Bond Price Volatility
“Well, Beethoven, what is this?”
— Attributed to Prince Anton Esterházy
Price Volatility

- Volatility measures how bond prices respond to interest rate changes.
- It is key to the risk management of interest rate-sensitive securities.
Price Volatility (concluded)

- What is the sensitivity of the percentage price change to changes in interest rates?

- Define price volatility by

  \[- \frac{\partial P}{\partial y} \frac{1}{P} \]  

  \((14)\)
Price Volatility of Bonds

• The price volatility of a level-coupon bond is

\[
- \frac{(C/y) n - (C/y^2) ((1 + y)^{n+1} - (1 + y))}{(C/y) ((1 + y)^{n+1} - (1 + y)) + F(1 + y)} - nF.
\]

- $F$ is the par value.
- $C$ is the coupon payment per period.
- Formula can be simplified a bit with $C = Fc/m$.

• For the above bond,

\[
- \frac{\partial P}{\partial y} \frac{P}{P} > 0.
\]
Macaulay Duration

- The Macaulay duration (MD) is a weighted average of the times to an asset’s cash flows.

- The weights are the cash flows’ PVs divided by the asset’s price.

- Formally,

\[ \text{MD} \triangleq \frac{1}{P} \sum_{i=1}^{n} \frac{C_i}{(1+y)^i} i. \]

- The Macaulay duration, in periods, is equal to

\[ \text{MD} = -(1+y) \frac{\partial P}{\partial y} \frac{1}{P}. \]  

\[ (15) \]

\(^a\)Macaulay (1938).
MD of Bonds

- The MD of a level-coupon bond is

\[
MD = \frac{1}{P} \left[ \sum_{i=1}^{n} \frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n} \right].
\] (16)

- It can be simplified to

\[
MD = \frac{c(1+y) [(1+y)^n - 1] + ny(y - c)}{cy [(1+y)^n - 1] + y^2},
\]

where \( c \) is the period coupon rate.

- The MD of a zero-coupon bond equals \( n \), its term to maturity.

- The MD of a level-coupon bond is less than \( n \).
Remarks

• Formulas (15) on p. 96 and (16) on p. 97 hold only if the coupon \( C \), the par value \( F \), and the maturity \( n \) are all independent of the yield \( y \).
  - That is, if the cash flow is independent of yields.

• To see this point, suppose the market yield declines.

• The MD will be lengthened.

• But for securities whose maturity actually decreases as a result, the price volatility\(^a\) may decrease.

\(^a\)As originally defined in formula (14) on p. 94.
How Not To Think about MD

- The MD has its origin in measuring the length of time a bond investment is outstanding.
- But it should be seen mainly as measuring price volatility.
- Duration of a security can be longer than its maturity or negative!
- Neither makes sense under the maturity interpretation.
- Many, if not most, duration-related terminology can only be comprehended as measuring volatility.
Conversion

- For the MD to be year-based, modify formula (16) on p. 97 to

\[
\frac{1}{P} \left[ \sum_{i=1}^{n} \frac{i}{k} \frac{C}{i} \left(1 + \frac{y}{k}\right)^i + \frac{n}{k} \frac{F}{k} \left(1 + \frac{y}{k}\right)^n \right],
\]

where \( y \) is the annual yield and \( k \) is the compounding frequency per annum.

- Formula (15) on p. 96 also becomes

\[
\text{MD} = - \left(1 + \frac{y}{k}\right) \frac{\partial P}{\partial y} \frac{1}{P}.
\]

- By definition, \( \text{MD (in years)} = \frac{\text{MD (in periods)}}{k} \).
Modified Duration

- Modified duration is defined as
  
  \[ \text{modified duration} \triangleq -\frac{\partial P}{\partial y} \frac{1}{P} = \frac{\text{MD}}{(1+y)}. \]  \hspace{1cm} (17)

  - Modified duration equals MD under continuous compounding.

- By the Taylor expansion,
  
  percent price change \approx -\text{modified duration} \times \text{yield change}. 

Example

• Consider a bond whose modified duration is 11.54 with a yield of 10%.

• If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

\[-11.54 \times 0.001 = -0.01154 = -1.154\%.\]
Modified Duration of a Portfolio

- By calculus, the modified duration of a portfolio equals

\[ \sum_i \omega_i D_i. \]

- \( D_i \) is the modified duration of the \( i \)th asset.
- \( \omega_i \) is the market value of that asset expressed as a percentage of the market value of the portfolio.
Effective Duration

• Yield changes may alter the cash flow or the cash flow may be too complex for simple formulas.
• We need a general numerical formula for volatility.
• The effective duration is defined as

\[ \frac{P_- - P_+}{P_0(y_+ - y_-)}. \]

- \( P_- \) is the price if the yield is decreased by \( \Delta y \).
- \( P_+ \) is the price if the yield is increased by \( \Delta y \).
- \( P_0 \) is the initial price, \( y \) is the initial yield.
- \( \Delta y \) is small.
\[ P_+ \quad P_0 \quad P_- \]

\[ y_- \quad y \quad y_+ \]
Effective Duration (concluded)

• One can compute the effective duration of just about any financial instrument.

• An alternative is to use

\[
\frac{P_0 - P_+}{P_0 \Delta y}.
\]

  – More economical but theoretically less accurate.
The Practices

• Duration is usually expressed in percentage terms — call it $D\%$ — for quick mental calculation.\(^a\)

• The percentage price change expressed in percentage terms is then approximated by

$$-D\% \times \Delta r$$

when the yield increases instantaneously by $\Delta r\%$.

  - Price will drop by 20% if $D\% = 10$ and $\Delta r = 2$ because $10 \times 2 = 20$.

• $D\%$ in fact equals modified duration (prove it!).

\(^a\)Neftci (2008), “Market professionals do not like to use decimal points.”
Hedging

• Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.

• Define dollar duration as

\[
\text{modified duration} \times \text{price} = -\frac{\partial P}{\partial y}.
\]

• The approximate dollar price change is

\[
\text{price change} \approx -\text{dollar duration} \times \text{yield change}.
\]

• One can hedge a bond portfolio with a dollar duration \( D \) by bonds with a dollar duration \( -D \).
Convexity

- Convexity is defined as

\[ \text{convexity (in periods)} \triangleq \frac{\partial^2 P}{\partial y^2} \frac{1}{P}. \]

- The convexity of a level-coupon bond is positive (prove it!).

- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude (see plot next page).

- So between two bonds with the same price and duration, the one with a higher convexity is more valuable.\(^a\)

\(^a\)Do you spot a problem here (Christensen & Sørensen, 1994)?
Convexity (concluded)

- Suppose there are $k$ periods per annum.
- Convexity measured in periods and convexity measured in years are related by

$$\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}.$$
Use of Convexity

• The approximation \( \frac{\Delta P}{P} \approx -\text{duration} \times \text{yield change} \) works for small yield changes.

• For larger yield changes, use

\[
\frac{\Delta P}{P} \approx \frac{\partial P}{\partial y} \frac{1}{P} \Delta y + \frac{1}{2} \frac{\partial^2 P}{\partial y^2} \frac{1}{P} (\Delta y)^2
\]

\[= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.\]

• Recall the figure on p. 110.
The Practices

• Convexity is usually expressed in percentage terms — call it $C\%$ — for quick mental calculation.

• The percentage price change expressed in percentage terms is approximated by

$$-D\% \times \Delta r + C\% \times (\Delta r)^2 / 2$$

when the yield increases instantaneously by $\Delta r\%$.

– Price will drop by 17% if $D\% = 10$, $C\% = 1.5$, and $\Delta r = 2$ because

$$-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.$$  

• $C\%$ equals convexity divided by 100 (prove it!).
Effective Convexity

- The effective convexity is defined as

\[
\frac{P_+ + P_- - 2P_0}{P_0 \left(0.5 \times (y_+ - y_-)\right)^2},
\]

- \( P_- \) is the price if the yield is decreased by \( \Delta y \).
- \( P_+ \) is the price if the yield is increased by \( \Delta y \).
- \( P_0 \) is the initial price, \( y \) is the initial yield.
- \( \Delta y \) is small.

- Effective convexity is most relevant when a bond’s cash flow is interest rate sensitive.

- How to choose the right \( \Delta y \) is a delicate matter.
Approximate \( d^2 f(x)^2 / dx^2 \) at \( x = 1 \), Where \( f(x) = x^2 \)

- The difference of \( [(1 + \Delta x)^2 + (1 - \Delta x)^2 - 2]/(\Delta x)^2 \) and 2:

- This numerical issue is common in financial engineering but does not admit general solutions yet (see pp. 869ff).
Interest Rates and Bond Prices: Which Determines Which?\textsuperscript{a}

- If you have one, you have the other.
- So they are just two names given to the same thing: cost of fund.
- Traders most likely work with prices.
- Banks most likely work with interest rates.

\textsuperscript{a}Contributed by Mr. Wang, Cheng (R01741064) on March 5, 2014.
Term Structure of Interest Rates
Why is it that the interest of money is lower, when money is plentiful?
— Samuel Johnson (1709–1784)

If you have money, don’t lend it at interest.
Rather, give [it] to someone from whom you won’t get it back.
— Thomas Gospel 95
Term Structure of Interest Rates

- Concerned with how interest rates change with maturity.
- The set of yields to maturity for bonds form the term structure.
  - The bonds must be of equal quality.
  - They differ solely in their terms to maturity.
- The term structure is fundamental to the valuation of fixed-income securities.
Term Structure of Interest Rates (concluded)

- The term “term structure” often refers exclusively to the yields of zero-coupon bonds.

- A yield curve plots the yields to maturity of coupon bonds against maturity.

- A par yield curve is constructed from bonds trading near par.
Yield Curve of U.S. Treasuries as of July 24, 2015

Yield (%) vs. Year
Four Typical Shapes

• A normal yield curve is upward sloping.
• An inverted yield curve is downward sloping.
• A flat yield curve is flat.
• A humped yield curve is upward sloping at first but then turns downward sloping.
Spot Rates

- The $i$-period spot rate $S(i)$ is the yield to maturity of an $i$-period zero-coupon bond.

- The PV of one dollar $i$ periods from now is by definition

$$[1 + S(i)]^{-i}.$$  

  - It is the price of an $i$-period zero-coupon bond.$^a$

- The one-period spot rate is called the short rate.

- Spot rate curve:$^b$ Plot of spot rates against maturity:

  $$S(1), S(2), \ldots, S(n).$$

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$^a$Recall Eq. (9) on p. 69.

$^b$That is, term structure.
Problems with the PV Formula

• In the bond price formula (4) on p. 41,

\[
\sum_{i=1}^{n} \frac{C}{(1 + y)^i} + \frac{F}{(1 + y)^n},
\]

every cash flow is discounted at the same yield \(y\).

• Consider two riskless bonds with different yields to maturity because of their different cash flows:

\[
PV_1 = \sum_{i=1}^{n_1} \frac{C}{(1 + y_1)^i} + \frac{F}{(1 + y_1)^{n_1}},
\]

\[
PV_2 = \sum_{i=1}^{n_2} \frac{C}{(1 + y_2)^i} + \frac{F}{(1 + y_2)^{n_2}}.
\]
Problems with the PV Formula (concluded)

- The yield-to-maturity methodology discounts their *contemporaneous* cash flows with *different* rates.
- But shouldn’t they be discounted at the *same* rate?
Spot Rate Discount Methodology

- A cash flow $C_1, C_2, \ldots, C_n$ is equivalent to a package of zero-coupon bonds with the $i$th bond paying $C_i$ dollars at time $i$. 

\[ \begin{align*} 
C_1 & \quad 1 \\
C_2 & \quad 2 \\
C_3 & \quad 3 \\
\vdots & \quad \vdots \\
C_n & \quad n 
\end{align*} \]
Spot Rate Discount Methodology (concluded)

• So a level-coupon bond has the price

\[ P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}. \]  \hspace{1cm} (18)

• This pricing method incorporates information from the term structure.

• It discounts each cash flow at the matching spot rate.
Discount Factors

• In general, any riskless security having a cash flow $C_1, C_2, \ldots, C_n$ should have a market price of

$$P = \sum_{i=1}^{n} C_i d(i).$$

– Above, $d(i) \triangleq [1 + S(i)]^{-i}$, $i = 1, 2, \ldots, n$, are called the discount factors.
– $d(i)$ is the PV of one dollar $i$ periods from now.
– The above formula will be justified on p. 222.

• The discount factors are often interpolated to form a continuous function called the discount function.
Extracting Spot Rates from Yield Curve

- Start with the short rate $S(1)$.
  - Note that short-term Treasuries are zero-coupon bonds.

- Compute $S(2)$ from the two-period coupon bond price $P$ by solving

$$P = \frac{C}{1 + S(1)} + \frac{C + 100}{[1 + S(2)]^2}.$$
Extracting Spot Rates from Yield Curve (concluded)

- Inductively, we are given the market price $P$ of the $n$-period coupon bond and $S(1), S(2), \ldots, S(n-1)$.

- Then $S(n)$ can be computed from Eq. (18) on p. 127, repeated below,

$$P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}.$$  

- The running time can be made to be $O(n)$ (see text).

- The procedure is called bootstrapping.
Some Problems

- Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).
- Some maturities might be missing from the data points (the incompleteness problem).
- Treasuries might not be of the same quality.
- Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.\textsuperscript{a}

\textsuperscript{a}Often without economic justifications.
Which One (from P. 121)?
Yield Spread

• Consider a *risky* bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for $P$.

• Calculate the IRR of the risky bond.

• Calculate the IRR of a riskless bond with comparable maturity.

• Yield spread is their difference.
Static Spread

- Were the risky bond riskless, it would fetch

\[ P^* = \sum_{t=1}^{n} \frac{C_t}{[1 + S(t)]^t}. \]

- But as risk must be compensated, in reality \( P < P^* \).

- The static spread is the amount \( s \) by which the spot rate curve has to shift *in parallel* to price the risky bond:

\[ P = \sum_{t=1}^{n} \frac{C_t}{[1 + s + S(t)]^t}. \]

- Unlike the yield spread, the static spread explicitly incorporates information from the term structure.
Of Spot Rate Curve and Yield Curve

- $y_k$: yield to maturity for the $k$-period coupon bond.
- $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
- $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
- $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
- $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).

- If the yield curve is flat, the spot rate curve coincides with the yield curve.
Shapes

- The spot rate curve often has the same shape as the yield curve.
  - If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).

- But this is only a trend not a mathematical truth.\(^a\)

\(^a\)See a counterexample in the text.
Forward Rates

• The yield curve contains information regarding future interest rates currently “expected” by the market.

• Invest $1 for $j$ periods to end up with \([1 + S(j)]^j\) dollars at time $j$.
  – The maturity strategy.

• Invest $1 in bonds for $i$ periods and at time $i$ invest the proceeds in bonds for another $j - i$ periods where $j > i$.

• Will have \([1 + S(i)]^i[1 + S(i, j)]^{j-i}\) dollars at time $j$.
  – $S(i, j)$: $(j - i)$-period spot rate $i$ periods from now.
  – The rollover strategy.
Forward Rates (concluded)

- When $S(i, j)$ equals

$$f(i, j) \triangleq \left[ \frac{(1 + S(j))^j}{(1 + S(i))^i} \right]^{1/(j-i)} - 1,$$

we will end up with $[1 + S(j)]^j$ dollars again.

- As expected,

$$f(0, j) = S(j).$$

- The $f(i, j)$ are the (implied) forward (interest) rates.
  - More precisely, the $(j - i)$-period forward rate $i$ periods from now.
Time Line

$\begin{align*}
&f(0, 1) \quad f(1, 2) \quad f(2, 3) \quad f(3, 4) \\
&\text{Time 0} \quad S(1) \quad S(2) \quad S(3) \quad S(4)
\end{align*}$
Forward Rates and Future Spot Rates

• We did not assume any a priori relation between $f(i, j)$ and future spot rate $S(i, j)$.
  – This is the subject of the term structure theories.

• We merely looked for the future spot rate that, *if realized*, will equate the two investment strategies.

• The $f(i, i + 1)$ are the *instantaneous* forward rates or one-period forward rates.
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,

\[ f(i, j) > S(j) > \cdots > S(i). \]

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

\[ f(i, j) < S(j) < \cdots < S(i). \]
Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curve

- The FV of $1$ at time $n$ can be derived in two ways.
- Buy $n$-period zero-coupon bonds and receive

$[1 + S(n)]^n$.

- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is

$[1 + S(1)][1 + f(1, 2)] \cdots [1 + f(n - 1, n)]$. 
Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curves (concluded)

- Since they are identical,

$$S(n) = \{[1 + S(1)] [1 + f(1, 2)]$$

$$\cdots [1 + f(n - 1, n)]\}^{1/n} - 1. \quad (20)$$

- Hence, the forward rates (specifically the one-period forward rates) determine the spot rate curve.

- Other equivalencies can be derived similarly, such as

$$f(T, T + 1) = \frac{d(T)}{d(T + 1)} - 1. \quad (21)$$
Locking in the Forward Rate \( f(n, m) \)

- Buy one \( n \)-period zero-coupon bond for \( 1/(1 + S(n))^n \) dollars.

- Sell \( (1 + S(m))^m/(1 + S(n))^n \) \( m \)-period zero-coupon bonds.\(^a\)

- No net initial investment because the cash inflow equals the cash outflow: \( 1/(1 + S(n))^n \).

- At time \( n \) there will be a cash inflow of $1.

- At time \( m \) there will be a cash outflow of \( (1 + S(m))^m/(1 + S(n))^n \) dollars.

\(^a\)Note that \( (1 + S(m))^m/(1 + S(n))^n = (1 + f(n, m))^{m-n} \) by formula (19) on p. 138.
Locking in the Forward Rate $f(n, m)$ (concluded)

- This implies the interest rate between times $n$ and $m$ equals $f(n, m)$ by formula (19) on p. 138.

$$\frac{(1 + S(m))^m}{(1 + S(n))^n}$$
Forward Loans

• We had generated the cash flow of a type of forward contract called the forward loan.

• Agreed upon today, it enables one to
  – Borrow money at time $n$ in the future, and
  – Repay the loan at time $m > n$ with an interest rate equal to the forward rate $f(n, m)$.

• Can the spot rate curve be arbitrarily drawn?\textsuperscript{a}

\textsuperscript{a}Contributed by Mr. Dai, Tian-Shyr (B82506025, R86526008, D88526006) in 1998.
Synthetic Bonds

• We had seen that

\[
\text{forward loan} = n\text{-period zero} - [1 + f(n, m)]^{m-n} \times m\text{-period zero}.
\]

• Thus

\[
n\text{-period zero} = \text{forward loan} + [1 + f(n, m)]^{m-n} \times m\text{-period zero}.
\]

• We have created a \textit{synthetic} zero-coupon bond with forward loans and other zero-coupon bonds.

• Useful if the \textit{n}-period zero is unavailable or illiquid.
Spot and Forward Rates under Continuous Compounding

• The pricing formula:

\[ P = \sum_{i=1}^{n} Ce^{-iS(i)} + Fe^{-nS(n)}. \]

• The market discount function:

\[ d(n) = e^{-nS(n)}. \]

• The spot rate is an arithmetic average of forward rates,\(^a\)

\[ S(n) = \frac{f(0, 1) + f(1, 2) + \cdots + f(n - 1, n)}{n}. \]

\(^a\)Compare it with formula (20) on p. 144.
Spot and Forward Rates under Continuous Compounding (continued)

- The formula for the forward rate:

\[ f(i, j) = \frac{jS(j) - iS(i)}{j - i}. \]  \hspace{1cm} (22)

- Compare the above formula with (19) on p. 138.

- The one-period forward rate:\(^a\)

\[ f(j, j + 1) = -\ln \frac{d(j + 1)}{d(j)}. \]

\(^a\)Compare it with formula (21) on p. 144.
Spot and Forward Rates under Continuous Compounding (concluded)

• Now, the (instantaneous) forward rate curve is:

\[
f(T) \triangleq \lim_{\Delta T \to 0} f(T, T + \Delta T) = S(T) + T \frac{\partial S}{\partial T}.
\]  

(23)

• So \( f(T) > S(T) \) if and only if \( \partial S/\partial T > 0 \) (i.e., a normal spot rate curve).

• If \( S(T) < -T(\partial S/\partial T) \), then \( f(T) < 0 \).\(^a\)

\(^a\)Consistent with the plot on p. 142. Contributed by Mr. Huang, Hsien-Chun (R03922103) on March 11, 2015.
An Example

- Let the interest rates be continuously compounded.
- Suppose the spot rate curve is $a$
  \[
  S(T) \triangleq 0.08 - 0.05 e^{-0.18T}.
  \]
- Then by Eq. (23) on p. 151, the forward rate curve is
  \[
  f(T) = S(T) + TS'(T)
  = 0.08 - 0.05 e^{-0.18T} + 0.009T e^{-0.18T}.
  \]

\(^{a}\text{Hull & White (1994).}\)
Unbiased Expectations Theory

- Forward rate equals the average future spot rate,
  \[ f(a, b) = E[S(a, b)]. \]  
  \[ (24) \]

- It does not imply that the forward rate is an accurate predictor for the future spot rate.

- It implies the maturity strategy and the rollover strategy produce the same result at the horizon “on average.”
Unbiased Expectations Theory and Spot Rate Curve

- It implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  - $f(j, j + 1) > S(j + 1)$ if and only if $S(j + 1) > S(j)$ from formula (19) on p. 138.
  - So
    \[ E[S(j, j + 1)] > S(j + 1) > \cdots > S(1) \]
    if and only if $S(j + 1) > \cdots > S(1)$.
- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.
A “Bad” Expectations Theory

• The expected returns\(^a\) on all possible riskless bond strategies are equal for all holding periods.

• So

\[
(1 + S(2))^2 = (1 + S(1)) \cdot E[1 + S(1, 2)] \tag{25}
\]

because of the equivalency between buying a two-period bond and rolling over one-period bonds.

• After rearrangement,

\[
\frac{1}{E[1 + S(1, 2)]} = \frac{1 + S(1)}{(1 + S(2))^2}.
\]

---

\(^a\)More precisely, the one-plus returns.
A “Bad” Expectations Theory (continued)

- Now consider two one-period strategies.
  - Strategy one buys a two-period bond for \((1 + S(2))^{-2}\) dollars and sells it after one period.
  - The expected return is
    \[
    E\left[ (1 + S(1, 2))^{-1} \right] / (1 + S(2))^{-2}.
    \]
  - Strategy two buys a one-period bond with a return of \(1 + S(1)\).
A “Bad” Expectations Theory (continued)

• The theory says the returns are equal:
  \[
  \frac{1 + S(1)}{(1 + S(2))^2} = E \left[ \frac{1}{1 + S(1, 2)} \right].
  \]

• Combine this with Eq. (25) on p. 155 to obtain
  \[
  E \left[ \frac{1}{1 + S(1, 2)} \right] = \frac{1}{E[1 + S(1, 2)]}.
  \]
A “Bad” Expectations Theory (concluded)

• But this is impossible save for a certain economy.
  – Jensen’s inequality states that $E[g(X)] > g(E[X])$
    for any nondegenerate random variable $X$ and
    strictly convex function $g$ (i.e., $g''(x) > 0$).
  – Use
    \[ g(x) \overset{\Delta}{=} (1 + x)^{-1} \]
    to prove our point.
Local Expectations Theory

• The expected rate of return of any bond over a single period equals the prevailing one-period spot rate:

\[
E \left[ \frac{(1 + S(1, n))^{-(n-1)}}{(1 + S(n))^{-n}} \right] = 1 + S(1) \quad \text{for all } n > 1.
\]

• This theory is the basis of many interest rate models.
Duration, in Practice

• We had assumed parallel shifts in the spot rate curve.

• To handle more general shifts, define a vector $[c_1, c_2, \ldots, c_n]$ that characterizes the shift.
  
  – Parallel shift: $[1, 1, \ldots, 1]$.
  
  – Twist: $[1, 1, \ldots, 1, -1, \ldots, -1]$, $[1.8, 1.6, 1.4, 1, 0, -1, -1.4, \ldots]$, etc.
  
  – …. 

• At least one $c_i$ should be 1 as the reference point.
Duration in Practice (concluded)

• Let

\[ P(y) \triangleq \sum_i C_i / (1 + S(i) + yc_i)^i \]

be the price associated with the cash flow \( C_1, C_2, \ldots \).

• Define duration as

\[ - \frac{\partial P(y)/P(0)}{\partial y} \bigg|_{y=0} \quad \text{or} \quad - \frac{P(\Delta y) - P(-\Delta y)}{2P(0)\Delta y}. \]

• Modified duration equals the above when

\[ [c_1, c_2, \ldots, c_n] = [1, 1, \ldots, 1], \]

\[ S(1) = S(2) = \cdots = S(n). \]
Some Loose Ends on Dates

- Holidays.
- Weekends.
- Business days ($T + 2$, etc.).
- Shall we treat a year as 1 year whether it has 365 or 366 days?