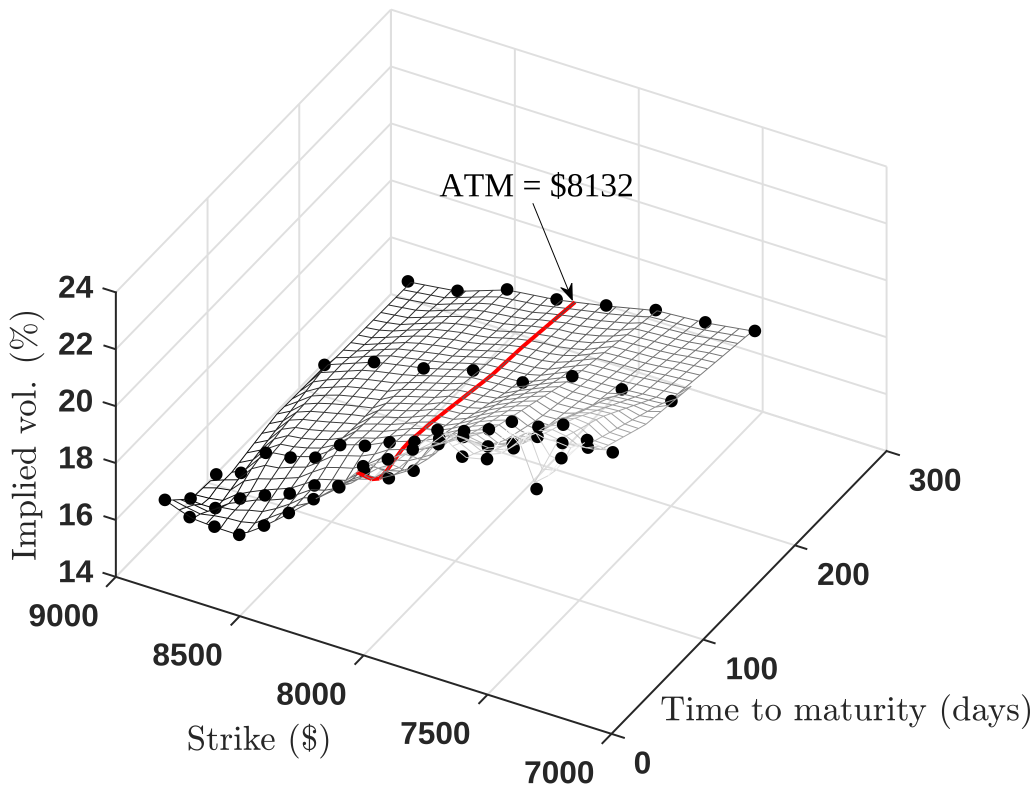
Problems; the Smile

- Options written on the same underlying asset usually do not produce the same implied volatility.
- A typical pattern is a "smile" in relation to the strike price.
  - The implied volatility is lowest for at-the-money options.
  - It becomes higher the further the option is in- or out-of-the-money.
- Other patterns have also been observed.

TXO Calls (September 25, 2015)[a]



[a]The underlying Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) closed at 8132. Plot supplied by Mr. Lok, U Hou (D99922028) on December 6, 2017.

Solutions to the Smile

- To address this issue, volatilities are often combined to produce a composite implied volatility.
- This practice is not sound theoretically.
- The existence of different implied volatilities for options on the same underlying asset shows the Black-Scholes model cannot be literally true.

Binomial Tree Algorithms for American Puts

- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs
$$\max(0, X - Su^j d^{n-j})$$
and applies backward induction.
- At each intermediate node, it compares the payoff if exercised and the *continuation value.*
- It keeps the larger one.

Bermudan Options

- Some American options can be exercised only at discrete time points instead of continuously.
- They are called Bermudan options.
- Their pricing algorithm is identical to that for American options.
- But early exercise is considered for only those nodes when early exercise is permitted.

Time-Dependent Instantaneous Volatility[a]

- Suppose the (instantaneous) volatility can change over time but otherwise predictable: $\sigma(t)$ instead of σ.
- In the limit, the variance of $\ln(S_\tau/S)$ is
$$\int_0^\tau \sigma^2(t)\,dt$$
rather than $\sigma^2\tau$.
- The annualized volatility to be used in the Black-Scholes formula should now be
$$\sqrt{\frac{\int_0^\tau \sigma^2(t)\,dt}{\tau}}\,.$$

[a]Merton (1973).

Time-Dependent Instantaneous Volatility (concluded)

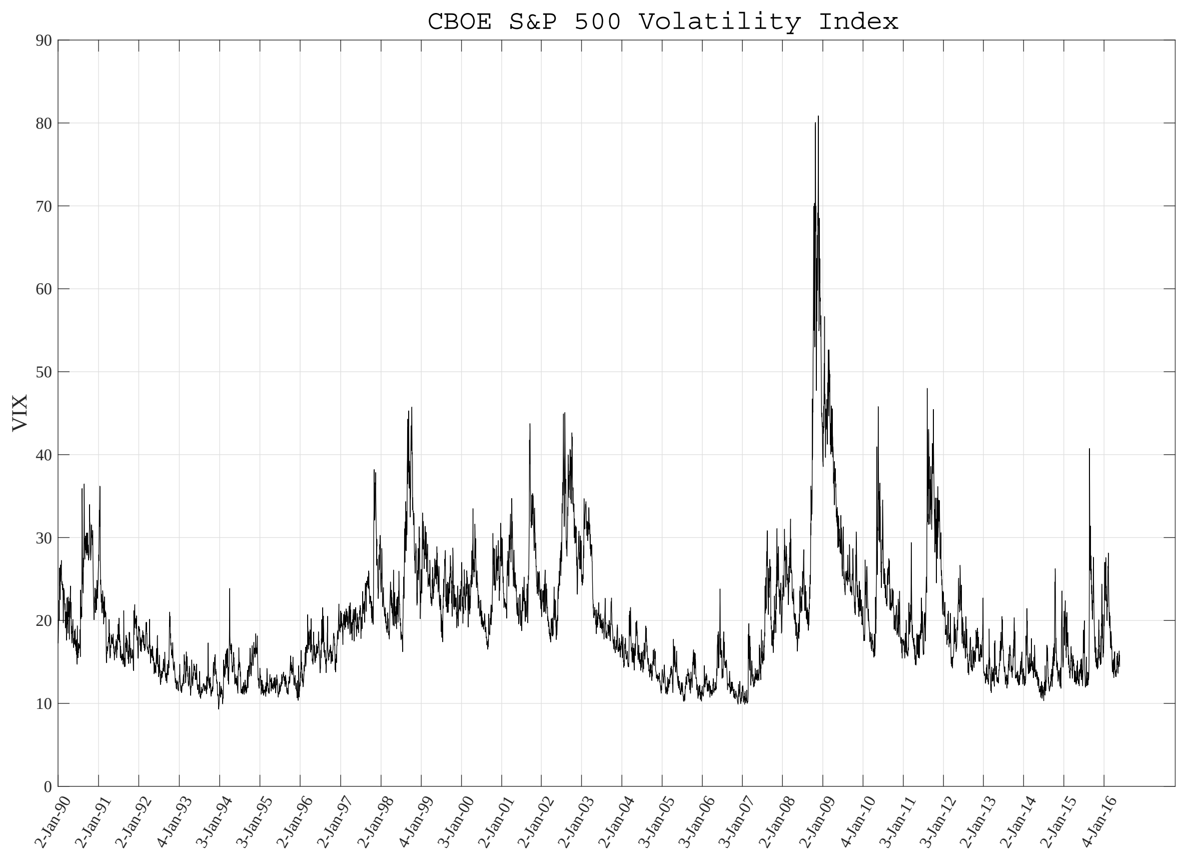
- For the binomial model, u and d depend on time:

$$
\begin{aligned}
u &= e^{\sigma(t)\sqrt{\tau/n}}, \\
d &= e^{-\sigma(t)\sqrt{\tau/n}}.
\end{aligned}
$$

- But how to make the binomial tree combine?[a]

[a] Amin (1991); C. I. Chen (R98922127) (2011).

Volatility (1990–2016)[a]



[a]Supplied by Mr. Lok, U Hou (D99922028) on July 17, 2017.

Time-Dependent Short Rates

- Suppose the short rate (i.e., the one-period spot rate or forward rate) changes over time but predictable.
- The annual riskless rate r in the Black-Scholes formula should be the spot rate with a time to maturity equal to τ.
- In other words,
$$r = \frac{\sum_{i=0}^{n-1} r_i}{\tau},$$
where r_i is the continuously compounded short rate measured in periods for period i.[a]
- Will the binomial tree fail to combine?

[a]That is, one-period forward rate.

Trading Days and Calendar Days

- Interest accrues based on the calendar day.
- But σ is usually calculated based on trading days only.
  - Stock price seems to have lower volatilities when the exchange is closed.[a]
- How to harmonize these two different times into the Black-Scholes formula and binomial tree algorithms?[b]

---

[a]Fama (1965); K. French (1980); K. French & Roll (1986).

[b]Recall p. 162 about dating issues.

Trading Days and Calendar Days (continued)

- Think of σ as measuring the *annualized* volatility of stock price *one year from now*.
- Suppose a year has m (say 253) trading days.
- We can replace σ in the Black-Scholes formula with[a]

$$\sigma \sqrt{\frac{365}{m} \times \frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}}.$$

[a] D. French (1984).

Trading Days and Calendar Days (concluded)

- This works only for European options.
- How about binomial tree algorithms?[a]

[a]Contributed by Mr. Lu, Zheng-Liang (`D00922011`) in 2015.

Options on a Stock That Pays Dividends

- Early exercise must be considered.
- Proportional dividend payout model is tractable (see text).
  - The dividend amount is a constant proportion of the *prevailing* stock price.
- In general, the corporate dividend policy is a complex issue.

Known Dividends

- Constant dividends introduce complications.
- Use D to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$.
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: $(Su - D)\, u$, $(Su - D)\, d$, $(Sd - D)\, u$, $(Sd - D)\, d$.
  - The binomial tree no longer combines.

$$
\begin{tikzcd}
 & & (Su-D)\,u \\
 & Su-D \arrow[ur] \arrow[dr] & \\
 & & (Su-D)\,d \\
S \arrow[uur] \arrow[ddr] & & \\
 & & (Sd-D)\,u \\
 & Sd-D \arrow[ur] \arrow[dr] & \\
 & & (Sd-D)\,d
\end{tikzcd}
$$

An Ad-Hoc Approximation

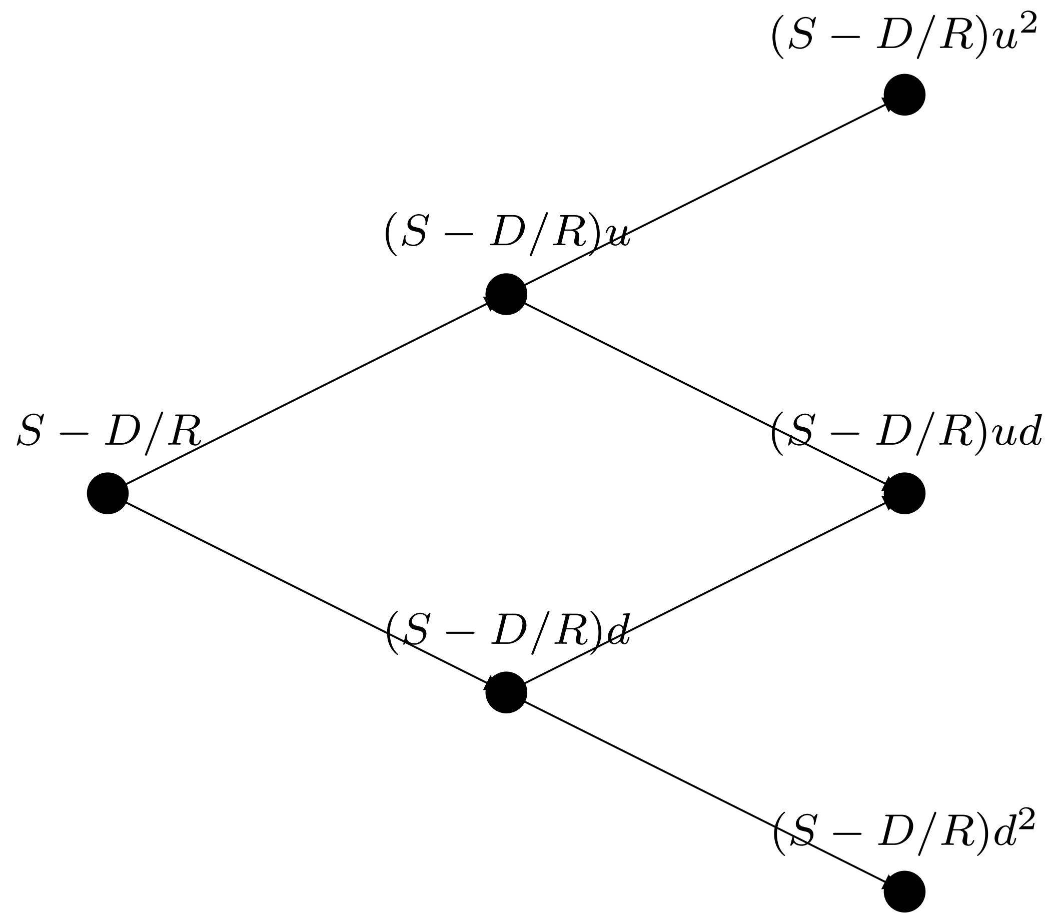
- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends.[a]
- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.
- The riskless component at any time is the PV of future dividends during the life of the option.
  - Then, σ is the volatility of the process followed by the *risky* component.
- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

[a]Roll (1977); Heath & Jarrow (1988).

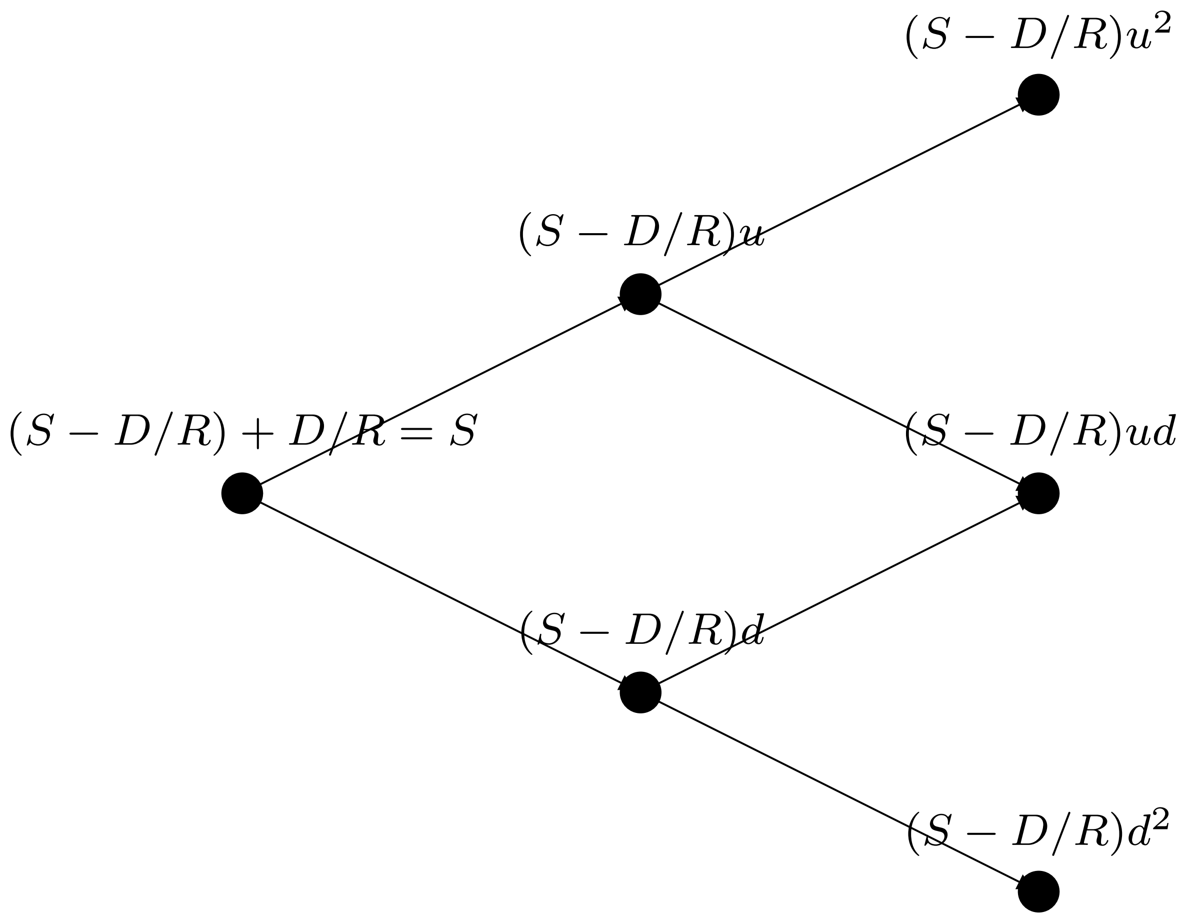
An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all *future* dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.

The Ad-Hoc Approximation vs. P. 320 (Step 1)



The Ad-Hoc Approximation vs. P. 320 (Step 2)



The Ad-Hoc Approximation vs. P. 320[a]

- The trees are different.
- The stock prices at maturity are also different.
  - $(Su - D)\,u, (Su - D)\,d, (Sd - D)\,u, (Sd - D)\,d$ (p. 320).
  - $(S - D/R)u^2, (S - D/R)ud, (S - D/R)d^2$ (ad hoc).
- Note that, as $d < R < u$,

$$\begin{aligned} (Su - D)\,u &> (S - D/R)u^2, \\ (Sd - D)\,d &< (S - D/R)d^2, \end{aligned}$$

[a]Contributed by Mr. Yang, Jui-Chung (`D97723002`) on March 18, 2009.

The Ad-Hoc Approximation vs. P. 320 (concluded)

- So the ad hoc approximation has a smaller dynamic range.
- This explains why in practice the volatility is usually *increased* when using the ad hoc approximation.

A General Approach[a]

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases (see pp. 775ff).
- Other approaches include adjusting σ and approximating the known dividend with a dividend yield.[b]

---

[a] Dai (B82506025, R86526008, D8852600) & Lyuu (2004). Also Arealy & Rodrigues (2013).

[b] Geske & Shastri (1985). It works well for American options but not European ones (Dai, 2009).

Continuous Dividend Yields

- Dividends are paid continuously.
  - Approximates a broad-based stock market portfolio.
- The payment of a continuous dividend yield at rate q reduces the growth rate of the stock price by q.
  - A stock that grows from S to S_τ with a continuous dividend yield of q would grow from S to $S_\tau e^{q\tau}$ without the dividends.
- A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays *no* dividends.[a]

[a] In pricing European options, only the distribution of S_τ matters.

Continuous Dividend Yields (continued)

- So the Black-Scholes formulas hold with S replaced by $Se^{-q\tau}$:[a]

$$C = Se^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{43}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau} N(-x), \tag{43'}$$

where

$$x \triangleq \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

- Formulas (43) and (43′) remain valid as long as the dividend yield is predictable.

[a] Merton (1973).

Continuous Dividend Yields (continued)

- To run binomial tree algorithms, replace u with $ue^{-q\Delta t}$ and d with $de^{-q\Delta t}$, where $\Delta t \stackrel{\Delta}{=} \tau/n$.
  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.
- Other than the changes, binomial tree algorithms stay the same.
  - In particular, p should use the *original* u and d![a]

---

[a]Contributed by Ms. Wang, Chuan-Ju (`F95922018`) on May 2, 2007.

Continuous Dividend Yields (concluded)

- Alternatively, pick the risk-neutral probability as

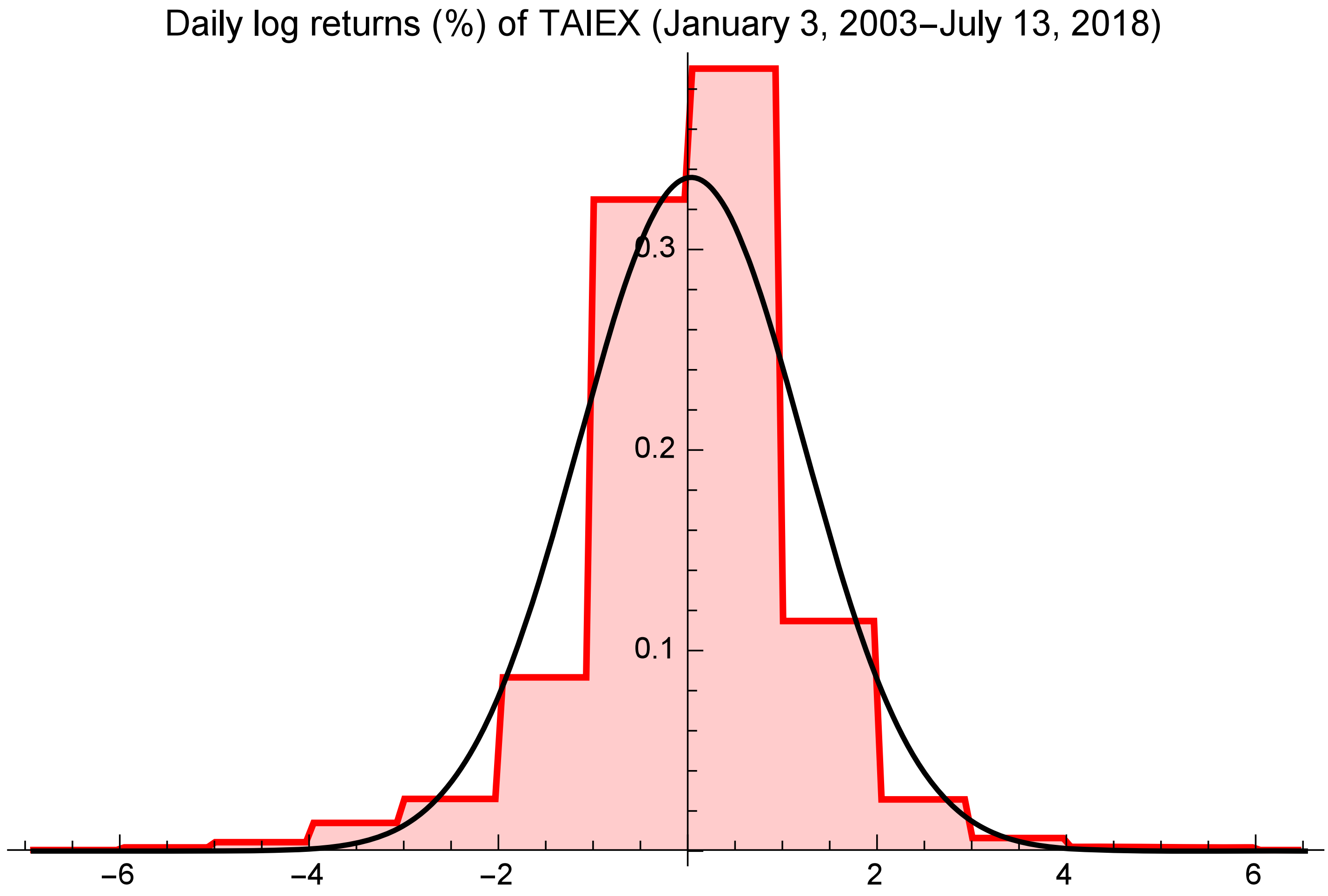
$$\frac{e^{(r-q)\,\Delta t} - d}{u - d}, \tag{44}$$

where $\Delta t \triangleq \tau/n$.

  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.

- The u and d remain unchanged.
- Other than the change in Eq. (44), binomial tree algorithms stay the same *as if there were no dividends*.

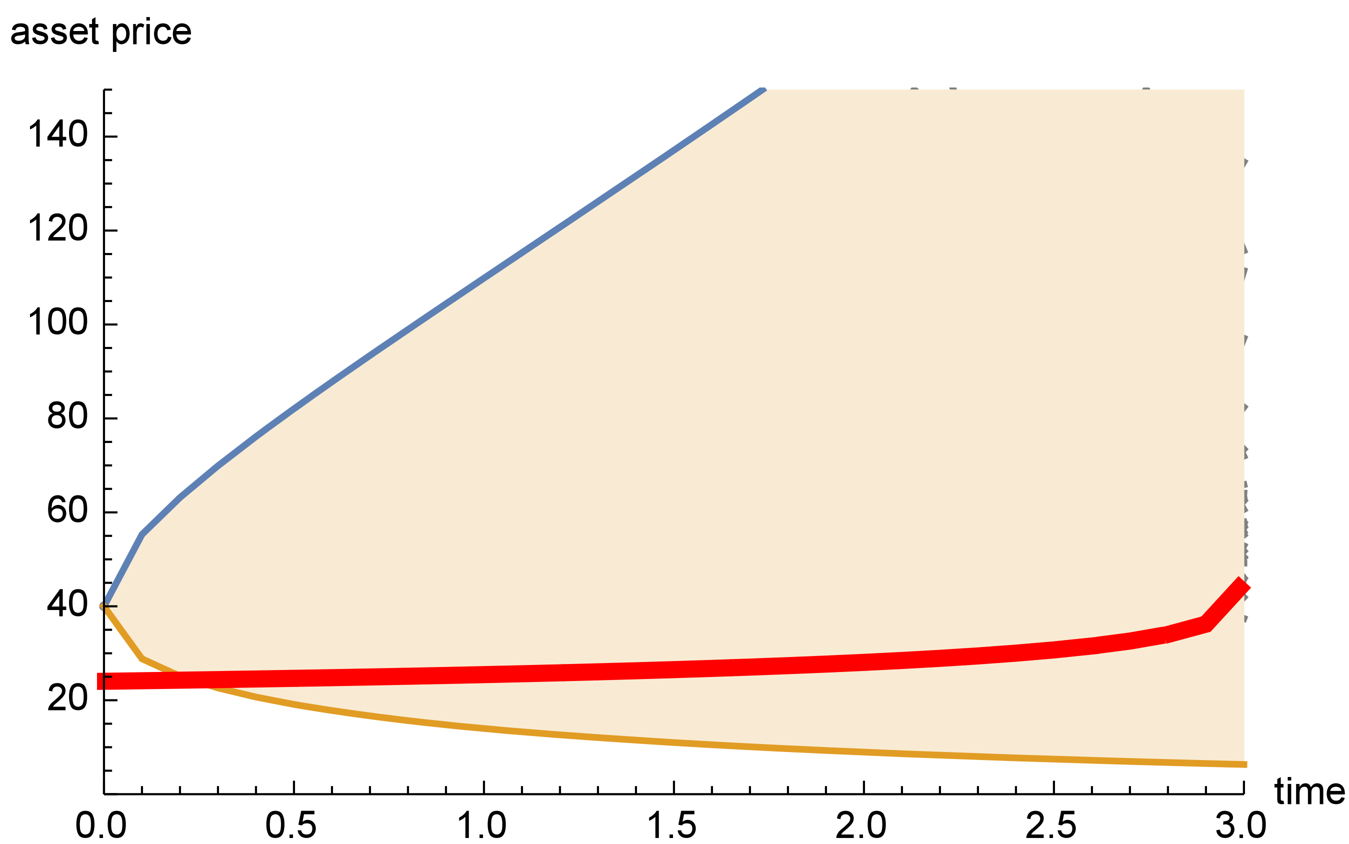
Distribution of Logarithmic Returns of TAIEX



Exercise Boundaries of American Options (in the Continuous-Time Model)[a]

- The exercise boundary is a nondecreasing function of t for American *puts* (see the plot next page).
- The exercise boundary is a nonincreasing function of t for American calls.

[a]See Section 9.7 of the textbook for the tree analog.


asset price
140
120
100
80
60
40
20
0.0
0.5
1.0
1.5
2.0
2.5
3.0
time

Sensitivity Analysis of Options

Cleopatra's nose, had it been shorter,
the whole face of the world
would have been changed.
— Blaise Pascal (1623–1662)

Sensitivity Measures ("The Greeks")

- How the value of a security changes relative to changes in a given parameter is key to hedging.
  - Duration, for instance.
- Let $x \stackrel{\Delta}{=} \frac{\ln(S/X)+(r+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$ (recall p. 299).
- Recall that

$$N'(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} > 0,$$

the density function of standard normal distribution.

Delta

- Defined as
$$\Delta \triangleq \frac{\partial f}{\partial S}.$$
  - f is the price of the derivative.
  - S is the price of the underlying asset.
- The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.[a]
- The delta used in the BOPM (p. 246) is the discrete analog.
- The delta of a long stock is apparently 1.

[a]Elementary calculus.

Delta (continued)

- The delta of a European call on a non-dividend-paying stock equals

$$\frac{\partial C}{\partial S} = N(x) > 0.$$

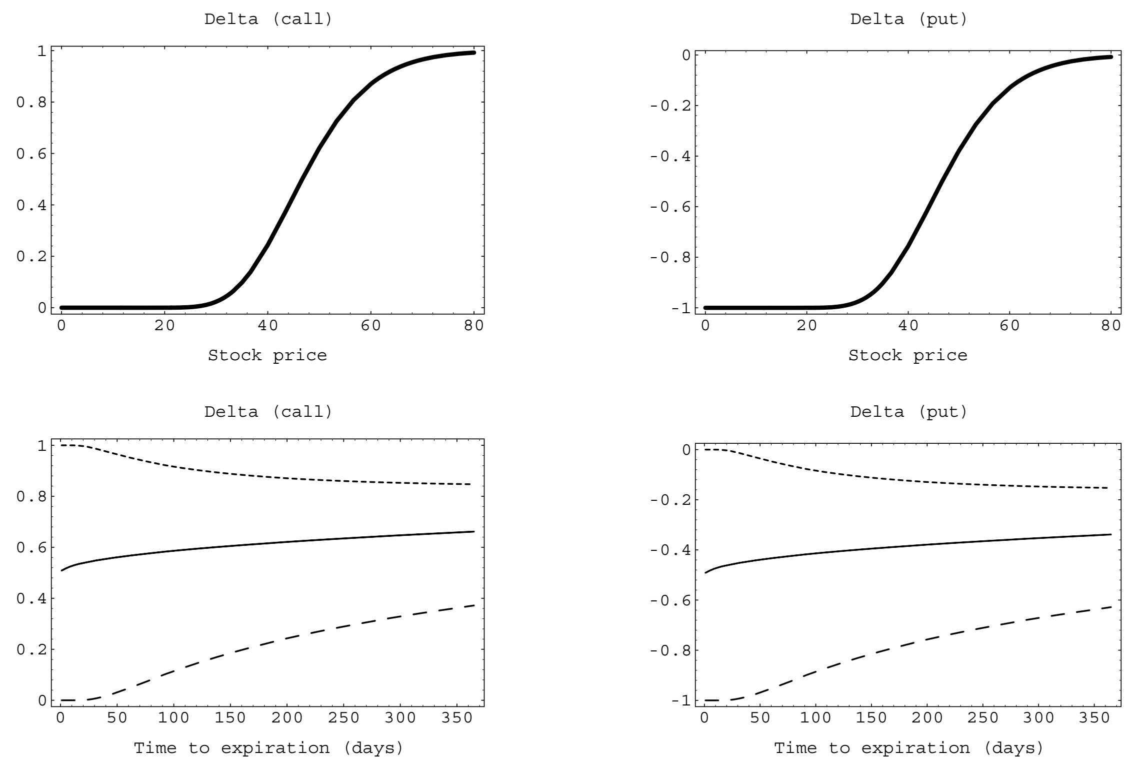
- The delta of a European put equals

$$\frac{\partial P}{\partial S} = N(x) - 1 = -N(-x) < 0.$$

- So the deltas of a call and an otherwise identical put cancel each other when $N(x) = 1/2$, i.e., when[a]

$$X = Se^{(r+\sigma^2/2)\,\tau}. \tag{45}$$

[a]The straddle (p. 213) $C + P$ then has zero delta!



Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money options.

Dashed curves: out-of-the-money calls or in-the-money puts.

Delta (continued)

- Suppose the stock pays a continuous dividend yield of q.
- Let
$$x \triangleq \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}} \tag{46}$$
(recall p. 329).
- Then
$$\begin{aligned} \frac{\partial C}{\partial S} &= e^{-q\tau} N(x) > 0, \\ \frac{\partial P}{\partial S} &= -e^{-q\tau} N(-x) < 0. \end{aligned}$$

Delta (continued)

- Consider an X_1-strike call and an X_2-strike put, $X_1 \geq X_2$.
- They are otherwise identical.
- Let

$$x_i \triangleq \frac{\ln(S/X_i) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}. \tag{47}$$

- Then their deltas sum to zero when $x_1 = -x_2$.[a]
- That implies

$$\frac{S}{X_1} = \frac{X_2}{S}\, e^{-(2r-2q+\sigma^2)\,\tau}. \tag{48}$$

[a]The strangle (p. 215) $C + P$ then has zero delta!

Delta (concluded)

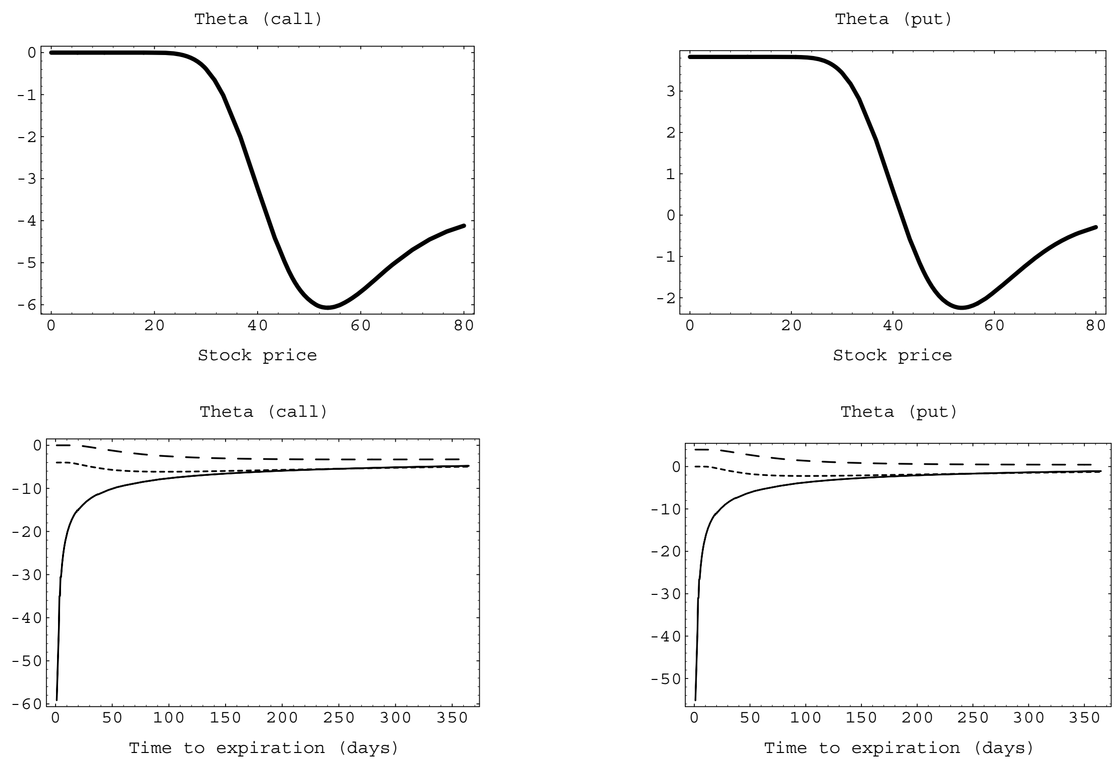
- Suppose we demand $X_1 = X_2 = X$ and have a straddle.
- Then
$$X = Se^{(r-q+\sigma^2/2)\,\tau}$$
leads to a straddle with zero delta.
  - This generalizes Eq. (45) on p. 339.
- When $C(X_1)$'s delta and $P(X_2)$'s delta sum to zero, does the portfolio $C(X_1) - P(X_2)$ have zero value?
- In general, no.

Delta Neutrality

- A position with a total delta equal to 0 is delta-neutral.
  - A delta-neutral portfolio is immune to *small* price changes in the underlying asset.
- Creating one serves for hedging purposes.
  - A portfolio consisting of a call and $-\Delta$ shares of stock is delta-neutral.
  - Short Δ shares of stock to hedge a long call.
  - Long Δ shares of stock to hedge a short call.
- In general, hedge a position in a security with delta Δ_1 by shorting Δ_1/Δ_2 units of a security with delta Δ_2.

Theta (Time Decay)

- Defined as the rate of change of a security's value with respect to time, or $\Theta \stackrel{\Delta}{=} -\partial f/\partial\tau = \partial f/\partial t$.
- For a European call on a non-dividend-paying stock,
$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(x-\sigma\sqrt{\tau}) < 0.$$
  - The call loses value with the passage of time.
- For a European put,
$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-x+\sigma\sqrt{\tau}).$$
  - Can be negative or positive.



Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money options.

Dashed curve: out-of-the-money call or in-the-money put.

Theta (concluded)

- Suppose the stock pays a continuous dividend yield of q.
- Define x as in Eq. (46) on p. 341.
- For a European call, add an extra term to the earlier formula for the theta:

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(x-\sigma\sqrt{\tau}) + qSe^{-q\tau}\,N(x).$$

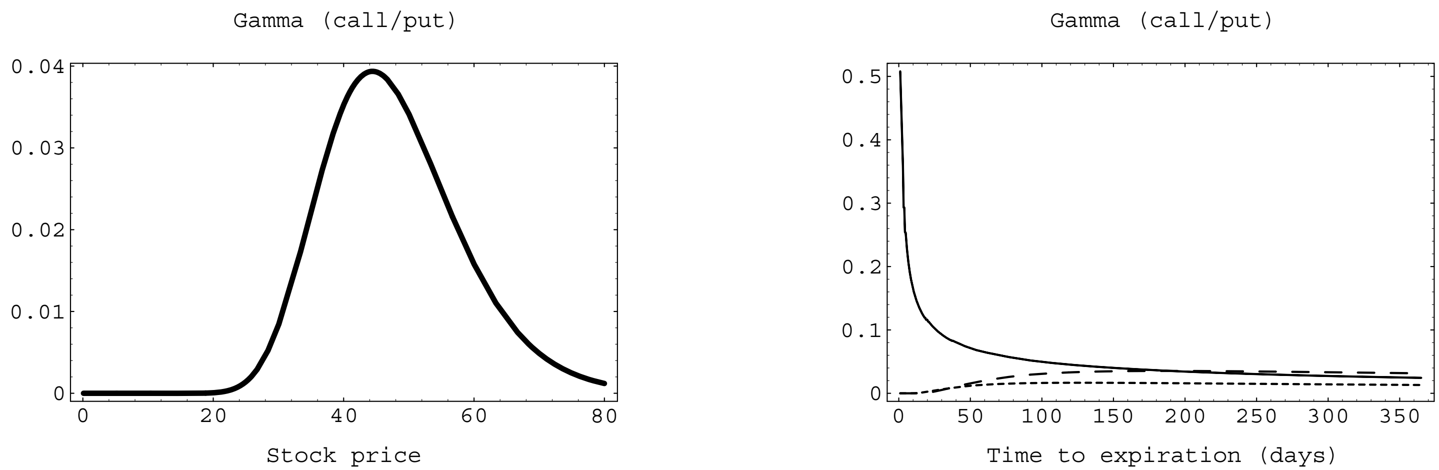
- For a European put, add an extra term to the earlier formula for the theta:

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-x+\sigma\sqrt{\tau}) - qSe^{-q\tau}\,N(-x).$$

Gamma

- Defined as the rate of change of its delta with respect to the price of the underlying asset, or $\Gamma \stackrel{\Delta}{=} \partial^2\Pi/\partial S^2$.
- Measures how sensitive delta is to changes in the price of the underlying asset.
- In practice, a portfolio with a high gamma needs be rebalanced more often to maintain delta neutrality.
- Roughly, delta $\sim$ duration, and gamma $\sim$ convexity.
- The gamma of a European call or put on a non-dividend-paying stock is

$$N'(x)/(S\sigma\sqrt{\tau}) > 0.$$



Dotted lines: in-the-money call or out-of-the-money put.

Solid lines: at-the-money option.

Dashed lines: out-of-the-money call or in-the-money put.

Vega[a] (Lambda, Kappa, Sigma)

- Defined as the rate of change of a security's value with respect to the volatility of the underlying asset

$$\Lambda \triangleq \frac{\partial f}{\partial \sigma}.$$

- Volatility often changes over time.
- A security with a high vega is very sensitive to small changes or estimation error in volatility.
- The vega of a European call or put on a non-dividend-paying stock is $S\sqrt{\tau}\, N'(x) > 0$.
  - So higher volatility always increases the option value.

[a] Vega is not Greek.

Vega (continued)

- Note that[a]

$$\Lambda = \tau \sigma S^2 \Gamma.$$

- If the stock pays a continuous dividend yield of q, then

$$\Lambda = S e^{-q\tau} \sqrt{\tau}\, N'(x),$$

where x is defined in Eq. (46) on p. 341.

- Vega is maximized when $x = 0$, i.e., when

$$S = X e^{-(r-q+\sigma^2/2)\,\tau}.$$

- Vega declines very fast as S moves away from that peak.

---

[a]Reiss & Wystup (2001).

Vega (continued)

- Now consider a portfolio consisting of an X_1-strike call C and a short X_2-strike put P, $X_1 \geq X_2$.
- The options' vegas cancel out when

$$x_1 = -x_2,$$

  where x_i are defined in Eq. (47) on p. 342.
- This also leads to Eq. (48) on p. 342.
  - Recall the same condition led to zero delta for the strangle $C + P$ (p. 342).

Vega (concluded)

- Note that if $S \neq X$, $\tau \to 0$ implies

$$\Lambda \to 0$$

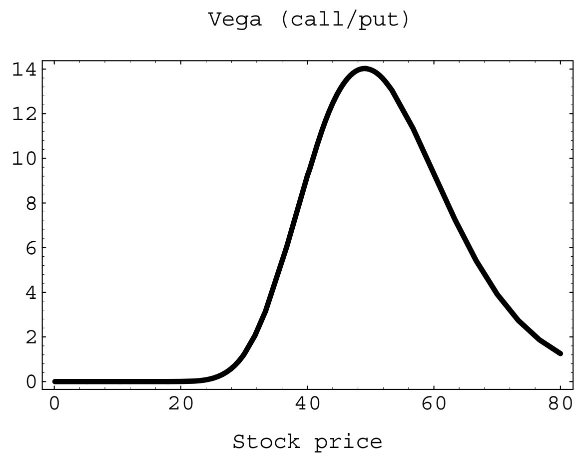
(which answers the question on p. 304 for the Black-Scholes model).

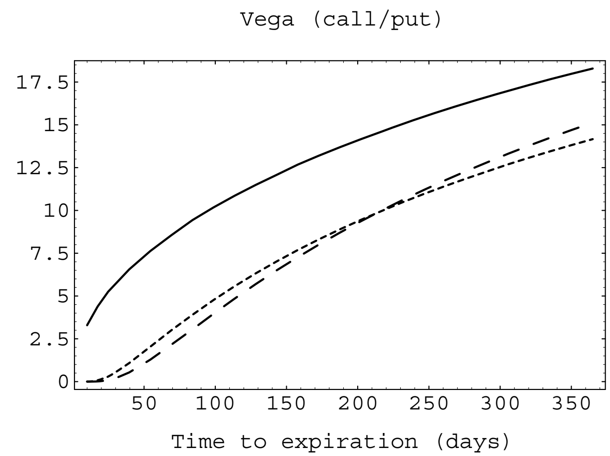
- The Black-Scholes formula (p. 299) implies

$$\begin{aligned} C &\to S, \\ P &\to Xe^{-r\tau}, \end{aligned}$$

as $\sigma \to \infty$.

- These boundary conditions may be handy for certain numerical methods.





Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money option.

Dashed curve: out-of-the-money call or in-the-money put.

Variance Vega[a]

- Defined as the rate of change of a security's value with respect to the variance (square of volatility) of the underlying asset

$$\text{variance vega} \triangleq \frac{\partial f}{\partial \sigma^2}.$$

  - Note that it is not defined as $\partial^2 f / \partial \sigma^2$!

- It is easy to verify that

$$\text{variance vega} = \frac{\Lambda}{2\sigma}.$$

[a]Demeterfi, Derman, Kamal, & Zou (1999).

Volga (Vomma, Volatility Gamma, Vega Convexity)

- Defined as the rate of change of a security's vega with respect to the volatility of the underlying asset

$$\text{volga} \stackrel{\Delta}{=} \frac{\partial \Lambda}{\partial \sigma} = \frac{\partial^2 f}{\partial \sigma^2}.$$

- It can be shown that

$$\begin{aligned} \text{volga} &= \Lambda \frac{x(x - \sigma\sqrt{\tau})}{\sigma} \\ &= \frac{\Lambda}{\sigma} \left[\frac{\ln^2(S/X)}{\sigma^2 \tau} - \frac{\sigma^2 \tau}{4} \right], \end{aligned}$$

where x is defined in Eq. (46) on p. 341.[a]

[a]Derman & M. B. Miller (2016).

Volga (concluded)

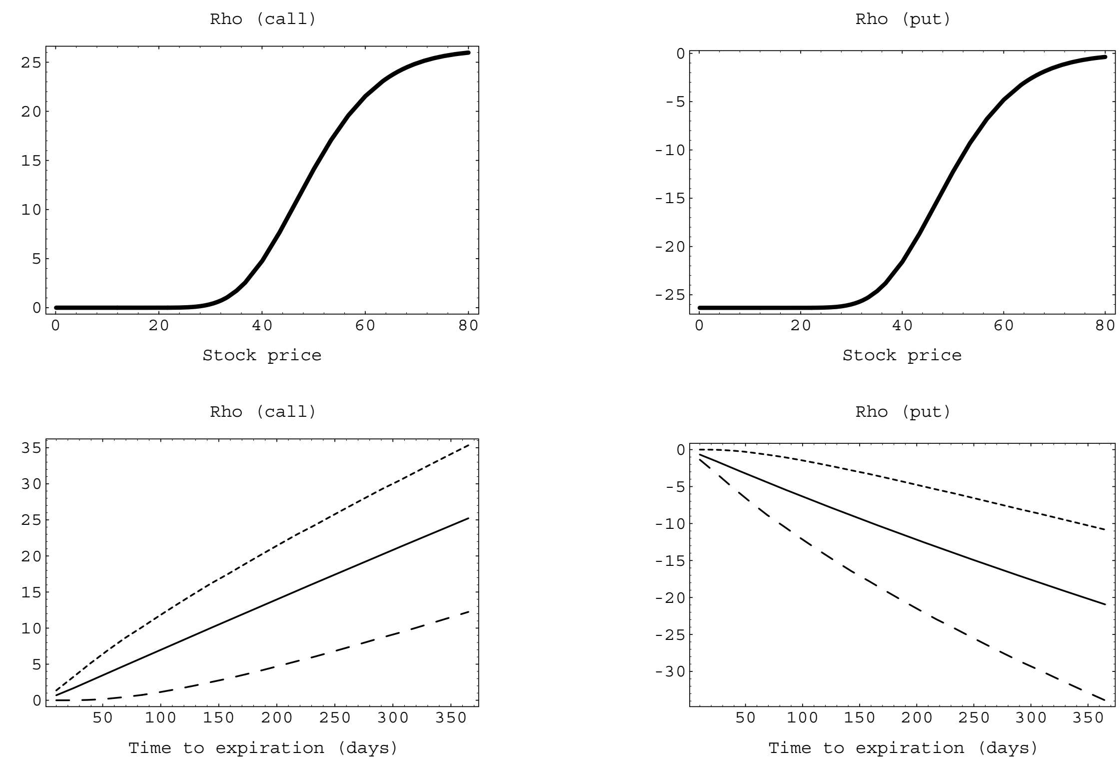
- Volga is zero when $S = Xe^{\pm\sigma^2\tau/2}$.
- For typical values of σ and τ, volga is positive except where $S \approx X$.
- Volga can be used to measure the 4th moment of the underlying asset and the smile of implied volatility at the same maturity.[a]

---

[a] Bennett (2014).

Rho

- Defined as the rate of change in its value with respect to interest rates
$$\rho \stackrel{\Delta}{=} \frac{\partial f}{\partial r}.$$
- The rho of a European call on a non-dividend-paying stock is
$$X\tau e^{-r\tau} N(x - \sigma\sqrt{\tau}) > 0.$$
- The rho of a European put on a non-dividend-paying stock is
$$-X\tau e^{-r\tau} N(-x + \sigma\sqrt{\tau}) < 0.$$



Dotted curves: in-the-money call or out-of-the-money put.

Solid curves: at-the-money option.

Dashed curves: out-of-the-money call or in-the-money put.

Numerical Greeks

- Needed when closed-form formulas do not exist.
- Take delta as an example.
- A standard method computes the finite difference,

$$\frac{f(S+\Delta S)-f(S-\Delta S)}{2\Delta S}.$$

- The computation time roughly doubles that for evaluating the derivative security itself.

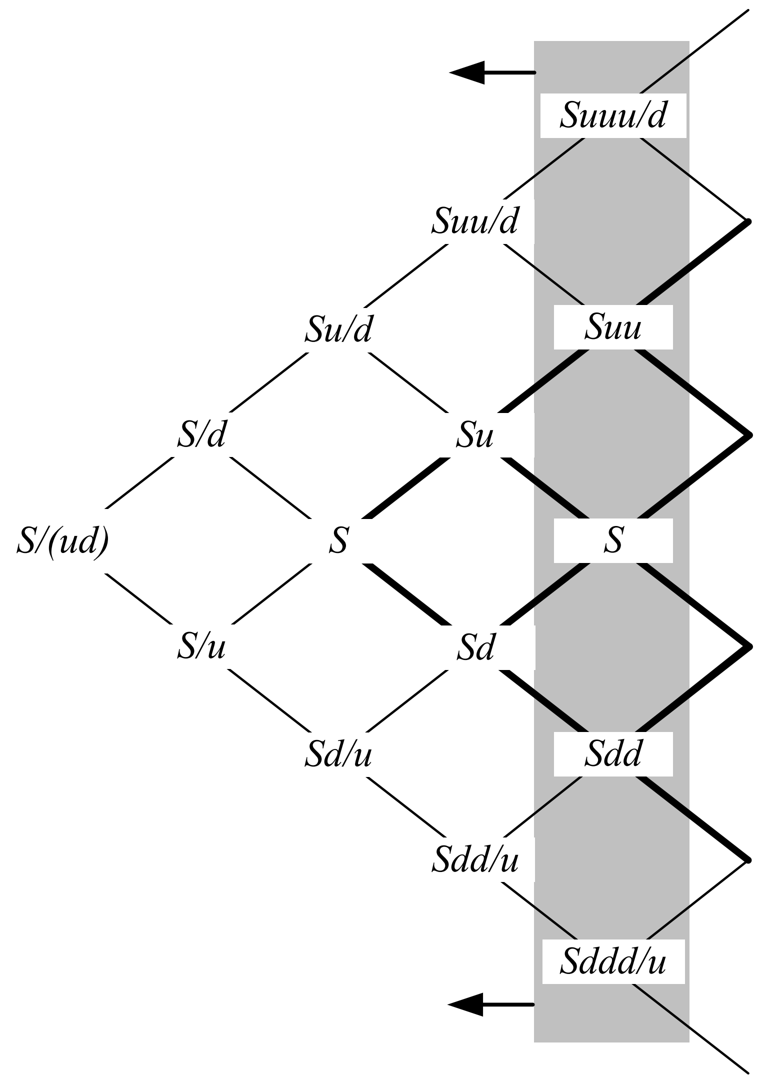
An Alternative Numerical Delta[a]

- Use intermediate results of the binomial tree algorithm.
- When the algorithm reaches the end of the first period, f_u and f_d are computed.
- These values correspond to derivative values at stock prices Su and Sd, respectively.
- Delta is approximated by

$$\frac{f_u - f_d}{Su - Sd}. \tag{49}$$

- Almost zero extra computational effort.

[a]Pelsser & Vorst (1994).


Suuu/d
Suu/d
Suu
Su/d
S/d
Su
S/(ud)
S
S
S/u
Sd
Sd/u
Sdd
Sdd/u
Sddd/u

Numerical Gamma

- At the stock price $(Suu + Sud)/2$, delta is approximately $(f_{uu} - f_{ud})/(Suu - Sud)$.
- At the stock price $(Sud + Sdd)/2$, delta is approximately $(f_{ud} - f_{dd})/(Sud - Sdd)$.
- Gamma is the rate of change in deltas between $(Suu + Sud)/2$ and $(Sud + Sdd)/2$, that is,

$$\frac{\frac{f_{uu}-f_{ud}}{Suu-Sud} - \frac{f_{ud}-f_{dd}}{Sud-Sdd}}{(Suu - Sdd)/2}. \tag{50}$$

Alternative Numerical Delta and Gamma[a]

- Let $\epsilon \equiv \ln u$.
- Think in terms of $\ln S$.
- Then
$$\left(\frac{f_u - f_d}{2\epsilon} \right) \frac{1}{S}$$
approximates the numerical delta.
- And
$$\left(\frac{f_{uu} - 2f_{ud} + f_{dd}}{\epsilon^2} - \frac{f_{uu} - f_{dd}}{2\epsilon} \right) \frac{1}{S^2}$$
approximates the numerical gamma.

[a]See p. 675.

Finite Difference Fails for Numerical Gamma

- Numerical differentiation gives

$$\frac{f(S+\Delta S)-2f(S)+f(S-\Delta S)}{(\Delta S)^2}.$$

- It does not work (see text for the reason).
- In general, calculating gamma is a hard problem numerically.
- But why did the binomial tree version work?

Other Numerical Greeks

- The theta can be computed as
$$\frac{f_{ud} - f}{2(\tau/n)}.$$
  - In fact, the theta of a European option can be derived from delta and gamma.[a]
- The vega of a European option can be derived from gamma.[b]
- For rho, there seems no alternative but to run the binomial tree algorithm twice.[c]

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[a] See p. 674.

[b] See p. 351.

[c] But see p. 856 and pp. 1049ff.

Extensions of Options Theory

As I never learnt mathematics,
so I have had to think.
— Joan Robinson (1903–1983)

Pricing Corporate Securities[a]

- Interpret the underlying asset as the firm's total value.[b]
- The option pricing methodology can be applied to price corporate securities.
- The result is called the structural model.
- Assumptions:
  - A firm can finance payouts by the sale of assets.
  - If a promised payment to an obligation other than stock is missed, the claim holders take ownership of the firm and the stockholders get nothing.

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[a] Black & Scholes (1973); Merton (1974).

[b] More realistic models posit firm value = asset value + tax benefits − bankruptcy costs (Leland & Toft, 1996).

Risky Zero-Coupon Bonds and Stock

- Consider XYZ.com.
- Capital structure:
  - n shares of its own common stock, S.
  - Zero-coupon bonds with an aggregate par value of X.
- What is the value of the bonds, B?
- What is the value of the XYZ.com stock?

Risky Zero-Coupon Bonds and Stock (continued)

- On the bonds' maturity date, suppose the total value of the firm V^* is less than the bondholders' claim X.
- Then the firm declares bankruptcy, and the stock becomes worthless.
- If $V^* > X$, then the bondholders obtain X and the stockholders $V^* - X$.

	$V^* \leq X$	$V^* > X$
Bonds	V^*	X
Stock	0	$V^* - X$

Risky Zero-Coupon Bonds and Stock (continued)

- The stock has the same payoff as a call!
- It is a call on the total value of the firm with a strike price of X and an expiration date equal to the bonds'.
  - This call provides the limited liability for the stockholders.
- The bonds are a covered call[a] on the total value of the firm.
- Let V stand for the total value of the firm.
- Let C stand for a call on V.

[a] Recall p. 202.

Risky Zero-Coupon Bonds and Stock (continued)

- Thus

$$\begin{aligned} nS &= C \,(\text{market capitalization of XYZ.com}), \\ B &= V - C. \end{aligned}$$

- Knowing C amounts to knowing how the value of the firm is divided between stockholders and bondholders.
- Whatever the value of C, the total value of the stock and bonds at maturity remains V^*.
- The relative size of debt and equity is irrelevant to the firm's current value V.

Risky Zero-Coupon Bonds and Stock (continued)

- From Theorem 10 (p. 299) and the put-call parity,[a]

$$nS = VN(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}), \tag{51}$$

$$B = VN(-x) + Xe^{-r\tau}N(x - \sigma\sqrt{\tau}). \tag{52}$$

  – Above,

$$x \triangleq \frac{\ln(V/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

- The continuously compounded yield to maturity of the firm's bond is

$$\frac{\ln(X/B)}{\tau}.$$

[a]This is sometimes called Merton's (1974) structural model.

Risky Zero-Coupon Bonds and Stock (continued)

- Define the credit spread or default premium as the yield difference between risky and riskless bonds,

$$\begin{aligned} &\frac{\ln(X/B)}{\tau} - r \\ = \quad & -\frac{1}{\tau}\ln\left(N(-z) + \frac{1}{\omega}\,N(z - \sigma\sqrt{\tau})\right). \end{aligned}$$

  - $\omega \stackrel{\Delta}{=} Xe^{-r\tau}/V$.
  - $z \stackrel{\Delta}{=} (\ln\omega)/(\sigma\sqrt{\tau}) + (1/2)\,\sigma\sqrt{\tau} = -x + \sigma\sqrt{\tau}$.
  - Note that ω is the debt-to-total-value ratio.

Risky Zero-Coupon Bonds and Stock (concluded)

- In general, suppose the firm has a dividend yield at rate q and the bankruptcy costs are a constant proportion α of the remaining firm value.
- Then Eqs. (51)–(52) on p. 374 become, respectively,

$$\begin{aligned} nS &= Ve^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\ B &= (1-\alpha) Ve^{-q\tau} N(-x) + Xe^{-r\tau} N(x - \sigma\sqrt{\tau}). \end{aligned}$$

  – Above,

$$x \triangleq \frac{\ln(V/X) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

A Numerical Example

- XYZ.com's assets consist of 1,000 shares of Merck as of March 20, 1995.
  - Merck's market value per share is \$44.5.
- It issues 1,000 shares of XYZ.com common stock and 30 zero-coupon bonds maturing on July 21, 1995.
- Each bond promises to pay \$1,000 at maturity.
- $n = 1,000$, $V = 44.5 \times n = 44,500$, and $X = 30 \times 1,000 = 30,000$.
- As Merck calls are being traded, we do not need option formulas to price them.

			—Call—		—Put—	
Option	Strike	Exp.	Vol.	Last	Vol.	Last
Merck	30	Jul	328	15 1/4	...	...
44 1/2	35	Jul	150	9 1/2	10	1/16
44 1/2	40	Apr	887	4 3/4	136	1/16
44 1/2	40	Jul	220	5 1/2	297	1/4
44 1/2	40	Oct	58	6	10	1/2
44 1/2	45	Apr	3050	7/8	100	1 1/8
44 1/2	45	May	462	1 3/8	50	1 3/8
44 1/2	45	Jul	883	1 15/16	147	1 3/4
44 1/2	45	Oct	367	2 3/4	188	2 1/16

A Numerical Example (continued)

- The Merck option relevant for pricing is the July call with a strike price of $X/n = 30$ dollars.
- Such a call is selling for \$15.25.
- So XYZ.com's stock is worth $15.25 \times n = 15,250$ dollars.
- The entire bond issue is worth

$$B = 44,500 - 15,250 = 29,250$$

  dollars.

  – Or \$975 per bond.

A Numerical Example (continued)

- The XYZ.com bonds are equivalent to a default-free zero-coupon bond with $\$X$ par value plus n written European puts on Merck at a strike price of \$30.
  - By the put-call parity.[a]
- The difference between B and the price of the default-free bond is the value of these puts.
- The next table shows the total market values of the XYZ.com stock and bonds under various debt amounts X.

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[a] See p. 228.

Promised payment to bondholders X	Current market value of bonds B	Current market value of stock nS	Current total value of firm V
30,000	29,250.0	15,250.0	44,500
35,000	35,000.0	9,500.0	44,500
40,000	39,000.0	5,500.0	44,500
45,000	42,562.5	1,937.5	44,500

A Numerical Example (continued)

- Suppose the promised payment to bondholders is \$45,000.
- Then the relevant option is the July call with a strike price of $45,000/n = 45$ dollars.
- Since that option is selling for \$1 15/16, the market value of the XYZ.com stock is $(1 + 15/16) \times n = 1,937.5$ dollars.
- The market value of the stock decreases as the debt-equity ratio increases.

A Numerical Example (continued)

- There are conflicts between stockholders and bondholders.
- An option's terms cannot be changed after issuance.
- But a firm can change its capital structure.
- There lies one key difference between options and corporate securities.
  - Parameters such volatility,[a] dividend, and strike price are under partial control of the stockholders or their boards.

[a]This is called the asset substitution problem (Myers, 1977).

A Numerical Example (continued)

- Suppose XYZ.com issues 15 more bonds with the same terms to buy back stock.
- The total debt is now $X = 45{,}000$ dollars.
- The table on p. 381 says the total market value of the bonds should be \$42,562.5.
- The *new* bondholders pay

$$42,562.5 \times (15/45) = 14,187.5$$

dollars.

- The remaining stock is worth \$1,937.5.

A Numerical Example (continued)

- The stockholders therefore gain

$$14,187.5 + 1,937.5 - 15,250 = 875$$

  dollars.

- The *original* bondholders lose an equal amount,

$$29,250 - \frac{30}{45} \times 42,562.5 = 875.$$

  - This is called claim dilution.[a]

[a] Fama & M. H. Miller (1972).

A Numerical Example (continued)

- Suppose the stockholders sell $(1/3) \times n$ Merck shares to fund a \$14,833.3 cash dividend.
- The stockholders now have \$14,833.3 in cash plus a call on $(2/3) \times n$ Merck shares.
- The strike price remains $X = 30,000$.
- This is equivalent to owning 2/3 of a call on n Merck shares with a strike price of \$45,000.
- n such calls are worth \$1,937.5 (p. 381).
- So the total market value of the XYZ.com stock is $(2/3) \times 1,937.5 = 1,291.67$ dollars.

A Numerical Example (concluded)

- The market value of the XYZ.com bonds is hence

$$(2/3) \times n \times 44.5 - 1,291.67 = 28,375$$

  dollars.

- Hence the stockholders gain

$$14,833.3 + 1,291.67 - 15,250 \approx 875$$

  dollars.

- The bondholders watch their value drop from \$29,250 to \$28,375, a loss of \$875.

Further Topics

- Other examples:[a]
  - Stock as compound call when company issues coupon bonds.
  - Subordinated debts as bull call spreads.
  - Warrants as calls.
  - Callable bonds as American calls with 2 strike prices.
  - Convertible bonds.
  - Bonds with safety covenants as barrier options.

[a] Cox & Rubinstein (1985); Geske (1977).

Further Topics (concluded)

- Securities issued by firms with a complex capital structure must be solved by trees.[a]

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[a] Dai (B82506025, R86526008, D8852600), Lyuu, & C. Wang (F95922018) (2010).

Distance to Default (DTD)[a]

- Let μ be the total value V's rate of expected return.
- From Eq. (51), on p. 374, the probability of default τ years from now equals
$$N(-\mathrm{DTD}),$$
where
$$\mathrm{DTD} \triangleq \frac{\ln(V/X) + (\mu - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$
- V/X is called the leverage ratio.

[a]Merton (1974).