### $\mathsf{Barrier}\ \mathsf{Options}^{\mathrm{a}}$

- Their payoff depends on whether the underlying asset's price reaches a certain price level *H* throughout its life.
- A knock-out (KO) option is an ordinary European option which ceases to exist if the barrier *H* is reached by the price of its underlying asset.
- A call knock-out option is sometimes called a down-and-out option if H < S.
- A put knock-out option is sometimes called an up-and-out option when H > S.

<sup>a</sup>A former MBA student in finance told me on March 26, 2004, that she did not understand why I covered barrier options until she started working in a bank. She was working for Lehman Brothers in Hong Kong as of April, 2006.



# Barrier Options (concluded)

- A knock-in (KI) option comes into existence if a certain barrier is reached.
- A down-and-in option is a call knock-in option that comes into existence only when the barrier is reached and H < S.
- An up-and-in is a put knock-in option that comes into existence only when the barrier is reached and H > S.
- Formulas exist for all the possible barrier options mentioned above.<sup>a</sup>

<sup>a</sup>Haug (2006).

#### A Formula for Down-and-In Calls $^{\rm a}$

- Assume  $X \ge H$ .
- The value of a European down-and-in call on a stock paying a dividend yield of q is

$$Se^{-q\tau} \left(\frac{H}{S}\right)^{2\lambda} N(x) - Xe^{-r\tau} \left(\frac{H}{S}\right)^{2\lambda-2} N(x - \sigma\sqrt{\tau}),$$
(53)

$$-x \stackrel{\Delta}{=} \frac{\ln(H^2/(SX)) + (r-q+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$
$$-\lambda \stackrel{\Delta}{=} (r-q+\sigma^2/2)/\sigma^2.$$

• A European down-and-out call can be priced via the in-out parity (see text).

<sup>a</sup>Merton (1973). See Exercise 17.1.6 of the textbook for a proof.

#### A Formula for Up-and-In $\mathsf{Puts}^\mathrm{a}$

- Assume  $X \leq H$ .
- The value of a European up-and-in put is

$$Xe^{-r\tau}\left(\frac{H}{S}\right)^{2\lambda-2}N(-x+\sigma\sqrt{\tau})-Se^{-q\tau}\left(\frac{H}{S}\right)^{2\lambda}N(-x).$$

• Again, a European up-and-out put can be priced via the in-out parity.

<sup>a</sup>Merton (1973).

#### Barrier Options: Popularity

- Knock-out options were issued in the U.S. in 1967.<sup>a</sup>
- Knock-in puts are the most popular barrier options.<sup>b</sup>
- Knock-out puts are the second most popular barrier options.<sup>c</sup>
- Knock-out calls are the most popular among barrier call options.<sup>d</sup>

<sup>a</sup>Cox & Rubinstein (1985). <sup>b</sup>Bennett (2014). <sup>c</sup>Bennett (2014). <sup>d</sup>Bennett (2014).

### Are American Options Barrier Options?<sup>a</sup>

- American options are barrier options with the exercise boundary as the barrier and the payoff as the rebate?
- One salient difference is that the exercise boundary must be found by backward induction.
- It cannot be specified in an arbitrary way.

<sup>a</sup>Contributed by Mr. Yang, Jui-Chung (D97723002) on March 25, 2009.

### Are American Options Barrier Options? (concluded)

- In conrast, the barrier in a barrier option is fixed by a contract.<sup>a</sup>
  - The option remains European-style, without the right to early exercise.<sup>b</sup>
- One can also have American barrier options.
  - Need to specify whether one can exercise the option early if the stock price has not touched the barrier.<sup>c</sup>

<sup>b</sup>Contributed by Ms. Chen, Sin-Huei (Amber) (P00922005) on March 31, 2021.

c<br/>Contributed by Mr. Lu, Yu-Ming ( $R06723032,\,D08922008)$  on March 31, 2021.

<sup>&</sup>lt;sup>a</sup>Cox & Rubinstein (1985).

#### Interesting Observations

- Assume H < X.
- Replace S in the Merton pricing formula Eq. (43) on p. 329 for the call with  $H^2/S$ .
  - Equation (53) on p. 394 for the down-and-in call becomes Eq. (43) when  $r q = \sigma^2/2$ .
  - Equation (53) becomes S/H times Eq. (43) when r-q=0.

#### Interesting Observations (concluded)

- Replace S in the pricing formula for the down-and-in call, Eq. (53), with  $H^2/S$ .
  - Equation (53) becomes Eq. (43) when  $r q = \sigma^2/2$ .
  - Equation (53) becomes H/S times Eq. (43) when  $r-q=0.^{a}$
- Why?

<sup>a</sup>Contributed by Mr. Chou, Ming-Hsin (R02723073) on April 24, 2014.

# Binomial Tree Algorithms

- Barrier options can be priced by binomial tree algorithms.
- Below is for the down-and-out option.





### Binomial Tree Algorithms (continued)

- But convergence is erratic because *H* is not at a price level on the tree (see plot on next page).<sup>a</sup>
  - The barrier H is moved lower (or higher, if you so choose) to a close-by node price.
  - This "effective barrier" thus changes as n increases.
- In fact, the binomial tree is  $O(1/\sqrt{n})$  convergent.<sup>b</sup>
- Solutions will be presented later.

<sup>&</sup>lt;sup>a</sup>Boyle & Lau (1994).

<sup>&</sup>lt;sup>b</sup>Tavella & Randall (2000); J. Lin (R95221010) (2008); J. Lin (R95221010) & Palmer (2013).



### Other Types of Barrier $\mathsf{Options}^\mathrm{a}$

- Partial barrier options.
- Forward-starting barrier options.
- Window barrier options.
- Rolling barrier options.
- Moving barrier options.

<sup>a</sup>Armtrong (2001); Carr & A. Chou (1997); Davydov & Linetsky (2001/2002); Haug (1998).

# Daily Monitoring

- Many barrier options monitor the barrier only for daily *closing prices*.
- If so, only nodes at the end of a day need to check for the barrier condition.
- We can even remove intraday nodes to create a multinomial tree.
  - A node is then followed by d + 1 nodes if each day is partitioned into d periods.
- Does this save time or space?<sup>a</sup>

 $<sup>^{\</sup>rm a}{\rm Contributed}$  by Ms. Chen, Tzu-Chun (R94922003) and others on April 12, 2006.



### Discrete Monitoring vs. Continuous Monitoring

- Discrete barriers are more expensive for knock-out options than continuous ones.
- But discrete barriers are less expensive for knock-in options than continuous ones.
- Discrete barriers are far less popular than continuous ones for individual stocks.<sup>a</sup>
- They are equally popular for indices.<sup>b</sup>

<sup>a</sup>Bennett (2014). <sup>b</sup>Bennett (2014). Data! data! data! — Arthur Conan Doyle (1892), The Adventures of Sherlock Holmes

### Foreign Currencies

- S denotes the spot exchange rate in domestic/foreign terms.
  - By that we mean the number of domestic currencies per unit of foreign currency.<sup>a</sup>
- $\sigma$  denotes the volatility of the exchange rate.
- r denotes the domestic interest rate.
- $\hat{r}$  denotes the foreign interest rate.

<sup>a</sup>The market convention is the opposite: A/B = x means one unit of currency A (the reference currency or base currency) is equal to x units of currency B (the counter-value currency).

# Foreign Currencies (concluded)

- A foreign currency is analogous to a stock paying a known dividend yield.
  - For eign currencies pay a "continuous dividend yield" equal to  $\hat{r}$  in the for eign currency.























# Foreign Exchange Options

- In 2000 the total notional volume of foreign exchange options was US\$13 trillion.<sup>a</sup>
  - 38.5% were vanilla calls and puts with a maturity less than one month.
  - 52.5% were vanilla calls and puts with a maturity between one and 18 months.
  - -4% were barrier options.
  - 1.5% were vanilla calls and puts with a maturity more than 18 months.
  - 1% were binary options (recall p. 207 or see p. 859).
  - -0.7% were Asian options (see p. 434).

<sup>a</sup>Lipton (2002).

# Foreign Exchange Options (continued)

- Foreign exchange options are settled via delivery of the underlying currency.
- A primary use of foreign exchange (or forex) options is to hedge currency risk.
- Consider a U.S. company expecting to receive 100 million Japanese yen in March 2000.
- Those 100 million Japanese yen will be exchanged for U.S. dollars.

# Foreign Exchange Options (continued)

- The contract size for the Japanese yen option is JPY6,250,000.
- The company purchases

$$\frac{100,000,000}{6,250,000} = 16$$

puts on the Japanese yen with a strike of \$.0088/JPY1 and an exercise month in March 2000.

• This put is in the money if the JPY-USD exchange rate drops below 0.0088.

# Foreign Exchange Options (continued)

• These puts provide the company the right to sell 100,000,000 Japanese yen for

 $100,000,000 \times .0088 = 880,000$ 

U.S. dollars.

- Note that these puts are equivalent to the right to buy 880,000 U.S. dollars with 100,000,000 Japanese yen.
  - From this angle, they become *calls*.
### Foreign Exchange Options (concluded)

- Assume the exchange rate S is lognormally distributed.
- The formulas derived for stock index options in Eqs. (43) on p. 329 apply with the dividend yield equal to  $\hat{r}$ :

$$C = Se^{-\hat{r}\tau}N(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}), \qquad (54)$$
$$P = Xe^{-r\tau}N(-x + \sigma\sqrt{\tau}) - Se^{-\hat{r}\tau}N(-x). \qquad (54')$$

– Above,

$$x \stackrel{\Delta}{=} \frac{\ln(S/X) + (r - \hat{r} + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}.$$







Bar the roads! Bar the paths! Wert thou to flee from here, wert thou to find all the roads of the world, the way thou seekst the path to that thou'dst find not[.] — Richard Wagner (1813–1883), Parsifal

### Path-Dependent Derivatives

- Let  $S_0, S_1, \ldots, S_n$  denote the prices of the underlying asset over the life of the option.
- $S_0$  is the known price at time zero.
- $S_n$  is the price at expiration.
- The standard European call has a terminal value depending only on the last price,  $\max(S_n X, 0)$ .
- Its value thus depends only on the underlying asset's terminal price regardless of how it gets there.<sup>a</sup>

<sup>a</sup>Called simple claims (Björk, 2009).

#### Path-Dependent Derivatives (continued)

- Some derivatives are path-dependent in that their terminal payoff depends *explicitly* on the path.
- The (arithmetic) average-rate call has this terminal value:

$$\max\left(\frac{1}{n+1}\sum_{i=0}^{n}S_{i}-X,0\right).$$

• The average-rate put's terminal value is given by

$$\max\left(X - \frac{1}{n+1}\sum_{i=0}^{n} S_i, 0\right).$$

## Path-Dependent Derivatives (continued)

- Average-rate options are also called Asian options.
- They are very popular.<sup>a</sup>
- They are useful hedging tools for firms that will make a stream of purchases over a time period because the costs are likely to be linked to the average price.
- They are mostly European.
- The averaging clause is also common in convertible bonds and structured notes.

<sup>a</sup>As of the late 1990s, the outstanding volume was in the range of 5–10 billion U.S. dollars (Nielsen & Sandmann, 2003).

## Path-Dependent Derivatives (continued)

• A lookback call option on the minimum has a terminal payoff of

$$S_n - \min_{0 \le i \le n} S_i.$$

• A lookback put on the maximum has a terminal payoff of

$$\max_{0 \le i \le n} S_i - S_n.$$

## Path-Dependent Derivatives (concluded)

- The fixed-strike lookback option provides a payoff of
  - $-\max(\max_{0\leq i\leq n} S_i X, 0)$  for the call.
  - $-\max(X \min_{0 \le i \le n} S_i, 0)$  for the put.
- Lookback calls and puts on the average (instead of a constant X) are called average-strike options.

### Average-Rate Options

- Average-rate options are notoriously hard to price.
- The binomial tree for the averages does not combine (see next page).
- A naive algorithm enumerates the  $2^n$  paths for an *n*-period binomial tree and then averages the payoffs.
- But the complexity is exponential.<sup>a</sup>
- The Monte Carlo method<sup>b</sup> and approximation algorithms are some of the alternatives left.

<sup>b</sup>See pp. 846ff.

<sup>&</sup>lt;sup>a</sup>Dai (B82506025, R86526008, D8852600) & Lyuu (2007) reduce it to  $2^{O(\sqrt{n})}$ .



### States and Their Transitions

• The tuple

(i, S, P)

captures the state<sup>a</sup> for the Asian option.

-i: the time.

- S: the prevailing stock price.

- P: the running sum.<sup>b</sup>

<sup>a</sup>A "sufficient statistic," if you will.

<sup>b</sup>When the average is a moving average, a different technique is needed (C. Kao (R89723057) & Lyuu, 2003).



### Pricing Some Path-Dependent Options

- Not all path-dependent derivatives are hard to price.
  - Barrier options are easy to price.
- When averaging is done *geometrically*, the option payoffs are

$$\max\left((S_0 S_1 \cdots S_n)^{1/(n+1)} - X, 0\right),\\ \max\left(X - (S_0 S_1 \cdots S_n)^{1/(n+1)}, 0\right).$$

## Pricing Some Path-Dependent Options (concluded)

• The limiting analytical solutions are the Black-Scholes formulas:<sup>a</sup>

$$C = Se^{-q_{a}\tau}N(x) - Xe^{-r\tau}N(x - \sigma_{a}\sqrt{\tau}),$$

$$P = Xe^{-r\tau}N(-x + \sigma_{a}\sqrt{\tau}) - Se^{-q_{a}\tau}N(-x),$$
(55)
(55)

- With the volatility set to 
$$\sigma_{\rm a} \stackrel{\Delta}{=} \sigma / \sqrt{3}$$
.

- With the dividend yield set to 
$$q_{\rm a} \stackrel{\Delta}{=} (r+q+\sigma^2/6)/2$$
.  
-  $x \stackrel{\Delta}{=} \frac{\ln(S/X) + (r-q_{\rm a}+\sigma_{\rm a}^2/2)\tau}{\sigma_{\rm a}\sqrt{\tau}}$ .

<sup>a</sup>See Angus (1999), for example.

An Approximate Formula for Asian Calls<sup>a</sup>  

$$C = e^{-r\tau} \left[ \frac{S}{\tau} \int_{0}^{\tau} e^{\mu t + \sigma^{2} t/2} N \left( \frac{-\gamma + (\sigma t/\tau)(\tau - t/2)}{\sqrt{\tau/3}} \right) dt - XN \left( \frac{-\gamma}{\sqrt{\tau/3}} \right) \right],$$

where

• 
$$\mu \stackrel{\Delta}{=} r - \sigma^2/2.$$

•  $\gamma$  is the unique value that satisfies

$$\frac{S}{\tau} \int_0^\tau e^{3\gamma\sigma t(\tau - t/2)/\tau^2 + \mu t + \sigma^2 [t - (3t^2/\tau^3)(\tau - t/2)^2]/2} dt = X.$$

<sup>a</sup>Rogers & Shi (1995); Thompson (1999); K. Chen (**R92723061**) (2005); K. Chen (**R92723061**) & Lyuu (2006).

#### Approximation Algorithm for Asian Options

- Based on the BOPM.
- Consider a node at time j with the underlying asset price equal to  $S_0 u^{j-i} d^i$ .
- Name such a node N(j, i).
- The running sum  $\sum_{m=0}^{j} S_m$  at this node has a maximum value of

$$S_0(1 + u + u^2 + \dots + u^{j-i} + u^{j-i}d + \dots + u^{j-i}d^i)$$
  
=  $S_0 \frac{1 - u^{j-i+1}}{1 - u} + S_0 u^{j-i}d \frac{1 - d^i}{1 - d}.$ 



- Divide this value by j+1 and call it  $A_{\max}(j,i)$ .
- Similarly, the running sum has a minimum value of

$$S_0(1 + d^2 + \dots + d^i + d^i u + \dots + d^i u^{j-i})$$
  
=  $S_0 \frac{1 - d^{i+1}}{1 - d} + S_0 d^i u \frac{1 - u^{j-i}}{1 - u}.$ 

- Divide this value by j+1 and call it  $A_{\min}(j,i)$ .
- $A_{\min}$  and  $A_{\max}$  are running averages.

• The number of paths to N(j,i) are far too many:  $\binom{j}{i}$ . - For example,

$$\binom{j}{j/2} \sim 2^j \sqrt{2/(\pi j)} \,.$$

• The number of distinct running averages for the nodes at any given time step n seems to be bimodal for n big enough.<sup>a</sup>

- In the plot on the next page, u = 5/4 and d = 4/5.

<sup>a</sup>Contributed by Mr. Liu, Jun (R99944027) on April 15, 2014.



- But all averages must lie between  $A_{\min}(j,i)$  and  $A_{\max}(j,i)$ .
- Pick k + 1 equally spaced values in this range and treat them as the true and only running averages:

$$A_m(j,i) \stackrel{\Delta}{=} \left(\frac{k-m}{k}\right) A_{\min}(j,i) + \left(\frac{m}{k}\right) A_{\max}(j,i)$$

for m = 0, 1, ..., k.



- Such "bucketing" or "binning " introduces errors, but it works reasonably well in practice.<sup>a</sup>
- A better alternative picks values whose logarithms are equally spaced.<sup>b</sup>
- Still other alternatives are possible (considering the distribution of averages on p. 448).

 $<sup>^{\</sup>rm a}$ Hull & White (1993); Ritchken, Sankarasubramanian, & Vijh (1993).  $^{\rm b}$ Called log-linear interpolation.

- Backward induction calculates the option values at each node for the k + 1 running averages.
- Suppose the current node is N(j, i) and the running average is a.
- Assume the next node is N(j+1,i), after an up move.
- As the asset price there is  $S_0 u^{j+1-i} d^i$ , we seek the option value corresponding to the new running average

$$A_{\mathbf{u}} \stackrel{\Delta}{=} \frac{(j+1)\,a + S_0 u^{j+1-i} d^i}{j+2}$$

- But  $A_u$  is not likely to be one of the k + 1 running averages at N(j + 1, i)!
- Find the 2 running averages that bracket it:

$$A_{\ell}(j+1,i) \le A_{u} < A_{\ell+1}(j+1,i).$$

• In "most" cases, the fastest way to nail  $\ell$  is via

$$\ell = \left\lfloor \frac{A_{\rm u} - A_{\rm min}(j+1,i)}{[A_{\rm max}(j+1,i) - A_{\rm min}(j+1,i)]/k} \right\rfloor$$



• But watch out for the rare case where

$$A_{\rm u} = A_\ell(j+1,i)$$

for some  $\ell$ .

• Also watch out for the case where

$$A_{\rm u} = A_{\rm max}(j,i).$$

- Finally, watch out for the degenerate case where  $A_0(j+1,i) = \cdots = A_k(j+1,i).$ 
  - It will happen along extreme paths!

• Express  $A_{\rm u}$  as a linearly interpolated value of the two running averages,

$$A_{\rm u} = x A_{\ell}(j+1,i) + (1-x) A_{\ell+1}(j+1,i), \quad 0 < x \le 1.$$

• Obtain the approximate option value given the running average  $A_{\rm u}$  via

$$C_{\rm u} \stackrel{\Delta}{=} x C_{\ell}(j+1,i) + (1-x) C_{\ell+1}(j+1,i).$$

 $-C_{\ell}(t,s)$  denotes the option value at node N(t,s)with running average  $A_{\ell}(t,s)$ .

- This interpolation introduces the second source of error.
  Alternatives to linear interpolation exist.
- The same steps are repeated for the down node N(j+1, i+1) to obtain another approximate option value  $C_{\rm d}$ .
- Finally obtain the option value as

$$[pC_{\rm u} + (1-p)C_{\rm d}]e^{-r\Delta t}.$$

- For the calculations at time step n-1, no interpolation is needed.<sup>a</sup>
  - The running time is  $O(kn^2)$ .
    - \* There are  $O(n^2)$  nodes.
    - \* Each node has O(k) buckets.
  - The option values are simply (for calls):

$$C_{\rm u} = \max(A_{\rm u} - X, 0),$$
  
 $C_{\rm d} = \max(A_{\rm d} - X, 0).$ 

- That saves O(nk) calculations.

<sup>a</sup>Contributed by Mr. Chen, Shih-Hang (R02723031) on April 9, 2014.

- Arithmetic average-rate options were assumed to be newly issued: no historical average to deal with.
- This problem can be easily addressed.<sup>a</sup>
- How about the Greeks?<sup>b</sup>

<sup>&</sup>lt;sup>a</sup>See Exercise 11.7.4 of the textbook.

<sup>&</sup>lt;sup>b</sup>Thanks to lively class discussions on March 31, 2004, and April 9, 2014.

## A Numerical Example

- Consider a European arithmetic average-rate call with strike price 50.
- Assume zero interest rate in order to dispense with discounting.
- The minimum running average at node A in the figure on p. 461 is 48.925.
- The maximum running average at node A in the same figure is 51.149.



## A Numerical Example (continued)

- Each node picks k = 3 for 4 equally spaced running averages.
- The same calculations are done for node A's successor nodes B and C.
- Suppose node A is 2 periods from the root node.
- Consider the up move from node A with running average 49.666.
## A Numerical Example (continued)

• Because the stock price at node B is 53.447, the new running average will be

$$\frac{3 \times 49.666 + 53.447}{4} \approx 50.612.$$

• With 50.612 lying between 50.056 and 51.206 at node B, we solve

 $50.612 = x \times 50.056 + (1 - x) \times 51.206$ 

to obtain  $x \approx 0.517$ .

## A Numerical Example (continued)

- The option value corresponding to running average 50.056 at node B is 0.056.
- The option values corresponding to running average 51.206 at node B is 1.206.
- Their contribution to the option value corresponding to running average 49.666 at node A is weighted linearly as

 $x \times 0.056 + (1 - x) \times 1.206 \approx 0.611.$ 

# A Numerical Example (continued)

- Now consider the down move from node A with running average 49.666.
- Because the stock price at node C is 46.775, the new running average will be

$$\frac{3 \times 49.666 + 46.775}{4} \approx 48.944.$$

• With 48.944 lying between 47.903 and 48.979 at node C, we solve

$$48.944 = x \times 47.903 + (1 - x) \times 48.979$$

to obtain  $x \approx 0.033$ .

# A Numerical Example (concluded)

- The option values corresponding to running averages 47.903 and 48.979 at node C are both 0.0.
- Their contribution to the option value corresponding to running average 49.666 at node A is 0.0.
- Finally, the option value corresponding to running average 49.666 at node A equals

 $p \times 0.611 + (1-p) \times 0.0 \approx 0.2956,$ 

where p = 0.483.

• The remaining three option values at node A can be computed similarly.



## Remarks on Asian Option Pricing

- Asian option pricing is an active research area.
- The above algorithm overestimates the "true" value.<sup>a</sup>
- To guarantee convergence, k needs to grow with n at least.<sup>b</sup>
- There is a convergent approximation algorithm that does away with interpolation with a running time of<sup>c</sup>

#### $2^{O(\sqrt{n})}$ .

<sup>a</sup>Dai (B82506025, R86526008, D8852600), G. Huang (F83506075), & Lyuu (2002).

<sup>b</sup>Dai (B82506025, R86526008, D8852600), G. Huang (F83506075), & Lyuu (2002).

<sup>c</sup>Dai (B82506025, R86526008, D8852600) & Lyuu (2002, 2004).

# Remarks on Asian Option Pricing (continued)

- There is an  $O(kn^2)$ -time algorithm with an error bound of O(Xn/k) from the naive  $O(2^n)$ -time binomial tree algorithm in the case of European Asian options.<sup>a</sup>
  - -k can be varied for trade-off between time and accuracy.
  - If we pick  $k = O(n^2)$ , then the error is O(1/n), and the running time is  $O(n^4)$ .

<sup>a</sup>Aingworth, Motwani (1962–2009), & Oldham (2000).

# Remarks on Asian Option Pricing (continued)

- Another approximation algorithm reduces the error to  $O(X\sqrt{n}/k)$ .<sup>a</sup>
  - It varies the number of buckets per node.
  - If we pick k = O(n), the error is  $O(n^{-0.5})$ .
  - If we pick  $k = O(n^{1.5})$ , then the error is O(1/n), and the running time is  $O(n^{3.5})$ .
- Under "reasonable assumptions," an  $O(n^2)$ -time algorithm with an error bound of O(1/n) exists.<sup>b</sup>

<sup>a</sup>Dai (B82506025, R86526008, D8852600), G. Huang (F83506075), & Lyuu (2002).

<sup>b</sup>Hsu (**R7526001**, **D89922012**) & Lyuu (2004).

#### Remarks on Asian Option Pricing (concluded)

- The basic idea is a nonuniform allocation of running averages instead of a uniform k.
- It strikes a tight balance between error and complexity.



## A Grand Comparison $^{\rm a}$

<sup>a</sup>Hsu (**R7526001**, **D89922012**) & Lyuu (2004); J. E. Zhang (2001,2003); K. Chen (**R92723061**) & Lyuu (2006).

X	$\sigma$	r	Exact	AA2	AA3	Hsu-Lyuu	Chen-Lyuu
95	0.05	0.05	7.1777275	7.1777244	7.1777279	7.178812	7.177726
100			2.7161745	2.7161755	2.7161744	2.715613	2.716168
105			0.3372614	0.3372601	0.3372614	0.338863	0.337231
95		0.09	8.8088392	8.8088441	8.8088397	8.808717	8.808839
100			4.3082350	4.3082253	4.3082331	4.309247	4.308231
105			0.9583841	0.9583838	0.9583841	0.960068	0.958331
95		0.15	11.0940944	11.0940964	11.0940943	11.093903	11.094094
100			6.7943550	6.7943510	6.7943553	6.795678	6.794354
105			2.7444531	2.7444538	2.7444531	2.743798	2.744406
90	0.10	0.05	11.9510927	11.9509331	11.9510871	11.951610	11.951076
100			3.6413864	3.6414032	3.6413875	3.642325	3.641344
110			0.3312030	0.3312563	0.3311968	0.331348	0.331074
90		0.09	13.3851974	13.3851165	13.3852048	13.385563	13.385190
100			4.9151167	4.9151388	4.9151177	4.914254	4.915075
110			0.6302713	0.6302538	0.6302717	0.629843	0.630064
90		0.15	15.3987687	15.3988062	15.3987860	15.398885	15.398767
100			7.0277081	7.0276544	7.0277022	7.027385	7.027678
110			1.4136149	1.4136013	1.4136161	1.414953	1.413286

X	$\sigma$	r	Exact	AA2	AA3	Hsu-Lyuu	Chen-Lyuı
90	0.20	0.05	12.5959916	12.5957894	12.5959304	12.596052	12.595602
100			5.7630881	5.7631987	5.7631187	5.763664	5.762708
110			1.9898945	1.9894855	1.9899382	1.989962	1.989242
90		0.09	13.8314996	13.8307782	13.8313482	13.831604	13.831220
100			6.7773481	6.7775756	6.7773833	6.777748	6.776999
110			2.5462209	2.5459150	2.5462598	2.546397	2.545459
90		0.15	15.6417575	15.6401370	15.6414533	15.641911	15.64159
100			8.4088330	8.4091957	8.4088744	8.408966	8.40851
110			3.5556100	3.5554997	3.5556415	3.556094	3.55468
90	0.30	0.05	13.9538233	13.9555691	13.9540973	13.953937	13.95242
100			7.9456288	7.9459286	7.9458549	7.945918	7.94435
110			4.0717942	4.0702869	4.0720881	4.071945	4.07011
90		0.09	14.9839595	14.9854235	14.9841522	14.984037	14.98278
100			8.8287588	8.8294164	8.8289978	8.829033	8.827548
110			4.6967089	4.6956764	4.6969698	4.696895	4.69490
90		0.15	16.5129113	16.5133090	16.5128376	16.512963	16.51202
100			10.2098305	10.2110681	10.2101058	10.210039	10.20872
110			5.7301225	5.7296982	5.7303567	5.730357	5.72816

# A Grand Comparison (concluded)