Call on a Non-Dividend-Paying Stock: Single Period

- The expiration date is only one period from now.
- $C_u$ is the call price at time 1 if the stock price moves to $S_u$.
- $C_d$ is the call price at time 1 if the stock price moves to $S_d$.
- Clearly,
  
  $$C_u = \max(0, S_u - X),$$
  $$C_d = \max(0, S_d - X).$$
\[ C_u = \max(0, Su - X) \]

\[ C_d = \max(0, Sd - X) \]
Call on a Non-Dividend-Paying Stock: Single Period (continued)

- Set up a portfolio of $h$ shares of stock and $B$ dollars in riskless bonds.
  - This costs $hS + B$.
  - We call $h$ the hedge ratio or delta.

- The value of this portfolio at time one is
  
  \[
  hSu + RB, \quad \text{up move},
  \]
  
  \[
  hSd + RB, \quad \text{down move}.
  \]
Call on a Non-Dividend-Paying Stock: Single Period (continued)

- Choose $h$ and $B$ such that the portfolio replicates the payoff of the call,

\[
\begin{align*}
    hS_u + RB &= C_u, \\
    hS_d + RB &= C_d.
\end{align*}
\]
Call on a Non-Dividend-Paying Stock: Single Period (concluded)

- Solve the above equations to obtain

\[ h = \frac{C_u - C_d}{S u - S d} \geq 0, \quad (32) \]

\[ B = \frac{u C_d - d C_u}{(u - d) R}. \quad (33) \]

- By the no-arbitrage principle, the European call should cost the same as the equivalent portfolio,\(^a\)

\[ C = h S + B. \]

- As \( u C_d - d C_u < 0 \), the equivalent portfolio is a *levered* long position in stocks.

\(^a\)Or the replicating portfolio, as it replicates the option.
American Call Pricing in One Period

• Have to consider immediate exercise.

• $C = \max(hS + B, S - X)$.
  
  – When $hS + B \geq S - X$, the call should not be exercised immediately.
  
  – When $hS + B < S - X$, the option should be exercised immediately.

• For non-dividend-paying stocks, early exercise is not optimal by Theorem 5 (p. 232).

• So

\[
C = hS + B.
\]
Put Pricing in One Period

- Puts can be similarly priced.
- The delta for the put is \( \frac{(P_u - P_d)}{(S_u - S_d)} \leq 0 \), where
  \[
  P_u = \max(0, X - S_u),
  \]
  \[
  P_d = \max(0, X - S_d).
  \]
- Let \( B = \frac{uP_d - dP_u}{(u-d)R} \).
- The European put is worth \( hS + B \).
- The American put is worth \( \max(hS + B, X - S) \).
  - Early exercise is possible with American puts.
Risk

• Surprisingly, the option value is independent of $q$.\(^a\)

• Hence it is independent of the expected value of the stock,

$$qS_u + (1 - q) S_d.$$  

• The option value depends on the sizes of price changes, $u$ and $d$, which the investors must agree upon.

• Then the set of possible stock prices is the same whatever $q$ is.

\(^a\)More precisely, not directly dependent on $q$. Thanks to a lively class discussion on March 16, 2011.
Pseudo Probability

- After substitution and rearrangement,

$$hS + B = \frac{\left(\frac{R-d}{u-d}\right) C_u + \left(\frac{u-R}{u-d}\right) C_d}{R}.$$ 

- Rewrite it as

$$hS + B = \frac{pC_u + (1 - p) C_d}{R},$$

where

$$p \triangleq \frac{R - d}{u - d}.$$  \hspace{1cm} (34)

- As $0 < p < 1$, it may be interpreted as a probability.
Risk-Neutral Probability

- The expected rate of return for the stock is equal to the riskless rate $\hat{r}$ under $p$ as
  \[ pS_u + (1 - p) S_d = R_S. \]  
  \[ (35) \]

- The expected rates of return of all securities must be the riskless rate when investors are risk-neutral.

- For this reason, $p$ is called the risk-neutral probability.

- The value of an option is the expectation of its discounted future payoff in a risk-neutral economy.

- So the rate used for discounting the FV is the riskless rate\(^a\) in a risk-neutral economy.

\(^a\)Recall the question on p. 238.
Option on a Non-Dividend-Paying Stock: Multi-Period

- Consider a call with two periods remaining before expiration.

- Under the binomial model, the stock can take on 3 possible prices at time two: $S_{uu}$, $S_{ud}$, and $S_{dd}$.
  - There are 4 paths.
  - But the tree combines or recombines; hence there are only 3 terminal prices.

- At any node, the next two stock prices only depend on the current price, not the prices of earlier times.$^a$

---

$^a$It is Markovian.
Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- Let $C_{uu}$ be the call’s value at time two if the stock price is $S_{uu}$.
- Thus,
  \[ C_{uu} = \max(0, S_{uu} - X). \]
- $C_{ud}$ and $C_{dd}$ can be calculated analogously,
  \[ C_{ud} = \max(0, S_{ud} - X), \]
  \[ C_{dd} = \max(0, S_{dd} - X). \]
$C_{uu} = \max(0, S_{uu} - X)$

$C_{ud} = \max(0, S_{ud} - X)$

$C_{dd} = \max(0, S_{dd} - X)$
Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- The call values at time 1 can be obtained by applying the same logic:

\[
C_u = \frac{pC_{uu} + (1 - p)C_{ud}}{R},
\]

\[
C_d = \frac{pC_{ud} + (1 - p)C_{dd}}{R}.
\]

- Deltas can be derived from Eq. (32) on p. 246.

- For example, the delta at \( C_u \) is

\[
\frac{C_{uu} - C_{ud}}{S_{uu} - S_{ud}}.
\]
Option on a Non-Dividend-Paying Stock: Multi-Period (concluded)

- We now reach the current period.
- Compute

\[
\frac{pC_u + (1 - p)C_d}{R}
\]

as the option price.
- The values of delta \( h \) and \( B \) can be derived from Eqs. (32)–(33) on p. 246.
Early Exercise

- Since the call will not be exercised at time 1 even if it is American, $C_u \geq Su - X$ and $C_d \geq Sd - X$.

- Therefore,

$$hS + B = \frac{pC_u + (1 - p)C_d}{R} \geq \frac{[pu + (1 - p)d]}{R} S - X$$

$$= S - \frac{X}{R} > S - X.$$  

- The call again will not be exercised at present.\(^a\)

- So

$$C = hS + B = \frac{pC_u + (1 - p)C_d}{R}.$$  

\(^a\)Consistent with Theorem 5 (p. 232).
Backward Induction

• The above expression calculates $C$ from the two successor nodes $C_u$ and $C_d$ and none beyond.
• The same computation happened at $C_u$ and $C_d$, too, as demonstrated in Eq. (36) on p. 256.
• This recursive procedure is called backward induction.
• $C$ equals

\[
[p^2 C_{uu} + 2p(1-p)C_{ud} + (1-p)^2 C_{dd}](1/R^2)
= [p^2 \max (0, S_u^2 - X) + 2p(1-p) \max (0, S_{ud} - X) \\
+ (1-p)^2 \max (0, S_d^2 - X)]/R^2.
\]

\(^{a}\)Ernst Zermelo (1871–1953).
\[ S_0 u^2 \]
\[ p^2 \]
\[ S_0 u \]
\[ p \]
\[ S_0 ud \]
\[ 2p(1 - p) \]
\[ S_0 d \]
\[ 1 - p \]
\[ S_0 d^2 \]
\[ (1 - p)^2 \]

\[ S_0 \]
\[ 1 \]
Backward Induction (continued)

• In the $n$-period case,

$$C = \sum_{j=0}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \times \max \left( 0, S u^j d^{n-j} - X \right) \frac{R^n}{R^n}. \tag{1}$$

The value of a call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.

• Similarly,

$$P = \sum_{j=0}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \times \max \left( 0, X - S u^j d^{n-j} \right) \frac{R^n}{R^n}. \tag{2}$$
Backward Induction (concluded)

• Note that

\[ p_j \triangleq \frac{\binom{n}{j} p^j (1 - p)^{n-j}}{R^n} \]

is the state price\(^a\) for the state \(Su^j d^{n-j}\), \(j = 0, 1, \ldots, n\).

• In general,

\[ \text{option price} = \sum_j (p_j \times \text{payoff at state } j). \]

\(^a\)Recall p. 210. One can obtain the undiscounted state price \(\binom{n}{j} p^j (1 - p)^{n-j}\)—the risk-neutral probability—for the state \(Su^j d^{n-j}\) with \((X_M - X_L)^{-1}\) units of the butterfly spread where \(X_L = Su^{j-1} d^{n-j+1}\), \(X_M = Su^j d^{n-j}\), and \(X_H = Su^{j-1+1} d^{n-j-1}\). See Bahra (1997).
Risk-Neutral Pricing Methodology

- Every derivative can be priced as if the economy were risk-neutral.

- For a European-style derivative with the terminal payoff function $D$, its value is

$$e^{-r^n}E^\pi[D].$$

(37)

- $E^\pi$ means the expectation is taken under the risk-neutral probability.

- The “equivalence” between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) fundamental theorem of asset pricing.\(^a\)

---

\(^a\)Dybvig & Ross (1987).
Self-Financing

- Delta changes over time.
- The maintenance of an equivalent portfolio is dynamic.
- But it does not depend on predicting future stock prices.
- The portfolio’s value at the end of the current period is precisely the amount needed to set up the next portfolio.
- The trading strategy is self-financing because there is neither injection nor withdrawal of funds throughout.\(^\text{a}\)
  - Changes in value are due entirely to capital gains.

\(^\text{a}\)Except at the beginning, of course, when you have to put up the option value \(C\) or \(P\) before the replication starts.
Binomial Distribution

- Denote the binomial distribution with parameters $n$ and $p$ by
  \[
  b(j; n, p) \triangleq \binom{n}{j} p^j (1 - p)^{n-j} = \frac{n!}{j! (n-j)!} p^j (1 - p)^{n-j}.
  \]
  - $n! = 1 \times 2 \times \cdots \times n$.
  - Convention: $0! = 1$.

- Suppose you flip a coin $n$ times with $p$ being the probability of getting heads.

- Then $b(j; n, p)$ is the probability of getting $j$ heads.
The Binomial Option Pricing Formula

• The stock prices at time $n$ are
  
  $S_u^n, S_u^{n-1}d, \ldots, S_d^n$.

• Let $a$ be the minimum number of upward price moves for the call to finish in the money.

• So $a$ is the smallest nonnegative integer $j$ such that
  
  $S_u^j d^{n-j} \geq X$,

  or, equivalently,

  $$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil.$$
The Binomial Option Pricing Formula (concluded)

• Hence,

\[
C = \frac{\sum_{j=a}^{n} \binom{n}{j} p^j (1 - p)^{n-j} (S u^j d^{n-j} - X)}{R^n}
\]

\[
= S \sum_{j=a}^{n} \binom{n}{j} (pu)^j [(1 - p) d]^{n-j} \frac{1}{R^n}
\]

\[
- \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1 - p)^{n-j}
\]

\[
= S \sum_{j=a}^{n} b(j; n, pu/R) - X e^{-\hat{r}n} \sum_{j=a}^{n} b(j; n, p). \quad (39)
\]
Numerical Examples

• A non-dividend-paying stock is selling for $160.
• $u = 1.5$ and $d = 0.5$.
• $r = 18.232\%$ per period ($R = e^{0.18232} = 1.2$).
  - Hence $p = (R - d)/(u - d) = 0.7$.
• Consider a European call on this stock with $X = 150$ and $n = 3$.
• The call value is $85.069$ by backward induction.
• Or, the PV of the expected payoff at expiration:
  \[
  \frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.
  \]
Binomial process for the stock price  
(probabilities in parentheses)

Binomial process for the call price  
(hedge ratios in parentheses)
Numerical Examples (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for $90 instead.
- Sell the call for $90.
- Invest $85.069 in the replicating portfolio with 0.82031 shares of stock as required by the delta.
- Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.
- The fund that remains,

$$90 - 85.069 = 4.931$$ dollars,

is the arbitrage profit, as we will see.
Numerical Examples (continued)

Time 1:

• Suppose the stock price moves to $240.

• The new delta is 0.90625.

• Buy

\[0.90625 - 0.82031 = 0.08594\]

more shares at the cost of \(0.08594 \times 240 = 20.6256\) dollars financed by borrowing.

• Debt now totals \(20.6256 + 46.1806 \times 1.2 = 76.04232\) dollars.
Numerical Examples (continued)

- The trading strategy is self-financing because the portfolio has a value of

  \[0.90625 \times 240 - 76.04232 = 141.45768.\]

- It matches the corresponding call value by backward induction!\(^a\)

\(^a\)See p. 269.
Numerical Examples (continued)

Time 2:

- Suppose the stock price plunges to $120.
- The new delta is 0.25.
- Sell $0.90625 - 0.25 = 0.65625$ shares.
- This generates an income of $0.65625 \times 120 = 78.75$ dollars.
- Use this income to reduce the debt to $76.04232 \times 1.2 - 78.75 = 12.5$ dollars.
Numerical Examples (continued)

Time 3 (the case of rising price):

- The stock price moves to $180.
- The call we wrote finishes in the money.
- Close out the call’s short position by buying back the call or buying a share of stock for delivery.
- This results in a loss of $180 − $150 = 30 dollars.
- Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.
- It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.
Numerical Examples (concluded)

Time 3 (the case of declining price):

- The stock price moves to $60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of

\[0.25 \times 60 = 15\]

dollars.
- Use it to repay the debt of \[12.5 \times 1.2 = 15\] dollars.
Applications besides Exploiting Arbitrage Opportunities\textsuperscript{a}

- Replicate an option using stocks and bonds.
  - Set up a portfolio to replicate the call with $85.069.

- Hedge the options we issued.
  - Use $85.069 to set up a portfolio to replicate the call to counterbalance its values exactly.\textsuperscript{b}

- ... 

- Without hedge, one may end up forking out $390 in the worst case (see p. 269)!\textsuperscript{c}

\textsuperscript{a} Thanks to a lively class discussion on March 16, 2011.

\textsuperscript{b} Hedging and replication are mirror images.

\textsuperscript{c} Thanks to a lively class discussion on March 16, 2016.
Binomial Tree Algorithms for European Options

- The BOPM implies the binomial tree algorithm that applies backward induction.

- The total running time is $O(n^2)$ because there are $\sim n^2/2$ nodes.

- The memory requirement is $O(n^2)$.
  - Can be easily reduced to $O(n)$ by reusing space.\(^a\)

- To find the hedge ratio, apply formula (32) on p. 246.

- To price European puts, simply replace the payoff.

\(^a\)But watch out for the proper updating of array entries.
Further Time Improvement for Calls
Optimal Algorithm

• We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.

• Note that

$$b(j; n, p) = \frac{p(n - j + 1)}{(1 - p) j} b(j - 1; n, p).$$
Optimal Algorithm (continued)

• The following program computes $b(j; n, p)$ in $b[j]$:

• It runs in $O(n)$ steps.

1: $b[a] := \binom{n}{a} p^a (1 - p)^{n-a}$;
2: for $j = a + 1, a + 2, \ldots, n$ do
3: $b[j] := b[j - 1] \times p \times (n - j + 1)/((1 - p) \times j)$;
4: end for
Optimal Algorithm (concluded)

- With the $b(j; n, p)$ available, the risk-neutral valuation formula (38) on p. 267 is trivial to compute.
- But we only need a single variable to store the $b(j; n, p)$s as they are being sequentially computed.
- This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.
- The above technique cannot be applied to American options because of early exercise.
- So binomial tree algorithms for American options usually run in $O(n^2)$ time.
Toward the Black-Scholes Formula

- The binomial model seems to suffer from two unrealistic assumptions.
  - The stock price takes on only two values in a period.
  - Trading occurs at discrete points in time.

- As $n$ increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.\(^a\)

- Need to calibrate the BOPM’s parameters $u$, $d$, and $R$ to make it converge to the continuous-time model.

- We now skim through the proof.

\(^a\)Continuous-time trading may create arbitrage opportunities in practice (Budish, Cramton, & Shim, 2015)!
Toward the Black-Scholes Formula (continued)

- Let $\tau$ denote the time to expiration of the option measured in years.
- Let $r$ be the continuously compounded annual rate.
- With $n$ periods during the option’s life, each period represents a time interval of $\tau/n$.
- Need to adjust the period-based $u$, $d$, and interest rate $\hat{r}$ to match the empirical results as $n \to \infty$. 
Toward the Black-Scholes Formula (continued)

• First, $\hat{r} = r\tau/n$.
  - Each period is $\Delta t \equiv \tau/n$ years long.
  - The period gross return $R = e^{\hat{r}}$.

• Let
  
  \[ \hat{\mu} \triangleq \frac{1}{n} E \left[ \ln \frac{S_\tau}{S} \right] \]

  denote the expected value of the continuously compounded rate of return per period of the BOPM.

• Let
  
  \[ \hat{\sigma}^2 \triangleq \frac{1}{n} \text{Var} \left[ \ln \frac{S_\tau}{S} \right] \]

  denote the variance of that return.
Toward the Black-Scholes Formula (continued)

• Under the BOPM, it is not hard to show that

$$\hat{\mu} = q \ln(u/d) + \ln d,$$
$$\hat{\sigma}^2 = q(1 - q) \ln^2(u/d).$$

• Assume the stock’s true continuously compounded rate of return over $\tau$ years has mean $\mu\tau$ and variance $\sigma^2\tau$.

• Call $\sigma$ the stock’s (annualized) volatility.

---

^aRecall the Bernoulli distribution.
Toward the Black-Scholes Formula (continued)

• The BOPM converges to the distribution only if

\[
\hat{n}\mu = n[q \ln(u/d) + \ln d] \to \mu \tau, \quad (40)
\]

\[
\hat{n}\sigma^2 = nq(1 - q) \ln^2(u/d) \to \sigma^2 \tau. \quad (41)
\]

• We need one more condition to have a solution for \(u, d, q\).
Toward the Black-Scholes Formula (continued)

• Impose

\[ ud = 1. \]

− It makes nodes at the same horizontal level of the tree have identical price (review p. 279).
− Other choices are possible (see text).

• Exact solutions for \( u, d, q \) are feasible if Eqs. (40)–(41) are replaced by equations: 3 equations for 3 variables.\(^a\)

\(^a\)Chance (2008).
Toward the Black-Scholes Formula (continued)

- The above requirements can be satisfied by

\[ u = e^{\sigma \sqrt{\Delta t}}, \quad d = e^{-\sigma \sqrt{\Delta t}}, \quad q = \frac{1}{2} + \frac{1}{2} \frac{\mu}{\sigma} \sqrt{\Delta t}. \] (42)

- With Eqs. (42), it can be checked that

\[
\begin{align*}
\hat{n}\mu &= \mu \tau, \\
\hat{n}\sigma^2 &= \left[ 1 - \left( \frac{\mu}{\sigma} \right)^2 \Delta t \right] \sigma^2 \tau \rightarrow \sigma^2 \tau.
\end{align*}
\]
Toward the Black-Scholes Formula (continued)

• The choices (42) result in the CRR binomial model.\textsuperscript{a}
  – Black (1992), “This method is probably used more than the original formula in practical situations.”

• With the above choice, even if $\sigma$ is not calibrated correctly, the mean is still matched!\textsuperscript{b}

• The CRR model is best seen in logarithmic price:

$$\ln S \to \begin{cases} 
\ln S + \sigma \sqrt{\Delta t}, & \text{up move}, \\
\ln S - \sigma \sqrt{\Delta t}, & \text{down move}.
\end{cases}$$

\textsuperscript{a}Cox, Ross, & Rubinstein (1979).

\textsuperscript{b}Recall Eq. (35) on p. 251. So $u$ and $d$ are related to volatility exclusively in the CRR model. They do not depend on $r$ or $\mu$. 
Toward the Black-Scholes Formula (continued)

• The no-arbitrage inequalities \( d < R < u \) may not hold under Eqs. (42) on p. 290 or Eq. (34) on p. 250.
  – If this happens, the probabilities lie outside \([0, 1]\).\(^a\)

• The problem disappears when \( n \) satisfies

\[
e^{\sigma \sqrt{\Delta t}} > e^{r \Delta t},
\]

i.e., when \( n > r^2 \tau / \sigma^2 \) (check it).
  – So it goes away if \( n \) is large enough.
  – Other solutions can be found in the textbook\(^b\) or will be presented later.

\(^a\)Many papers and programs forget to check this condition!
\(^b\)See Exercise 9.3.1 of the textbook.
Toward the Black-Scholes Formula (continued)

- The central limit theorem says $\ln(S_\tau/S)$ converges to $N(\mu\tau, \sigma^2\tau)$.

- So $\ln S_\tau$ approaches $N(\mu\tau + \ln S, \sigma^2\tau)$.

- Conclusion: $S_\tau$ has a lognormal distribution in the limit.

---

The normal distribution with mean $\mu\tau$ and variance $\sigma^2\tau$. 
Toward the Black-Scholes Formula (continued)

**Lemma 9** The continuously compounded rate of return 
\[ \ln(S_\tau/S) \] approaches the normal distribution with mean \((r - \sigma^2/2)\tau\) and variance \(\sigma^2\tau\) in a risk-neutral economy.

- Let \( q \) equal the risk-neutral probability
  \[ p \overset{\Delta}{=} \frac{(e^{r\tau/n} - d)}{(u - d)}. \]

- Let \( n \to \infty \).

- Then \( \mu = r - \sigma^2/2 \).

\( ^a\text{See Lemma 9.3.3 of the textbook.} \)
Toward the Black-Scholes Formula (continued)

• The expected stock price at expiration in a risk-neutral economy is \( S e^{r\tau} \).

• The stock’s expected annual rate of return \(^b\) is thus the riskless rate \( r \).

---

\(^a\)By Lemma 9 (p. 295) and Eq. (29) on p. 180.

\(^b\)In the sense of \((1/\tau) \ln E[S_{\tau}/S]\) (arithmetic average rate of return) not \((1/\tau)E[\ln(S_{\tau}/S)]\) (geometric average rate of return). In the latter case, it would be \( r - \sigma^2/2 \) by Lemma 9.
Toward the Black-Scholes Formula (continued)\(^a\)

Theorem 10 (The Black-Scholes Formula, 1973)

\[
C = SN(x) - X e^{-r \tau} N(x - \sigma \sqrt{\tau}), \\
P = X e^{-r \tau} N(-x + \sigma \sqrt{\tau}) - SN(-x),
\]

where

\[
x \triangleq \frac{\ln(S/X) + (r + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}.
\]

\(^a\)On a United flight from San Francisco to Tokyo on March 7, 2010, a real-estate manager mentioned this formula to me!
Toward the Black-Scholes Formula (concluded)

• See Eq. (39) on p. 267 for the meaning of $x$.

• See Exercise 13.2.12 of the textbook for an interpretation of the probability associated with $N(x)$ and $N(-x)$. 
BOPM and Black-Scholes Model

• The Black-Scholes formula needs 5 parameters: \( S, X, \sigma, \tau, \) and \( r. \)

• Binomial tree algorithms take 6 inputs: \( S, X, u, d, \hat{r}, \) and \( n. \)

• The connections are

\[
\begin{align*}
  u &= e^{\sigma \sqrt{\frac{\tau}{n}}}, \\
  d &= e^{-\sigma \sqrt{\frac{\tau}{n}}}, \\
  \hat{r} &= \frac{r \tau}{n}.
\end{align*}
\]
- $S = 100$, $X = 100$ (left), and $X = 95$ (right).
BOPM and Black-Scholes Model (concluded)

- The binomial tree algorithms converge reasonably fast.
- The error is $O(1/n)$.$^a$
- Oscillations are inherent, however.
- Oscillations can be dealt with by the judicious choices of $u$ and $d$.\textsuperscript{b}

\textsuperscript{a}L. Chang & Palmer (2007).
\textsuperscript{b}See Exercise 9.3.8 of the textbook.
Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market’s opinion of the volatility.\(^a\)
  - Solve for \(\sigma\) given the option price, \(S\), \(X\), \(\tau\), and \(r\) with numerical methods.
  - How about American options?\(^b\)

---

\(^a\)Implied volatility is hard to compute when \(\tau\) is small (why?).
Implied Volatility (concluded)

• Implied volatility is the wrong number to put in the wrong formula to get the right price of plain-vanilla options.\(^a\)

• Just think of it as an alternative to quoting option prices.

• Implied volatility is often preferred to historical volatility in practice.
  - Using the historical volatility is like driving a car with your eyes on the rearview mirror?\(^b\)

\(^a\)Rebonato (2004).
\(^b\)E.g., 1:16:23 of [https://www.youtube.com/watch?v=8TJQhQ2GZ0Y](https://www.youtube.com/watch?v=8TJQhQ2GZ0Y)