*Stochastic Processes and Brownian Motion*

Of all the intellectual hurdles which the human mind
has confronted and has overcome in the last
fifteen hundred years, the one which seems to me
to have been the most amazing in character and
the most stupendous in the scope of its
consequences is the one relating to
the problem of motion.
— Herbert Butterfield (1900–1979)

## Stochastic Processes

- A stochastic process

$$X = \{ X(t) \}$$

is a time series of random variables.

- $X(t)$ (or $X_t$) is a random variable for each time $t$ and is usually called the state of the process at time $t$.

- A realization of $X$ is called a sample path.

## Stochastic Processes (concluded)

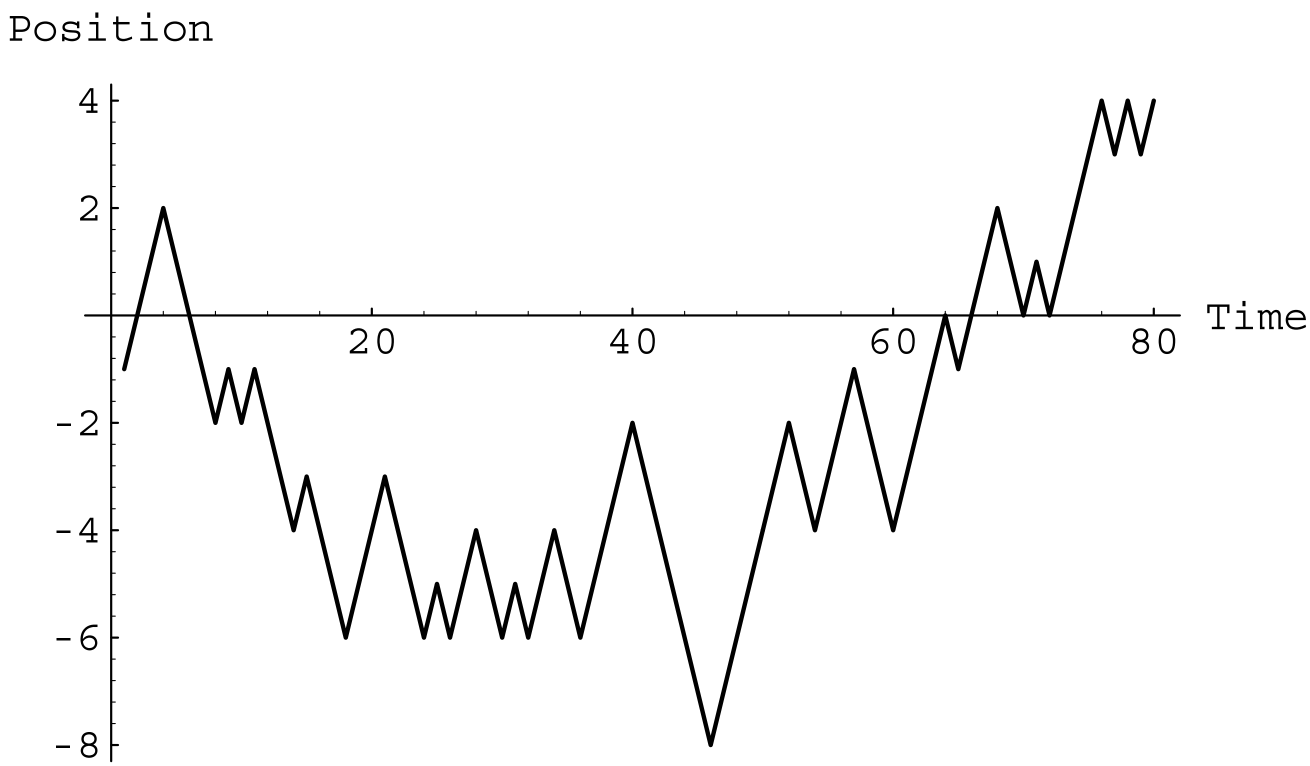
- If the times $t$ form a countable set, $X$ is called a discrete-time stochastic process or a time series.
- In this case, subscripts rather than parentheses are usually employed, as in

$$X = \{ X_n \}.$$

- If the times form a continuum, $X$ is called a continuous-time stochastic process.

## Random Walks

- The binomial model is a random walk in disguise.
- Consider a particle on the integer line, $0, \pm 1, \pm 2, \ldots$.
- In each time step, it can make one move to the right with probability $p$ or one move to the left with probability $1 - p$.
  - This random walk is symmetric when $p = 1/2$.
- Connection with the BOPM: The particle's position denotes the number of up moves minus that of down moves up to that time.


Position
4
2
-2
-4
-6
-8
20
40
60
80
Time

# Random Walk with Drift

$$X_n = \mu + X_{n-1} + \xi_n.$$

- $\xi_n$ are independent and identically distributed with zero mean.
- Drift $\mu$ is the expected change per period.
- Note that this process is continuous in space.

# Martingales[a]

- $\{ X(t), t \geq 0 \}$ is a martingale if $E[\,|\,X(t)\,|\,] < \infty$ for $t \geq 0$ and

$$E[\,X(t)\,|\,X(u), 0 \leq u \leq s\,] = X(s), \quad s \leq t. \tag{68}$$

- In the discrete-time setting, a martingale means

$$E[\,X_{n+1}\,|\,X_1, X_2, \ldots, X_n\,] = X_n. \tag{69}$$

- $X_n$ can be interpreted as a gambler's fortune after the $n$th gamble.
- Identity (69) then says the expected fortune after the $(n+1)$th gamble equals the fortune after the $n$th gamble regardless of what may have occurred before.

[a] The origin of the name is somewhat obscure.

## Martingales (concluded)

- A martingale is therefore a notion of fair games.
- Apply the law of iterated conditional expectations to both sides of Eq. (69) on p. 544 to yield

$$E[\, X_n \,] = E[\, X_1 \,] \tag{70}$$

  for all $n$.
- Similarly,

$$E[\, X(t) \,] = E[\, X(0) \,]$$

  in the continuous-time case.

## Still a Martingale?

- Suppose we replace Eq. (69) on p. 544 with

$$E[\, X_{n+1} \,|\, X_n \,] = X_n.$$

- It also says past history cannot affect the future.
- But is it equivalent to the original definition (69) on p. 544?[a]

[a]Contributed by Mr. Hsieh, Chicheng (`M9007304`) on April 13, 2005.

## Still a Martingale? (continued)

- Well, no.[a]
- Consider this random walk with drift:

$$X_i = \begin{cases} X_{i-1} + \xi_i, & \text{if } i \text{ is even,} \\ X_{i-2}, & \text{otherwise.} \end{cases}$$

- Above, $\xi_n$ are random variables with zero mean.

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[a]Contributed by Mr. Zhang, Ann-Sheng (B89201033) on April 13, 2005.

# Still a Martingale? (concluded)

- It is not hard to see that

$$E[\,X_i \mid X_{i-1}\,] = \begin{cases} X_{i-1}, & \text{if } i \text{ is even,} \\ X_{i-1}, & \text{otherwise.} \end{cases}$$

  – It is a martingale by the "new" definition.

- But

$$E[\,X_i \mid \ldots, X_{i-2}, X_{i-1}\,] = \begin{cases} X_{i-1}, & \text{if } i \text{ is even,} \\ X_{i-2}, & \text{otherwise.} \end{cases}$$

  – It is not a martingale by the original definition.

## Example

- Consider the stochastic process

$$\left\{ Z_n \stackrel{\Delta}{=} \sum_{i=1}^{n} X_i, n \geq 1 \right\},$$

where $X_i$ are independent random variables with zero mean.

- This process is a martingale because

$$\begin{aligned} & E[\, Z_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n \,] \\ = \;& E[\, Z_n + X_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n \,] \\ = \;& E[\, Z_n \,|\, Z_1, Z_2, \ldots, Z_n \,] + E[\, X_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n \,] \\ = \;& Z_n + E[\, X_{n+1} \,] = Z_n. \end{aligned}$$

# Probability Measure

- A probability measure assigns probabilities to states of the world.[a]
- A martingale is defined with respect to a probability measure, under which the expectation is taken.
- Second, a martingale is defined with respect to an information set.
  - In the characterizations (68)–(69) on p. 544, the information set contains the current and past values of $X$ by default.
  - But it need not be so.

[a]Only certain sets such as Borel sets receive probabilities (Feller, 1972).

## Probability Measure (continued)

- A stochastic process $\{ X(t), t \geq 0 \}$ is a martingale with respect to information sets $\{ I_t \}$ if, for all $t \geq 0$, $E[\,|\,X(t)\,|\,] < \infty$ and

$$E[\, X(u) \,|\, I_t \,] = X(t)$$

for all $u > t$.

- The discrete-time version: For all $n > 0$,

$$E[\, X_{n+1} \,|\, I_n \,] = X_n,$$

given the information sets $\{ I_n \}$.

## Probability Measure (concluded)

- The above implies

$$E[\,X_{n+m} \,|\, I_n\,] = X_n$$

for any $m > 0$ by Eq. (26) on p. 169.

  - A typical $I_n$ is the price information up to time $n$.
  - Then the above identity says the FVs of $X$ will not deviate systematically from today's value given the price history.

## Example

- Consider the stochastic process $\{ Z_n - n\mu, n \geq 1 \}$.
  - $Z_n \stackrel{\Delta}{=} \sum_{i=1}^{n} X_i$.
  - $X_1, X_2, \ldots$ are independent random variables with mean $\mu$.
- Now,

$$
\begin{aligned}
& E[\, Z_{n+1} - (n+1)\,\mu \,|\, X_1, X_2, \ldots, X_n \,] \\
= \; & E[\, Z_{n+1} \,|\, X_1, X_2, \ldots, X_n \,] - (n+1)\,\mu \\
= \; & E[\, Z_n + X_{n+1} \,|\, X_1, X_2, \ldots, X_n \,] - (n+1)\,\mu \\
= \; & Z_n + \mu - (n+1)\,\mu \\
= \; & Z_n - n\mu.
\end{aligned}
$$

## Example (concluded)

- Define
$$I_n \triangleq \{ X_1, X_2, \ldots, X_n \}.$$
- Then
$$\{ Z_n - n\mu, n \geq 1 \}$$
is a martingale with respect to $\{ I_n \}$.

## Martingale Pricing

- Stock prices and zero-coupon bond prices are expected to rise, while option prices are expected to fall.
- They are *not* martingales.
- Why is then martingale useful?
- Recall a martingale is defined with respect to some information set *and* some probability measure.
- By modifying the probability measure, we can convert a price process into a martingale.

## Martingale Pricing (continued)

- The price of a European option is the expected discounted payoff in a risk-neutral economy.[a]
- This principle can be generalized using the concept of martingale.
- Recall the recursive valuation of European option via

$$C = [\, pC_u + (1-p)\, C_d \,]/R.$$

  - $p$ is the risk-neutral probability.
  - \$1 grows to \$$R$ in a period.

---

[a] Recall Eq. (37) on p. 265.

## Martingale Pricing (continued)

- Let $C(i)$ denote the value of the option at time $i$.
- Consider the discount process

$$\left\{ \frac{C(i)}{R^i}, i = 0, 1, \ldots, n \right\}.$$

- Then,

$$\begin{aligned} E\left[ \frac{C(i+1)}{R^{i+1}} \,\middle|\, S(i) \right] &= \frac{E[\,C(i+1) \,|\, S(i)\,]}{R^{i+1}} \\ &= \frac{pC_u + (1-p)\,C_d}{R^{i+1}} \\ &= \frac{C(i)}{R^i}. \end{aligned}$$

## Martingale Pricing (continued)

- It is easy to show that

$$E\left[\left.\frac{C(k)}{R^k}\,\right|\,S(i)\right]=\frac{C(i)}{R^i},\quad i\leq k.$$

- This simplified formulation assumes:[a]
  1. The model is Markovian: The distribution of the future is determined by the present (time $i$) and not the past.
  2. The payoff depends only on the terminal price of the underlying asset[b] (Asian options do not qualify).

[a]Contributed by Mr. Wang, Liang-Kai (Ph.D. student, ECE, University of Wisconsin-Madison) and Mr. Hsiao, Huan-Wen (B90902081) on May 3, 2006.

[b]Recall they are called simple claims.

# Martingale Pricing (continued)

- In general, the discount process is a martingale in that[a]

$$E_i^{\pi}\left[\frac{C(k)}{R^k}\right] = \frac{C(i)}{R^i}, \quad i \leq k. \tag{71}$$

  - $E_i^{\pi}$ is taken under the risk-neutral probability conditional on the price information *up to time* $i$.

- This risk-neutral probability is also called the EMM, or the equivalent[b] martingale (probability) measure.

[a] In this general formulation, Asian options do qualify.

[b] Two probability measures are said to be equivalent if they assign nonzero probabilities to the same set of states.

## Martingale Pricing (continued)

- Equation (71) holds for all assets, not just options.
- When interest rates are stochastic, the equation becomes

$$\frac{C(i)}{M(i)} = E_i^{\pi}\left[\frac{C(k)}{M(k)}\right], \quad i \leq k. \tag{72}$$

  - $M(j)$ is the balance in the money market account at time $j$ using the rollover strategy with an initial investment of \$1.
  - It is called the bank account process.
- It says the discount process is a martingale under $\pi$.

## Martingale Pricing (continued)

- If interest rates are stochastic, then $M(j)$ is a random variable.
  - $M(0) = 1$.
  - $M(j)$ is known at time $j - 1$.[a]
- Identity (72) on p. 560 is the general formulation of risk-neutral valuation.

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[a] Because the interest rate for the *next* period has been revealed then.

## Martingale Pricing (concluded)

**Theorem 16** *A discrete-time model is arbitrage-free if and only if there exists an equivalent probability measure*[a] *such that the discount process is a martingale.*

---

[a] Called the risk-neutral probability measure.

# Futures Price under the BOPM

- Futures prices form a martingale under the risk-neutral probability.
  - The expected futures price in the next period is[a]

$$p_{\mathrm{f}} F u + (1 - p_{\mathrm{f}})\, F d = F \left( \frac{1-d}{u-d}\, u + \frac{u-1}{u-d}\, d \right) = F.$$

- Can be generalized to

$$F_i = E_i^{\pi}[\, F_k \,], \quad i \leq k,$$

  where $F_i$ is the futures price at time $i$.

- This equation holds under stochastic interest rates, too.[b]

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[a] Recall p. 517.

[b] See Exercise 13.2.11 of the textbook.

## Martingale Pricing and Numeraire[a]

- The martingale pricing formula (72) on p. 560 uses the money market account as numeraire.
  - It expresses the price of any asset *relative to* the money market account.[b]
- The money market account is not the only choice for numeraire.
- Suppose asset $S$'s value is *positive* at all time.

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[a] John Law (1671–1729), "Money to be qualified for exchaning goods and for payments need not be certain in its value."

[b] Leon Walras (1834–1910).

## Martingale Pricing and Numeraire (concluded)

- Choose $S$ as numeraire.
- Martingale pricing says there exists a risk-neutral probability $\pi$ under which the relative price of any asset $C$ is a martingale:

$$\frac{C(i)}{S(i)} = E_i^{\pi}\left[\frac{C(k)}{S(k)}\right], \quad i \leq k.$$

  - $S(j)$ denotes the price of $S$ at time $j$.
- So the discount process remains a martingale.[a]

[a] This result is related to Girsanov's theorem (1960).

## Example

- Take the binomial model with two assets.
- In a period, asset one's price can go from $S$ to $S_1$ or $S_2$.
- In a period, asset two's price can go from $P$ to $P_1$ or $P_2$.
- Both assets must move up or down at the same time.
- Assume

$$\frac{S_1}{P_1} < \frac{S}{P} < \frac{S_2}{P_2} \tag{73}$$

to rule out arbitrage opportunities.

## Example (continued)

- For any derivative security, let $C_1$ be its price at time one if asset one's price moves to $S_1$.
- Let $C_2$ be its price at time one if asset one's price moves to $S_2$.
- Replicate the derivative by solving

$$\begin{aligned} \alpha S_1 + \beta P_1 &= C_1, \\ \alpha S_2 + \beta P_2 &= C_2, \end{aligned}$$

using $\alpha$ units of asset one and $\beta$ units of asset two.

## Example (continued)

- By inequalities (73) on p. 566, $\alpha$ and $\beta$ have unique solutions.
- In fact,

$$\alpha = \frac{P_2 C_1 - P_1 C_2}{P_2 S_1 - P_1 S_2} \quad \text{and} \quad \beta = \frac{S_2 C_1 - S_1 C_2}{S_2 P_1 - S_1 P_2}.$$

- The derivative costs

$$\begin{aligned} C &= \alpha S + \beta P \\ &= \frac{P_2 S - P S_2}{P_2 S_1 - P_1 S_2} C_1 + \frac{P S_1 - P_1 S}{P_2 S_1 - P_1 S_2} C_2. \end{aligned}$$

## Example (continued)

- It is easy to verify that

$$\frac{C}{P} = p\,\frac{C_1}{P_1} + (1-p)\,\frac{C_2}{P_2}$$

with

$$p \triangleq \frac{(S/P) - (S_2/P_2)}{(S_1/P_1) - (S_2/P_2)}.$$

- By inequalities (73) on p. 566, $0 < p < 1$.
- $C$'s price using asset two as numeraire (i.e., $C/P$) is a martingale under the risk-neutral probability $p$.
- The expected returns of the two assets are *irrelevant*.

## Example (concluded)

- In the BOPM, $S$ is the stock and $P$ is the bond.
- Furthermore, $p$ assumes the bond is the numeraire.
- In the binomial option pricing formula (39) on p. 269, $S \sum b(j; n, pu/R)$ uses *stock* as the numeraire.
  - Its probability measure $pu/R$ differs from $p$.
- $SN(x)$ for the call and $SN(-x)$ for the put in the Black-Scholes formulas (p. 299) use stock as the numeraire as well.[a]

[a] See Exercise 13.2.12 of the textbook.

## Brownian Motion[a]

- Brownian motion is a stochastic process $\{ X(t), t \geq 0 \}$ with the following properties.

  **1.** $X(0) = 0$, unless stated otherwise.

  **2.** for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables

  $$X(t_k) - X(t_{k-1})$$

  for $1 \leq k \leq n$ are independent.[b]

  **3.** for $0 \leq s < t$, $X(t) - X(s)$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$, where $\mu$ and $\sigma \neq 0$ are real numbers.

---

[a] Robert Brown (1773–1858).

[b] So $X(t) - X(s)$ is independent of $X(r)$ for $r \leq s < t$.

# Brownian Motion (concluded)

- The existence and uniqueness of such a process is guaranteed by Wiener's theorem.[a]
- This process will be called a $(\mu, \sigma)$ Brownian motion with drift $\mu$ and variance $\sigma^2$.
- Although Brownian motion is a continuous function of $t$ with probability one, it is almost nowhere differentiable.
- The $(0, 1)$ Brownian motion is called the Wiener process.
- If condition 3 is replaced by "$X(t) - X(s)$ depends only on $t - s$," we have the more general Levy process.[b]

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[a] Norbert Wiener (1894–1964). He received his Ph.D. from Harvard in 1912.

[b] Paul Levy (1886–1971).

## Example

- If $\{ X(t), t \geq 0 \}$ is the Wiener process, then

$$X(t) - X(s) \sim N(0, t-s).$$

- A $(\mu, \sigma)$ Brownian motion $Y = \{ Y(t), t \geq 0 \}$ can be expressed in terms of the Wiener process:

$$Y(t) = \mu t + \sigma X(t). \tag{74}$$

- Note that

$$Y(t+s) - Y(t) \sim N(\mu s, \sigma^2 s).$$

## Brownian Motion as Limit of Random Walk

**Claim 1** *A* $(\mu, \sigma)$ *Brownian motion is the limiting case of random walk.*

- A particle moves $\Delta x$ to the right with probability $p$ after $\Delta t$ time.
- It moves $\Delta x$ to the left with probability $1 - p$.
- Define

$$X_i \stackrel{\Delta}{=} \begin{cases} +1 & \text{if the } i\text{th move is to the right,} \\ -1 & \text{if the } i\text{th move is to the left.} \end{cases}$$

  - $X_i$ are independent with

$$\text{Prob}[\, X_i = 1\,] = p = 1 - \text{Prob}[\, X_i = -1\,].$$

## Brownian Motion as Limit of Random Walk (continued)

- Recall

$$
\begin{aligned}
E[\,X_i\,] &= 2p-1, \\
\mathrm{Var}[\,X_i\,] &= 1-(2p-1)^2.
\end{aligned}
$$

- Assume $n \triangleq t/\Delta t$ is an integer.
- Its position at time $t$ is

$$
Y(t) \triangleq \Delta x\,(X_1 + X_2 + \cdots + X_n)\,.
$$

## Brownian Motion as Limit of Random Walk (continued)

- Therefore,

$$E[\,Y(t)\,] = n(\Delta x)(2p-1),$$
$$\mathrm{Var}[\,Y(t)\,] = n(\Delta x)^2 \left[\,1-(2p-1)^2\,\right].$$

- With $\Delta x \stackrel{\Delta}{=} \sigma\sqrt{\Delta t}$ and $p \stackrel{\Delta}{=} [\,1+(\mu/\sigma)\sqrt{\Delta t}\,]/2$,[a]

$$\begin{aligned} E[\,Y(t)\,] &= n\sigma\sqrt{\Delta t}\,(\mu/\sigma)\sqrt{\Delta t} = \mu t, \\ \mathrm{Var}[\,Y(t)\,] &= n\sigma^2\Delta t\left[\,1-(\mu/\sigma)^2\Delta t\,\right] \to \sigma^2 t, \end{aligned}$$

as $\Delta t \to 0$.

[a] Identical to Eq. (42) on p. 292!

## Brownian Motion as Limit of Random Walk (concluded)

- Thus, $\{ Y(t), t \geq 0 \}$ converges to a $(\mu, \sigma)$ Brownian motion by the central limit theorem.
- Brownian motion with zero drift is the limiting case of symmetric random walk by choosing $\mu = 0$.
- Similarity to the the BOPM: The $p$ is identical to the probability in Eq. (42) on p. 292 and $\Delta x = \ln u$.
- Note that

$$\begin{aligned} & \mathrm{Var}[\, Y(t + \Delta t) - Y(t)\,] \\ =& \mathrm{Var}[\, \Delta x\, X_{n+1}\,] = (\Delta x)^2 \times \mathrm{Var}[\, X_{n+1}\,] \to \sigma^2 \Delta t. \end{aligned}$$

# Geometric Brownian Motion

- Let $X \stackrel{\Delta}{=} \{ X(t), t \geq 0 \}$ be a Brownian motion process.
- The process

$$\{ Y(t) \stackrel{\Delta}{=} e^{X(t)}, t \geq 0 \},$$

is called geometric Brownian motion.

- Suppose further that $X$ is a $(\mu, \sigma)$ Brownian motion.
- By assumption, $Y(0) = e^0 = 1$.

# Geometric Brownian Motion (concluded)

- $X(t) \sim N(\mu t, \sigma^2 t)$ with moment generating function

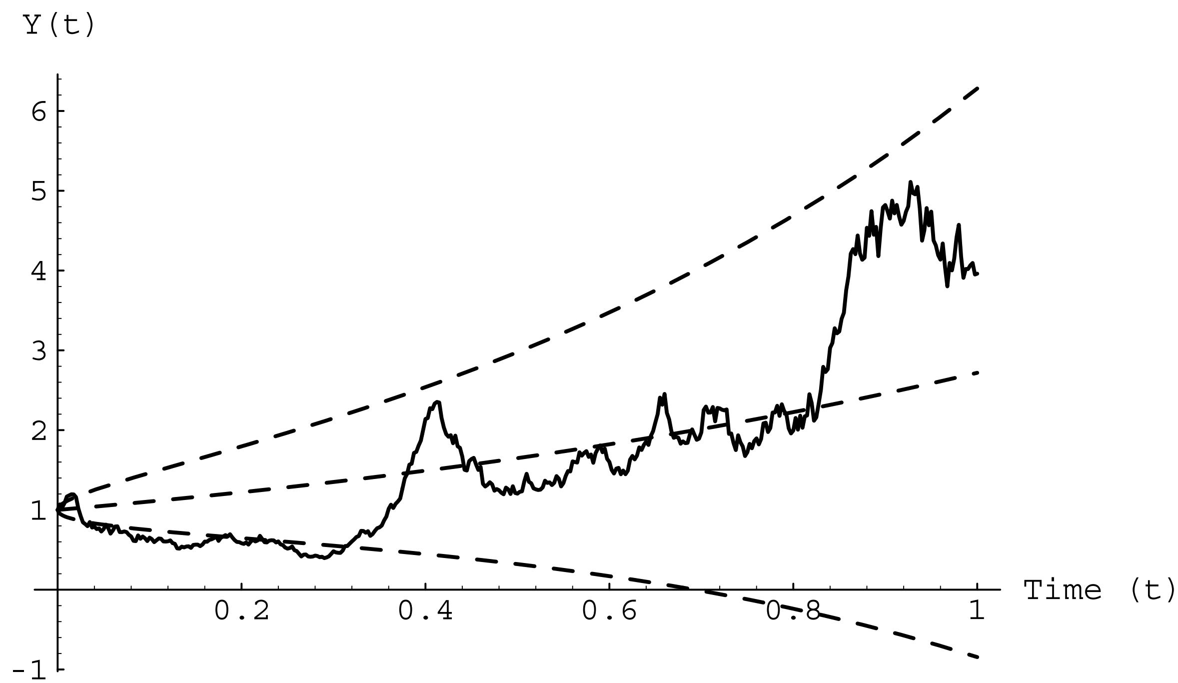
$$E\left[e^{sX(t)}\right] = E\left[\,Y(t)^s\,\right] = e^{\mu ts+(\sigma^2 ts^2/2)}$$

from Eq. (27) on p 171.

- In particular,[a]

$$\begin{aligned} E[\,Y(t)\,] &= e^{\mu t+(\sigma^2 t/2)}, \\ \mathrm{Var}[\,Y(t)\,] &= E\left[\,Y(t)^2\,\right] - E[\,Y(t)\,]^2 \\ &= e^{2\mu t+\sigma^2 t}\left(e^{\sigma^2 t} - 1\right). \end{aligned}$$

[a]Recall Eqs. (29) on p. 180.


Y(t)
6
5
4
3
2
1
-1
0.2
0.4
0.6
0.8
1
Time (t)

## A Case for Long-Term Investment[a]

- Suppose the stock follows the geometric Brownian motion

$$S(t) = S(0)\, e^{N(\mu t, \sigma^2 t)} = S(0)\, e^{tN(\mu, \sigma^2/t)}, \quad t \geq 0,$$

where $\mu > 0$.

- The annual rate of return has a normal distribution:

$$N\left(\mu, \frac{\sigma^2}{t}\right).$$

- The larger the $t$, the likelier the return is positive.
- The smaller the $t$, the likelier the return is negative.

[a] Contributed by Dr. King, Gow-Hsing on April 9, 2015. See http://www.cb.idv.tw/phpbb3/viewtopic.php?f=7&t=1025

## *Continuous-Time Financial Mathematics*

A proof is that which convinces a reasonable man;
a rigorous proof is that which convinces an
unreasonable man.
— Mark Kac (1914–1984)

The pursuit of mathematics is a
divine madness of the human spirit.
— Alfred North Whitehead (1861–1947),
*Science and the Modern World*

# Stochastic Integrals

- Use $W \stackrel{\Delta}{=} \{ W(t), t \geq 0 \}$ to denote the Wiener process.
- The goal is to develop integrals of $X$ from a class of stochastic processes,[a]

$$\mathrm{I}_t(X) \stackrel{\Delta}{=} \int_0^t X \, dW, \quad t \geq 0.$$

- $\mathrm{I}_t(X)$ is a random variable called the stochastic integral of $X$ with respect to $W$.
- The stochastic process $\{ \mathrm{I}_t(X), t \geq 0 \}$ will be denoted by $\int X \, dW$.

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[a] Kiyoshi Ito (1915–2008).

## Stochastic Integrals (concluded)

- Typical requirements for $X$ in financial applications are:
  - $\text{Prob}[\int_0^t X^2(s)\, ds < \infty] = 1$ for all $t \geq 0$ or the stronger $\int_0^t E[X^2(s)]\, ds < \infty$.
  - The information set at time $t$ includes the history of $X$ and $W$ up to that point in time.
  - But it contains nothing about the evolution of $X$ or $W$ after $t$ (nonanticipating, so to speak).
  - The future cannot influence the present.

# Ito Integral

- A theory of stochastic integration.
- As with calculus, it starts with step functions.
- A stochastic process $\{ X(t) \}$ is simple if there exist

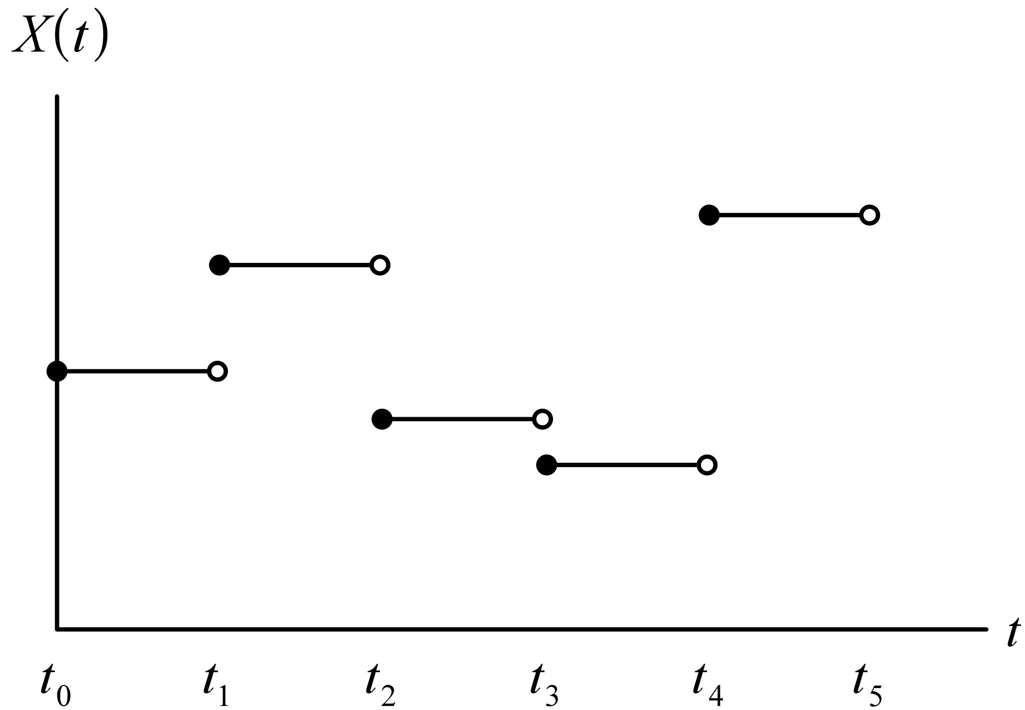
$$0 = t_0 < t_1 < \cdots$$

such that

$$X(t) = X(t_{k-1}) \text{ for } t \in [\, t_{k-1}, t_k), \, k = 1, 2, \ldots$$

for any realization (see figure on next page).[a]

[a] It is right-continuous.



X(t)
t
t0
t1
t2
t3
t4
t5

# Ito Integral (continued)

- The Ito integral of a simple process is defined as

$$\mathrm{I}_t(X) \triangleq \sum_{k=0}^{n-1} X(t_k)[\,W(t_{k+1}) - W(t_k)\,], \tag{75}$$

  where $t_n = t$.

  – The integrand $X$ is evaluated at $t_k$, not $t_{k+1}$.

- Define the Ito integral of more general processes as a limiting random variable of the Ito integral of simple stochastic processes.

# Ito Integral (continued)

- Let $X = \{ X(t), t \geq 0 \}$ be a general stochastic process.
- Then there exists a random variable $\mathrm{I}_t(X)$, unique almost certainly, such that $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ for each sequence of simple stochastic processes $X_1, X_2, \ldots$ such that $X_n$ converges in probability to $X$.
- If $X$ is continuous with probability one, then $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ as

$$\delta_n \stackrel{\Delta}{=} \max_{1 \leq k \leq n} (t_k - t_{k-1})$$

goes to zero.

# Ito Integral (concluded)

- It is a fundamental fact that $\int X\,dW$ is continuous almost surely.
- The following theorem says the Ito integral is a martingale.[a]

**Theorem 17** *The Ito integral* $\int X\,dW$ *is a martingale.*

- A corollary is the mean value formula

$$E\left[\int_a^b X\,dW\right]=0.$$

[a] See Exercise 14.1.1 for simple stochastic processes.

# Discrete Approximation

- Recall Eq. (75) on p. 588.
- The following *simple* stochastic process $\{ \widehat{X}(t) \}$ can be used in place of $X$ to approximate $\int_0^t X\, dW$,

$$\widehat{X}(s) \stackrel{\Delta}{=} X(t_{k-1}) \text{ for } s \in [\, t_{k-1}, t_k), \, k = 1, 2, \ldots, n.$$

- Note the *nonanticipating* feature of $\widehat{X}$.
  - The information up to time $s$,

$$\{ \widehat{X}(t), W(t), 0 \leq t \leq s \},$$

    cannot determine the future evolution of $X$ or $W$.

# Discrete Approximation (concluded)

- Suppose, unlike Eq. (75) on p. 588, we defined the stochastic integral from

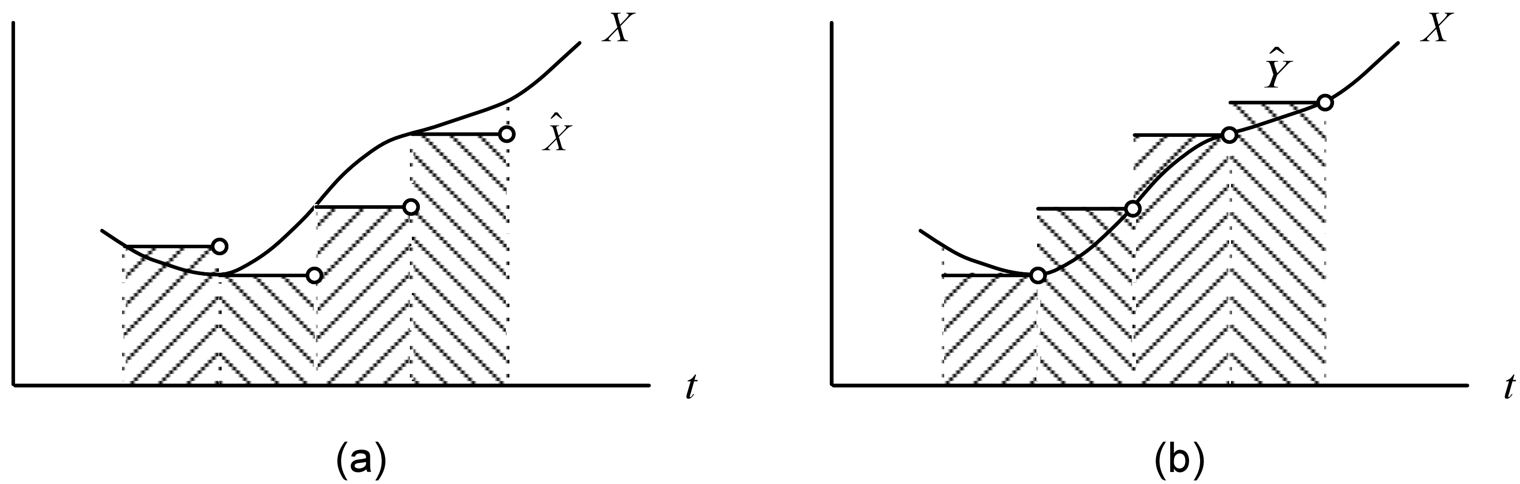
$$\sum_{k=0}^{n-1} X(t_{k+1})[\, W(t_{k+1}) - W(t_k)\,].$$

- Then we would be using the following different simple stochastic process in the approximation,

$$\widehat{Y}(s) \stackrel{\Delta}{=} X(t_k) \quad \text{for } s \in [\, t_{k-1}, t_k), \, k = 1, 2, \ldots, n.$$

- This clearly anticipates the future evolution of $X$.[a]

[a]See Exercise 14.1.2 of the textbook for an example where it matters.



X
$\hat{X}$
t
(a)
X
$\hat{Y}$
t
(b)

# Ito Process

- The stochastic process $X = \{ X_t, t \geq 0 \}$ that solves

$$X_t = X_0 + \int_0^t a(X_s, s)\, ds + \int_0^t b(X_s, s)\, dW_s, \quad t \geq 0$$

  is called an Ito process.

  - $X_0$ is a scalar starting point.
  - $\{ a(X_t, t) : t \geq 0 \}$ and $\{ b(X_t, t) : t \geq 0 \}$ are stochastic processes satisfying certain regularity conditions.
  - $a(X_t, t)$: the drift.
  - $b(X_t, t)$: the diffusion.

# Ito Process (continued)

- Typical regularity conditions are:[a]
  - For all $T > 0$, $x \in \mathbb{R}^n$, and $0 \leq t \leq T$,

    $$|\, a(x,t) \,| + |\, b(x,t) \,| \leq C(1 + |\, x \,|)$$

    for some constant $C$.[b]
  - (Lipschitz continuity) For all $T > 0$, $x \in \mathbb{R}^n$, and $0 \leq t \leq T$,

    $$|\, a(x,t) - a(y,t) \,| + |\, b(x,t) - b(y,t) \,| \leq D \,|\, x - y \,|$$

    for some constant $D$.

---

[a] Øksendal (2007).

[b] This condition is not needed in *time-homogeneous* cases, where $a$ and $b$ do not depend on $t$.

## Ito Process (continued)

- A shorthand[a] is the following stochastic differential equation[b] (SDE) for the Ito differential $dX_t$,

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t. \tag{76}$$

  - Or simply

$$dX_t = a_t\, dt + b_t\, dW_t.$$

  - This is Brownian motion with an *instantaneous* drift $a_t$ and an *instantaneous* variance $b_t^2$.

- $X$ is a martingale if $a_t = 0$.[c]

---

[a]Paul Langevin (1872–1946) in 1904.

[b]Like any equation, an SDE contains an unknown, the process $X_t$.

[c]Recall Theorem 17 (p. 590).

# Ito Process (concluded)

- From calculus, we would expect $\int_0^t W\, dW = W(t)^2/2$.
- But $W(t)^2/2$ is not a martingale, hence wrong!
- The correct answer is $[\, W(t)^2 - t\,]/2$.
- A popular representation of Eq. (76) is

$$dX_t = a_t\, dt + b_t \sqrt{dt}\; \xi, \tag{77}$$

where $\xi \sim N(0,1)$.

# Euler Approximation

- Define $t_n \stackrel{\Delta}{=} n\Delta t$.
- The following approximation follows from Eq. (77),

$$\begin{aligned}&\widehat{X}(t_{n+1})\\ =&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\,\Delta W(t_n). \end{aligned} \tag{78}$$

- It is called the Euler or Euler-Maruyama method.
- Recall that $\Delta W(t_n)$ should be interpreted as

$$W(t_{n+1}) - W(t_n),$$

not $W(t_n) - W(t_{n-1})$![a]

[a]Recall Eq. (75) on p. 588.

## Euler Approximation (concluded)

- With the Euler method, one can obtain a sample path

$$\widehat{X}(t_1), \widehat{X}(t_2), \widehat{X}(t_3), \ldots$$

  from a sample path

$$W(t_0), W(t_1), W(t_2), \ldots .$$

- Under mild conditions, $\widehat{X}(t_n)$ converges to $X(t_n)$.

# More Discrete Approximations

- Under fairly loose regularity conditions, Eq. (78) on p. 598 can be replaced by

$$\begin{aligned} &\widehat{X}(t_{n+1}) \\ =&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\sqrt{\Delta t}\,Y(t_n). \end{aligned}$$

  - $Y(t_0), Y(t_1), \ldots$ are independent and identically distributed with zero mean and unit variance.

## More Discrete Approximations (concluded)

- An even simpler discrete approximation scheme:

$$\begin{aligned}&\widehat{X}(t_{n+1})\\ =&\widehat{X}(t_n)+a(\widehat{X}(t_n),t_n)\,\Delta t+b(\widehat{X}(t_n),t_n)\sqrt{\Delta t}\,\xi.\end{aligned}$$

  - $\text{Prob}[\,\xi=1\,]=\text{Prob}[\,\xi=-1\,]=1/2$.
  - Note that $E[\,\xi\,]=0$ and $\text{Var}[\,\xi\,]=1$.
- This is a binomial model.
- As $\Delta t$ goes to zero, $\widehat{X}$ converges to $X$.[a]

[a]He (1990).

## Trading and the Ito Integral

- Consider an Ito process

$$d\boldsymbol{S}_t = \mu_t\, dt + \sigma_t\, dW_t.$$

  – $\boldsymbol{S}_t$ is the vector of security prices at time $t$.

- Let $\boldsymbol{\phi}_t$ be a trading strategy denoting the quantity of each type of security held at time $t$.

  – Hence the stochastic process $\boldsymbol{\phi}_t \boldsymbol{S}_t$ is the value of the portfolio $\boldsymbol{\phi}_t$ at time $t$.

- $\boldsymbol{\phi}_t\, d\boldsymbol{S}_t \stackrel{\Delta}{=} \boldsymbol{\phi}_t(\mu_t\, dt + \sigma_t\, dW_t)$ represents the change in the value from security price changes occurring at time $t$.

## Trading and the Ito Integral (concluded)

- The equivalent Ito integral,

$$G_T(\boldsymbol{\phi}) \stackrel{\Delta}{=} \int_0^T \boldsymbol{\phi}_t \, d\boldsymbol{S}_t = \int_0^T \boldsymbol{\phi}_t \mu_t \, dt + \int_0^T \boldsymbol{\phi}_t \sigma_t \, dW_t,$$

measures the gains realized by the trading strategy over the period $[\,0, T\,]$.

# Ito's Lemma[a]

A smooth function of an Ito process is itself an Ito process.

**Theorem 18** *Suppose* $f : R \to R$ *is twice continuously differentiable and* $dX = a_t\, dt + b_t\, dW$. *Then* $f(X)$ *is the Ito process,*

$$\begin{aligned} & f(X_t) \\ = \; & f(X_0) + \int_0^t f'(X_s)\, a_s\, ds + \int_0^t f'(X_s)\, b_s\, dW \\ & + \frac{1}{2} \int_0^t f''(X_s)\, b_s^2\, ds \end{aligned}$$

*for* $t \geq 0$.

---

[a]Ito (1944).

# Ito's Lemma (continued)

- In differential form, Ito's lemma becomes

$$\begin{aligned} & df(X) \\ = \; & f'(X)\, a\, dt + f'(X)\, b\, dW + \frac{1}{2}\, f''(X)\, b^2\, dt \qquad (79) \\ = \; & \left[ f'(X)\, a + \frac{1}{2}\, f''(X)\, b^2 \right] dt + f'(X)\, b\, dW. \end{aligned}$$

- Compared with calculus, the interesting part is the third term on the right-hand side of Eq. (79).
- A convenient formulation of Ito's lemma is

$$df(X) = f'(X)\, dX + \frac{1}{2}\, f''(X)(dX)^2. \qquad (80)$$

## Ito's Lemma (continued)

- We are supposed to multiply out $(dX)^2 = (a\,dt + b\,dW)^2$ symbolically according to

| $\times$ | $dW$ | $dt$ |
|---|---|---|
| $dW$ | $dt$ | 0 |
| $dt$ | 0 | 0 |

  - The $(dW)^2 = dt$ entry is justified by a known result.

- Hence $(dX)^2 = (a\,dt + b\,dW)^2 = b^2\,dt$ in Eq. (80).

- This form is easy to remember because of its similarity to the Taylor expansion.

# Ito's Lemma (continued)

**Theorem 19 (Higher-Dimensional Ito's Lemma)** *Let* $W_1, W_2, \ldots, W_n$ *be independent Wiener processes and* $X \stackrel{\Delta}{=} (X_1, X_2, \ldots, X_m)$ *be a vector process. Suppose* $f : R^m \to R$ *is twice continuously differentiable and* $X_i$ *is an Ito process with* $dX_i = a_i\, dt + \sum_{j=1}^{n} b_{ij}\, dW_j$. *Then* $df(X)$ *is an Ito process with the differential,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X)\, dX_i\, dX_k,$$

*where* $f_i \stackrel{\Delta}{=} \partial f / \partial X_i$ *and* $f_{ik} \stackrel{\Delta}{=} \partial^2 f / \partial X_i \partial X_k$.

# Ito's Lemma (continued)

- The multiplication table for Theorem 19 is

| $\times$ | $dW_i$ | $dt$ |
|---|---|---|
| $dW_k$ | $\delta_{ik}\, dt$ | 0 |
| $dt$ | 0 | 0 |

in which

$$\delta_{ik} = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{otherwise.} \end{cases}$$

## Ito's Lemma (continued)

- In applying the higher-dimensional Ito's lemma, usually one of the variables, say $X_1$, is time $t$ and $dX_1 = dt$.
- In this case, $b_{1j} = 0$ for all $j$ and $a_1 = 1$.
- As an example, let

$$dX_t = a_t\, dt + b_t\, dW_t.$$

- Consider the process $f(X_t, t)$.

# Ito's Lemma (continued)

- Then

$$
\begin{aligned}
& df \\
= \quad & \frac{\partial f}{\partial X_t}\, dX_t + \frac{\partial f}{\partial t}\, dt + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\, (dX_t)^2 \\
= \quad & \frac{\partial f}{\partial X_t}\, (a_t\, dt + b_t\, dW_t) + \frac{\partial f}{\partial t}\, dt \\
& + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\, (a_t\, dt + b_t\, dW_t)^2 \\
= \quad & \left( \frac{\partial f}{\partial X_t}\, a_t + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\, b_t^2 \right) dt + \frac{\partial f}{\partial X_t}\, b_t\, dW_t. \quad (81)
\end{aligned}
$$

# Ito's Lemma (continued)

**Theorem 20 (Alternative Ito's Lemma)** *Let* $W_1, W_2, \ldots, W_m$ *be Wiener processes and* $X \triangleq (X_1, X_2, \ldots, X_m)$ *be a vector process. Suppose* $f : R^m \to R$ *is twice continuously differentiable and* $X_i$ *is an Ito process with* $dX_i = a_i\, dt + b_i\, dW_i$. *Then* $df(X)$ *is the following Ito process,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X)\, dX_i\, dX_k.$$

## Ito's Lemma (concluded)

- The multiplication table for Theorem 20 is

| $\times$ | $dW_i$ | $dt$ |
|---|---|---|
| $dW_k$ | $\rho_{ik}\, dt$ | $0$ |
| $dt$ | $0$ | $0$ |

- Above, $\rho_{ik}$ denotes the correlation between $dW_i$ and $dW_k$.

# Geometric Brownian Motion

- Consider geometric Brownian motion
$$Y(t) \stackrel{\Delta}{=} e^{X(t)}.$$
  - $X(t)$ is a $(\mu, \sigma)$ Brownian motion.
  - By Eq. (74) on p. 573,
$$dX = \mu\, dt + \sigma\, dW.$$
- Note that
$$\begin{aligned} \frac{\partial Y}{\partial X} &= Y, \\ \frac{\partial^2 Y}{\partial X^2} &= Y. \end{aligned}$$

# Geometric Brownian Motion (continued)

- Ito's formula (79) on p. 605 implies

$$\begin{aligned} dY &= Y\,dX + (1/2)\,Y\,(dX)^2 \\ &= Y\,(\mu\,dt + \sigma\,dW) + (1/2)\,Y\,(\mu\,dt + \sigma\,dW)^2 \\ &= Y\,(\mu\,dt + \sigma\,dW) + (1/2)\,Y\sigma^2\,dt. \end{aligned}$$

- Hence

$$\frac{dY}{Y} = \left(\mu + \sigma^2/2\right) dt + \sigma\,dW. \tag{82}$$

- The annualized *instantaneous* rate of return is $\mu + \sigma^2/2$ (not $\mu$).[a]

[a] Consistent with Lemma 9 (p. 297).

## Geometric Brownian Motion (continued)

- Alternatively, from Eq. (74) on p. 573,

$$X_t = X_0 + \mu t + \sigma W_t,$$

an explicit (strong) solution.

- Hence

$$Y_t = Y_0 \, e^{\mu t + \sigma W_t},$$

a strong solution to the SDE (82) where $Y_0 = e^{X_0}$.

## Geometric Brownian Motion (concluded)

- On the other hand, suppose

$$\frac{dY}{Y} = \mu\, dt + \sigma\, dW.$$

- Then $X(t) \stackrel{\Delta}{=} \ln Y(t)$ follows

$$dX = \left(\mu - \sigma^2/2\right) dt + \sigma\, dW.$$

## Exponential Martingale

- The Ito process

$$dX_t = b_t X_t \, dW_t$$

  is a martingale.[a]

- It is called an exponential martingale.

- By Ito's formula (79) on p. 605,

$$X(t) = X(0) \exp\left[ -\frac{1}{2} \int_0^t b_s^2 \, ds + \int_0^t b_s \, dW_s \right].$$

---

[a] Recall Theorem 17 (p. 590).