Risk

• Surprisingly, the option value is independent of $q$.\(^{a}\)

• Hence it is independent of the expected value of the stock,

\[
qSu + (1 - q) Sd.
\]

• The option value depends on the sizes of price changes, $u$ and $d$, which the investors must agree upon.

• Then the set of possible stock prices is the same whatever $q$ is.

\(^{a}\)More precisely, not directly dependent on $q$. Thanks to a lively class discussion on March 16, 2011.
Pseudo Probability

• After substitution and rearrangement,

\[ hS + B = \frac{\left( \frac{R-d}{u-d} \right) C_u + \left( \frac{u-R}{u-d} \right) C_d}{R}. \]

• Rewrite it as

\[ hS + B = \frac{pC_u + (1-p) C_d}{R}, \]

where

\[ p \triangleq \frac{R-d}{u-d}. \]  \( (34) \)
Pseudo Probability (concluded)

- As $0 < p < 1$, it may be interpreted as probability.

- Alternatively,

$$
\left( \frac{R - d}{u - d} \right) C_u + \left( \frac{u - R}{u - d} \right) C_d
$$

interpolates the value at $SR$ through points $(Su, C_u)$ and $(Sd, C_d)$. 
Risk-Neutral Probability

• The expected rate of return for the stock is equal to the riskless rate $\hat{r}$ under $p$ as

$$pS_u + (1 - p) S_d = RS. \quad (35)$$

• The expected rates of return of all securities must be the riskless rate when investors are risk-neutral.

• For this reason, $p$ is called the risk-neutral probability.

• The value of an option is the expectation of its discounted future payoff in a risk-neutral economy.

• So the rate used for discounting the FV is the riskless rate\(^a\) in a risk-neutral economy.

\(^a\)Recall the question on p. 240.
Option on a Non-Dividend-Paying Stock: Multi-Period

- Consider a call with two periods remaining before expiration.

- Under the binomial model, the stock can take on 3 possible prices at time two: $S_{uu}$, $S_{ud}$, and $S_{dd}$.
  - There are 4 paths.
  - But the tree combines or recombines; hence there are only 3 terminal prices.

- At any node, the next two stock prices only depend on the current price, not the prices of earlier times.$^a$

---

$^a$It is Markovian.
Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- Let \( C_{uu} \) be the call’s value at time two if the stock price is \( S_{uu} \).
- Thus,
  \[
  C_{uu} = \max(0, S_{uu} - X).
  \]
- \( C_{ud} \) and \( C_{dd} \) can be calculated analogously,
  \[
  C_{ud} = \max(0, S_{ud} - X),
  \]
  \[
  C_{dd} = \max(0, S_{dd} - X).
  \]
\[ C_{uu} = \max(0, S_{uu} - X) \]
\[ C_{ud} = \max(0, S_{ud} - X) \]
\[ C_{dd} = \max(0, S_{dd} - X) \]
Option on a Non-Dividend-Paying Stock: Multi-Period
(continued)

- The call values at time 1 can be obtained by applying the same logic:

\[
C_u = \frac{pC_{uu} + (1 - p)C_{ud}}{R}, \\
C_d = \frac{pC_{ud} + (1 - p)C_{dd}}{R}.
\]

- Deltas can be derived from Eq. (32) on p. 251.
- For example, the delta at \( C_u \) is

\[
\frac{C_{uu} - C_{ud}}{S_{uu} - S_{ud}}.
\]
Option on a Non-Dividend-Paying Stock: Multi-Period (concluded)

- We now reach the current period.
- Compute

\[
\frac{pC_u + (1 - p)C_d}{R}
\]

as the option price.
- The values of delta \( h \) and \( B \) can be derived from Eqs. (32)–(33) on p. 251.
Early Exercise

- Since the call will not be exercised at time 1 even if it is American, \( C_u \geq Su - X \) and \( C_d \geq Sd - X \).

- Therefore,

\[
\begin{align*}
    hS + B &= \frac{pC_u + (1-p)C_d}{R} \\
    &= \frac{[pu + (1-p)d]S - X}{R} \\
    &= S - \frac{X}{R} > S - X.
\end{align*}
\]

- The call again will not be exercised at present.\(^a\)

- So

\[
C = hS + B = \frac{pC_u + (1-p)C_d}{R}.
\]

\(^a\)Consistent with Theorem 5 (p. 234).
Backward Induction\footnote{Ernst Zermelo (1871–1953).}

- The above expression calculates $C$ from the two successor nodes $C_u$ and $C_d$ and none beyond.
- The same computation happened at $C_u$ and $C_d$, too, as demonstrated in Eq. (36) on p. 262.
- This recursive procedure is called backward induction.
- $C$ equals

\[
\begin{align*}
&\left[ p^2 C_{uu} + 2p(1-p) C_{ud} + (1-p)^2 C_{dd} \right] \left(1/R^2\right) \\
= &\left[ p^2 \max (0, S_{u^2} - X) + 2p(1-p) \max (0, S_{ud} - X) \\
&+ (1-p)^2 \max (0, S_{d^2} - X) \right] / R^2.
\end{align*}
\]
Backward Induction (continued)

• In the $n$-period case,

$$C = \sum_{j=0}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \times \max (0, Su^j d^{n-j} - X) \frac{R^n}{R^n}.$$  

- The value of a call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.

• Similarly,

$$P = \sum_{j=0}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \times \max (0, X - Su^j d^{n-j}) \frac{R^n}{R^n}.$$
Backward Induction (concluded)

- Note that

\[ p_j \triangleq \frac{\binom{n}{j} p^j (1 - p)^{n-j}}{R^n} \]

is the state price\(^a\) for the state \(S u^j d^{n-j}\), \(j = 0, 1, \ldots, n\).

- In general,

\[
\text{option price} = \sum_j (p_j \times \text{payoff at state } j).
\]

\(^a\)Recall p. 212. One can obtain the undiscounted state price \(\binom{n}{j} p^j (1 - p)^{n-j}\)—the risk-neutral probability—for the state \(S u^j d^{n-j}\) with \((X_M - X_L)^{-1}\) units of the butterfly spread where \(X_L = S u^j - 1 d^{n-j+1}\), \(X_M = S u^j d^{n-j}\), and \(X_H = S u^{j-1+1} d^{n-j-1}\). See Bahra (1997).
Risk-Neutral Pricing Methodology

• Every derivative can be priced as if the economy were risk-neutral.

• For a European-style derivative with the terminal payoff function \( D \), its value is

\[
e^{-\hat{r}n} E^\pi[D].
\]  

(37)

– \( E^\pi \) means the expectation is taken under the risk-neutral probability.

• The “equivalence” between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) fundamental theorem of asset pricing.\(^a\)

\(^a\)Dybvig & Ross (1987).
Self-Financing

• Delta changes over time.

• The maintenance of an equivalent portfolio is dynamic.

• But it does not depend on predicting future stock prices.

• The portfolio’s value at the end of the current period is precisely the amount needed to set up the next portfolio.

• The trading strategy is self-financing because there is neither injection nor withdrawal of funds throughout.\(^a\)
  
  – Changes in value are due entirely to capital gains.

\(^a\)Except at the beginning, of course, when the option premium is paid before the replication starts.
Binomial Distribution

- Denote the binomial distribution with parameters $n$ and $p$ by

$$b(j; n, p) \triangleq \binom{n}{j} p^j (1 - p)^{n-j} = \frac{n!}{j! (n - j)!} p^j (1 - p)^{n-j}.$$  

- $n! = 1 \times 2 \times \cdots \times n$.  
- Convention: $0! = 1$.

- Suppose you flip a coin $n$ times with $p$ being the probability of getting heads.

- Then $b(j; n, p)$ is the probability of getting $j$ heads.
The Binomial Option Pricing Formula

- The stock prices at time $n$ are
  
  $$Su^n, Su^{n-1}d, \ldots, Sd^n.$$  

- Let $a$ be the minimum number of upward price moves for the call to finish in the money.

- So $a$ is the smallest nonnegative integer $j$ such that
  
  $$Su^j d^{n-j} \geq X,$$

  or, equivalently,

  $$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil.$$
The Binomial Option Pricing Formula (concluded)

- Hence,

\[
C = \sum_{j=a}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \left( S u^j d^{n-j} - X \right) \frac{R^n}{R^n} \\
= S \sum_{j=a}^{n} \binom{n}{j} (pu)^j (1 - p)^{n-j} \frac{R^n}{R^n} \\
- \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \\
= S \sum_{j=a}^{n} b(j; n, pu/R) - X e^{-\hat{v}n} \sum_{j=a}^{n} b(j; n, p). \tag{39}
\]
Numerical Examples

- A non-dividend-paying stock is selling for $160.
- $u = 1.5$ and $d = 0.5$.
- $r = 18.232\%$ per period ($R = e^{0.18232} = 1.2$).
  - Hence $p = (R - d)/(u - d) = 0.7$.
- Consider a European call on this stock with $X = 150$ and $n = 3$.
- The call value is $85.069$ by backward induction.
- Or, the PV of the expected payoff at expiration:
  \[
  \frac{390 	imes 0.343 + 30 	imes 0.441 + 0 	imes 0.189 + 0 	imes 0.027}{(1.2)^3} = 85.069.
  \]
Binomial process for the stock price
(probabilities in parentheses)

Binomial process for the call price
(hedge ratios in parentheses)
Numerical Examples (continued)

• Mispricing leads to arbitrage profits.

• Suppose the option is selling for $90 instead.

• Sell the call for $90.

• Invest $85.069 in the replicating portfolio with 0.82031 shares of stock as required by the delta.

• Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.

• The fund that remains,

\[ 90 - 85.069 = 4.931 \text{ dollars}, \]

is the arbitrage profit, as we will see.
Numerical Examples (continued)

Time 1:

• Suppose the stock price moves to $240.

• The new delta is 0.90625.

• Buy

\[
0.90625 - 0.82031 = 0.08594
\]

more shares at the cost of \(0.08594 \times 240 = 20.6256\) dollars financed by borrowing.

• Debt now totals \(20.6256 + 46.1806 \times 1.2 = 76.04232\) dollars.
Numerical Examples (continued)

- The trading strategy is self-financing because the portfolio has a value of
  \[0.90625 \times 240 - 76.04232 = 141.45768.\]

- It matches the corresponding call value by backward induction!\(^a\)

\(^a\)See p. 275.
Numerical Examples (continued)

Time 2:

- Suppose the stock price plunges to $120.
- The new delta is 0.25.
- Sell $0.90625 - 0.25 = 0.65625$ shares.
- This generates an income of $0.65625 \times 120 = 78.75$ dollars.
- Use this income to reduce the debt to

\[
76.04232 \times 1.2 - 78.75 = 12.5
\]

dollars.
Numerical Examples (continued)

Time 3 (the case of rising price):

- The stock price moves to $180.
- The call we wrote finishes in the money.
- Close out the call’s short position by buying back the call or buying a share of stock for delivery.
- This results in a loss of $180 – $150 = 30 dollars.
- Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.
- It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.
Numerical Examples (concluded)

Time 3 (the case of declining price):

- The stock price moves to $60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of

\[ 0.25 \times 60 = 15 \]

dollars.
- Use it to repay the debt of \(12.5 \times 1.2 = 15\) dollars.
Applications besides Exploiting Arbitrage Opportunities

- Replicate an option using stocks and bonds.
  - Set up a portfolio to replicate the call with $85.069.

- Hedge the options we issued.
  - Use $85.069 to set up a portfolio to replicate the call to counterbalance its values exactly.

- Without hedge, one may end up forking out $390 in the worst case (see p. 275)!  

---

a Thanks to a lively class discussion on March 16, 2011.

b Hedging and replication are mirror images.

c Thanks to a lively class discussion on March 16, 2016.
Binomial Tree Algorithms for European Options

- The BOPM implies the binomial tree algorithm that applies backward induction.

- The total running time is $O(n^2)$ because there are $\sim n^2/2$ nodes.

- The memory requirement is $O(n^2)$.
  - Can be easily reduced to $O(n)$ by reusing space.

- To find the hedge ratio, apply formula (32) on p. 251.

- To price European puts, simply replace the payoff.

\footnote{But watch out for the proper updating of array entries.}
Further Time Improvement for Calls
Optimal Algorithm

- We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.

- Note that

$$b(j; n, p) = \frac{p(n - j + 1)}{(1 - p) j} b(j - 1; n, p).$$
Optimal Algorithm (continued)

- The following program computes $b(j; n, p)$ in $b[j]$: 
- It runs in $O(n)$ steps.

1: $b[a] := \binom{n}{a} p^a (1 - p)^{n-a}$;
2: for $j = a + 1, a + 2, \ldots, n$ do
3: \hspace{1em} $b[j] := b[j - 1] \times p \times (n - j + 1) / ((1 - p) \times j)$;
4: end for
Optimal Algorithm (concluded)

- With the $b(j; n, p)$ available, the risk-neutral valuation formula (38) on p. 273 is trivial to compute.

- But we only need a single variable to store the $b(j; n, p)$s as they are being sequentially computed.

- This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.

- This forward-induction approach cannot be applied to American options because of early exercise.

- So binomial tree algorithms for American options usually run in $O(n^2)$ time.
The Bushy Tree

\[ S \rightarrow Su, Su^2, Su^3, Su^2d, Su^3d, Su^2d^2, Su^3d^2, Su^2d^3, Su^3d^3, \ldots \]

\[ 2^n \text{ nodes} \]

\[ Su^n, Su^{n-1}d, Su^n-1d \]
Toward the Black-Scholes Formula

- The binomial model seems to suffer from two unrealistic assumptions.
  - The stock price takes on only two values in a period.
  - Trading occurs at discrete points in time.
- As $n$ increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.$^a$
- Need to calibrate the BOPM’s parameters $u$, $d$, and $R$ to make it converge to the continuous-time model.
- We now skim through the proof.

$^a$Continuous-time trading may create arbitrage opportunities in practice (Budish, Cramton, & Shim, 2015)!
Toward the Black-Scholes Formula (continued)

- Let $\tau$ denote the time to expiration of the option measured in years.
- Let $r$ be the continuously compounded annual rate.
- With $n$ periods during the option’s life, each period represents a time interval of $\tau/n$.
- Need to adjust the period-based $u$, $d$, and interest rate $\hat{r}$ to match the empirical results as $n \to \infty$. 
Toward the Black-Scholes Formula (continued)

• First, $\hat{r} = r\tau/n$.
  - Each period is $\Delta t = \tau/n$ years long.
  - The period gross return $R = e^{\hat{r}}$.

• Let

$$\hat{\mu} \triangleq \frac{1}{n} E \left[ \ln \frac{S_\tau}{S} \right]$$

denote the expected value of the continuously compounded rate of return per period of the BOPM.

• Let

$$\hat{\sigma}^2 \triangleq \frac{1}{n} \text{Var} \left[ \ln \frac{S_\tau}{S} \right]$$

denote the variance of that return.
Toward the Black-Scholes Formula (continued)

• Under the BOPM, it is not hard to show that

\[ \hat{\mu} = q \ln(u/d) + \ln d, \]
\[ \hat{\sigma}^2 = q(1 - q) \ln^2(u/d). \]

• Assume the stock’s true continuously compounded rate of return over \( \tau \) years has mean \( \mu \tau \) and variance \( \sigma^2 \tau \).

• Call \( \sigma \) the stock’s (annualized) volatility.

\(^a\)It follows the Bernoulli distribution.
Toward the Black-Scholes Formula (continued)

• The BOPM converges to the distribution only if

\[ n \hat{\mu} = n \left[ q \ln(u/d) + \ln d \right] \to \mu \tau, \]  
\( n \hat{\sigma}^2 = nq(1 - q) \ln^2(u/d) \to \sigma^2 \tau. \)  

(40)  
(41)

• We need one more condition to have a solution for \( u, d, q. \)
Toward the Black-Scholes Formula (continued)

• Impose

\[ ud = 1. \]

– It makes nodes at the same horizontal level of the tree have identical price (review p. 285).
– Other choices are possible (see text).

• Exact solutions for \( u, d, q \) are feasible if Eqs. (40)–(41) are replaced by equations: 3 equations for 3 variables.\(^a\)

\(^a\)Chance (2008).
Toward the Black-Scholes Formula (continued)

- The above requirements can be satisfied by

\[ u = e^{\sigma \sqrt{\Delta t}}, \quad d = e^{-\sigma \sqrt{\Delta t}}, \quad q = \frac{1}{2} + \frac{1}{2} \frac{\mu}{\sigma} \sqrt{\Delta t}. \] (42)

- With Eqs. (42), it can be checked that

\[ n\hat{\mu} = \mu \tau, \]

\[ n\hat{\sigma}^2 = \left[ 1 - \left( \frac{\mu}{\sigma} \right)^2 \Delta t \right] \sigma^2 \tau \rightarrow \sigma^2 \tau. \]

- With the above choice, even if \( \sigma \) is not calibrated correctly, the mean is still matched!\(^a\)

\(^a\) Recall Eq. (35) on p. 257. So \( u \) and \( d \) are related to volatility exclusively in the CRR model. Both are independent of \( r \) and \( \mu \).
Toward the Black-Scholes Formula (continued)

• The choices (42) result in the CRR binomial model.\textsuperscript{a}
  
  – Black (1992), “This method is probably used more than the original formula in practical situations.”
  
  – OptionMetrics’s (2015) IvyDB uses the CRR model.\textsuperscript{b}

• The CRR model is best seen in logarithmic price:

\[
\ln S \rightarrow \begin{cases} 
\ln S + \sigma \sqrt{\Delta t}, & \text{up move}, \\
\ln S - \sigma \sqrt{\Delta t}, & \text{down move}.
\end{cases}
\]

\textsuperscript{a}Cox, Ross, & Rubinstein (1979).
Toward the Black-Scholes Formula (continued)

• The no-arbitrage inequalities \( d < R < u \) may not hold under Eqs. (42) on p. 296 or Eq. (34) on p. 255.
  – If this happens, the probabilities lie outside \([0, 1]\).\(^a\)

• The problem disappears when \( n \) satisfies \( e^{\sigma \sqrt{\Delta t}} > e^{r \Delta t} \), i.e., when

\[
  n > \frac{r^2}{\sigma^2} \tau. \tag{43}
\]

  – So it goes away if \( n \) is large enough.
  – Other solutions can be found in the textbook\(^b\) or will be presented later.

\(^a\)Many papers and programs forget to check this condition!
\(^b\)See Exercise 9.3.1 of the textbook.
Toward the Black-Scholes Formula (continued)

- The central limit theorem says $\ln(S_\tau/S)$ converges to $N(\mu \tau, \sigma^2 \tau)$.\(^a\)

- So $\ln S_\tau$ approaches $N(\mu \tau + \ln S, \sigma^2 \tau)$.

- Conclusion: $S_\tau$ has a lognormal distribution in the limit.

\(^a\)The normal distribution with mean $\mu \tau$ and variance $\sigma^2 \tau$. As our probabilities depend on $n$, this argument is heuristic.
Toward the Black-Scholes Formula (continued)

Lemma 10 The continuously compounded rate of return \( \ln(S_\tau/S) \) approaches the normal distribution with mean \( (r - \sigma^2/2)\tau \) and variance \( \sigma^2\tau \) in a risk-neutral economy.

- Let \( q \) equal the risk-neutral probability
  \[
  p \triangleq \frac{(e^{r\tau/n} - d)}{(u - d)}.
  \]

- Let \( n \to \infty. \)

- Then \( \mu = r - \sigma^2/2. \)

\(^a\)See Lemma 9.3.3 of the textbook.
Toward the Black-Scholes Formula (continued)

- The expected stock price at expiration in a risk-neutral economy is\footnote{By Lemma 10 (p. 301) and Eq. (29) on p. 180.}

\[ Se^{r\tau}. \]

- The stock’s expected annual rate of return is thus the riskless rate \( r \).
  - By rate of return we mean \( (1/\tau) \ln E[S_\tau/S] \) (arithmetic average rate of return) not \( (1/\tau)E[\ln(S_\tau/S)] \) (geometric average rate of return).
  - The latter would give \( r - \sigma^2/2 \) by Lemma 10.
Toward the Black-Scholes Formula (continued)\textsuperscript{a}

Theorem 11 (The Black-Scholes Formula, 1973)

\begin{align*}
C &= SN(x) - Xe^{-r\tau}N(x - \sigma \sqrt{\tau}), \\
P &= Xe^{-r\tau}N(-x + \sigma \sqrt{\tau}) - SN(-x),
\end{align*}

where

\begin{equation*}
x \triangleq \frac{\ln(S/X) + (r + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}.
\end{equation*}

\textsuperscript{a}On a United flight from San Francisco to Tokyo on March 7, 2010, a real-estate manager mentioned this formula to me!
Toward the Black-Scholes Formula (concluded)

• See Eq. (39) on p. 273 for the meaning of $x$.

• See Exercise 13.2.12 of the textbook for an interpretation of the probability associated with $N(x)$ and $N(-x)$.
BOPM and Black-Scholes Model

• The Black-Scholes formula needs 5 parameters: $S$, $X$, $\sigma$, $\tau$, and $r$.

• Binomial tree algorithms take 6 inputs: $S$, $X$, $u$, $d$, $\hat{r}$, and $n$.

• The connections are

\[
\begin{align*}
    u &= e^{\sigma \sqrt{\tau/n}}, \\
    d &= e^{-\sigma \sqrt{\tau/n}}, \\
    \hat{r} &= r\tau/n.
\end{align*}
\]

– This holds for the CRR model as well.
- $S = 100$, $X = 100$ (left), and $X = 95$ (right).
BOPM and Black-Scholes Model (concluded)

- The binomial tree algorithms converge reasonably fast.
- The error is $O(1/n)$.\(^a\)
- Oscillations are inherent, however.
- Oscillations can be dealt with by judicious choices of $u$ and $d$.\(^b\)

\(^a\)F. Diener & M. Diener (2004); L. Chang & Palmer (2007).
\(^b\)See Exercise 9.3.8 of the textbook.
Implied Volatility

- Volatility is the sole parameter not directly observable.

- The Black-Scholes formula can be used to compute the market’s opinion of the volatility.\(^a\)
  - Solve for \(\sigma\) given the option price, \(S\), \(X\), \(\tau\), and \(r\) with numerical methods.
  - How about American options?

\(^a\)Implied volatility is hard to compute when \(\tau\) is small (why?).
Implied Volatility (concluded)

- Implied volatility is
  the wrong number to put in the wrong formula to get the right price of plain-vanilla options.\textsuperscript{a}

- Just think of it as an alternative to quoting option prices.

- Implied volatility is often preferred to historical volatility in practice.
  - Using the historical volatility is like driving a car with your eyes on the rearview mirror?\textsuperscript{b}

\textsuperscript{a}Rebonato (2004).
\textsuperscript{b}E.g., 1:16:23 of https://www.youtube.com/watch?v=8TJQhQ2GZ0Y
Problems; the Smile

- Options written on the same underlying asset usually do not produce the same implied volatility.

- A typical pattern is a “smile” in relation to the strike price.
  - The implied volatility is lowest for at-the-money options.
  - It becomes higher the further the option is in- or out-of-the-money.

- Other patterns have also been observed.
The underlying Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) closed at 8132. Plot supplied by Mr. Lok, U Hou (D99922028) on December 6, 2017.
Tackling the Smile

• To address this issue, volatilities are often combined to produce a composite implied volatility.

• This practice is not sound theoretically.

• The existence of different implied volatilities for options on the same underlying asset shows the Black-Scholes model cannot be literally true.
Binomial Tree Algorithms for American Puts

- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs
  \[ \max(0, X - S u^j d^{m-j}) \]
  and applies backward induction.
- At each intermediate node, compare the payoff if exercised and the \textit{continuation value}.
- Keep the larger one.
Bermudan Options

• Some American options can be exercised only at discrete time points instead of continuously.

• They are called Bermudan options.

• Their pricing algorithm is identical to that for American options.

• But early exercise is considered for only those nodes when early exercise is permitted.
Time-Dependent Volatility\textsuperscript{a}

- Suppose the (instantaneous) volatility can change over time but otherwise predictable: $\sigma(t)$ instead of $\sigma$.

- In the limit, the variance of $\ln(S_\tau/S)$ is

\[
\int_0^\tau \sigma^2(t) \, dt
\]

rather than $\sigma^2 \tau$.

- The annualized volatility to be used in the Black-Scholes formula should now be

\[
\sqrt{\frac{\int_0^\tau \sigma^2(t) \, dt}{\tau}}.
\]

\textsuperscript{a}Merton (1973).
Time-Dependent Instantaneous Volatility (concluded)

• For the binomial model, $u$ and $d$ depend on time:

\[
\begin{align*}
    u &= e^{\sigma(t)\sqrt{\tau/n}}, \\
    d &= e^{-\sigma(t)\sqrt{\tau/n}}.
\end{align*}
\]

• But how to make the binomial tree combine?\(^a\)

\(^a\)Amin (1991); C. I. Chen (R98922127) (2011).
Volatility (1990–2016)\textsuperscript{a}

\textsuperscript{a}Supplied by Mr. Lok, U Hou (D99922028) on July 17, 2017.
Time-Dependent Short Rates

- Suppose the short rate (i.e., the one-period spot rate or forward rate) changes over time but predictable.

- The annual riskless rate $r$ in the Black-Scholes formula should be the spot rate with a time to maturity equal to $\tau$.

- In other words,

$$r = \frac{\sum_{i=0}^{n-1} r_i}{\tau},$$

where $r_i$ is the continuously compounded short rate measured in periods for period $i$.\(^a\)

- Will the binomial tree fail to combine?

\(^a\)That is, one-period forward rate.
Trading Days and Calendar Days

- Interest accrues based on the calendar day.
- But $\sigma$ is usually calculated based on trading days only.
  - Stock price seems to have lower volatilities when the exchange is closed.\(^a\)

- How to harmonize these two different times into the Black-Scholes formula and binomial tree algorithms?\(^b\)

\(^a\)Fama (1965); K. French (1980); K. French & Roll (1986).
\(^b\)Recall p. 162 about dating issues.
Trading Days and Calendar Days (continued)

- Think of $\sigma$ as measuring the *annualized* volatility of stock price *one year from now*.

- Suppose a year has $m$ (say 253) trading days.

- We can replace $\sigma$ in the Black-Scholes formula with$^a$
  \[
  \sigma \sqrt{\frac{365}{m}} \times \frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}.
  \]

Trading Days and Calendar Days (concluded)

- This works only for European options.

- How about binomial tree algorithms?\textsuperscript{a}

\textsuperscript{a}Contributed by Mr. Lu, Zheng-Liang (D00922011) in 2015.