Spread of Nonbenchmark Bonds

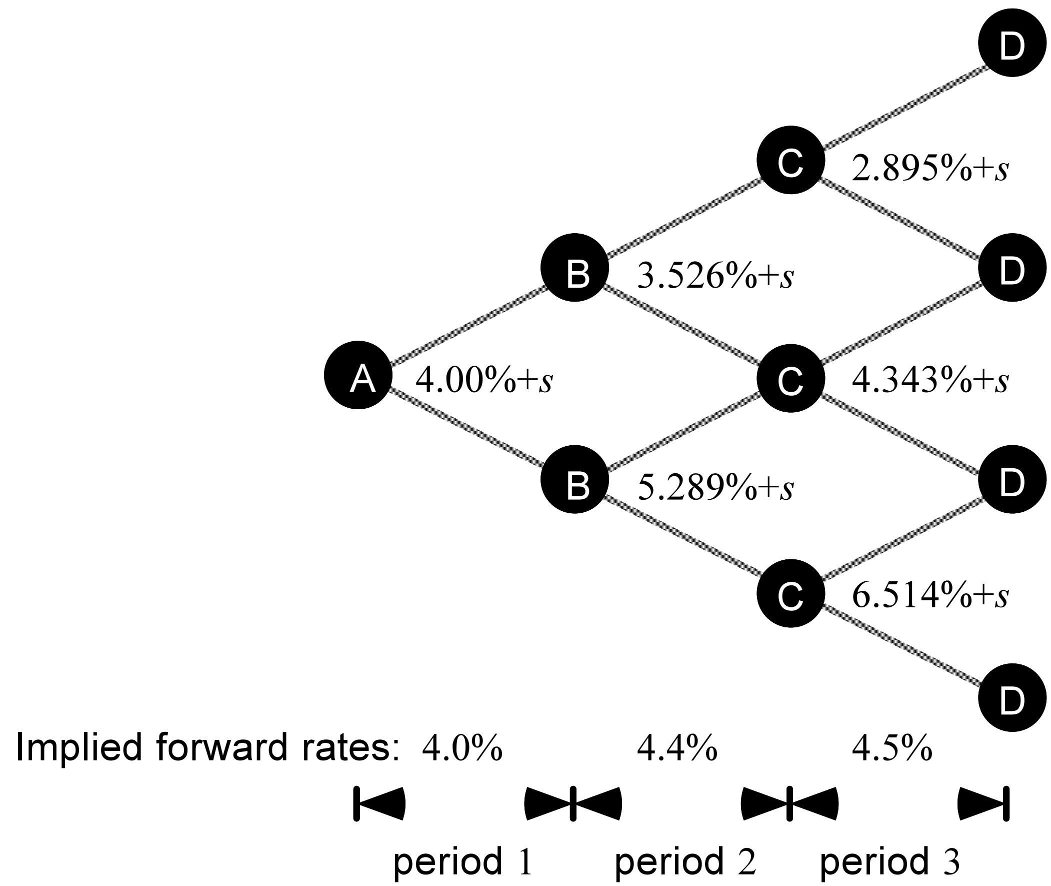
- Model prices calculated by the calibrated tree as a rule do not match market prices of nonbenchmark bonds.
- The incremental return over the benchmark bonds is called spread.
- If we add the spread uniformly over the short rates in the tree, the model price will equal the market price.
- We will apply the spread concept to option-free bonds next.

Spread of Nonbenchmark Bonds (continued)

- We illustrate the idea with an example.
- Start with the tree on p. 1049.
- Consider a security with cash flow C_i at time i for $i = 1, 2, 3$.
- Its model price is $p(s)$, which is equal to

$$\frac{1}{1.04+s} \times \left[C_1 + \frac{1}{2} \times \frac{1}{1.03526+s} \times \left(C_2 + \frac{1}{2} \left(\frac{C_3}{1.02895+s} + \frac{C_3}{1.04343+s} \right) \right) + \frac{1}{2} \times \frac{1}{1.05289+s} \times \left(C_2 + \frac{1}{2} \left(\frac{C_3}{1.04343+s} + \frac{C_3}{1.06514+s} \right) \right) \right].$$

- Given a market price of P, the spread is the s that solves $P = p(s)$.


D
C
2.895%+s
B
3.526%+s
D
A
4.00%+s
C
4.343%+s
B
5.289%+s
D
C
6.514%+s
D
Implied forward rates: 4.0%
4.4%
4.5%
period 1
period 2
period 3

Spread of Nonbenchmark Bonds (continued)

- The model price $p(s)$ is a monotonically decreasing, convex function of s.
- We will employ the Newton-Raphson root-finding method to solve
$$p(s) - P = 0$$
for s.
- But a quick look at the equation for $p(s)$ reveals that evaluating $p'(s)$ directly is infeasible.
- Fortunately, the tree can be used to evaluate both $p(s)$ and $p'(s)$ during backward induction.

Spread of Nonbenchmark Bonds (continued)

- Consider an arbitrary node A in the tree associated with the short rate r.
- In the process of computing the model price $p(s)$, a price $p_{\mathrm{A}}(s)$ is computed at A.
- Prices computed at A's two successor nodes B and C are discounted by $r + s$ to obtain $p_{\mathrm{A}}(s)$ as follows,

$$p_{\mathrm{A}}(s) = c + \frac{p_{\mathrm{B}}(s) + p_{\mathrm{C}}(s)}{2(1 + r + s)},$$

where c denotes the cash flow at A.

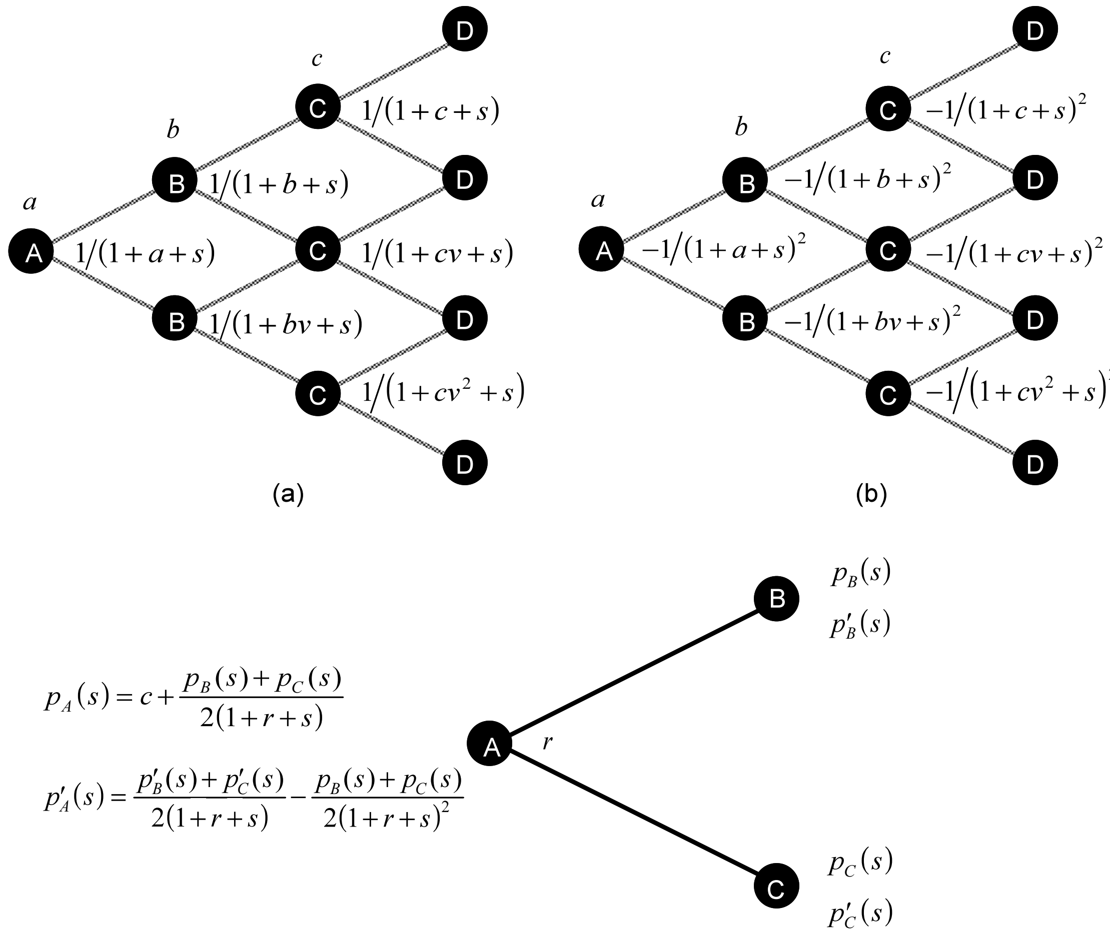
Spread of Nonbenchmark Bonds (continued)

- To compute $p'_{\mathrm{A}}(s)$ as well, node A calculates

$$p'_{\mathrm{A}}(s) = \frac{p'_{\mathrm{B}}(s) + p'_{\mathrm{C}}(s)}{2(1+r+s)} - \frac{p_{\mathrm{B}}(s) + p_{\mathrm{C}}(s)}{2(1+r+s)^2}. \tag{138}$$

- This is easy if $p'_{\mathrm{B}}(s)$ and $p'_{\mathrm{C}}(s)$ are also computed at nodes B and C.
- When A is a terminal node, simply use the payoff function for $p_{\mathrm{A}}(s)$.[a]

[a]Contributed by Mr. Chou, Ming-Hsin (R02723073) on May 28, 2014.



a
b
c
$1/(1+a+s)$
$1/(1+b+s)$
$1/(1+bv+s)$
$1/(1+c+s)$
$1/(1+cv+s)$
$1/(1+cv^2+s)$
$-1/(1+a+s)^2$
$-1/(1+b+s)^2$
$-1/(1+bv+s)^2$
$-1/(1+c+s)^2$
$-1/(1+cv+s)^2$
$-1/(1+cv^2+s)^2$
$p_A(s) = c + \frac{p_B(s)+p_C(s)}{2(1+r+s)}$
$p'_A(s) = \frac{p'_B(s)+p'_C(s)}{2(1+r+s)} - \frac{p_B(s)+p_C(s)}{2(1+r+s)^2}$
r
$p_B(s)$
$p'_B(s)$
$p_C(s)$
$p'_C(s)$

(a) (b) (c)

Spread of Nonbenchmark Bonds (continued)

- Apply the above procedure inductively to yield $p(s)$ and $p'(s)$ at the root (p. 1053).
- This is called the differential tree method.[a]
  - Similar ideas can be found in automatic differentiation (AD)[b] and backpropagation[c] in artificial neural networks.
- The total running time is $O(n^2)$.
- The memory requirement is $O(n)$.

---

[a] Lyuu (1999).
[b] Rall (1981).
[c] Werbos (1974); Rumelhart, Hinton, & Williams (1986).

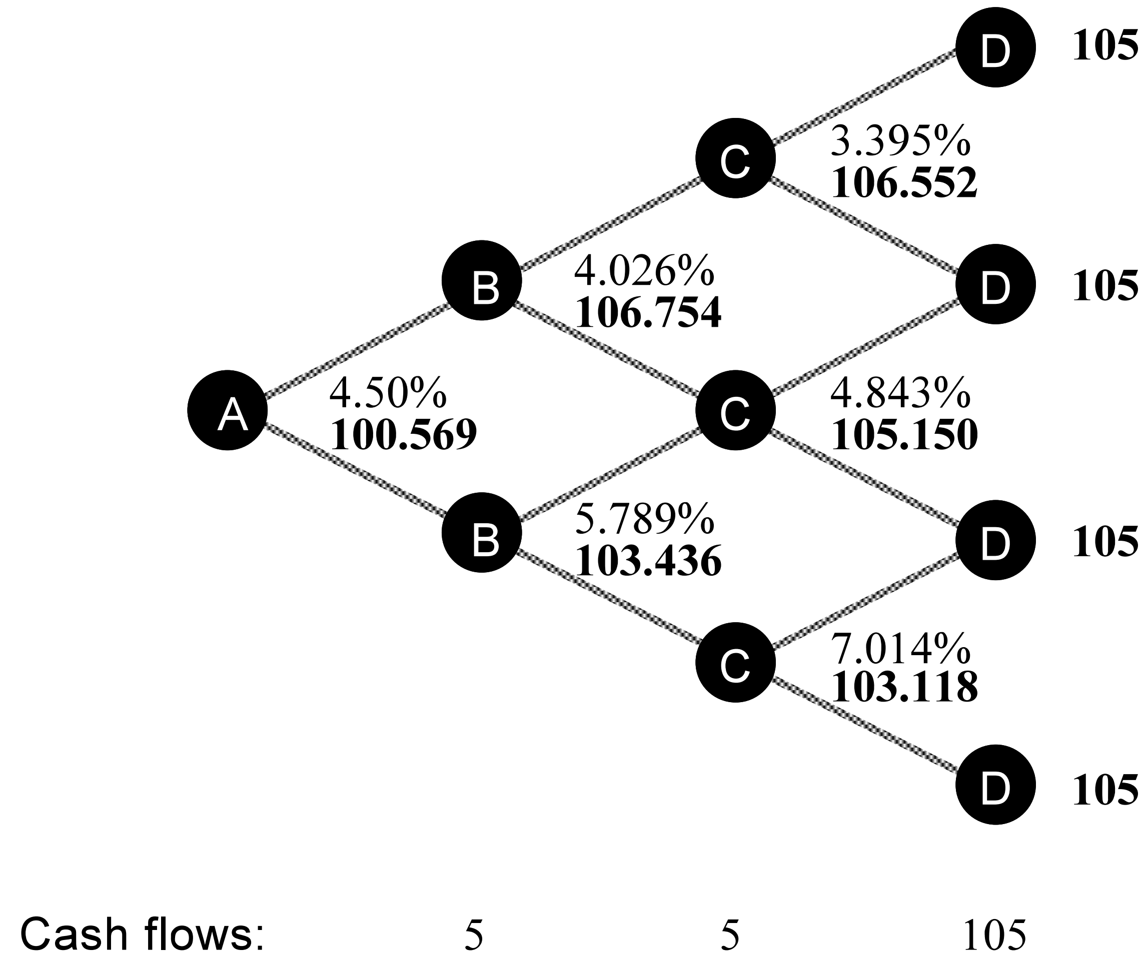
Spread of Nonbenchmark Bonds (continued)

Number of partitions n	Running time (s)	Number of iterations	Number of partitions	Running time (s)	Number of iterations
500	7.850	5	10500	3503.410	5
1500	71.650	5	11500	4169.570	5
2500	198.770	5	12500	4912.680	5
3500	387.460	5	13500	5714.440	5
4500	641.400	5	14500	6589.360	5
5500	951.800	5	15500	7548.760	5
6500	1327.900	5	16500	8502.950	5
7500	1761.110	5	17500	9523.900	5
8500	2269.750	5	18500	10617.370	5
9500	2834.170	5	....	....	....

75MHz Sun SPARCstation 20.

Spread of Nonbenchmark Bonds (concluded)

- Consider a three-year, 5% bond with a market price of 100.569.
- Assume the bond pays annual interest.
- The spread can be shown to be 50 basis points over the tree (p. 1057).
- Note that the idea of spread does not assume parallel shifts in the term structure.
- It also differs from the yield spread (p. 133) and static spread (p. 134) of the nonbenchmark bond over an otherwise identical benchmark bond.


D
105
C
3.395%
106.552
B
4.026%
106.754
D
105
A
4.50%
100.569
C
4.843%
105.150
B
5.789%
103.436
D
105
C
7.014%
103.118
D
105
Cash flows:
5
5
105

More Applications of the Differential Tree: Calculating Implied Volatility (in seconds)[a]

American call			American put		
Number of partitions	Running time	Number of iterations	Number of partitions	Running time	Number of iterations
100	0.008210	2	100	0.013845	3
200	0.033310	2	200	0.036335	3
300	0.072940	2	300	0.120455	3
400	0.129180	2	400	0.214100	3
500	0.201850	2	500	0.333950	3
600	0.290480	2	600	0.323260	2
700	0.394090	2	700	0.435720	2
800	0.522040	2	800	0.569605	2

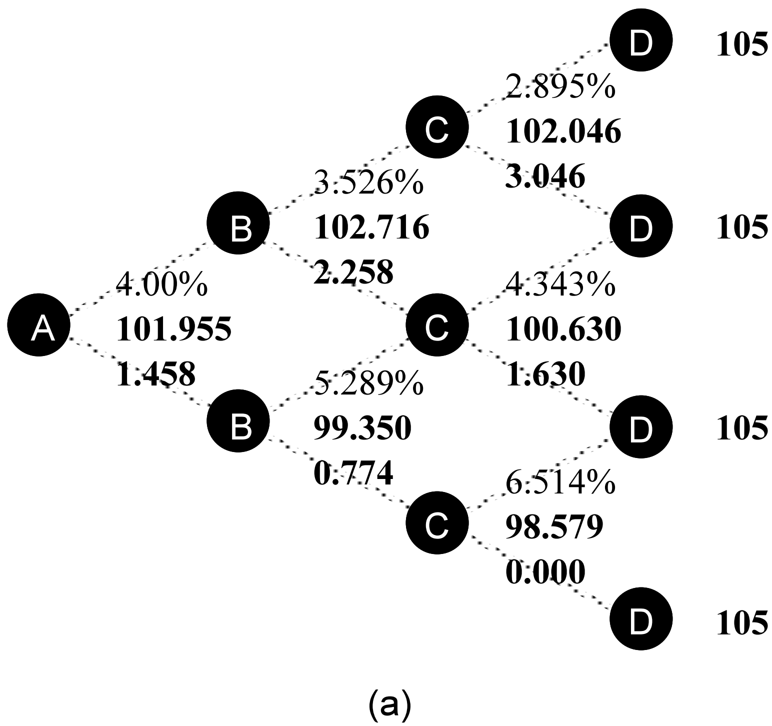
Intel 166MHz Pentium, running on Microsoft Windows 95.

---

[a] Lyuu (1999).

Fixed-Income Options

- Consider a 2-year 99 European call on the 3-year, 5% Treasury.
- Assume the Treasury pays annual interest.
- From p. 1060 the 3-year Treasury's price minus the \$5 interest at year 2 could be \$102.046, \$100.630, or \$98.579 two years from now.
  - The accrued interest is *not* included as it belongs to the original bondholder.
- Now compare the strike price against the bond prices.
- The call is in the money in the first two scenarios out of the money in the third.



A
4.00%
101.955
1.458
B
3.526%
102.716
2.258
B
5.289%
99.350
0.774
C
2.895%
102.046
3.046
C
4.343%
100.630
1.630
C
6.514%
98.579
0.000
D
105
D
105
D
105
D
105
(a)
A
4.00%
101.955
0.096
B
3.526%
102.716
0.000
B
5.289%
99.350
0.200
C
2.895%
102.046
0.000
C
4.343%
100.630
0.000
C
6.514%
98.579
0.421
D
105
D
105
D
105
D
105
(b)

Fixed-Income Options (continued)

- The option value is calculated to be \$1.458 on p. 1060(a).
- European interest rate puts can be valued similarly.
- Consider a two-year 99 European put on the same security.
- At expiration, the put is in the money only when the Treasury is worth \$98.579 without the accrued interest.
- The option value is computed to be \$0.096 on p. 1060(b).

Fixed-Income Options (concluded)

- The present value of the strike price is $\text{PV}(X) = 99 \times 0.92101 = 91.18$.
- The Treasury is worth $B = 101.955$.
- The present value of the interest payments during the life of the options is[a]

$$\text{PV}(I) = 5 \times 0.96154 + 5 \times 0.92101 = 9.41275.$$

- The call and the put are worth $C = 1.458$ and $P = 0.096$, respectively.
- Hence the put-call parity is preserved:

$$C = P + B - \text{PV}(I) - \text{PV}(X).$$

[a] There is no coupon today.

Delta or Hedge Ratio

- How much does the option price change in response to changes in the *price* of the underlying bond?
- This relation is called delta (or hedge ratio) defined as

$$\frac{O_{\mathrm{h}} - O_{\ell}}{P_{\mathrm{h}} - P_{\ell}}.$$

- In the above P_{h} and P_{ℓ} denote the bond prices if the short rate moves up and down, respectively.
- Similarly, O_{h} and O_{ℓ} denote the option values if the short rate moves up and down, respectively.

Delta or Hedge Ratio (concluded)

- Delta measures the sensitivity of the option value to changes in the underlying bond price.
- So it shows how to hedge one with the other.
- Take the call and put on p. 1060 as examples.
- Their deltas are

$$\frac{0.774 - 2.258}{99.350 - 102.716} = 0.441,$$
$$\frac{0.200 - 0.000}{99.350 - 102.716} = -0.059,$$

respectively.

Volatility Term Structures

- The binomial interest rate tree can be used to calculate the yield volatility of zero-coupon bonds.
- Consider an n-period zero-coupon bond.
- First find its yield to maturity y_{h} (y_ℓ, respectively) at the end of the initial period if the short rate rises (declines, respectively).
- The yield volatility for our model is defined as

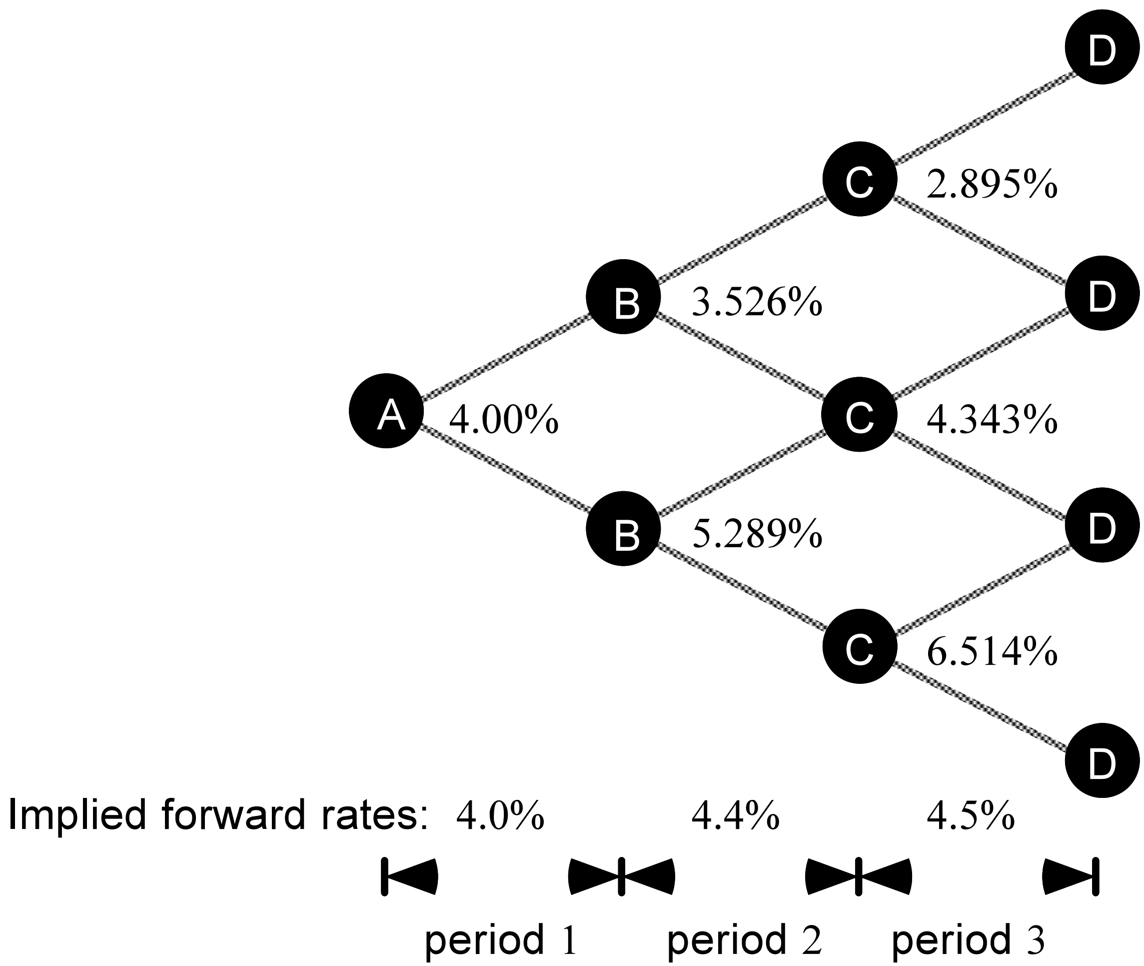
$$\frac{1}{2} \ln\left(\frac{y_{\text{h}}}{y_\ell}\right). \tag{139}$$

Volatility Term Structures (continued)

- For example, take the tree on p. 1043 (repeated on next page).
- The two-year zero's yield at the end of the first period is 5.289% if the rate rises and 3.526% if the rate declines.
- Its yield volatility is therefore

$$\frac{1}{2}\ln\left(\frac{0.05289}{0.03526}\right) = 20.273\%.$$

Volatility Term Structures (continued)



Volatility Term Structures (continued)

- Consider the three-year zero-coupon bond.
- If the short rate rises, the price of the zero one year from now will be

$$\frac{1}{2} \times \frac{1}{1.05289} \times \left(\frac{1}{1.04343} + \frac{1}{1.06514} \right) = 0.90096.$$

- Thus its yield is $\sqrt{\frac{1}{0.90096}} - 1 = 0.053531$.
- If the short rate declines, the price of the zero one year from now will be

$$\frac{1}{2} \times \frac{1}{1.03526} \times \left(\frac{1}{1.02895} + \frac{1}{1.04343} \right) = 0.93225.$$

Volatility Term Structures (continued)

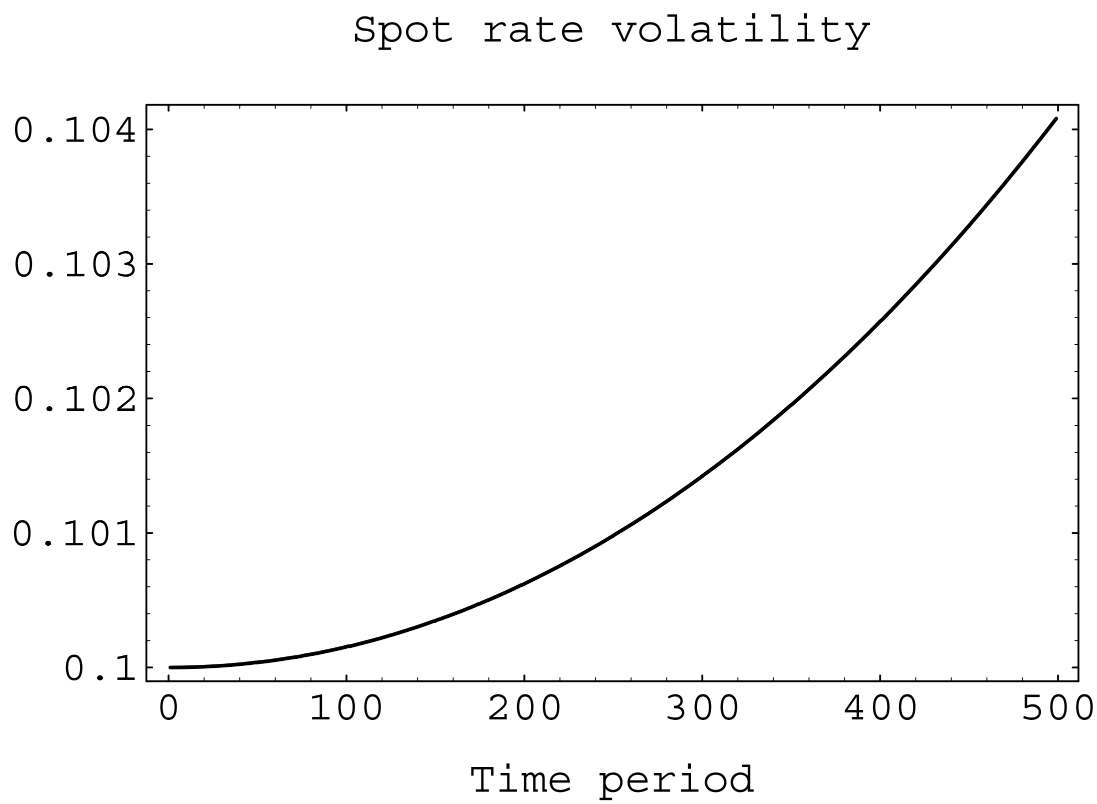
- Thus its yield is $\sqrt{\frac{1}{0.93225}} - 1 = 0.0357$.
- The yield volatility is hence

$$\frac{1}{2} \ln\left(\frac{0.053531}{0.0357}\right) = 20.256\%,$$

  slightly less than the one-year yield volatility.

- This is consistent with the reality that longer-term bonds typically have lower yield volatilities than shorter-term bonds.[a]
- The procedure can be repeated for longer-term zeros to obtain their yield volatilities.

[a]The relation is reversed for *price* volatilities (duration).



Short rate volatility given a flat %10 volatility structure.

Volatility Term Structures (concluded)

- We started with v_i and then derived the volatility term structure.
- In practice, the steps are reversed.
- The volatility term structure is supplied by the user along with the term structure.
- The v_i—hence the short rate volatilities via Eq. (134) on p. 1020—and the r_i are then simultaneously determined.
- The result is the Black-Derman-Toy model of Goldman Sachs.[a]

[a] Black, Derman, & Toy (1990).

Foundations of Term Structure Modeling

[Meriwether] scoring especially high marks
in mathematics — an indispensable subject
for a bond trader.
— Roger Lowenstein,
When Genius Failed (2000)

[The] fixed-income traders I knew
seemed smarter than the equity trader [$\cdots$]
there's no competitive edge to
being smart in the equities business[.]
— Emanuel Derman,
My Life as a Quant (2004)

Bond market terminology was designed less
to convey meaning than to bewilder outsiders.
— Michael Lewis, *The Big Short* (2011)

Terminology

- A period denotes a unit of elapsed time.
  - Viewed at time t, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.
- Bonds will be assumed to have a par value of one — unless stated otherwise.
- The time unit for continuous-time models will usually be measured by the year.

Standard Notations

The following notation will be used throughout.

t**:** a point in time.

$r(t)$**:** the one-period riskless rate prevailing at time t for repayment one period later.[a]

$P(t,T)$**:** the present value at time t of one dollar at time T.

---

[a] Alternatively, the instantaneous spot rate, or short rate, at time t.

Standard Notations (continued)

$r(t,T)$**:** the $(T-t)$-period interest rate prevailing at time t stated on a per-period basis and compounded once per period.[a]

$F(t,T,M)$**:** the forward price at time t of a forward contract that delivers at time T a zero-coupon bond maturing at time $M \geq T$.

---

[a] In other words, the $(T-t)$-period spot rate at time t.

Standard Notations (concluded)

$f(t, T, L)$**:** the L-period forward rate at time T implied at time t stated on a per-period basis and compounded once per period.

$f(t, T)$**:** the one-period or instantaneous forward rate at time T as seen at time t stated on a per period basis and compounded once per period.

- It is $f(t, T, 1)$ in the discrete-time model and $f(t, T, dt)$ in the continuous-time model.
- Note that $f(t, t)$ equals the short rate $r(t)$.

Fundamental Relations

- The price of a zero-coupon bond equals

$$P(t,T) = \begin{cases} (1+r(t,T))^{-(T-t)}, & \text{in discrete time,} \\ e^{-r(t,T)(T-t)}, & \text{in continuous time.} \end{cases} \tag{140}$$

- $r(t,T)$ as a function of T defines the spot rate curve at time t.
- By definition,

$$f(t,t) = \begin{cases} r(t,t+1), & \text{in discrete time,} \\ r(t,t), & \text{in continuous time.} \end{cases}$$

Fundamental Relations (continued)

- Forward prices and zero-coupon bond prices are related:

$$F(t, T, M) = \frac{P(t, M)}{P(t, T)}, \quad T \leq M. \tag{141}$$

  - The forward price equals the future value at time T of the underlying asset.[a]

- Equation (141) holds whether the model is discrete-time or continuous-time.

[a]See Exercise 24.2.1 of the textbook for proof.

Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by

$$f(t,T,L) = \left(\frac{1}{F(t,T,T+L)}\right)^{1/L} - 1 = \left(\frac{P(t,T)}{P(t,T+L)}\right)^{1/L} - 1 \tag{142}$$

  in discrete time.

- The analog to Eq. (142) under simple compounding is

$$f(t,T,L) = \frac{1}{L}\left(\frac{P(t,T)}{P(t,T+L)} - 1\right).$$

Fundamental Relations (continued)

- In continuous time,

$$f(t,T,L) = -\frac{\ln F(t,T,T+L)}{L} = \frac{\ln(P(t,T)/P(t,T+L))}{L} \tag{143}$$

by Eq. (141) on p. 1080.

- Furthermore,

$$\begin{aligned} f(t,T,\Delta t) &= \frac{\ln(P(t,T)/P(t,T+\Delta t))}{\Delta t} \to -\frac{\partial \ln P(t,T)}{\partial T} \\ &= -\frac{\partial P(t,T)/\partial T}{P(t,T)}. \end{aligned}$$

Fundamental Relations (continued)

- So

$$f(t,T) \stackrel{\Delta}{=} -\frac{\partial \ln P(t,T)}{\partial T} = -\frac{\partial P(t,T)/\partial T}{P(t,T)}, \quad t \leq T. \tag{144}$$

- Because Eq. (144) is equivalent to

$$P(t,T) = e^{-\int_t^T f(t,s)\,ds}, \tag{145}$$

  the spot rate curve is

$$r(t,T) = \frac{\int_t^T f(t,s)\,ds}{T-t}.$$

Fundamental Relations (concluded)

- The discrete analog to Eq. (145) is

$$P(t,T) = \frac{1}{(1+r(t))(1+f(t,t+1))\cdots(1+f(t,T-1))}.$$

- The short rate and the market discount function are related by

$$r(t) = -\left.\frac{\partial P(t,T)}{\partial T}\right|_{T=t}.$$

Risk-Neutral Pricing

- Assume the local expectations theory.
- The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  - For all $t + 1 < T$,

$$\frac{E_t[\,P(t+1,T)\,]}{P(t,T)} = 1 + r(t). \tag{146}$$

  - Relation (146) in fact follows from the risk-neutral valuation principle.[a]

---

[a]Theorem 16 on p. 562.

Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability π.
- Equation (146) on p. 1085 can also be expressed as

$$E_t[\,P(t+1,T)\,] = F(t,t+1,T).$$

  - Verify that with, e.g., Eq. (141) on p. 1080.
- Hence the forward price for the next period is an unbiased estimator of the expected bond price.[a]

[a]Under the local expectations theory. But the forward rate is *not* an unbiased estimator of the expected future short rate (p. 1034).

Risk-Neutral Pricing (continued)

- Rewrite Eq. (146) on p. 1085 as

$$\frac{E_t^{\pi}[\,P(t+1,T)\,]}{1+r(t)} = P(t,T). \tag{147}$$

  – It says the current market discount function equals the expected market discount function one period from now discounted by the short rate.

Risk-Neutral Pricing (concluded)

- Apply the above equality iteratively to obtain

$$
\begin{aligned}
& P(t,T) \\
= \;& E_t^{\pi}\left[\,\frac{P(t+1,T)}{1+r(t)}\,\right] \\
= \;& E_t^{\pi}\left[\,\frac{E_{t+1}^{\pi}[\,P(t+2,T)\,]}{(1+r(t))(1+r(t+1))}\,\right] = \cdots \\
= \;& E_t^{\pi}\left[\,\frac{1}{(1+r(t))(1+r(t+1))\cdots(1+r(T-1))}\,\right].
\end{aligned}
$$

Continuous-Time Risk-Neutral Pricing

- In continuous time, the local expectations theory implies

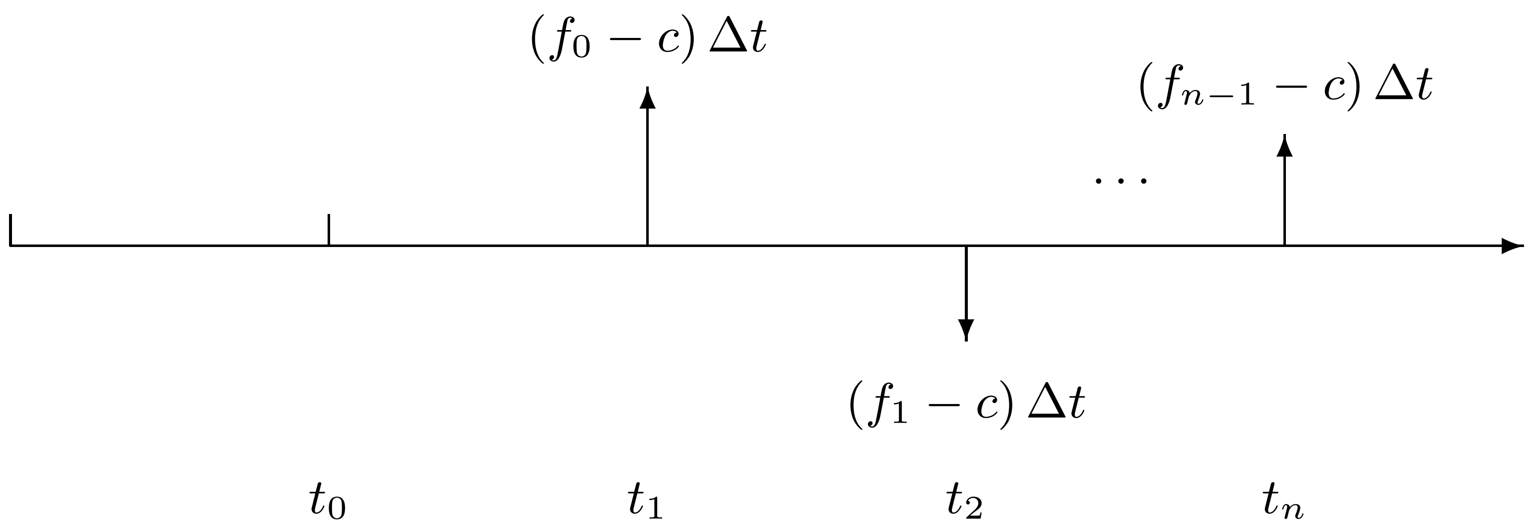
$$P(t,T) = E_t \left[e^{-\int_t^T r(s)\,ds} \right], \quad t < T. \tag{148}$$

- Note that $e^{\int_t^T r(s)\,ds}$ is the bank account process, which denotes the rolled-over money market account.

Interest Rate Swaps

- Consider an interest rate swap made at time t (now) with payments to be exchanged at times $t_1, t_2, \ldots, t_n$.
- For simplicity, assume $t_{i+1} - t_i$ is a fixed constant Δt for all i, and the notional principal is one dollar.
- The fixed rate is c per annum.
- The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.
- The payoff at time t_{i+1} for the *fixed-rate payer* is $(f_i - c)\,\Delta t$.

Interest Rate Swaps (continued)



Interest Rate Swaps (continued)

- Simple rates are adopted here.
- Hence f_i satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$

- If $t < t_0$, we have a forward interest rate swap.
- The ordinary swap corresponds to $t = t_0$.

Interest Rate Swaps (continued)

- The value of the swap at time t is thus

$$
\begin{aligned}
& \sum_{i=1}^{n} E_t^{\pi}\left[e^{-\int_t^{t_i} r(s)\,ds} (f_{i-1} - c)\,\Delta t \right] \\
= & \sum_{i=1}^{n} E_t^{\pi}\left[e^{-\int_t^{t_i} r(s)\,ds} \left(\frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right) \right] \\
= & \sum_{i=1}^{n} E_t^{\pi}\left[e^{-\int_t^{t_i} r(s)\,ds} \left(e^{\int_{t_{i-1}}^{t_i} r(s)\,ds} - (1 + c\Delta t) \right) \right] \\
= & \sum_{i=1}^{n} \left[\, P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i) \,\right] \\
= & \; P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).
\end{aligned}
$$

Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present-value calculations.

Swap Rate

- The swap rate, which gives the swap zero value, equals

$$S_n(t) \triangleq \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} P(t, t_i)\, \Delta t}. \tag{149}$$

- The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.
- For an ordinary swap, $P(t, t_0) = 1$.
- The swap rate is called a forward swap rate if $t_0 > t$.

The Term Structure Equation[a]

- Let us start with the zero-coupon bonds and the money market account.
- Let the zero-coupon bond price $P(r, t, T)$ follow
$$\frac{dP}{P} = \mu_p \, dt + \sigma_p \, dW.$$
- At time t, short one unit of a bond maturing at time s_1 and buy α units of a bond maturing at time s_2.

[a]Vasicek (1977).

The Term Structure Equation (continued)

- The net wealth change follows

$$\begin{aligned} & -dP(r,t,s_1) + \alpha\, dP(r,t,s_2) \\ = \; & (-P(r,t,s_1)\, \mu_p(r,t,s_1) + \alpha P(r,t,s_2)\, \mu_p(r,t,s_2))\, dt \\ & + (-P(r,t,s_1)\, \sigma_p(r,t,s_1) + \alpha P(r,t,s_2)\, \sigma_p(r,t,s_2))\, dW. \end{aligned}$$

- Pick

$$\alpha \stackrel{\Delta}{=} \frac{P(r,t,s_1)\, \sigma_p(r,t,s_1)}{P(r,t,s_2)\, \sigma_p(r,t,s_2)}.$$

The Term Structure Equation (continued)

- Then the net wealth has no volatility and must earn the riskless return:

$$\frac{-P(r,t,s_1)\,\mu_p(r,t,s_1)+\alpha P(r,t,s_2)\,\mu_p(r,t,s_2)}{-P(r,t,s_1)+\alpha P(r,t,s_2)}=r.$$

- Simplify the above to obtain

$$\frac{\sigma_p(r,t,s_1)\,\mu_p(r,t,s_2)-\sigma_p(r,t,s_2)\,\mu_p(r,t,s_1)}{\sigma_p(r,t,s_1)-\sigma_p(r,t,s_2)}=r.$$

- This becomes

$$\frac{\mu_p(r,t,s_2)-r}{\sigma_p(r,t,s_2)}=\frac{\mu_p(r,t,s_1)-r}{\sigma_p(r,t,s_1)}$$

after rearrangement.

The Term Structure Equation (continued)

- Since the above equality holds for any s_1 and s_2,

$$\frac{\mu_p(r,t,s)-r}{\sigma_p(r,t,s)} \stackrel{\Delta}{=} \lambda(r,t) \tag{150}$$

for some λ independent of the bond maturity s.

- As $\mu_p = r + \lambda\sigma_p$, all assets are expected to appreciate at a rate equal to the sum of the short rate and a constant times the asset's volatility.

- The term $\lambda(r,t)$ is called the market price of risk.

- The market price of risk must be the same for all bonds to preclude arbitrage opportunities.

The Term Structure Equation (continued)

- Assume a Markovian short rate model,

$$dr = \mu(r,t)\,dt + \sigma(r,t)\,dW.$$

- Then the bond price process is also Markovian.
- By Eq. (14.15) on p. 202 of the textbook,

$$\mu_p = \left(-\frac{\partial P}{\partial T} + \mu(r,t)\,\frac{\partial P}{\partial r} + \frac{\sigma(r,t)^2}{2}\,\frac{\partial^2 P}{\partial r^2} \right) / P, \tag{151}$$

$$\sigma_p = \left(\sigma(r,t)\,\frac{\partial P}{\partial r} \right) / P, \tag{151$'$}$$

subject to $P(\,\cdot\,, T, T) = 1$.

The Term Structure Equation (concluded)

- Substitute μ_p and σ_p into Eq. (150) on p. 1099 to obtain

$$-\frac{\partial P}{\partial T} + [\,\mu(r,t) - \lambda(r,t)\,\sigma(r,t)\,]\,\frac{\partial P}{\partial r} + \frac{1}{2}\,\sigma(r,t)^2\,\frac{\partial^2 P}{\partial r^2} = rP. \tag{152}$$

- This is called the term structure equation.
- It applies to all interest rate derivatives: The differences are the terminal and boundary conditions.
- Once P is available, the spot rate curve emerges via

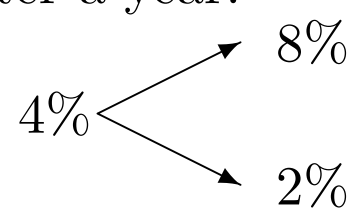
$$r(t,T) = -\frac{\ln P(t,T)}{T-t}.$$

Numerical Examples

- Assume this spot rate curve:

Year	1	2
Spot rate	4%	5%

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:



Numerical Examples (continued)

- *No* real-world probabilities are specified.
- The prices of one- and two-year zero-coupon bonds are, respectively,

$$\begin{aligned} 100/1.04 &= 96.154, \\ 100/(1.05)^2 &= 90.703. \end{aligned}$$

- They follow the binomial processes on p. 1104.

Numerical Examples (continued)

$$90.703 \begin{cases} \nearrow\ 92.593\ (= 100/1.08) \\ \searrow\ 98.039\ (= 100/1.02) \end{cases} \qquad 96.154 \begin{cases} \nearrow\ 100 \\ \searrow\ 100 \end{cases}$$

The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.

Numerical Examples (continued)

- The pricing of derivatives can be simplified by assuming investors are risk-neutral.
- Suppose all securities have the same expected one-period rate of return, the riskless rate.
- Then

$$(1-p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\%,$$

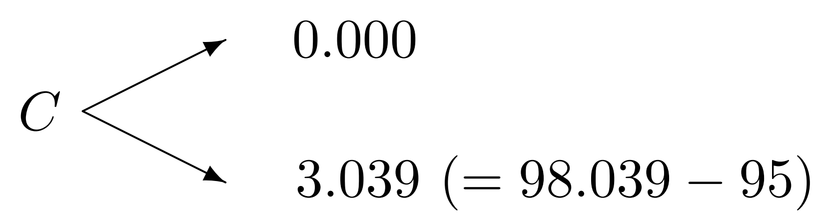
where p denotes the risk-neutral probability of a down move in rates.

Numerical Examples (concluded)

- Solving the equation leads to $p = 0.319$.
- Interest rate contingent claims can be priced under this probability.

Numerical Examples: Fixed-Income Options

- A one-year European call on the two-year zero with a \$95 strike price has the payoffs,



- To solve for the option value C, we replicate the call by a portfolio of x one-year and y two-year zeros.

Numerical Examples: Fixed-Income Options (continued)

- This leads to the simultaneous equations,

$$\begin{aligned} x \times 100 + y \times 92.593 &= 0.000, \\ x \times 100 + y \times 98.039 &= 3.039. \end{aligned}$$

- They give $x = -0.5167$ and $y = 0.5580$.
- Consequently,

$$C = x \times 96.154 + y \times 90.703 \approx 0.93$$

to prevent arbitrage.

Numerical Examples: Fixed-Income Options (continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.

Numerical Examples: Fixed-Income Options (concluded)

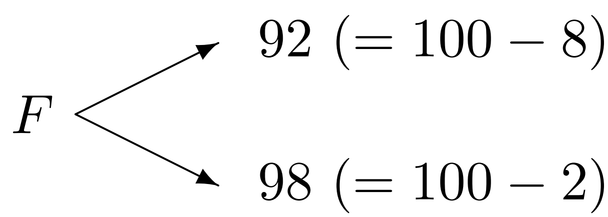
- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth

$$C = \frac{(1-p) \times 0 + p \times 3.039}{1.04} \approx 0.93,$$

  the same as before.
- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.

Numerical Examples: Futures and Forward Prices

- A one-year futures contract on the one-year rate has a payoff of $100 - r$, where r is the one-year rate at maturity:



- As the futures price F is the expected future payoff,[a]

$$F = (1 - p) \times 92 + p \times 98 = 93.914.$$

[a]See Exercise 13.2.11 of the textbook or p. 563.

Numerical Examples: Futures and Forward Prices (concluded)

- The forward price for a one-year forward contract on a one-year zero-coupon bond is[a]

$$90.703/96.154 = 94.331\%.$$

- The forward price exceeds the futures price.[b]

---

[a]By Eq. (141) on p. 1080.

[b]Unlike the nonstochastic case on p. 505.

Equilibrium Term Structure Models

The nature of modern trade
is to give to those who have much
and take from those who have little.
— Walter Bagehot (1867),
The English Constitution

8. What's your problem? Any moron
can understand bond pricing models.
— *Top Ten Lies Finance Professors
Tell Their Students*

Introduction

- We now survey equilibrium models.
- Recall that the spot rates satisfy

$$r(t,T) = -\frac{\ln P(t,T)}{T-t}$$

by Eq. (140) on p. 1079.

- Hence the discount function $P(t,T)$ suffices to establish the spot rate curve.
- All models to follow are short rate models.
- Unless stated otherwise, the processes are risk-neutral.

The Vasicek Model[a]

- The short rate follows

$$dr = \beta(\mu - r)\, dt + \sigma\, dW.$$

- The short rate is pulled to the long-term mean level μ at rate β.
- Superimposed on this "pull" is a normally distributed stochastic term $\sigma\, dW$.
- Since the process is an Ornstein-Uhlenbeck process,

$$E[\, r(T) \,|\, r(t) = r\,] = \mu + (r - \mu)\, e^{-\beta(T-t)}$$

from Eq. (86) on p. 630.

[a]Vasicek (1977). Vasicek co-founded KMV, which was sold to Moody's for USD$210 million in 2002.

The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

$$P(t,T) = A(t,T)\, e^{-B(t,T)\, r(t)}, \tag{153}$$

where

$$A(t,T) = \begin{cases} \exp\left[\frac{(B(t,T)-T+t)(\beta^2\mu-\sigma^2/2)}{\beta^2} - \frac{\sigma^2 B(t,T)^2}{4\beta} \right], & \text{if } \beta \neq 0, \\ \exp\left[\frac{\sigma^2 (T-t)^3}{6} \right], & \text{if } \beta = 0, \end{cases}$$

and

$$B(t,T) = \begin{cases} \frac{1-e^{-\beta(T-t)}}{\beta}, & \text{if } \beta \neq 0, \\ T-t, & \text{if } \beta = 0. \end{cases}$$

The Vasicek Model (continued)

- If $\beta = 0$, then P goes to infinity as $T \to \infty$.
- Sensibly, P goes to zero as $T \to \infty$ if $\beta \neq 0$.
- But even if $\beta \neq 0$, P may exceed one for a finite T.
- The long rate $r(t, \infty)$ is the constant

$$\mu - \frac{\sigma^2}{2\beta^2},$$

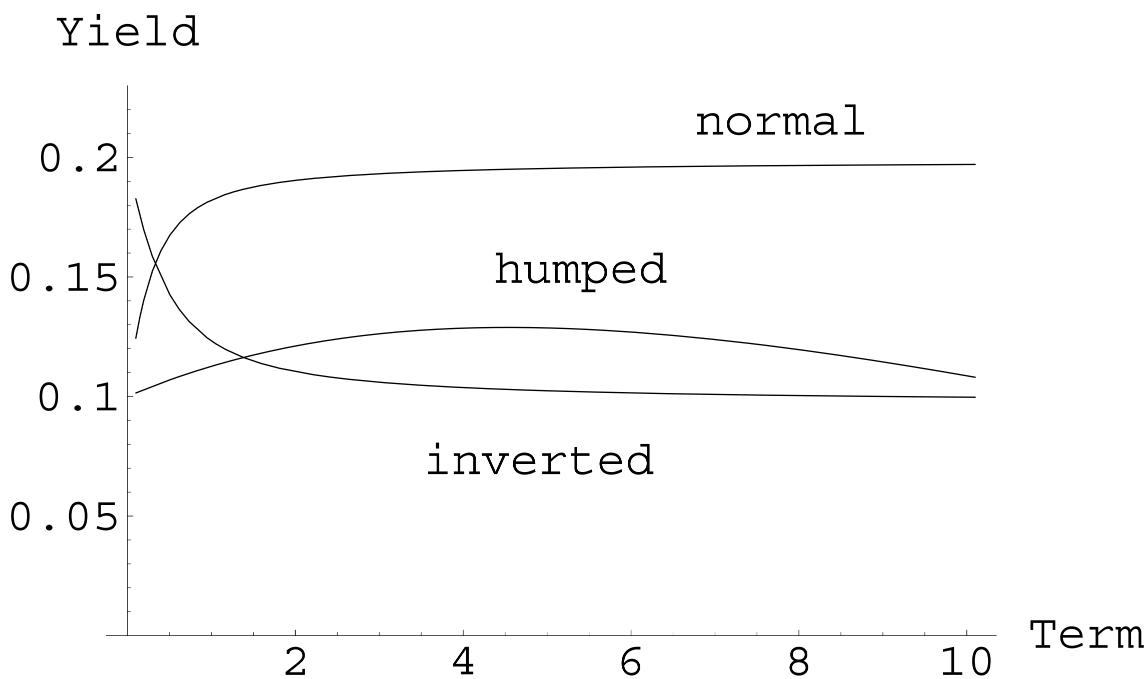
independent of the current short rate.

The Vasicek Model (concluded)

- The spot rate volatility structure is the curve

$$\sigma \, \frac{\partial r(t,T)}{\partial r} = \frac{\sigma B(t,T)}{T-t}.$$

- As it depends only on $T-t$ not on t by itself, the same curve is maintained for any future time t.
- When $\beta > 0$, the curve tends to decline with maturity.
  - The long rate's volatility is zero unless $\beta = 0$.
- The speed of mean reversion, β, controls the shape of the curve.
- Higher β leads to greater attenuation of volatility with maturity.


Yield
normal
humped
inverted
0.2
0.15
0.1
0.05
2
4
6
8
10
Term

The Vasicek Model: Options on Zeros[a]

- Consider a European call with strike price X expiring at time T on a zero-coupon bond with par value \$1 and maturing at time $s > T$.
- Its price is given by

$$P(t, s)\, N(x) - X P(t, T)\, N(x - \sigma_v).$$

[a] Jamshidian (1989).

The Vasicek Model: Options on Zeros (concluded)

- Above

$$
\begin{aligned}
x &\stackrel{\Delta}{=} \frac{1}{\sigma_v} \ln\left(\frac{P(t,s)}{P(t,T)\,X}\right) + \frac{\sigma_v}{2}, \\
\sigma_v &\equiv v(t,T)\,B(T,s), \\
v(t,T)^2 &\stackrel{\Delta}{=} \begin{cases} \frac{\sigma^2\left[1-e^{-2\beta(T-t)}\right]}{2\beta}, & \text{if } \beta \neq 0 \\ \sigma^2(T-t), & \text{if } \beta = 0 \end{cases}.
\end{aligned}
$$

- By the put-call parity, the price of a European put is

$$
XP(t,T)\,N(-x+\sigma_v) - P(t,s)\,N(-x).
$$

Binomial Vasicek[a]

- Consider a binomial model for the short rate in the time interval $[\,0, T\,]$ divided into n identical pieces.
- Let $\Delta t \stackrel{\Delta}{=} T/n$ and[b]

$$p(r) \stackrel{\Delta}{=} \frac{1}{2} + \frac{\beta(\mu - r)\sqrt{\Delta t}}{2\sigma}.$$

- The following binomial model converges to the Vasicek model,[c]

$$r(k+1) = r(k) + \sigma\sqrt{\Delta t}\;\xi(k), \quad 0 \leq k < n.$$

---

[a]Nelson & Ramaswamy (1990).

[b]The same form as Eq. (42) on p. 292 for the BOPM.

[c]Same as the CRR tree except that the probabilities vary here.

Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with

$$\text{Prob}[\,\xi(k) = 1\,] = \begin{cases} p(r(k)), & \text{if } 0 \leq p(r(k)) \leq 1 \\ 0, & \text{if } p(r(k)) < 0, \\ 1, & \text{if } 1 < p(r(k)). \end{cases}$$

- Observe that the probability of an up move, p, is a decreasing function of the interest rate r.
- This is consistent with mean reversion.

Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its *constant* volatility, σ.

The Cox-Ingersoll-Ross Model[a]

- It is the following square-root short rate model:

$$dr = \beta(\mu - r)\,dt + \sigma\sqrt{r}\,dW. \tag{154}$$

- The diffusion differs from the Vasicek model by a multiplicative factor $\sqrt{r}$.
- The parameter β determines the speed of adjustment.
- If $r(0) > 0$, then the short rate can reach zero *only if*

$$2\beta\mu < \sigma^2.$$

  - This is called the Feller (1951) condition.
- See text for the bond pricing formula.

[a]Cox, Ingersoll, & Ross (1985).

Binomial CIR

- We want to approximate the short rate process in the time interval $[\,0, T\,]$.
- Divide it into n periods of duration $\Delta t \triangleq T/n$.
- Assume $\mu, \beta \geq 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will *not* combine.

Binomial CIR (continued)

- Instead, consider the transformed process[a]

$$x(r) \triangleq 2\sqrt{r}/\sigma.$$

- By Ito's lemma (p. 605),

$$dx = m(x)\, dt + dW,$$

where

$$m(x) \triangleq 2\beta\mu/(\sigma^2 x) - (\beta x/2) - 1/(2x).$$

- This new process has a *constant* volatility.
- Thus its binomial tree combines.

[a]See pp. 1138ff for justification.

Binomial CIR (continued)

- Construct the combining tree for r as follows.
- First, construct a tree for x.
- Then transform each node of the tree into one for r via the inverse transformation (see next page)

$$r = f(x) \triangleq \frac{x^2 \sigma^2}{4}.$$

- But when $x \approx 0$ (so $r \approx 0$), the moments may not be matched well.[a]

[a]Nawalkha & Beliaeva (2007).

$$
\begin{array}{ccccc}
 & & & & x + 2\sqrt{\Delta t} \\
 & & & \nearrow & \\
 & & x + \sqrt{\Delta t} & & \\
 & \nearrow & & \searrow & \\
x & & & & x \\
 & \searrow & & \nearrow & \\
 & & x - \sqrt{\Delta t} & & \\
 & & & \searrow & \\
 & & & & x - 2\sqrt{\Delta t}
\end{array}
\qquad
\begin{array}{ccccc}
 & & & & f(x + 2\sqrt{\Delta t}) \\
 & & & \nearrow & \\
 & & f(x + \sqrt{\Delta t}) & & \\
 & \nearrow & & \searrow & \\
f(x) & & & & f(x) \\
 & \searrow & & \nearrow & \\
 & & f(x - \sqrt{\Delta t}) & & \\
 & & & \searrow & \\
 & & & & f(x - 2\sqrt{\Delta t})
\end{array}
$$

Binomial CIR (continued)

- The probability of an up move at each node r is

$$p(r) \stackrel{\Delta}{=} \frac{\beta(\mu - r)\,\Delta t + r - r^-}{r^+ - r^-}.$$

  - $r^+ \stackrel{\Delta}{=} f(x + \sqrt{\Delta t})$ denotes the result of an up move from r.
  - $r^- \stackrel{\Delta}{=} f(x - \sqrt{\Delta t})$ the result of a down move.

- Finally, set the probability $p(r)$ to one as r goes to zero to make the probability stay between zero and one.

Binomial CIR (concluded)

- It can be shown that

$$p(r) = \left(\beta\mu - \frac{\sigma^2}{4}\right)\sqrt{\frac{\Delta t}{r}} - B\sqrt{r\Delta t} + C,$$

for some $B \geq 0$ and $C > 0$.[a]

- If $\beta\mu - (\sigma^2/4) \geq 0$, the up-move probability $p(r)$ decreases if and only if short rate r increases.
- Even if $\beta\mu - (\sigma^2/4) < 0$, $p(r)$ *tends* to decrease as r increases and decrease as r declines.
- This phenomenon agrees with mean reversion.

[a] Thanks to a lively class discussion on May 28, 2014.

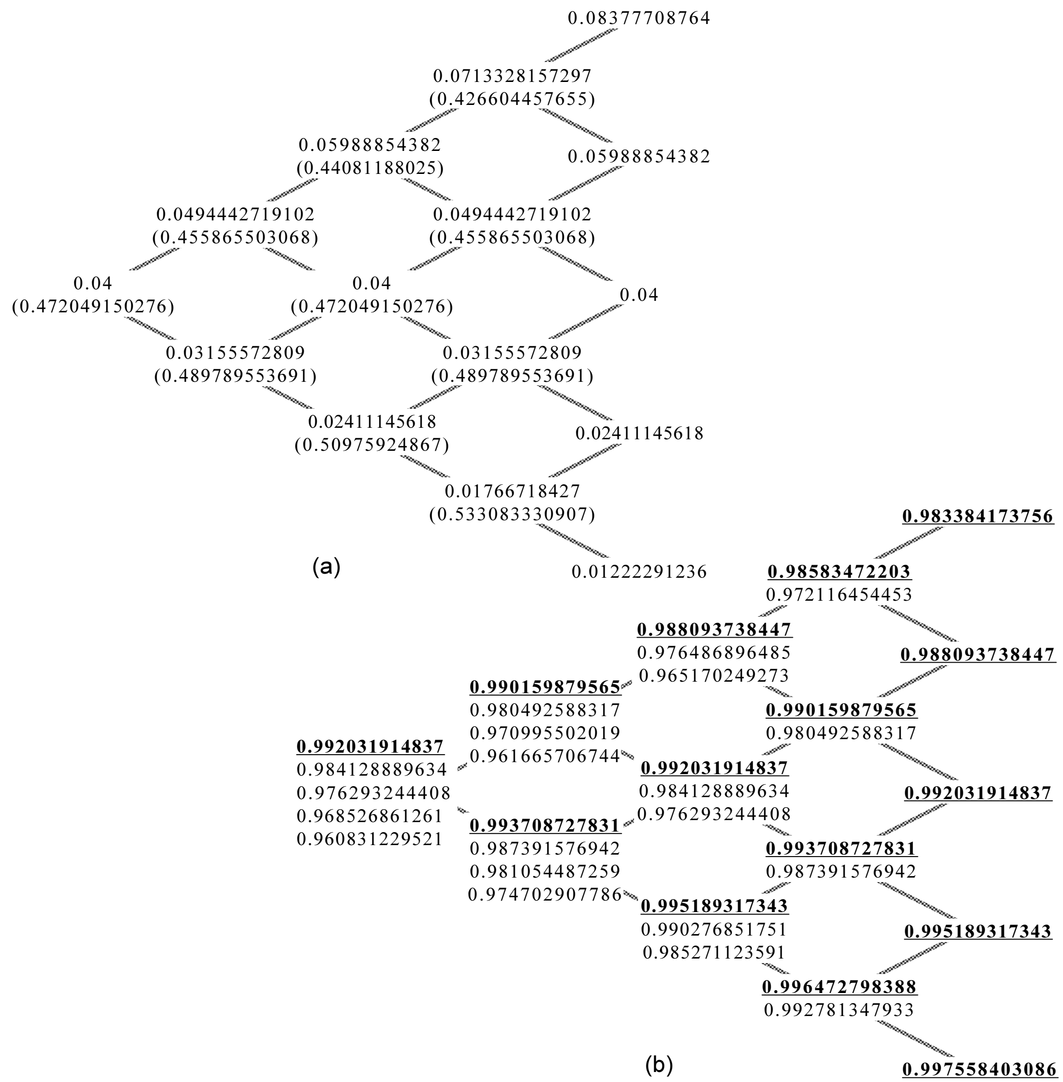
Numerical Examples

- Consider the process,

$$0.2\,(0.04 - r)\,dt + 0.1\sqrt{r}\,dW,$$

for the time interval $[\,0, 1\,]$ given the initial rate $r(0) = 0.04$.

- We shall use $\Delta t = 0.2$ (year) for the binomial approximation.
- See p. 1134(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.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(0.44081188025)
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0.990159879565
0.980492588317
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0.990276851751
0.985271123591
0.995189317343
0.996472798388
0.992781347933
0.997558403086
(b)

Numerical Examples (concluded)

- Consider the node which is the result of an up move from the root.
- Since the root has $x = 2\sqrt{r(0)}/\sigma = 4$, this particular node's x value equals $4 + \sqrt{\Delta t} = 4.4472135955$.
- Use the inverse transformation to obtain the short rate

$$\frac{x^2 \times (0.1)^2}{4} \approx 0.0494442719102.$$

- Once the short rates are in place, computing the probabilities is easy.
- Convergence is quite good.[a]

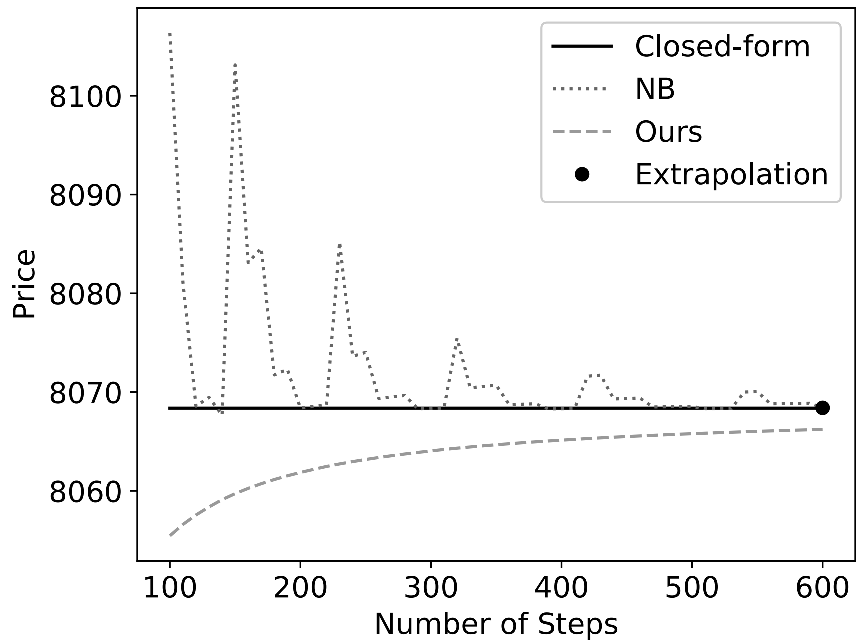
[a] See p. 369 of the textbook.

Trinomial CIR

- The binomial CIR tree does not have the degree of freedom to match the mean and variance exactly.
- It actually fails to match them at very low x.
- A trinomial tree for the CIR model with $O(n^{1.5})$ nodes that matches the mean and variance exactly is recently obtained using the ideas on pp. 792ff and others.[a]

[a] Z. Lu (D00922011) & Lyuu (2018); H. Huang (R03922103) (2019).

A Comparison[a]



$r(0) = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\beta = 1.2$, $T = 5$, principal is 10,000.

[a]Plot from H. Huang (R03922103) (2019).