow S dollars for τ years.
  - Buy the underlying asset.
  - Short the forward contract with delivery price F.

Proof (concluded)

- At maturity:
  - Deliver the asset for F.
  - Use $Se^{r\tau}$ to repay the loan, leaving an arbitrage profit of
$$F - Se^{r\tau} > 0.$$
- If $F < Se^{r\tau}$, do the opposite.

Example: Zero-Coupon Bonds

- r is the annualized 3-month riskless interest rate.
- S is the spot price of the 6-month zero-coupon bond.
- A new 3-month forward contract on that 6-month zero-coupon bond should command a delivery price of $Se^{r/4}$.
- So if $r = 6\%$ and $S = 970.87$, then the delivery price is

$$970.87 \times e^{0.06/4} = 985.54.$$

Example: Options

- Suppose S is the spot price of the *European call* that expires at some time later than T.
- A τ-year forward contract on that call commands a delivery price of $Se^{r\tau}$.
- So it equals the future value of the Black-Scholes formula on p. 303.

Contract Value: The Underlying Pays No Income

The value of a forward contract is

$$f = S - Xe^{-r\tau}. \tag{61}$$

- Consider a portfolio consisting of:
  - One long forward contract;
  - Cash amount $Xe^{-r\tau}$;
  - One short position in the underlying asset.

Contract Value: The Underlying Pays No Income (concluded)

- The cash will grow to X at maturity, which can be used to take delivery of the forward contract.
- The delivered asset will then close out the short position.
- Since the value of the portfolio is zero at maturity, its PV must be zero.[a]
- So a forward contract can be replicated by a long position in the underlying and a loan of $Xe^{-r\tau}$ dollars.

---

[a]Recall p. 221.

Lemma 12 (p. 491) Revisited

- Set $f = 0$ in Eq. (61) on p. 495.
- Then $X = Se^{r\tau}$, the forward price.

Forward Price: Underlying Pays Predictable Income

Lemma 13 *For a forward contract on an underlying asset providing a predictable income with a PV of* I,

$$F = (S - I)\, e^{r\tau}. \tag{62}$$

- If $F > (S - I)\, e^{r\tau}$, borrow S dollars for τ years, buy the underlying asset, and short the forward contract with delivery price F.
- Use the income to repay part of the loan.

The Proof (concluded)

- At maturity, the asset is delivered for F, and $(S - I)\, e^{r\tau}$ is used to repay the *remaining* loan.
- That leaves an arbitrage profit of

$$F - (S - I)\, e^{r\tau} > 0.$$

- If $F < (S - I)\, e^{r\tau}$, reverse the above.

Example

- Consider a 10-month forward contract on a \$50 stock.
- The stock pays a dividend of \$1 every 3 months.
- The forward price is

$$\left(50 - e^{-r_3/4} - e^{-r_6/2} - e^{-3\times r_9/4}\right) e^{r_{10}\times(10/12)}.$$

  - r_i is the annualized i-month interest rate.

Underlying Pays a Continuous Dividend Yield of q

- The value of a forward contract at any time prior to T is[a]

$$f = Se^{-q\tau} - Xe^{-r\tau}. \tag{63}$$

- One consequence of Eq. (63) is that the forward price is

$$F = Se^{(r-q)\,\tau}. \tag{64}$$

---

[a]See p. 160 of the textbook for proof.